Subrectangular Macdonald polynomials and the alphabet $\mathbb{X}^\vee$

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Abstract

We prove that subrectangular skew Macdonald polynomials are Macdonald polynomials.

1 Introduction

Macdonald polynomials are a (q,t)-deformation of the Schur functions and appear in the representation theory of the affine Hecke algebra (see *e.g.* [4, 6, 7]). The Macdonald polynomials considered in this paper are the homogeneous symmetric polynomials $P_\lambda(\mathbb{X};q,t)$ defined by orthogonality condition *w.r.t.* a deformation of the usual scalar product on symmetric functions. Our aim consists in proving that the skew Macdonald polynomial $P_{[r^n]/\lambda}(\mathbb{Y};q,t)$ is equal (up to an explicit multiplicative constant) to the polynomial $P_{[r^n]-\overleftarrow{\lambda}^n}$ where $\overleftarrow{\lambda}^n$ denotes the partition $(\lambda_n,\dots,\lambda_1)$ if $\lambda=(\lambda_1,\dots,\lambda_n)$. We show that this equality is a consequence of properties relying the Macdonald polynomials on a finite alphabet $\mathbb{X}=\{x_1,\dots,x_n\}$ and the alphabet of the opposite variables $\mathbb{X}^\vee:=\{x_1^{-1},\dots,x_n^{-1}\}$.

The paper is organized as follows. After recalling the classical definition and properties of Macdonald polynomials. We repeat, in Section 2 a theorem shown in [5]. In Section 3, we investigate the polynomials $P_\lambda(\mathbb{X}^\vee;q,t)$ for a finite alphabet $\mathbb{X}$. Finally, Section 4 is devoted to our main theorem.

2 Background and notations

One considers the (q,t)-deformation (see *e.g.* [6]) of the usual scalar product on symmetric functions defined for a pair of power sum functions Ψ^λ and Ψ^μ (in the notation of [3] by

$$\langle \Psi^\lambda, \Psi^\mu \rangle_{q,t} = \delta_{\lambda,\mu} z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}. \tag{1}$$

The family of Macdonald polynomials $(P_\lambda(\mathbb{X};q,t))_\lambda$ is the unique basis of symmetric functions orthogonal *w.r.t.* $\langle\ ,\ \rangle_{q,t}$ verifying

$$P_\lambda(\mathbb{X};q,t) = m_\lambda(\mathbb{X}) + \sum_{\mu \leq \lambda} u_{\lambda\mu} m_\mu(\mathbb{X}), \tag{2}$$

where m_λ denote, as usual, a monomial function [3, 6]. Denote by $Q_\lambda(\mathbb{X};q,t)$ the dual basis of $P_\lambda(\mathbb{Y};q,t)$ for $\langle\ ,\ \rangle_{q,t}$. One has

$$Q_\lambda(\mathbb{X};q,t) = \langle P_\lambda, P_\lambda \rangle_{q,t}^{-1} P_\lambda(\mathbb{X};q,t). \tag{3}$$

The coefficient $b_\lambda(q,t) = \langle P_\lambda, P_\lambda \rangle_{q,t}^{-1}$ is known to be

$$b_\lambda(q,t) = \prod_{(i,j)\in\lambda} \frac{1-q^{\lambda_j - i+1} t^{\lambda'_i - j}}{1-q^{\lambda_j - i} t^{\lambda'_i - j+1}} \tag{4}$$

see [6] VI.6.

Let us define as in [6] VI 7, the skew Q functions by

$$\langle Q_{\lambda/\mu}, P_\nu \rangle_{q,t} := \langle Q_\lambda, P_\mu P_\nu \rangle_{q,t}. \tag{5}$$

Straightforwardly, one has

$$Q_{\lambda/\mu}(\mathbb{X};q,t) = \sum_\nu \langle Q_\lambda, P_\nu P_\mu \rangle_{q,t} Q_\nu(\mathbb{X};q,t). \tag{6}$$

Let $\mathbb{X} = \{x_1, \ldots, x_n\}$ be a finite alphabet and $\mathbb{Y}$ be an other (potentially infinite) alphabet. Let us define as in [1] and [5] the transformation

$$\int_{\mathbb{Y}} x^p = S^p(\mathbb{Y}), \tag{7}$$

for each $x \in \mathbb{X}$ and each $p \in \mathbb{Z}$. Set $\mathbb{Y}^{tq} = \frac{1-t}{1-q}\mathbb{Y}$ and consider the substitution

$$\int_{\mathbb{Y}^{tq}} x^p = S^p\left(\mathbb{Y}^{tq}\right) = Q_p(\mathbb{Y}; q, t). \tag{8}$$

Setting

$$\mathfrak{H}^{n,k}_{\lambda/\mu}(\mathbb{Y}; q, t) := \frac{1}{n!}\int_{\mathbb{Y}} P_\lambda(\mathbb{X}; q, t) Q_\mu(\mathbb{X}^\vee; q, t)\Delta(\mathbb{X}, q, t) \tag{9}$$

where $\mathbb{X}^\vee = \{x_1^{-1}, \dots, x_n^{-1}\}$. In [5], the following property is shown.

Theorem 2.1 *Let $\mathbb{X} = \{x_1, \dots, x_n\}$ be an alphabet and $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition and $\mu \subset \lambda$. The polynomial $\mathfrak{H}^{n,k}_{\lambda/\mu}(\mathbb{Y}^{tq}; q, t)$ is the Macdonald polynomial*

$$\mathfrak{H}^{n,k}_{\lambda/\mu}(\mathbb{Y}^{tq}; q, t) = \frac{1}{n!}\prod_{(i,j)\in\lambda}\frac{1-q^{i-1}t^{n-j+1}}{1-q^i t^{n-j}}\mathrm{C.T.}\{\Delta(\mathbb{X}, q, t)\}Q_{\lambda/\mu}(\mathbb{Y}, q, t) \tag{10}$$

3 Macdonald polynomials for the alphabet $\mathbb{X}^\vee$

In this section $\mathbb{X} = \{x_1, \dots, x_n\}$ will denote an alphabet of size n. If λ and μ are two partitions of length at most n, we denotes by $(\lambda \ddagger \mu)_n$ the partition defined by

$$(\lambda \ddagger \mu)_n := \mathrm{sort}(\lambda_1 + \mu_n, \lambda_2 + \mu_{n-1}, \dots, \lambda_n + \mu_1)$$

where $\lambda_i = \mu_j = 0$ if $l(\lambda) + 1 \le i \le n$ and $l(\mu) + 1 \le j \le n$ and $\mathrm{sort}(v)$ is the unique (decreasing) partition obtained by a permutation of the elements of v. One need the following result.

Proposition 3.1

$$P_\lambda(\mathbb{X}; q, t)P_\mu(\mathbb{X}; q, t) = \sum_{(\lambda\ddagger\mu)_n \le \nu} f^\nu_{\mu\lambda}(q, t)P_\nu(\mathbb{X}; q, t). \tag{11}$$

Proof First, let us prove the similar identity for Schur functions. That is,

$$S_\lambda(\mathbb{X})S_\mu(\mathbb{X}) = \sum_{(\lambda\ddagger\mu)_n \le \nu} f^\nu_{\mu\lambda}S_\nu(\mathbb{X}). \tag{12}$$

The product of the two Schur functions can be written as the determinant

$$S_\lambda(\mathbb{X})S_\mu(\mathbb{X}) = \det\left(S^{\lambda_i - i + \mu_{n-j+1} + j}(\mathbb{X})\right)_{1 \le i,j \le n}. \tag{13}$$

The complete function $S^{(\lambda \ddagger \mu)}(\mathbb{X})$ is the product of the diagonal elements and $(\lambda \ddagger \mu)$ is the minimal partition having a contribution in the expansion of the determinant. Hence, one has

$$S_\lambda(\mathbb{X})S_\mu(\mathbb{X}) = \sum_{(\lambda \ddagger \mu)_n \le \nu} (*) S^\nu(\mathbb{X}). \tag{14}$$

But, for each partition ν, $S^\nu(\mathbb{X}) = \sum_{\rho \ge \nu} S_\rho(\mathbb{X})$, hence Equality (12) holds.

Now, each polynomial $P_\lambda(\mathbb{X}; q, t)$ can be written as

$$P_\lambda(\mathbb{X}; q, t) = \sum_{\rho \ge \lambda} (*) S_\rho(\mathbb{X}^{qt}) \tag{15}$$

(see [2] for a determinantal expression). Hence, from (12),

$$P_\lambda(\mathbb{X}; q, t) P_\mu(\mathbb{X}; q, t) = \sum_{\rho \ge (\lambda \ddagger \mu)} (*) S_\rho(\mathbb{X}^{qt}). \tag{16}$$

The result follows.

□

Example 3.2 If $\mathbb{X} = \{x_1, x_2, x_3\}$, one has $(21 \ddagger 211) = [322]$ and

$$\begin{aligned} P_{21}(\mathbb{X}; q, t) P_{211}(\mathbb{X}; q, t) = \ & \frac{(-1+q)(t+1)\left(qt^3-1\right)\left(q^2t-1\right)}{(qt^2-1)(qt+1)(qt-1)^2} P_{322}(\mathbb{X}; q, t) \\ & + \frac{(-1+q)(t+1)}{qt-1} P_{331}(\mathbb{X}; q, t) + P_{421}(\mathbb{X}; q, t). \end{aligned}$$

Corollary 3.3 *Let $n, r \in \mathbb{N}$ and $\mathbb{X}$ be an alphabet of size n, for any partition $\lambda \subset [r^n]$, one has*

$$\Lambda^n(\mathbb{X})^r Q_\lambda(\mathbb{X}^\vee; q, t) = \prod_{(i,j) \in \lambda} \frac{1 - q^{\lambda_j - i + 1} t^{\lambda'_i - j}}{1 - q^{\lambda_j - i} t^{\lambda'_i - j + 1}} P_{[r^n] - \overleftarrow{\lambda}^n}(\mathbb{X}; q, t).$$

Proof Consider the scalar product

$$\langle \Lambda^n(\mathbb{X})^r Q_\lambda(\mathbb{X}^\vee; q, t), P_\mu(\mathbb{X}; q, t) \rangle = \langle \Lambda^n(\mathbb{X}), Q_\lambda(\mathbb{X}; q, t) P_\mu(\mathbb{X}; q, t) \rangle. \tag{17}$$

But $\Lambda^n(\mathbb{X})^r = P_{[r^n]}(\mathbb{X};q,t)$ and $[r^n]$ is the minimal partition of length n and weight rn for the dominance order. Hence, Lemma 3.1 implies that if $[r^n] \neq (\lambda \ddagger \mu)$ then $\langle \Lambda^n(\mathbb{X}), Q_\lambda(\mathbb{X};q,t)P_\mu(\mathbb{X};q,t)\rangle = 0$. It follows

$$\langle \Lambda^n(\mathbb{X})^r Q_\lambda(\mathbb{X}^\vee;q,t), P_\mu(\mathbb{X};q,t)\rangle = (*)\delta_{[r^n]-\overleftarrow{\lambda}^n,\mu}. \quad (18)$$

Hence, the polynomials $\Lambda^n(\mathbb{X})^r Q_\lambda(\mathbb{X}^\vee;q,t)$ and $P_{[r^n]-\overleftarrow{\lambda}^n}(\mathbb{X};q,t)$ are proportional. Computing the coefficient of $m_{[r^n]-\overleftarrow{\lambda}^n}$ in the expansion of the two polynomials, one finds

$$\Lambda^n(\mathbb{X})^r Q_\lambda(\mathbb{X}^\vee;q,t) = \langle P_\lambda, P_\lambda\rangle P_{[r^n]-\overleftarrow{\lambda}^n}(\mathbb{X};q,t). \quad (19)$$

The result follows. □

4 Subrectangular skew-Macdonald polynomials are Macdonald polynomials

Theorem 4.1

$$\begin{aligned} Q_{[r^n]/\lambda}(\mathbb{Y};q,t) = & \prod_{(i,j)\in\lambda} \frac{1-t^{\lambda_i-j}q^{\lambda'_j-i+1}}{1-t^{\lambda_i-j+1}q^{\lambda_j-i}} \times \\ & \times \prod_{(i,j)\in[r^n]/[r^n]-\overleftarrow{\lambda}^n} \frac{1-q^{r-j}t^i}{1-q^{r-j+1}t^{i-1}} Q_{[r^n]-\overleftarrow{\lambda}^n}(\mathbb{Y};q,t). \end{aligned} \quad (20)$$

Proof From Theorem 2.1 the proportionality of $Q_{[r^n]/\lambda}(\mathbb{Y})$ and $Q_{[r^n]-\overleftarrow{\lambda}^n}(\mathbb{Y})$ is equivalent to the proportionality of $\mathfrak{H}^{m,k}_{[r^n]/\lambda}(\mathbb{Y}^{tq};q,t)$ and $\mathfrak{H}^{m,k}_{[r^n]-\overleftarrow{\lambda}^n}(\mathbb{Y}^{tq};q,t)$ for a $m \geq n$. More precisely, it suffices to show the property when $n = m$. Writing

$$\mathfrak{H}^{n,k}_{[r^n]/\lambda}(\mathbb{Y}^{tq};q,t) = \int_{\mathbb{Y}^{tq}} P_{[r^n]}(\mathbb{X};q,t)Q_\lambda(\mathbb{X}^\vee;q,t)$$

and

$$\mathfrak{H}^{n,k}_{[r^n]/\lambda}(\mathbb{Y}^{tq};q,t) = \int_{\mathbb{Y}^{tq}} P_{[r^n]-\overleftarrow{\lambda}^n}(\mathbb{X};q,t),$$

where $\mathbb{X} = \{x_1,\dots,x_n\}$. Let us prove that

$$P_{[r^n]}(\mathbb{X};q,t)Q_\lambda(\mathbb{X}^\vee;q,t) = (*)P_{[r^n]-\overleftarrow{\lambda}^n}(\mathbb{X};q,t)$$

where $(*)$ is a constant coefficient. Since the size of $\mathbb{X}$ is n, $P_{[r^n]}(\mathbb{X};q,t) = (x_1 \dots x_n)^r$, we need only to prove

$$(x_1 \dots x_n)^r Q_\lambda(\mathbb{X}^\vee;q,t) = (*) P_{[r^n]-\overleftarrow{\lambda}^n}(\mathbb{X};q,t).$$

This is a consequence of Corollary 3.3. One obtains

$$\mathfrak{H}_{[r^n]/\lambda}(\mathbb{X}^\vee;q,t) = \prod_{(i,j)\in\lambda} \frac{1-q^{\lambda_j-i+1}t^{\lambda'_i-j}}{1-q^{\lambda_j-i}t^{\lambda'_i-j+1}} \mathfrak{H}_{[r^n]-\overleftarrow{\lambda}^n}(\mathbb{X};q,t). \tag{21}$$

And by Theorem 2.1, one has find the result. □

Example 4.2 Let us explain how to obtain the following result

$$Q_{[44]/[32]}(\mathbb{X};q,t) = \frac{Q_{[21]}(\mathbb{X};q,t)\,(-1+q)\,(t+1)}{qt-1}$$

The first product of Equality (20) can be graphically interpreted as

×	×		
×	×	×	

and each marked cell (i,j) by $(i,j) := \frac{1-t^{\lambda_i-j}q^{\lambda'_j-i+1}}{1-t^{\lambda_i-j+1}q^{\lambda'_j-i}}$.

Hence, the first product reads

$$(1,1)(2,1)(3,1)(1,2)(2,2) = \frac{(1-q^3t)(1-q^2t)(1-q)(1-q^2)(1-q)}{(1-q^2t^2)(1-qt^2)(1-t)(1-qt)(1-t)} \tag{22}$$

The second product of (20) can be interpreted as

	×	×	×
		×	×

and each cell $\langle i,j\rangle$ by $\langle i,j\rangle := \frac{1-q^{4-j}t^i}{1-q^{5-j}t^{i-1}}$. Hence, the second product is

$$\langle 2,2\rangle\langle 3,2\rangle\langle 4,2\rangle\langle 3,1\rangle\langle 4,1\rangle = \frac{(1-q^2t^2)(1-qt^2)(1-t^2)(1-qt)(1-t)}{(1-q^3t)(1-q^2t)(1-qt)(1-q^2)(1-q)} \tag{23}$$

Multiplying (22) and (23), one recovers the result after simplifications

$$(1,1)(2,1)(3,1)(1,2)(2,2)\langle 2,2\rangle\langle 3,2\rangle\langle 4,2\rangle\langle 3,1\rangle\langle 4,1\rangle = \frac{(-1+q)\,(t+1)}{qt-1}.$$

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