Catacaustics, caustics, curves of direction and orthogonal tangent transformation.
- Part XIII -

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Abstract

We give some properties of a tangential axial transformation used by d'Ocagne and Cesaro and others : the orthogonal tangents Transformation (OTT). It is a simple tool to study caustics by reflection for parallele rays of light. We present two associated transformations bisector and anti-bisector and apply these to some examples : parabolas/hyperbolas and pursuit curves. We expose the link with the curves of direction which have a special interesting property : they are algebraic with a rational expression of the arc length.

1 Catacaustics and Caustic by reflection for parllele rays of light.

Tschirnhausen defined in 1982 the caustic of a plane curve : the parabola. We will see that these caustics have, when algebraic, fascinating properties and gave a few examples that can be easily studied. An important part of this paper is inspired by the "Nouvelles annales of mathematics" (1880-1920) and Wieleitner's "Spezielle eben Kurve" of 1908.

2 The orthogonal tangents transformation (OTT):

2.1 Definition of orthogonal tangents transformation

It is a tangential transformation that makes the correspondance between lines in the plane and we apply this to find relations between curves defined as the envelope of their tangents. The x-axis is the axis of the transformation which maps a line L to a line L' of the euclidean plane in the following way :

1 - The point of intersection of L and L' lies on the x-axis.

2 - The two lines are orthogonal so :

$$\boxed{\theta \text{ - } \theta' = \pm\pi/2}$$

$$\boxed{\tan\theta \ . \ \tan\theta' = -1}$$

Here $\tan\theta$ is the slope of the first line $\tan\theta'$ the one of the second line.

Nota : Axial transformations are generally defined using semi-lines. These are oriented lines, as in Laguerre geometry, instead of lines and it plays an important role in the geometry of the curves of direction. We note that incident rays have a direction and the reflected rays in fig. 2 are the same tangent to the caustic but with opposite directions.
If x-axis is oriented so the two cycles (C1) and (C2) are not tangent at K (since oriented tangents at the contact point are opposite) but have for envelope the same catacaustic traveled in opposite directions. So this envelope is a kind of double superimposed curve.
The angle γ is the angle between x-axis and oriented tangent at current point. If a curve (C) is given by its parametric coordinates in an orthonormal system [x(t), y(t)], with axis the x-axis, then the above definition gives the following equations to compute the coordinates [X(t), Y(t)] of the new curve (C') envelope of the tangents mapped by the orthogonal tangents transformation (OTT) :

$$\boxed{X(t) = x(t) - 2.[\frac{x'(t)}{y'(t)}].y(t) = x(t) - 2.\frac{y(t)}{\tan\gamma}}$$

$$\boxed{Y(t) = [\frac{x'(t)}{y'(t)}]^2.y(t) = \frac{y(t)}{(\tan\gamma)^2}}$$

Since $(i).(\frac{-1}{i}) = -1$ isotrope lines are associated lines in the transformation and the OTT preserves the foci of algebraic curves.

3 Bisector of a plane curve with respect to an axis.

The bisector of a given curve wrt an axis is the locus of the center of the circles touching this curve and the axis. There are two curves since we don't take into account the direction of the curve. So as for the case of angles there are two bisectors. In the formulas this fact appears with the radical that has two branches $\pm$. For each couple of formulas [x(t), y(t)] we must

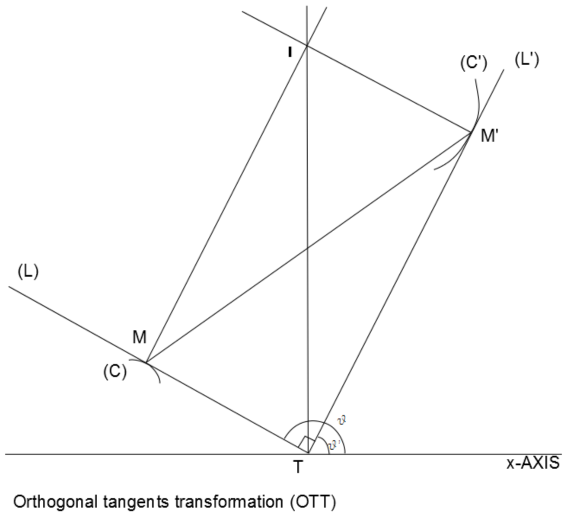

Figure 1: Orthogonal tangents transformation : L $\longmapsto$ L'

take the same sign to get the right curve. The angle γ_K is the angle between x-axis and oriented tangent at current point K.

$$\boxed{X_M = x_K + r\sin\gamma_K}$$

$$\boxed{Y_M = y_K - r\cos\gamma_K = \frac{y_K}{1 \pm \cos\gamma_k}}$$

Or using differentials of the parametric coordinates function of t (the ' is derivation wrt t):

$$\boxed{X_M = x_K + \frac{y_K . y'_K}{x'_K \pm \sqrt{x'^2_K + y'^2_K}}}$$

$$\boxed{Y_M = y_K - \frac{y_K . x'_K}{x'_K \pm \sqrt{x'^2_K + y'^2_K}}}$$

In the formulas the signs before square roots must be together + or together - and correspond to the two directions of the oriented envelope. The bisector of an anti-bisector is composed of two curves the initial one and the locus of centers of circles touching the anti-bisector on the other side of the tangent. If we consider oriented curves (as in Laguerre geometry of direction) then the double solution desappears and we have only one bisector just like for the case of the bisector of an angle with oriented sides.
In tangential coordinates of d'Ocagne (λ, θ), a length along x-axis and a line

with angle θ measured from this axis. If θ is the angle of a tangent of (C) with x-axis then λ is conserved and we have :

$$\boxed{\theta' = \frac{1}{2}.\theta \pm \frac{\pi}{2}}$$

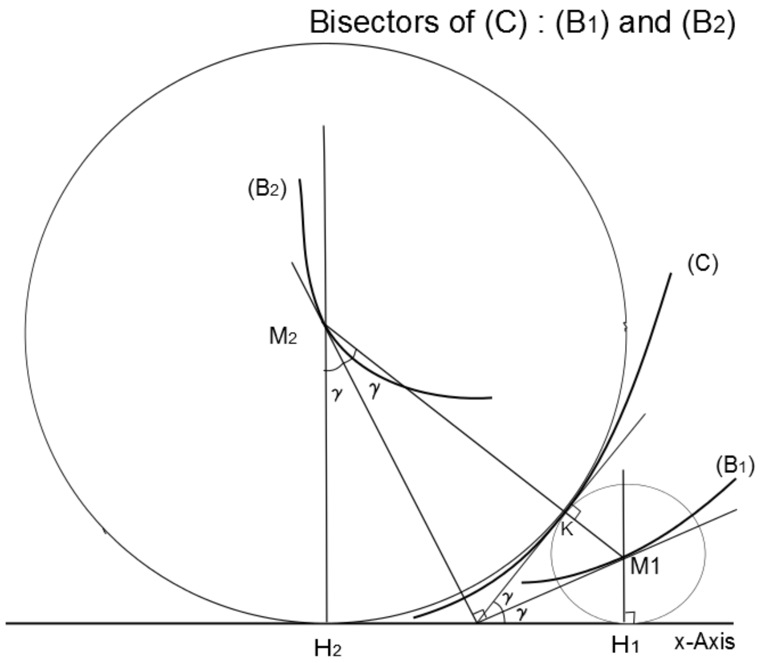

Figure 2: The two bisectors of a curve (C) wrt x-axis.

4 Anti-bisector of a plane curve with respect to an axis.

The anti-bisector of a given curve wrt an axis (x-axis) is the envelope of the circles centered on this curve and touching the x-axis. This has only one solution and there is no square root in the formulas.
The bisector of an algebraic curve is often called the catacaustic as we have seen in part XII that it is an involute of the caustic.

$$\boxed{X_K = x_M - 2.y_M.\frac{x'_M.y'_M}{x'^2_M + y'^2_M}}$$

$$\boxed{Y_K = 2.y_M.\frac{x_M'^2}{x_M'^2+y_M'^2}}$$

The anti-bisector of a bisector is the initial curve.

If we use tangential coordinates of d'Ocagne (λ, θ), a length along x-axis and a line with angle θ measured from this axis. If θ is the angle of a tangent of (C) with x-axis then λ is conserved and we have :

$$\boxed{\lambda' = \lambda \qquad \theta = 2.\theta'}$$

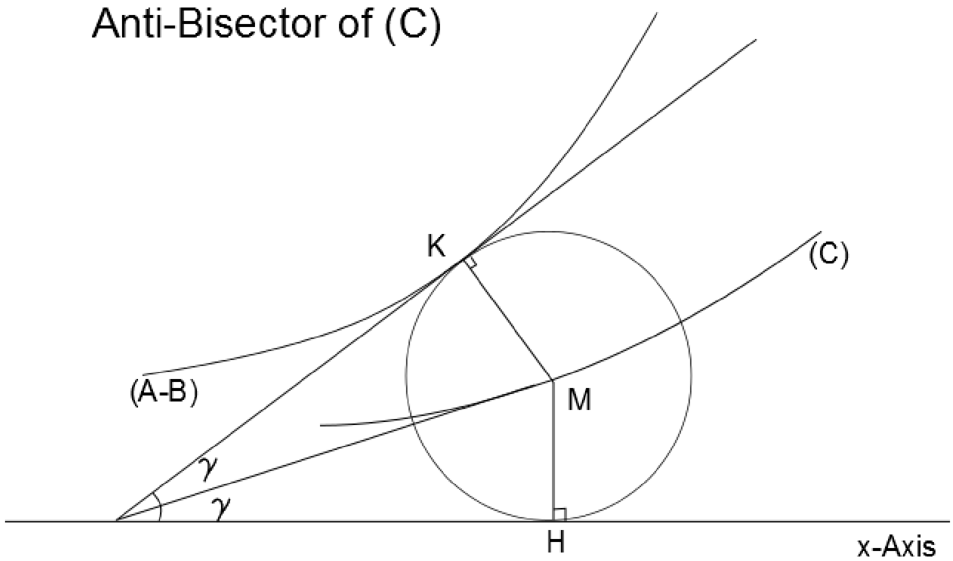

Figure 3: Anti-bisector of a curve (C) wrt x-axis.

5 Rational arc length of algebraic curves

In his paper "Sur les courbes algbriques planes rectifiable" [3] G. Humbert looks for :

1 - All rectifiable plane curves and tries to find the form of equation that links the arc length to values of coordinates.

2 - All algebraic plane curves with a rational expression for the arc length as function of the coordinates.

He mentions the following results :

- Any rectifiable algebraic curve is the evolute of an algebraic curve and

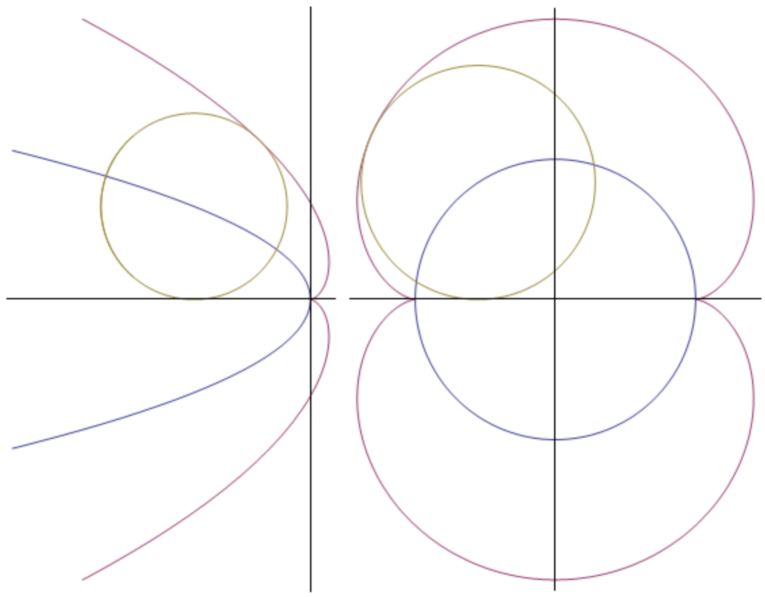

Figure 4: Anti-bisector of a parabola, of a circle $\longmapsto$ (C') $=$ involute of TC,$=$ Nephroid

reciprocally.
- An algebraic curve $f(x, y) = 0$ has a rational arc length if the radical $\sqrt{f_x'^2 + f_y'^2}$ is equal at every point of the curve to a rational function $Q(x, y)$. This property is characteristic of curves of direction, their general tangential equation is :

$$(u^2 + v^2)F^2(u, v) - G^2(u, v) = 0$$

where F and G are polynomials in u, v.
- If the evolute of a curve is a curve of direction then the initial curve is also of direction.

G. Humbert defines : a "simple curve" a curve cut orthogonally by its normals in one point, a "complex curve" a curve cut orthogonally by its normals in more than one point. The following properties are true :
The evolute of a simple curve of direction is a curve of direction.

Any complex curve is a curve of direction.

The evolute of a complex curve is never of direction.

Property 1 : The algebraic plane curves with a rational arc length as function of the coordinates are evolutes of simple algebraic curves of direction.

Property 2 : Algebraic plane curves for which the arc length is a rational function of coordinates are caustic by reflection of algebraic curves, for incident parallele light rays.

However if the reflecting curve is such that it is possible to trace two orthogonal tangents from all points of a line perpendicular to incident rays, then the caustic will have an arc expressible algebraically but not rationally.

Property 3 : Algebraic plane curves, the arc of which is expressible by an algebraic function of the coordinates, are caustics by reflection of algebraic curves, for convergent incident rays and conversely.

The geometric construction of catacaustic for parallele rays leads to an algebraic result : the arc length of this plane curve is a rational expression. It is important to recall the analogy of the previous result with the theorem of J. Haag (7) - see part XI - about unicursal curves with rational arc length. If ϕ and p are two rational functions of a parameter t and ϕ' and p' their derivatives wrt t. The plane curves defined as the envelope of the following line equation:

$$\frac{1-\phi^2}{1+\phi^2}.x + \frac{2.\phi}{1+\phi^2}.y = \frac{1+\phi^2}{2}.\frac{p'}{\phi'}$$

Using parameter $\phi = \tan\theta/2$, then the new equation for the line is :

$$x.\cos\theta - y.\sin\theta = \frac{dp}{d\theta}$$

The curves, envelopes of these lines, for all rational expressions of ϕ an p, as functions of a parameter t represent the most general unicursal rational curves with rational arc length (as a function of the coordinates).
We can recognize in the above equation the formulas of the double angle which is an analogic of the double angle of the geometric definition of the catacaustic.

Since the plane curves with a rational arc length are caustics of algebraic curves for parallele light rays, the two classes are very similar but the geometric construction is more general. If we apply the geometric construction to a caustic by reflection of a transcendental curve there is also an interesting expression for the arc length of the resulting caustic. This arc length, though not rational, can be computed with elementary functions used to define the initial reflecting curve.
Examples are the catenary as the caustic of the exponential for light rays perpendicular to the asymptote, the special pursuit curve as the caustic for light rays parallele to the asymptote or the cycloid as the caustic of a twice bigger cycloid for light rays perpendicular to the common base.

6 The OTT and the construction of catacaustics and caustics by reflection for parallele rays orthogonal to the x-axis.

The classical construction of the catacaustic for parallele light rays of a curve (C_1) can be completed in the following way. We transform (C_1) by OTT for x-axis that gives a curve (C_2). These two curves have the same catacaustic (C_a) with respect to x-axis for light rays parallele to y-axis. Trace any normal to this catacaustic : it cuts the two curves (C_1) and (C_2) at M_1 and M_2 The two circles centered at these points and tangent to x-axis are tangent to the catacaustic at point K. Since the two curves have the same catacaustic, they have the same caustic : the unique evolute of the catacaustic.

7 The example of catacaustics and caustics of parbolas / hyperbolas.

Wieleitner in [5] studied the case of caustics of general parabolas in the form $y = a.x^n$ which are pursuit curves and are algebraic when n is a rational number($n \neq 1$). He shows that the parametric equations (x as the parameter) of the pursuit curve are :

$$X = \frac{n-2}{n-1}.x \qquad \text{and } Y = \frac{a^2 n(n-2).x^{2n-2}+1}{2a.n(n-1)x^{n-2}}$$

$$Y = \frac{1}{2}[A.X^n + \frac{X^{2-n}}{A.n(n-2)}] \qquad \text{with : } A = \frac{a(n-1)^{n-1}}{(n-2)^{(n-1)}}$$

The form of this last equation suggests a duality between general parabolas of index n and (2-n). For the catacaustic in the last form he shows thatthe two parabolas $y = a.x^n$ and $y = a'.x^{2-n}$. It is possible to find a relation between a and a' with the constraint that : $A = \frac{a(n-1)^{n-1}}{(n-2)^{(n-1)}}$ and the coefficient of X^n and X^{n-2} be equal. This gives the relation :

$$a.a' = \frac{(n-2)^{n-2}}{n^n}$$

Wieleitner gives the example of the second degree parabola $y = a.x^{1/2}$ and associated Neil or semi-cubic parabola $y = a'.x^{3/2}$. So $a.a' = \frac{3i\sqrt{3}}{2}$.

8 Quadruplets of curves and caustics by reflection

These properties can be geometrically interpreted as the fact the parabola $y = a.x^n$ are transformed by OTT in another general parbola of equation

$y = a'.x^n$. Using the formulas above we can find the OTT of n-parabola has the following equation :

$$X = x(1 - \frac{2}{n}) \quad \text{and} \quad Y = \frac{x^{2-n}}{a.n^2}$$

So, eliminating x, we get another parabola :

$$Y = \frac{(n-2)^{n-2}}{a.n^n}.X^{2-n}$$

And we obtain this way the same result as Wieleitner :

$$a' = \frac{(n-2)^{n-2}}{a.n^n}$$

A general construction permits to define a quadruplet of curves. Starting with a given curve (C_1) and its tangential transformed curve (C_2) by OTT, their common catacaustic (C_3) and its evolute the caustic (C_4).

9 Case of a ponctual light O at finite distance.

In the case of caustic by reflection for a ponctual light O at finite distance (the origin) we don't have the couple of curves since catacaustic in this case is the envelope of the circle center on the reflecting curve and passing through O. One of the curve is reduced to the point O that plays an important role in the geometry of the caustic which is the evolute of the pedal dilated from O with ratio 2. In this case the initial curve is algebraic and the caustic is a rectifiable curve as evolute of an algebraic curve.

10 The limit case of exponentials :

In part VI we considered the catenary as a caustic of an exponential curve for parallele light rays in y direction. The curve $y = \exp \pm x$ is a limit case of parabolas/hyperbolas but is not algebraic. The catenary is the resulting caustic but is not a curve of direction in the usual meaning. Its arc length is an hyperbolic sine. The logarithmic $y = \log x$ has for caustic the special pursuit curve with equal speed of mobiles. This curve is not algebraic but its arc length has a simple expression by elementary transcendentals (real or complex exponentials and inverse functions) as for the catenary.

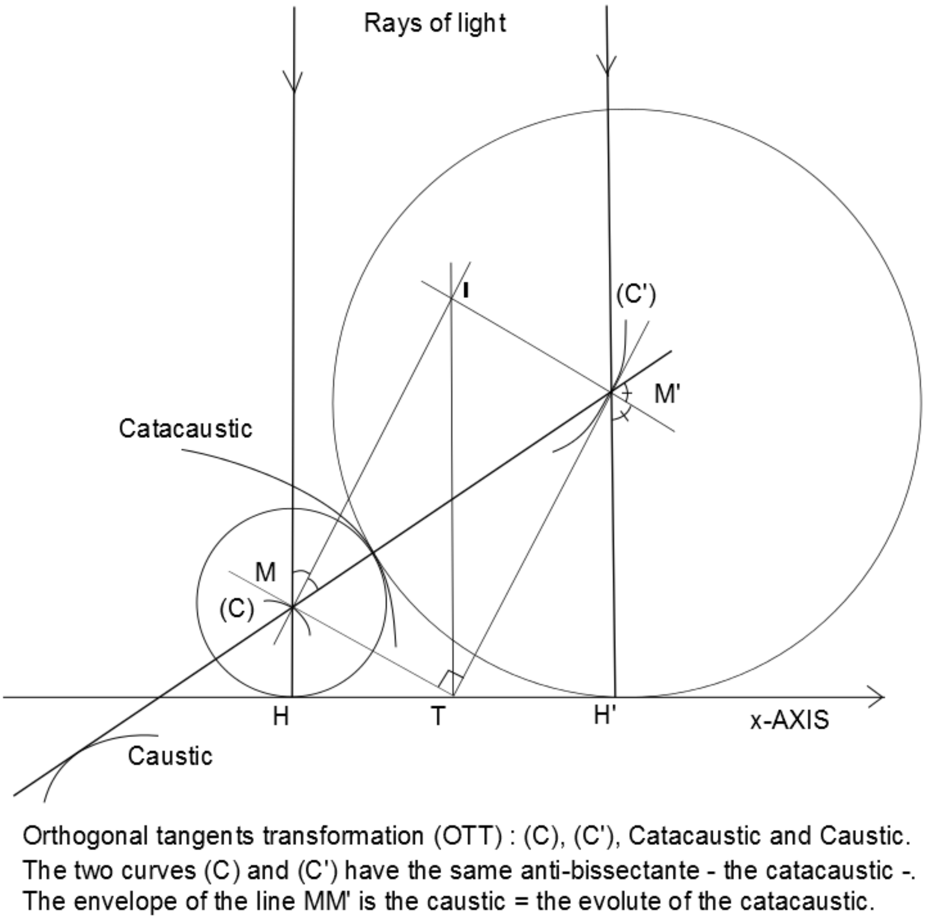

Figure 5: OTT (C) and (C’) $\leftrightarrow$ Catacaustic - Caustic.

11 Ribaucour curves. Catacaustics and caustics

It is possible to create exemples of quadruplets of curves using the well known Ribaucor curves. The circles with centers on a Ribaucour curve and tangent to the x-axis (the base-line) envelope the special sequence of curves of Cesaro (C) - see Part XII -, for which the osculator circle dilated from the current point is tangent to a fixed line in the plane : Nephroid (circle centered on xx’), Poleni’s curve (Catenary), a circle or a point for the parabola or a special evolute of the cycloid for the cycloid, etc.

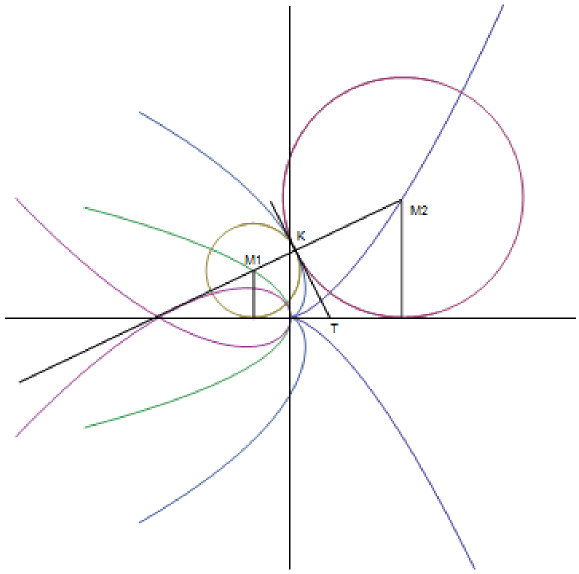

Figure 6: Quadruplet : parabolas (n=1/2), (n=3/2) $\leftrightarrow$ Catacaustic / Caustic = Tschirnhausen's cubic.

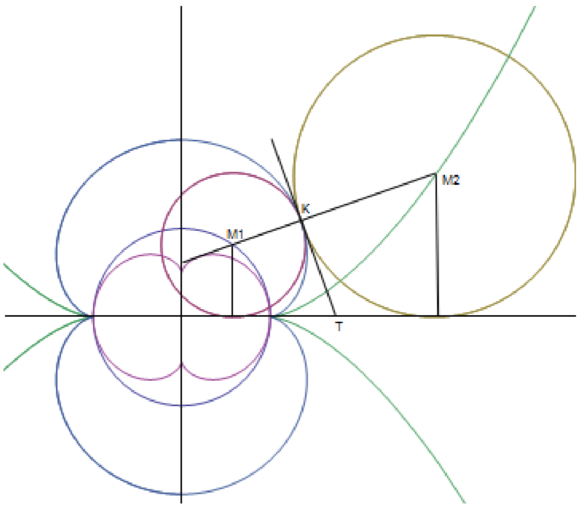

Figure 7: Quadruplet (C=circle), (C') $\leftrightarrow$ catacaustic = Nephroid = caustic

12 Curves of pursuit as caustic of parabolas or hyperbolas.

The above geometric construction of the catacaustic and caustic is a powerful general method to create as many curves with rational arc length as we need. The pursuit curves are curves of direction when the ratio of speeds is a rational number ($n = p/q$) with p and q $\in N$. The classic example of

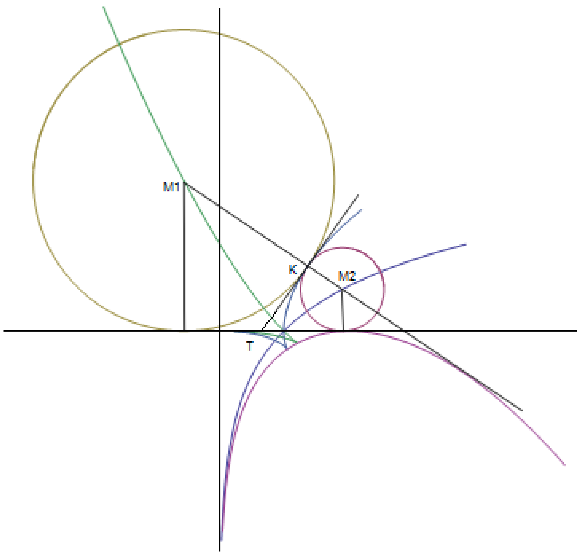

Figure 8: Quadruplet : logarithmic/ OTT $\leftrightarrow$ Catacaustic and caustic

Tschirnhausen's cubic corresponds to n=2. We give two graphic examples of caustics of general parabolas.

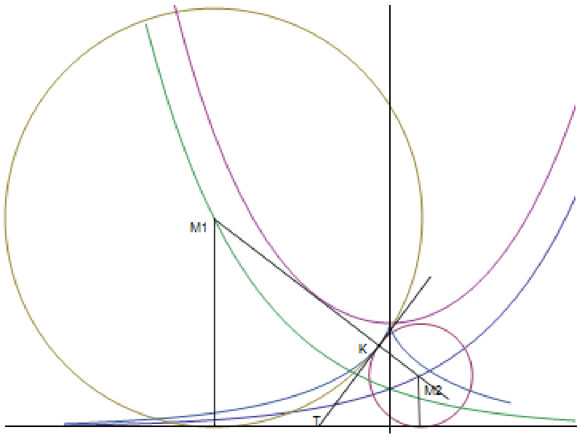

Figure 9: Quadruplet : two exponentials $\leftrightarrow$ Tractrix and Catenary=caustic.

13 Conclusion

The construction of caustics for parallele light rays or even for incident rays give a simple tool to find plane curves of direction with rational arc length. And this fact, known since Tschirnhausen and its cubic, has been used recently in GCAD design and the pythagorean hodograph. It would be useful to find other low order curves with rational arc length, or even rectifiable

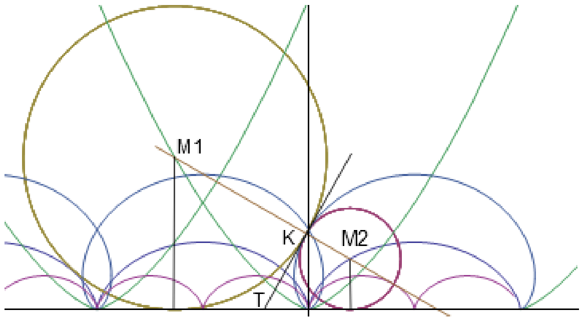

Figure 10: Quadruplet : cycloid/OTT $\leftrightarrow$ Involute of cycloid and caustic = Cycloid (half size).

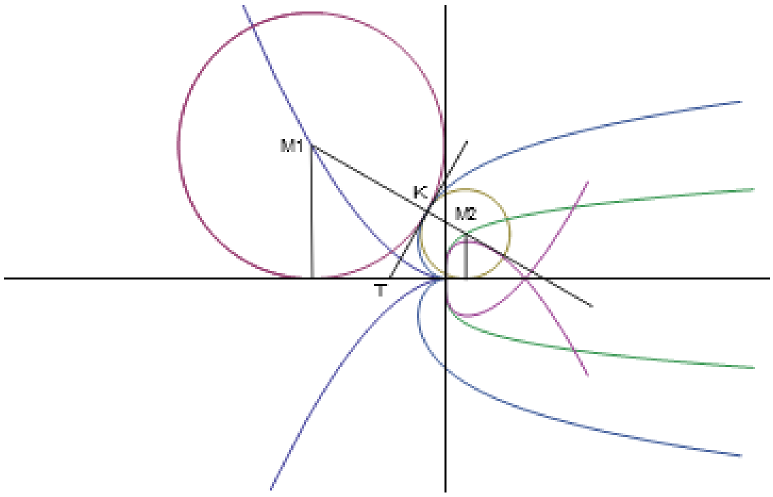

Figure 11: Quadruplet : Two parbolas $\leftrightarrow$ Catacaustic and caustic

arc length by elementary functions, because calculations can be treated algebraically.
For most of classical algebraic curves, like the ellipse or the lemniscate, the arc length is not simple integrable expression and need advanced theories and new functions. Evolutes of plane algebraic curves are rectifiable. The curves of direction, according to the naming of Laguerre, belong to an important class of curves with a rational arc and are solutions of a general tangential equation : $(u^2 + v^2)F^2(u, v) - G^2(u, v) = 0$, that should deserve the interest of designers.
The simplest examples, of class 4 and order 4 or 6, have been studied geometrically by Laguerre in papers around 1880 as a generalization of polarity of a circle. He called them hypercycles but I could not find in the literature a comprehensive survey on the subject.

This article is the $XIII^{th}$ part on a total of 13 papers on Gregory's transformation and related topics.
Part I : Gregory's transformation.
Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.
Part III : A generalization of sinusoidal spirals and Ribaucour curves
Part IV: Tschirnhausen's cubic.
Part V : Closed wheels and periodic grounds
Part VI : Catalan's curve.
Part VII : Anallagmatic spirals, Pursuit curves, Hyperbolic-Tangentoid spirals, β-curves.
Part VIII : Translations, rotations, orthogonal trajectories, differential equations, Gregory's transformation.
Part IX : Curves of Duporcq - Sturmian spirals.
Part X : Intrinsically defined plane curves, periodicity and Gregory's transformation.
Part XI : Inversion, Laguerre T.S.D.R., Euler polar tangential equation and d'Ocagne axial coordinates.
Part XII : Caustics by reflection, curves of direction, rational arc length.
Part XIII : Catacaustics, caustics, curves of direction and orthogonal tangent transformation.

Two papers in french :
1- Quand la roue ne tourne plus rond - Bulletin de l'IREM de Lille (no 15 Fevrier 1983).
2- Une generalisation de la roue - Bulletin de l'APMEP (no 364 juin 1988). There is an english adaptation.
Gregory's transformation : http://christophe.masurel.free.fr

References :

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(2) E. Cesaro - Remarques sur un article de M.d'Ocagne (Tome 4 NAM 1885).
(3) G. Humbert - Sur les courbes algbriques planes rectifiables JMPA (4e serie), tome IV-Fasc. II (1888).
(4) M. d'Ocagne - Sur un systeme special de coordonnees tangentielles et sur la transformation par tangentes orthogonales (serie 4 t.I NAM octobre 1901 p23-25).
(5) M. d'Ocagne - Sur les courbes a axe orthoptique et les courbes de direction (NAM 4eme serie t.XIX 1919 pp329-338).
(6) H. Wieleitner - Spezielle ebene Kurven" 1908 Leipzig.
(7) J. Haag - Sur les courbes unicursales a arc rationnel (NAM 1915).
- Nouvelles annales de mathematiques (1842-1927) Archives Gallica.

- Journal de mathematiques pures et appliquees (1836-1934) Archives Gallica.