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NONCOMMUTATIVE DUPLICATE AND JORDAN ALGEBRAS II

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Abstract

Let K be a commutative field of characteristic zero. Let A be a finite dimensional algebra over K, not necessarily commutative and $D(A)$ the noncommutative duplicate of A. Here we give necessary and suffisant conditions for $D(A)$ to be a Jordan algebra.

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Résumé

Soit K un corps commutatif de caractéristique zéro. Soient A une K-algèbre non nécessairement commutative de dimension finie et $D(A)$ sa dupliquée non commutative. Nous donnons ici des conditions nécessaires et suffisantes pour que $D(A)$ soit une algèbre de Jordan.

1 Introduction

This paper follows an article ([8]) where it is given a characterization of the subalgebra A^2 of A when the noncommutative duplicate $D(A)$ of A is Jordan algebra and A^2 is a flexible, power-associative, but not a nilalgebra; in this study, it is obtained the following result "The noncommutative duplicate $D(A)$ of A is a Jordan algebra if and and only if $\dim_K(A^2) = 1$". We give here a characterization of the subalgebra of A^2 when $D(A)$ is a Jordan algebra and the result mentioned will be a corollary.

2 Noncommutative duplicate

We recall here some general results on the duplicate ([6], [7]).

Definition 1. ([7]) Let K be a commutative field of characteristic zero and let A be a finite dimensional K-algebra. The noncommutative duplicate $D(A)$ of A is defined as the algebra whose underlying vector space is the tensor product $A \underset{K}{\otimes} A$ endowed with the product $(x \otimes y)(x' \otimes y') = xy \otimes x'y'$ for all x, y, x', y' in A.

Note that if $\dim_K(A) = n, (n \geq 1)$, then $\dim_K(D(A)) = n^2$. Moreover, if $(e_i)_{0 \leq i \leq n-1}$ is a basis of A, then $(e_i \otimes e_j)_{0 \leq i,j \leq n-1}$ is a basis of $D(A)$.

The linear mapping $\mu : D(A) \longrightarrow A^2$ with $\mu(x \otimes y) = xy$ defines an algebra epimorphism. If $N_D(A)$ is the kernel of μ, there exists an algebra isomorphism $D(A)/N_D(A) \cong A^2$ and $N_D(A)D(A) = \{0\} = D(A)N_D(A)$. Since the exact sequence $0 \longrightarrow N_D(A) \longrightarrow D(A) \overset{\mu}{\longrightarrow} A^2 \longrightarrow 0$ splits, there exists a K-linear application $\eta : A^2 \longrightarrow D(A)$ such that $\mu \circ \eta = Id_{A^2}$. The K-bilinear application $\varphi : A^2 \times A^2 \longrightarrow N_D(A)$ defined by $\varphi(x, y) = \eta(x)\eta(y) - \eta(xy)$ and the conditions $N_D(A)D(A) = \{0\} = D(A)N_D(A)$ are used to define on the product $A^2 \times N_D(A)$ a finite dimensional K-algebra structure where the multiplication is given by $(x, m)(y, n) = (xy, \varphi(x, y))$ for all x, y in A^2 and all m, n in $N_D(A)$.
We denote this algebra by $A^2 \underset{s.d.}{\times} N_D(A)$ (s.d. for semi-direct).

This allows us to state the following result:

Theorem 2. *(Etherington) ([7]) Let K be a commutative field and A a finite dimensional K-algebra. Then $D(A) \cong A^2 \underset{sd}{\times} N_D(A)$ an isomorphism of K-algebras.*

In the following we will adopt the notation $D(A) = A^2 \underset{\varphi}{\times} N_D(A)$, where the structure of the noncommutative duplicate of an algebra A is given by $(x, m)(y, n) = (xy, \varphi(x, y))$ and φ is defined above. It should be noted that the structure of $A^2 \underset{\varphi}{\times} N_D(A)$ is, up to isomorphism, independent of the choice of the linear application η.

Proposition 3. *([5]) Let $\mathcal{C}$ be a class of algebras and A a finite dimensional K-algebra. The algebra $D(A) = A^2 \underset{\varphi}{\times} N_D(A)$ is in the class $\mathcal{C}$ if and only if the algebra A^2 is in the class $\mathcal{C}$ and φ is a 2-cocycle with coefficients in $N_D(A)$.*

Then it is clear that the properties of $D(A)$ depend on the properties of A^2.

In the following, we examine the conditions that $D(A)$ is a Jordan algebra.

We mention now some results which are necessary.

Lemma 4. *For all z, z' in $D(A)$, $zz' = \mu(z) \otimes \mu(z')$.*

Using Lemma 4, we have the following result:

Lemma 5. *Let $\eta : A^2 \longrightarrow D(A)$ be a K-linear application such that $\mu \circ \eta = id_{A^2}$ and $D(A) = A^2 \underset{\varphi}{\times} N_D(A)$ where $\varphi(x, y) = \eta(x)\eta(y) - \eta(xy)$. Then for all $x, y \in A^2$, $\eta(x)\eta(y) = x \otimes y$.*

Remark 6. If the characteristic of the field K is zero, then in the tensor produit $A \underset{K}{\otimes} A$, $x \otimes y = 0$ if and only $x = 0$ or $y = 0$ ([4] p.257).

Corollary 7. *([3]) If the characteristic of the field K is zero, in the tensor product $A \underset{K}{\otimes} A$, if x is a nonzero vector in A and $x \otimes y = z \otimes x$ then there exists a linear form f on A such that $z = y = f(x)x$.*

Proof. If $x = 0$, then $x \otimes y = z \otimes x$ for all y, z in A. Suppose that x is not zero and let y, z be the vectors of A. If $x \otimes y = z \otimes x$, then $y \in Kx$ and $z \in Kx$ say that it exists forms K-linear f and g on A such that $y = f(x)x$ and $z = g(x)x$. Thus, we have $0 = f(x)x \otimes x - g(x)x \otimes x = (f(x) - g(x))x \otimes x$; as x is nonzero then $f(x) = g(x)$. □

3 Duplication and Jordan algebras

Let A be a finite dimensional K-algebra not necessarily commutative or associative whose multiplication is denoted by xy.

We say that A is *flexible* if $(xy)x = x(yx)$ for all x, y in A.

We say that A is a K-*Jordan algebra* if A is flexible and $x^2(yx) = (x^2y)x$ for all x, y in A ([10] p. 141).

Let K be a commutative field, A a K-algebras of finite dimensional and $D(A)$ the noncommutative duplicate of A.

Theorem 8. *The algebra $D(A)$ is a Jordan algebra if and only if A^2 is a Jordan algebra and $\varphi(xy, x) - \varphi(x, yx) = 0$, $\varphi(x^2y, x) - \varphi(x^2, yx) = 0$ for all x, y, z in A^2.*

Proof. That is a traduction of Proposition 3. □

We have these following results which are immediate ([8]).

Proposition 9. *If A^2 is a zero-algebra then, $D(A)$ is a Jordan algebra.*

Proposition 10. *If $\dim_K(A^2) = 1$, then $D(A)$ is a Jordan algebra.*

The following result will be widely used.

Lemma 11. *If A^2 is a Jordan algebra, the map $\varphi : A^2 \times A^2 \longrightarrow N_D(A)$ verifies $\varphi(xy, x) - \varphi(x, yx) = xy \otimes x - x \otimes yx$ and $\varphi(x^2y, x) - \varphi(x^2, yx) = x^2y \otimes x - x^2 \otimes yx$.*

Proof. Suppose that $D(A)$ is a Jordan algebra. We have:

$$\begin{aligned}\varphi(xy,x)-\varphi(x,yx) &= \eta(xy)\eta(x)-\eta((xy)x)-\eta(x)\eta(yx)+\eta(x(yx))\\ &= \eta(xy)\eta(x)-\eta(x)\eta(yx)-\eta((xy)x-x(yx))\\ &= \eta(xy)\eta(x)-\eta(x)\eta(yx)-\eta(0)\\ &= xy\otimes x-x\otimes yx.\end{aligned}$$

Similary we get the relation $\varphi(x^2y,x)-\varphi(x^2,yx)=x^2y\otimes x-x^2\otimes yx$. □

For the sequel, we denote:

$$\Phi_{J2}(x,y)=x^2y\otimes x-x^2\otimes yx$$

and

$$\Psi_{J2}(x,y)=xy\otimes x-x\otimes yx.$$

Corollary 12. *The algebra $D(A)$ is a Jordan algebra if and only if the three following conditions are satisfied: (i) A^2 is Jordan algebra, (ii) $\Phi_{J2}(x,y)=0$ and (iii) $\Psi_{J2}(x,y)=0$ for all x,y in A^2.*

Remark 13. Let A be the three dimensional K-algebra with multiplication $e^2=e, xy=yx=x$ with respect to the basis $\{e,x,y\}$, other products being zero. The subalgebra $A^2=<e,x>$ is a Jordan algebra, but $\Psi(e+x,y)=(e+x)y\otimes(e+x)=(e+x)\otimes y(e+x)=x\otimes e-e\otimes x$ and the noncommutative duplicate is not a Jordan algebra.

4 Algebras which the noncommutative duplicate is a Jordan algebra

These following results will be helpful.

Lemma 14. *([3]) Let A be finite dimensional K-algebra. Then the following conditions are equivalent for two vectors x and y in A.*

1. *$f(x)=0$ for all linear form f on A implies $f(y)=0$ for all linear form f on A.*

2. The two vectors x and y are linearly dependant.

Proof. The fact that the assertion 2 implies the assertion 1 is trivial. Let us show that assertion 1 implies assertion 2. Suppose that the vectors $e_1 = x$ and $y = e_2$ are K-linearly independent and complement the pair $e_1,\ e_2$ in a basis $B = \{e_1, e_2, e_3, ..., e_n\}$ of the K-vector space A. Let $\{e'_1, e'_2, e'_3, ..., e'_n\}$ be the dual basis of B, basis of A^* the dual algebra of A. This basis satisfies the conditions $e'_i(e_j) = \delta_{ij}$ $(i, j = 1, 2, ..., n)$ where δ_{ij} is the Kronecker symbol. We have the linear form e'_2 on A that satisfies $e'_2(e_1) = 0$ and $e'_2(e_2) = 1$. □

Corollary 15. *Let A be a algebra and $D(A)$ its noncommutative duplicate. If $D(A)$ is Jordan algebra, then $dim_K(A^2)^2 \leq 1$.*

Proof. Indeed, since $D(A)$ is a Jordan algebra, $xy \otimes x - x \otimes yx = 0$ for all x, y in A^2. Corollary 7 says that xy and x are linearly dependant for all x, y in A^2; then $\dim_K((A^2)^2) \leq 1$. □

Remark 16. Let A be the three dimensional K-algebra which the multiplication relative to the base $\{e_1, e_2, e_3\}$ is gived by: $e_2^2 = e_2$, $e_3^2 = e_1$ and the other products are nul. We have $A^2 =< e_1, e_2 >$ and $(A^2)^2 =< e_2 >$. In the noncommutative duplicate of A, we have:

$$\begin{aligned}\Psi_{J2}(e_1 + e_2, e_2) &= (e_1 + e_2)e_2 \otimes (e_1 + e_2) - (e_1 + e_2) \otimes e_2(e_1 + e_2)\\ &= e_2 \otimes (e_1 + e_2) - (e_1 + e_2) \otimes e_2\\ &= e_2 \otimes e_1 + e_2 \otimes e_2 - e_1 \otimes e_2 - e_2 \otimes e_2\\ &= e_2 \otimes e_1 - e_1 \otimes e_2.\end{aligned}$$

Then $\Psi_{J2}(e_1 + e_2, e_2)$ is not zeo and the noncommutative duplicate of A is not a Jordan algebra.

We consider the field of reel numbers $\mathbb{R}$ and recall the classification of the two dimensional commutative $\mathbb{R}$-Jordan algebras give in [2].

Theorem 17. *([2]) A two dimensional commutative $\mathbb{R}$-Jordan algebras is isomorphic at one of the following algebras.*
AJ1 : $(e_1)^2 = e_1, e_1e_2 = e_2, (e_2)^2 = e_1$;

AJ2 : $(e_1)^2 = e_1, e_1e_2 = e_2, (e_2)^2 = 0$;
AJ3 : $(e_1)^2 = 0, e_1e_2 = 0, (e_2)^2 = e_2$;
AJ4 : $(e_1)^2 = e_2, e_1e_2 = 0, (e_2)^2 = 0$;
AJ5 : $(e_1)^2 = e_1, e_1e_2 = \frac{1}{2}e_2, (e_2)^2 = 0$;
AJ6 : $(e_1)^2 = e_1, e_1e_2 = e_2, (e_2)^2 = -e_1$.
Moreover, the algebra AJ5 is a commutative simple Jordan algebra.

Remark 18. It is clear that in Theorem 17, the two dimensional zero-algebra must be added.

We use Theorem 17 to give a characterization of the two dimensional reel commutative Jordan algebras when their noncommutative duplicates are Jordan algebras.

Theorem 19. *Let A be a reel commutative algebra such that A^2 is two dimensional Jordan algebra. The noncommutative duplication of A is a Jordan algebra if an only if A^2 is a zero-algebra.*

Proof. Since A^2 is a two dimensional reel commutative Jordan algebra, A^2 is a zero-algebra or is isomorphic to one of the algebra quoted in Theorem 17. Then using Corollary 12 and a direct calculation we have the result. □

We have the following results.

Theorem 20. *Let K be a commutative field characteristic zero and A a K-algebra. Suppose that $dim_K(A^2) = 2$. Then the following assertions are equivalent.*

1. *$D(A)$ is Jordan algebra.*
2. *A^2 is a zero-algebra.*

Proof. Suppose that $D(A)$ is a Jordan algebra. Using Corollary 15, we have $\dim_K(A^2)^2 \leq 1$. Let {e, f} be a basis of A^2 such that $(A^2)^2 \subseteq Ke$ and write $e^2 = \alpha e,\ ef = \beta e,\ fe = \beta' e$ and $f^2 = \gamma e$. We have $\Psi_{J2}(f, e) = fe \otimes f - f \otimes ef = \beta' e \otimes f - f \otimes (\beta e)$. The relation $\Psi_{J2}(f, e) = 0$ says that $\beta = \beta' = 0$. $\Psi_{J2}(e + f, e) = (f + e)e \otimes (e + f) - (e+f) \otimes e(e+f) = \alpha e \otimes (e+f) - (e+f) \otimes (\alpha e) = \alpha(e \otimes f - f \otimes e)$. The relations $\Psi_{J2}(e + f, e) = 0$ says that $\alpha = 0$. Similary the relation $\Psi_{J2}(f, e + f) = 0$ implies $\gamma = 0$. Thus it is proved that assertion 1. implies assertion 2.. The converse is obvious. □

Theorem 21. *Let K be a commutative field characteristic zero and A a K-algebra. Then the following assertions are equivalent.*

1. *$D(A)$ is Jordan algebra.*
2. *One of these conditions is satisfied:*
   (a) *$dim_K(A^2) \leq 1$;*
   (b) *$dim_K(A^2) \geq 2$ and A^2 is a zero-algbra.*

Proof. Suppose that $D(A)$ is Jordan algebra. Corollary 15 says that $dim_K((A^2)^2) \leq 1$. If $dim_K(A^2) \leq 1$, then Proposition 10 allows to conclude. If $dim_K(A^2) = 2$, Theorem 20 allows to conclude. Suppose that $dim_K(A^2)^2 \leq 1$ and $dim_K(A^2) \geq 3$; consider a family $\{e, u, v\}$ of linearly independent vectors of A^2 such that $(A^2)^2 \subseteq Ke$. The condition $(A^2)^2 \subseteq Ke$ says that $B =< e, u, v >$ is a subalgebra of A^2 and that, $B_1 =< e, u >$ and $B_2 =< e, v >$ are subalgebras of A^2. Applying Theorem 20 to B_i $(i = 1, 2)$, we get: B_i $(i = 1, 2)$ is a zero-algebra. Then B is a zero-algebra or B has one of the following multiplication tables:

$\curvearrowright$	e	u	v
e	0	0	βe
u	0	0	γe
v	$\beta' e$	$\gamma' e$	0

$\curvearrowright$	e	u	v
e	0	δe	0
u	$\delta' e$	0	ρe
v	0	$\rho' e$	0

If B is a zero-algebra, as the vectors u and v are arbitrary such that e, u, v are lineary independant, then A^2 is a zero-algebra. Suppose that B has a multiplication gived by the first table. $\Psi_{J2}(v, e) = ve \otimes v - v \otimes ev = \beta' e \otimes v - \beta v \otimes e$ and then the relation $\Psi_{J2}(v, e) = 0$ say that $\beta = \beta' = 0$; $\Psi_{J2}(v, u) = vu \otimes v - v \otimes uv = \gamma' e \otimes v - \gamma v \otimes e$ and then the relation $\Psi_{J2}(v, u) = 0$ say that $\gamma = \gamma' = 0$. We conclude that A^2 is a zero-algebra. If B has a multiplication gived by the second table, using the relations $\Psi(u, e) = 0$ and $\Psi(u, v) = 0$, we get $\delta = \delta' = 0$ and $\rho = \rho' = 0$ and then we conclude that A^2 is a zero-algebra. So it is proved that the assertion 1. implies assertion 2.. The converse is immediate. □

In [8] having assumed that A is an algebra not necessearily commutative and A^2 is flexible, associative powers and non-nil, it is studied the conditions for the noncommutative duplicate of A to be a Jordan algebra using the Peirce decomposition given by in [1]; it is obtained the following result which is a corollary of Theorem 21.

Theorem 22. *([8]) Let A a K-algebra such that A^2 is flexible, power-associative and non nil. The noncommutative duplicate $D(A)$ of A is Jordan if and only* $\dim_K(A^2) = 1$.

Proof. The given conditions allow to say that A^2 possesses a nonzero idempotent e ([1]). The idempotent e is a nonzero vector of $(A^2)^2$ and finally Theorem 21 allows us to say that $A^2 = (A^2)^2 = Ke$.

□

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