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On the Resolvents of Nonconvolution Volterra Kernels

By

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§ 1. Introduction

The purpose of this paper is to study the resolvent kernel associated with the linear Volterra equation

$$x(t)+\int_0^t k(t, s)x(s)ds=f(t), \qquad t \in R^+=[0, \infty) \tag{1.1}$$

i.e. the solution of the equation

$$r(t, s)+\int_s^t k(t, u)r(u, s)du=k(t, s), \qquad 0\leq s\leq t. \tag{1.2}$$

The importance of the resolvent derives from the fact that the solution x of (1.1) is given by

$$x(t)=f(t)-\int_0^t r(t, s)f(s)ds, \qquad t \in R^+. \tag{1.3}$$

From equations (1.2) and (1.3) one may deduce a variation of constants formula for the solutions of nonlinear perturbed forms of (1.1) (see, e.g., [8, Chap. IV]). This formula shows why certain properties of the resolvent kernel r are crucial when the solution of (1.1) is compared with the solutions of such perturbed forms of (1.1), see [6, 9–11, 13].

The properties of the resolvent that we will investigate here are the following:

$$\sup_{t\geq 0}\int_0^t |r(t, s)|\, ds<\infty, \tag{1.4}$$

$$\lim_{t\to\infty}\int_0^T |r(t, s)|\, ds=0 \qquad \text{for all } T>0 \tag{1.5}$$

and

$$\lim_{h\to 0}\left(\int_t^{t+h} |r(t+h, s)|\, ds+\int_0^t |r(t+h, s)-r(t, s)|\, ds\right)=0, \qquad t \in R^+. \tag{1.6}$$

If the resolvent r is of convolution type, i.e. $r(t, s)=r(t-s)$ (this is the case if $k(t, s)$

$=k(t-s)$), then these conditions are satisfied if $r \in L^1(R^+)$. For sufficient assumptions on k for this to happen, see [1, 7, 12]. In the nonconvolution case some sufficient conditions can be found in [2, 9, 13].

Although the results in this paper are only formulated for real-valued functions it is obvious how some, but not all, of the assertions below can be extended to more abstract settings.

§ 2. Statement of results

First we consider the question of what assumptions on the kernel k are needed in order for r to be well defined. In [8, p. 197] it is shown that if $k(t, s)$ is measurable on $R^+ \times R^+$, $k(t, s)=0$ if $s>t$ and

$$\int_0^T \left(\int_0^T |k(t, s)|^p ds\right)^{q/p} dt < \infty, \quad \int_0^T \left(\int_0^T |k(t, s)|^q dt\right)^{p/q} ds < \infty$$

for some $p, q \in (1, \infty)$, $p^{-1}+q^{-1}=1$ and $T>0$, then r exists and satisfies the same conditions. In Theorem 1 we study the cases $p=1$ and $p=\infty$.

Theorem 1. *Assume that* $T>0$ *and that*

(2.1) $k(t, s)$ *is a measurable function on* $[0, T]\times[0, T]$ *and* $k(t, s)=0$ *if* $s>t$,

$$\int_0^t |k(t, s)|\, ds \in L^\infty(0, T), \tag{2.2}$$

$$\limsup_{h\to 0+} \left\| \int_{\max\{t-h,0\}}^t |k(t, s)|\, ds \right\|_{L^\infty(0,T)} < 1. \tag{2.3}$$

Then there exists a function $r(t, s)$ *such that*

(2.4) $r(t, s)$ *is measurable on* $[0, T]\times[0, T]$ *and* $r(t, s)=0$ *if* $s>t$,

$$\int_0^t |r(t, s)|\, ds \in L^\infty(0, T), \tag{2.5}$$

$$k(t, s)-r(t, s)=\int_s^t k(t, u)r(u, s)du=\int_s^t r(t, u)\, k(u, s)du \tag{2.6}$$

for a.e. $(t, s) \in [0, T]\times[0, T]$.

If instead of (2.2) *and* (2.3)

$$\left\| \operatorname*{ess\,sup}_{s\in(0,t)} |k(t, s)| \right\|_{L^1(0,T)} < \infty \tag{2.7}$$

holds, then $r(t, s)$ *exists and satisfies* (2.4), (2.6) *and*

(2.8) $$\left\| \operatorname*{ess\,sup}_{s\in(0,t)} |r(t,s)| \right\|_{L^1(0,T)} < \infty .$$

Concerning the assumption (2.3) we have

Proposition 1. *If* (2.1) *and* (2.2) *holds for some* $T>0$ *but* (2.3) *does not hold, then* (2.4)–(2.6) *do not necessarily hold.*

In the next easy theorem we consider the condition (1.6) and show that under certain assumptions it follows from the corresponding hypothesis on k.

Theorem 2. *Assume that for some* $T>0$, (2.1), (2.2) *and* (2.4)–(2.6) *hold and that*

(2.9) $$\lim_{\varepsilon\to 0+} \operatorname*{ess\,sup}_{|h|\in(0,\varepsilon)} \left(\int_t^{t+h} |k(t+h,s)|\,ds + \int_0^t |k(t+h,s)-k(t,s)|\,ds \right) = 0, \quad \text{a.e. } t\in(0,T).$$

Then

(2.10) $$\lim_{\varepsilon\to 0+} \operatorname*{ess\,sup}_{|h|\in(0,\varepsilon)} \left(\int_t^{t+h} |r(t+h,s)|\,ds + \int_0^t |r(t+h,s)-r(t,s)|\,ds \right) = 0, \quad \text{a.e. } t\in(0,T).$$

The following result is an extension of [13, Prop. 4] where it is assumed that

$$\sup_{t\geq 0} \int_0^t |k(t,s)|\,ds < 1.$$

Theorem 3. *Assume that* (2.1) *and* (2.3) *hold for all* $T>0$ *and that*

(2.11) $$\int_0^t |k(t,s)|\,ds \in L^\infty(R^+),$$

(2.12) $$\inf_{T>0} \limsup_{t\to\infty} \operatorname*{ess\,sup}_{\tau\in(t,\infty)} \int_T^\tau |k(\tau,s)|\,ds < 1.$$

Then the function $r(t,s)$ *that exists by Theorem* 1 *satisfies*

(2.13) $$\int_0^t |r(t,s)|\,ds \in L^\infty(R^+).$$

If moreover

(2.14) $$\lim_{t\to\infty} \operatorname*{ess\,sup}_{\tau\in(t,\infty)} \int_0^T |k(\tau,s)|\,ds = 0 \qquad \text{for all } T>0,$$

then

(2.15) $$\lim_{t\to\infty} \operatorname{ess\,sup}_{\tau\in(t,\infty)} \int_0^T |r(\tau, s)|\, ds = 0 \qquad \textit{for all } T>0.$$

The next theorem is inspired by [5].

Theorem 4. *Assume that* (2.1)–(2.3) *hold for all* $T>0$ *and that for some* $T_0 \geq 0$

(2.16) *$k(t+T_0, s+T_0)=k_1(t, s)+k_2(t, s)$, $0\leq s\leq t$, where the functions $k_1(t, s)$ and $k_2(t, s)$ satisfy* (2.1)–(2.3) *for all* $T>0$,

(2.17) $\int_0^t |r_1(t, s)|\, ds \in L^\infty(R^+)$, *where $r_1(t, s)$ is the resolvent associated with $k_1(t, s)$*,

(2.18) *the function* $k_3(t, s)=\int_s^t r_1(t, u)k_2(u, s)du - k_2(t, s)$ *satisfies* (2.11) *and* (2.12),

(2.19) $$\int_0^{T_0} |k(t, s)|\, ds \in L^\infty(R^+).$$

Then (2.13) *holds.*

Observe that the condition (2.18) is satisfied if

$$\left\| \int_0^t |k_2(t, s)|\, ds \right\|_{L^\infty(R^+)} \left(\left\| \int_0^t |r_1(t, s)|\, ds \right\|_{L^\infty(R^+)} + 1 \right) < 1.$$

From this observation it is clear that the assumptions of Theorem 4 are satisfied if $k(t, s)=a(t)b(s)c(t-s)$, where a and b are bounded functions that converge to 1 as $t\to\infty$, and where $c \in L^1(R^+)$ is such that $\int_0^\infty \exp(-zt)c(t)dt \neq -1$, $\operatorname{Re} z \geq 0$, cf. [5].

The next theorem is an extension to the nonconvolution case of the well known result that the resolvent associated with a nonnegative, nonincreasing and integrable convolution kernel belongs to $L^1(R^+)$, (but note that in the convolution case it is not necessary to assume that the kernel is bounded).

Theorem 5. *Assume that*

(2.20) $$k(t, s)=a(s)-\int_s^t b(u, s)du, \quad 0\leq s\leq t, \quad k(t, s)=0, \quad s>t$$

where

(2.21) $$a \in L^\infty(R^+),$$

(2.22) *$b(t, s)$ is measurable and nonnegative on $R^+\times R^+$*,

(2.23) *for any $T>0$ and a.e. $t \in (0, T)$*
$$a(t) \geq \max\left\{ \int_0^t b(t, s)ds, \int_t^T b(s, t)ds \right\},$$

$$\lim_{T\to\infty} \sup_{t\geq 0} \int_0^{t-T} |k(t, s)|\, ds=0. \tag{2.24}$$

Then (2.13) *and* (2.15) *hold.*

The proof of this theorem is an application of [2, Th. 5] and [4, Th. 2].

§ 3. Proof of Theorem 1

We proceed in the same way as in [8, Chap. IV] and define

$$r_1(t, s)=k(t, s) \quad \text{a.e.} \quad (t, s) \in [0, T]\times[0, T], \tag{3.1}$$

$$r_{n+1}(t, s)=-\int_s^t k(t, u)r_n(u, s)du,\ n\geq 1, \qquad \text{a.e. } (t, s) \in [0, T]\times[0, T]. \tag{3.2}$$

By (2.1), (2.2), Fubini's theorem and an easy induction argument one sees that

$$\int_0^t |r_n(t, s)|\, ds \in L^\infty(0, T)$$

and that the functions $r_n(t, s)$ are measurable for all n.

By (2.3) there exists $\varepsilon>0$ and $\gamma \in (0, 1)$ so that

$$\int_t^{t+h} |k(t+h, s)|\, ds\leq\gamma, \qquad t \in [0, T],\ \text{a.e. } h \in (0, \varepsilon). \tag{3.3}$$

Here we have defined $k(t, s)=0$ if $t>T$. From (3.2) and Fubini's theorem we obtain for $t \in [0, T]$ and a.e. $h>0$,

$$\begin{aligned}\int_t^{t+h} |r_{n+1}(t+h, s)|\, ds &\leq\int_t^{t+h} |k(t+h, u)| \int_t^u |r_n(u, s)|dsdu\\ &\leq\gamma \left\| \operatorname*{ess\,sup}_{h\in(0,\varepsilon)} \int_t^{t+h} |r_n(t+h, s)|\, ds\right\|_{L^\infty(0,T)},\end{aligned}$$

and so it follows from (3.1) and (3.3) by induction that

$$\left\| \operatorname*{ess\,sup}_{h\in(0,\varepsilon)} \int_t^{t+h} |r_n(t+h, s)|\, ds\right\|_{L^\infty(0,T)} \leq\gamma^n, \qquad n\geq 1. \tag{3.4}$$

Let

$$c_1=\gamma^{-1}\left\|\int_0^t |k(t, s)|\, ds\right\|_{L^\infty(0,T)}+1.$$

Assume that for some $m\geq 1$ and all $n\geq 1$ we have

$$\left\| \operatorname*{ess\,sup}_{h\in(0,m\varepsilon)} \int_t^{t+h} |r_n(t+h, s)|\, ds\right\|\leq nc_1^m\gamma^n, \qquad n\geq 1. \tag{3.5}$$

By (3.4) this is true when $m=1$. From Fubini's theorem, (3.2) and the definition of c_1 we deduce that for $t=[0, T]$ and a.e. $h \in (0, (m+1)\varepsilon)$

$$\int_t^{t+h} |r_{n+1}(t+h, s)|\, ds \leq \int_t^{t_1} |k(t+h, u)| \int_t^u |r_n(u, s)|\, dsdu$$
$$+\int_{t_1}^{t+h} |k(t+h, u)| \int_t^u |r_n(u, s)|\, dsdu \leq c_1 n c_1^m \gamma^{n+1}$$
$$+\gamma \left\| \operatorname*{ess\,sup}_{h \in (0,(m+1)\varepsilon)} \int_t^{t+h} |r_n(t+h, s)|\, ds \right\|_{L^\infty(0,T)}$$

where t_1 is a suitably chosen number so that $t_1 \leq t+m\varepsilon$ and $0 \leq t+h-t_1 \leq \varepsilon$. By induction we may conclude that (3.5) holds with m replaced by $m+1$. Consequently there exists a constant c_2 such that

$$\left\| \int_0^t |r_n(t, s)|\, ds \right\|_{L^\infty(0,T)} \leq c_2 n \gamma^n, \qquad n \geq 1.$$

This implies that the sum

$$r(t, s) = \sum_{n=1}^{\infty} r_n(t, s) \tag{3.6}$$

exists for a.e. (t, s) and is a measurable function. Moreover, it is easy to see that (2.5) holds.

Next we show that (2.6) holds. Let $f(t, s)$ be any C^∞-function with support in $(0, T)\times(0, T)$. By Fubini's theorem, the dominated convergence theorem, (3.1), (3.2) and (3.6) we obtain

$$\begin{aligned}
&\int_0^T \int_0^T f(t, s) \int_s^t k(t, u) r(u, s)\, dudsdt \\
&= \sum_{n=1}^{\infty} \int_0^T \int_0^T f(t, s) \int_s^t k(t, u) r_n(u, s)\, dudsdt \qquad (3.7)\\
&= -\sum_{n=2}^{\infty} \int_0^T \int_0^T f(t, s) r_n(t, s)\, dsdt = \int_0^T \int_0^T f(t, s)(-r(t, s)+k(t, s))\, dsdt
\end{aligned}$$

and the first equality in (2.6) follows.

To establish the second equality in (2.6) we assume that v and w are positive integers and that

$$r_{w+1}(t, s) = -\int_s^t r_w(t, u) k(u, s)\, du \qquad \text{a.e. } (t, s). \tag{3.8}$$

Let $f(t, s)$ be an arbitrary C^∞-function supported in $(0, T)\times(0, T)$. Then it follows from (3.2), (3.8) and Fubini's theorem that

$$\int_0^T \int_0^T f(t, s) \int_s^t r_w(t, u) r_{v+1}(u, s) du ds dt$$
$$= -\int_0^T \int_0^T \int_s^t \int_s^u f(t, s) r_w(t, u) k(u, z) r_v(z, s) dz du ds dt$$
$$= \int_0^T \int_0^T f(t, s) \int_s^t r_{w+1}(t, z) r_v(z, s) dz ds dt$$

or, since f was arbitrary,

$$\int_s^t r_w(t, u) r_{v+1}(u, s) du = \int_s^t r_{w+1}(t, u) r_v(u, s) du \qquad \text{a.e. } (t, s). \tag{3.9}$$

We recall that (3.8) holds if $w=1$. Hence by (3.2), (3.9) and an induction argument we see that (3.8) holds for all w. Now we get the desired equality if we change the order of k and r in (3.7). This completes the proof of the first part of Theorem 1.

To establish the second part of the theorem we define r_1 by (3.1) but instead of (3.2) we take

$$r_{n+1}(t, s) = -\int_s^t r_n(t, u) k(u, s) du, \ n \geq 1, \qquad \text{a.e. } (t, s) \in [0, T] \times [0, T]. \tag{3.10}$$

By (2.1), (2.7) and Fubini's theorem we again observe that the functions r_n are measurable and that now

$$\left\| \operatorname*{ess\,sup}_{s \in (0,T)} |r_n(t, s)| \right\|_{L^1(0,T)} < \infty$$

for all n. Define

$$a(t) = \operatorname*{ess\,sup}_{s \in (0,T)} |k(t, s)|, \quad t \in [0, T], \quad a(t) = 0 \ \text{ otherwise.} \tag{3.11}$$

It is a consequence of (2.7) that there exists a number $\varepsilon > 0$ so that

$$\int_{t-\varepsilon}^t a(s) ds \leq 2^{-1}, \qquad t \in [0, T]. \tag{3.12}$$

For $n \geq 1$ and a.e. $t \in (0, T)$ we have by (3.10)–(3.12)

$$\operatorname*{ess\,sup}_{s \in (t-\varepsilon, t)} |r_{n+1}(t, s)| \leq 2^{-1} \operatorname*{ess\,sup}_{s \in (t-\varepsilon, t)} |r_n(t, s)|$$

and so it follows from (3.1) that

$$\operatorname*{ess\,sup}_{s \in (t-\varepsilon, t)} |r_n(t, s) \leq 2^{-n+1} a(t), \qquad \text{a.e. } t \in (0, T). \tag{3.13}$$

Let

$$c_3=2\int_0^T a(s)ds+2.$$

Assume that for some $m\geq 1$ and all $n\geq 1$

$$\operatorname*{ess\,sup}_{s\in(t-m\varepsilon,t)} |r_n(t,s)|\leq nc_3^m 2^{-n}a(t), \qquad \text{a.e. } t\in(0,T). \tag{3.14}$$

This is the case when $m=1$ by (3.13). Now by (3.11) we obtain for a.e. $t\in(0,T)$, if necessary dividing the interval (s,t) into two parts $(s,t-m\varepsilon)$ and $(t-m\varepsilon,t)$,

$$\operatorname*{ess\,sup}_{s\in(t-(m+1)\varepsilon,t)} |r_{n+1}(t,s)|\leq nc_3^{m+1}2^{-(n+1)}a(t)+2^{-1}\operatorname*{ess\,sup}_{s\in(t-(m+1)\varepsilon,t)} |r_n(t,s)|.$$

(We assume $r_n(t,s)=0$ if $s<0$). From this inequality combined with (3.1) and (3.11) we see that (3.14) holds with m replaced by $m+1$. We conclude that there exists a constant c_4 such that

$$\operatorname*{ess\,sup}_{s\in(0,T)} |r_n(t,s)|\leq c_4 n2^{-n}a(t), \qquad \text{a.e. } t\in(0,T). \tag{3.15}$$

It follows from (2.7), (3.11) and (3.15) that the sum

$$r(t,s)=\sum_{n=1}^{\infty} r_n(t,s)$$

exists for a.e. (t,s), is a measurable function and satisfies (2.8).

To see that (2.6) holds in this case too, we proceed in the same way as in the first part of the proof but we reverse the order of r and k at appropriate points. This completes the proof of Theorem 1.

§ 4. Proofs of Proposition 1 and Theorem 2

Define the function $k(t,s)$ by

$$k(t,s)=-t^{-1}, \quad s\in[0,t], \quad t>0, \quad k(t,s)=0 \text{ otherwise.} \tag{4.1}$$

Then (2.1) and (2.2) are obviously satisfied for all $T>0$. Assume that there exists a function $r(t,s)$ so that (2.4)–(2.6) hold. Then we have by Fubini's theorem, (2.6) and (4.1) for a.e. $t\in(0,1)$ (choosing $T=1$)

$$\int_0^1 r(t,s)ds=\int_0^1 k(t,s)ds-\int_0^1\int_s^t r(t,u)k(u,s)duds$$
$$=-1+\int_0^1 r(t,u)\int_0^u u^{-1}dsdu=-1+\int_0^1 r(t,s)ds$$

and since this implies $0=-1$ we have a contradiction and the proof of Proposition 1 is completed.

To prove Theorem 2 we may by (2.5) choose c_1 to be a constant such that

$$\int_0^t |r(t, s)|\, ds \leq c_1, \qquad \text{a.e. } t \in (0, T). \tag{4.2}$$

Let $t \in [0, T]$ be arbitrary, except that it will not belong to a certain set with zero measure. From (2.1), (2.4), (2.6), (4.2) and Fubini's theorem we deduce that

$$\begin{aligned} \int_t^{t+h} |r(t+h, s)|\, ds &\leq \int_t^{t+h} |k(t+h, s)|\, ds + \int_t^{t+h} |k(t+h, u)| \int_t^u |r(u, s)|\, ds du \\ &\leq (1+c_1) \int_t^{t+h} |k(t+h, s)|\, ds, \qquad \text{a.e. } h>0. \end{aligned} \tag{4.3}$$

Obviously we also get in the same way

$$\begin{aligned} \int_0^t |r(t+h, s)-r(t, s)|\, ds &\leq \int_0^t |k(t+h, s)-k(t, s)|\, ds \\ &+ \int_t^{t+h} |k(t+h, u)| \int_0^t |r(u, s)|\, ds du \\ &+ \int_0^t |k(t+h, u)-k(t, u)| \int_0^u |r(u, s)|\, ds du \\ &\leq (c_1+1)\left(\int_t^{t+h} |k(t+h, s)|\, ds + \int_0^t |k(t+h, s)-k(t, s)|\, ds\right), \qquad \text{a.e. } h>0. \end{aligned} \tag{4.4}$$

Since the case $h<0$ can be handled in a similar manner, the desired result follows from (2.9), (4.3) and (4.4). This completes the proof of Theorem 2.

§ 5. Proofs Theorems 3 and 4

We observe that (2.4)–(2.6) (for all $T>0$) follow from Theorem 1. By (2.12) there exists positive constants T_0, t_0 and $\gamma, \gamma<1$, so that

$$\int_{T_0}^t |k(t, s)|\, ds \leq \gamma, \qquad \text{a.e. } t \geq t_0. \tag{5.1}$$

Applying Fubini's theorem and some obvious estimates we obtain with the aid of (5.1)

$$\begin{aligned} \int_0^t |r(t, s)|\, ds &\leq \int_0^t |k(t, s)|\, ds + \int_0^t |k(t, u)| \int_0^u |r(u, s)|\, ds du \\ &\leq \left\| \int_0^\tau |k(\tau, s)|\, ds \right\|_{L^\infty(R^+)} \left(1 + \left\| \int_0^\tau |r(\tau, s)|\, ds \right\|_{L^\infty(0, T_0)}\right) \\ &+ \gamma \left\| \int_0^\tau |r(\tau, s)|\, ds \right\|_{L^\infty(0, t)}, \qquad \text{a.e. } t \geq t_0. \end{aligned}$$

Since (2.5) and (2.11) hold and $\gamma<1$, this inequality implies that (2.13) holds.

Next assume that (2.14) holds and fix $T>0$. By (2.6), (2.11), (2.13) and Fubini's theorem we get for a.e. $t>T$

$$\int_0^T |r(t,s)|\,ds \le \int_0^T |k(t,s)|\,ds + \int_0^t |k(t,u)| \int_0^T |r(u,s)|\,dsdu. \tag{5.2}$$

Let T_0, t_0 and γ be as above. If (2.15) does not hold, then there exists a number $t_1 \ge \max\{T, T_0\}$ such that

$$\operatorname*{ess\,sup}_{\tau\in(t,\infty)} \int_0^T |r(\tau,s)|\,ds \ge 2^{-1}(1+\gamma) \operatorname*{ess\,sup}_{\tau\in(t_1,\infty)} \int_0^T |r(\tau,s)|\,ds \qquad \text{for all } t\ge t_1. \tag{5.3}$$

By (2.13) and (5.1) there exists a constant c_1 so that for a.e. $t\ge \max\{t_1, t_0\}$

$$\int_0^t |k(t,u)| \int_0^T |r(u,s)|\,dsdu \le c_1 \int_0^{t_1} |k(t,s)|\,ds + \gamma \operatorname*{ess\,sup}_{\tau\in(t_1,\infty)} \int_0^T |r(\tau,s)|\,ds.$$

But combining this inequality with (2.14), (5.2) and (5.3) we get a contradiction since $\gamma<1$. This completes the proof of Theorem 3.

To prove Theorem 4 we note that the resolvent $r(t,s)$ associated with $k(t,s)$ exists and satisfies (2.4)–(2.6) for all $T>0$. Let $r_{T_0}(t,s)=r(t+T_0, s+T_0)$, ($r_{T_0}(t,s)$, $T_0=1$ should not be confused with r_1). Then we have by (2.6) and (2.16)

$$r_{T_0}(t,s)+\int_s^t k_1(t,u)r_{T_0}(u,s)du = k_1(t,s)+k_2(t,s)-\int_s^t k_2(t,u)r_{T_0}(u,s)du,$$
$$\text{a.e. } 0\le s\le t.$$

Multiply this equation by $r_1(\tau,t)$ (by Theorem 1 we know that $r_1(\tau,t)$ exists and satisfies (2.4)–(2.6) for all $T>0$) and integrate over (s,τ). Then we obtain, using Fubini's theorem and (2.6) (with $r_1(t,s)$ and $k_1(t,s)$) that

$$\begin{aligned} r_{T_0}(t,s) &= k_1(t,s)+k_2(t,s)-\int_s^t k_2(t,u)r_{T_0}(u,s)du \\ &\quad -\int_s^t r_1(t,u)(k_1(u,s)+k_2(u,s))du + \int_s^t r_1(t,u)\int_s^u k_2(u,v)r_{T_0}(v,s)dvdu, \end{aligned}$$
$$\text{a.e. } 0\le s\le t.$$

Applying Fubini's theorem, (2.6) and the definition of $k_3(t,s)$ to this equation we see that

$$r_{T_0}(t,s)=r_1(t,s)-k_3(t,s)+\int_s^t k_3(t,u)r_{T_0}(u,s)du, \qquad \text{a.e. } 0\le s\le t. \tag{5.4}$$

Using (2.17) and (2.18) in (5.4) we can proceed in the same way as in the proof of Theorem 3 and show that

$$\operatorname*{ess\,sup}_{t\geq 0}\int_0^t |r_{T_0}(t,s)|\,ds<\infty. \tag{5.5}$$

By (2.3) we can choose positive numbers γ and ε, $\gamma<1$ such that

$$\left\|\int_{\max\{t-\varepsilon,0\}}^t |k(t,s)|\,ds\right\|_{L^\infty(0,T_0)}\leq\gamma. \tag{5.6}$$

Assume that $T\in(0,T_0]$ is such that

$$\operatorname*{ess\,sup}_{t\geq T_0}\int_T^t |r(t,s)|\,ds<\infty. \tag{5.7}$$

This is the case when $T=T_0$ by (5.5). Let $h=\min\{T,\varepsilon\}$. Now we have by (2.6) and Fubini's theorem for a.e. $t\geq T_0$

$$\int_{T-h}^T |r(t,s)|\,ds\leq\int_{T-h}^T |k(t,s)|\,ds+\int_{T-h}^T |r(t,u)|\int_{T-h}^u |k(u,s)|\,dsdu$$
$$+\int_T^t |r(t,u)|\int_{T-h}^T |k(u,s)|\,dsdu.$$

Combining (2.19), (5.6) and (5.7) with this inequality we see that

$$\operatorname*{ess\,sup}_{t\geq T_0}\int_{T-h}^T |r(t,s)|\,ds<\infty$$

and so (5.7) holds with T replaced by $T-h$. In view of (5.5) this implies that (5.7) holds with $T=0$ and this fact together with (2.5) gives (2.13) and the proof of Theorem 4 is completed.

§ 6. Proof of Theorem 5

It follows from [4, Th. 2] that if $f\in C(R^+)\cap L^\infty(R^+)$, then there exists a bounded continuous solution x of the equation (1.1) and it is easy to see that this solution is unique. Using the closed-graph theorem (which is possible by (2.20)–(2.23)) we see that there is a constant c_1 so that

$$\|x\|_{L^\infty(R^+)}\leq c_1\|f\|_{L^\infty(R^+)}. \tag{6.1}$$

Now we claim that

$$\int_0^t |r(t,s)|\,ds\leq c_1+1,\qquad t\in R^+\backslash E \tag{6.2}$$

where E is a certain set with Lebesgue measure zero. If this is not the case there exists $t_0\in R^+\backslash E$ and a continuous function f such that $|f(t)|\leq 1$ for all $t\geq 0$ and

(6.3) $$\int_0^{t_0} r(t_0, s)f(s)ds > c_1+1.$$

Using Fubini's theorem and (2.6) we see that (1.3) holds for a.e. $t \geq 0$. Clearly (2.9) is satisfied and we conclude from (1.3), (2.10) and (6.3) that (6.1) cannot hold. Hence (6.2) is established.

To prove (2.15) we note that it follows from (2.20)–(2.24) that for a.e. $s \geq 0$ we have, defining $f_s(t) = k(t, s)$,

(6.4) $$f_s(t) \in BV([s, \infty)) \cap C([s, \infty)), \qquad \lim_{t\to\infty} f_s(t) = 0.$$

Fix such an s and let x_s be the unique solution of the equation

(6.5) $$x_s(t) + \int_s^t k(t, u)x_s(u)du = f_s(t), \qquad t \geq s.$$

If one changes the variable one sees that (6.5) is equation (1.1) with a kernel satisfying the conditions (2.20)–(2.24). Hence it follows from [2, Th. 5] and (6.4) that

(6.6) $$\lim_{t\to\infty} x_s(t) = 0.$$

(Note that [2, line (1.46)] follows from the assumptions of Theorem 5, cf. [4, proof of Th. 3]).

It is clear from (2.6), (6.5) and the definition of $f_s(t)$ that $x_s(t) = r(t, s)$, a.e. $0 \leq s \leq t$. Since it is a consequence of (2.6), (2.20), (2.23) and (6.2) that $|r(t, s)| \leq (c_1+1)$ $a(s)$ for a.e. (t, s), we can apply the dominated convergence theorem and (2.15) follows from (6.6). This completes the proof of Theorem 5.

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