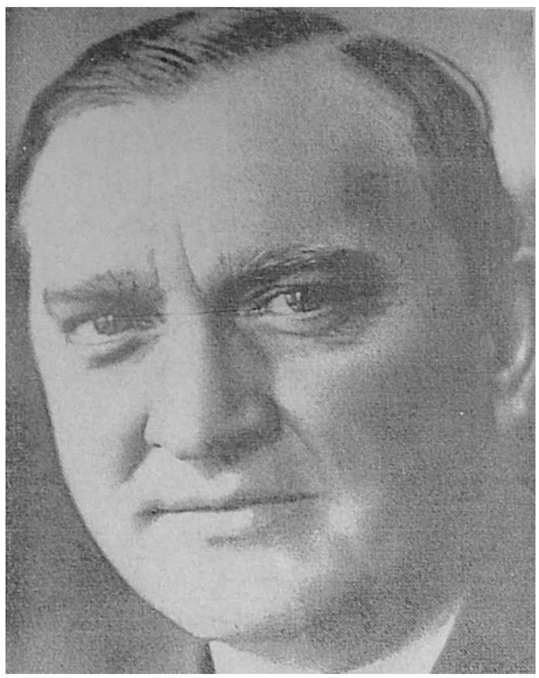
Stefan Banach
- Polish mathematician.
His life and achievements



Katarzyna Trabka

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1. Biography

Stefan Banach was born March 30, 1892 in Kraków (then Austria-Hungary, now Poland) and died August 31, 1945 in Lwów (then Poland occupied by Soviet Union, now Ukraine). His father, Stefan Greczek, a tax official was not married to Banach's mother. Banach's mother, Katarzyna Banach, disappeared after Stefan was baptised, when he was only four days old and nothing more is known about her. On baptism Banach was given his father's first name and mother's surname. He did not know his mother. He only met his father once in a while.

Figure 1. Stefan Banach

He was brought up by Franciszka Płowa who lived in Kraków with her daughter Maria. Banach attended primary school in Kraków. In 1902 he began his secondary education at the Henryk Sienkiewicz Gymnasium No. 4 in Kraków. From his early teens he was talented in mathematics and languages. During his first few years at the Gymnasium Banach achieved first class grades in mathematics, and natural sciences were his favourites subjects. A fellow school pupil recalled Banach: *he was pleasant in dealings with his colleagues, but outside of mathematics he was not interested in anything. If he spoke at all, he would speak very rapidly, as rapidly as he thought mathematically ... there was no mathematical problem that he could not speedily tackle.*

The excellent grades of his early years gave way to poorer grades as he approached his final school examination. He passed this examination in 1910 but he failed to achieve a pass with distinction, an honour which went to about one quarter of the students. On leaving school Banach wanted to study mathematics but he felt that nothing new could be discovered in mathematics so he chose to study engineering.

Banach's father had never given his son much support but now, once he left school, he quite openly told Banach that he was now on his own. Banach left Kraków and went to Lwów where he enrolled in the Faculty of Engineering at Lwów Technical University. Without any financial support, he had to support himself by tutoring and working in a bookshop. This occupied quite a lot of his time and he passed only partly-exam after two years studies. He did not complete the entire curriculum. He had returned to Kraków frequently during the period of his studies in Lwów. The outbreak of First World War in August caused that Banach left Lwów.

At the time Banach's studies, Lwów was under Austrian control as it had been from the partition of Poland in 1772. In Banach's youth Poland did not exist and Russia controlled much of the country. With the outbreak of First World War, the Russian troops occupied the city of Lwów. Banach was not physically fit for army service, having poor vision in his left eye. During the war he worked building roads. He also spent time in Kraków where he earned money by teaching in local schools. He also attended mathematics lectures at the Jagiellonian University in Kraków but mostly he studied mathematics by himself.

A chance event occurred in the spring of 1916 and it had a major impact on Banach's life. Profesor Hugo Steinhaus [1] , during his walk through the streets of Kraków, met Banach by accident. As he related it in his memories: *During one walk I overheard the words "Lebesgue measure". I approached the park bench and introduced myself to the two young apprentices of mathematics ... From then we would meet on a regular basis, and ... we decided to establish a mathematical society.* Steinhaus told Banach about a problem which he was working on without

[1] Hugo Dionizy Steinhaus (born January 14, 1887 Jasło, died February 25, 1972 Wrocław) was a Polish mathematician, educator and humanist. He was professor of Lwów and Wrocław Universities and then University of Notre Dame (USA), University of Sussex, corresponding member of Polish Academy of Abilities and Polish Academy of Sciences, and many international science societies and science academies. He was author of many works (over 170) in the fields of mathematical analysis, probability theory and statistics.

FIGURE 2. Stefan Banach

success. After a few days Banach had the general idea for the required counterexample and both men wrote a joint paper. The war delayed publication but the paper, Banach's first, appeared in the *Bulletin of the Kraków Academy* (Polish: Biuletyn Akademii Krakowskiej) in 1918. From the time that he produced these first results with Steinhaus, Banach started to produce important mathematics papers at a rapid rate. It was also through Steinhaus that Banach met his future wife Łucja Braus. They got married in the mountain resort Zakopane in 1920.

On Steinhaus's initiative, the Mathematical Society of Kraków was set up in 1919. Profesor Stanisław Zaremba [2] chaired the inaugural meeting and was elected as the first President of the Society. Banach lectured to the Society twice during 1919 and continued to produce top quality research papers. The Mathematical Society of Kraków went on to became the Polish Mathematical Society (Polish: Polskie Towarzystwo Matematyczne) [3] in 1920.

[2] Stanisław Zaremba (born October 3, 1863 Romanowka Ukraine, died November 23, 1942 Kraków) was a Polish mathematician. He studied engineering at the Institute of Technology in St Petersburg. After graduating he went to Paris where he studied mathematics for his doctorate at the Sorbonne. He published his results in French mathematical journals so his work became well known. He returned to Poland in 1900 where he was appointed to a chair in the Jagiellonian University in Kraków. He achieved much in teaching, writing textbooks and organising the progress of mathematics in Kraków. Much of his research work was in partial differential equations and potential theory. He also made major contributions to mathematical physics and to crystallography.

[3] See [14] and [15]

Banach was offered an assistantship to profesor Antoni Łomnicki [4] at Lwów Technical University in 1920. He lectured there in mathematics and submitted a dissertation for his doctorate under Łomnicki's supervision. This was not the standard route to a doctorate, because Banach had no university mathematics qualifications - he didn't graduate the studies. However, an exception was made to allow him to submit *On Operations on Abstract Sets and their Application to Integral Equations*. This thesis is said to mark *the birth of functional analysis*.

In 1922 the Jan Kazimierz University in Lwów awarded Banach his habilitation for a thesis on measure theory. In 1924 Banach was promoted to full professor and he spent the academic year 1924-25 in Paris. The years between the wars were extremely busy for Banach. He was a Dean of Faculty of Mathematics at the Jan Kazimierz University in Lwów. Apart from continuing to produce a stream of important papers, he wrote arithmetic, geometry and algebra textbooks for high schools. He was also very much involved with the publication of mathematics. A group of young talented mathematicians gathered around him. They would meet in cafes, mainly in famous Scottish Cafe [5] (figure 3) and they created The Lwów School of Mathematics. [6] In 1929 Banach together with Steinhaus started a new journal *Studia Mathematica* (figure 4) which *focus on research in functional analysis and related topics.*

Another important publishing venture, begun in 1931, was a new series of *Mathematical Monographs*. These were set up under the editorship of Banach and Steinhaus from Lwów and Kazimierz Kuratowski, Stefan Mazurkiewicz and Wacław Sierpinski from Warszawa. [7] The first volume in the series *Theorie des Operations Lineaires* was written by Banach and appeared in 1932 (figure 5). It was a French version of a volume he originally published in Polish in 1931 and quickly became a classic. In 1936 Banach gave a plenary address at the

[4] Antoni Marian Łomnicki (born January 17, 1881, died July 4, 1941 Lwów, then Poland) was a Polish mathematician. He studied at the Lwów University and the University in Gottingen. In 1920 he became professor of the Lwów University of Technology. Since 1938 he was a member of the Warsaw Science Society. He was murdered by the Germans during the Second World War in Lwów.

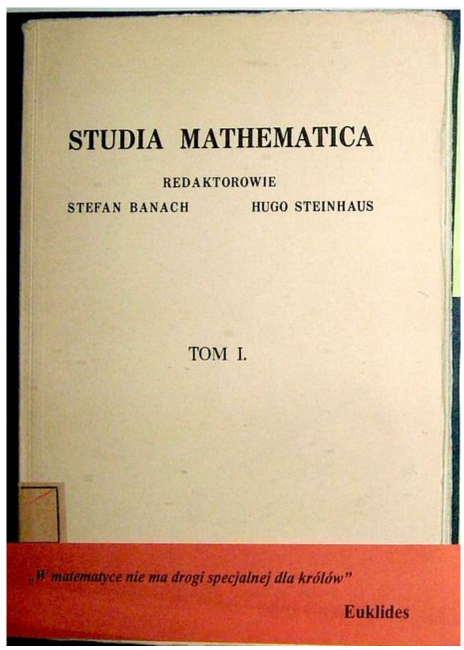
[5] The Scottish Cafe was the cafe in Lwów where, in the 1930s and 1940s, Polish mathematicians from the Lwów School met and spent their afternoons discussing mathematical problems.

[6] The Lwów School of Mathematics was a group of mathematicians working in Lwów (then Poland, now Ukraine) between the World Wars. They often met at the famous Scottish Cafe to discuss mathematical problems, and published in the journal *Studia Mathematica*.

[7] Polish mathematicians, members of Warsaw School of Mathematics. See more in [16]

FIGURE 3. Scottish Cafe in Lwów where Polish mathematicians met and spent their afternoons discussing mathematical problems

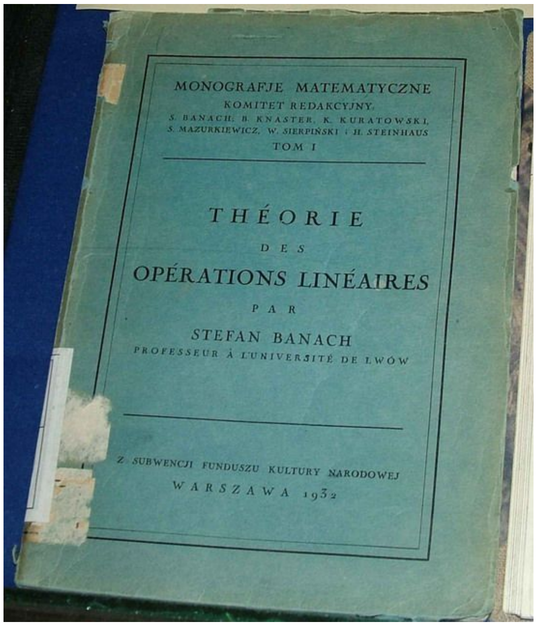
FIGURE 4. Journal *Studia Mathematica* edited by Stefan Banach and Hugo Steinhaus

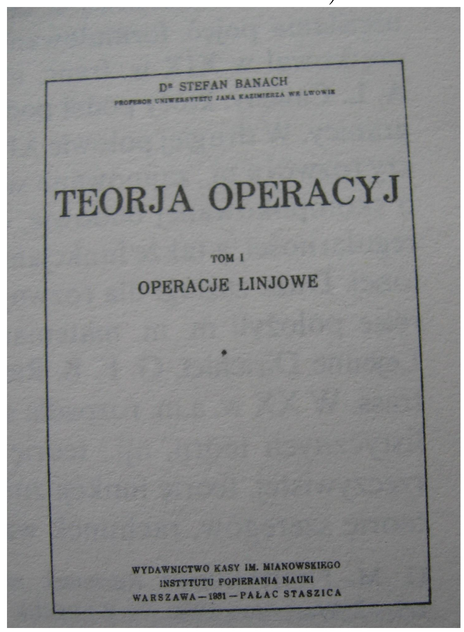


International Congress of Mathematicians in Oslo. In this address he described the work of the whole of the Lwów School and he also spoke about the plans which had to develop their ideas further.

In 1939, just before the start of Second World War, Banach was elected President of the Polish Mathematical Society. At the beginning of the war Soviet

FIGURE 5. The first volume in the series of *Mathematical Monographs* written by Stefan Banach (French and Polish version)





troops occupied Lwów. Banach had been on good terms with the Soviet mathematicians before the war started, having visited Moscow several times, and he was treated well by the new Soviet administration. He was allowed to continue to hold his chair at the university and he became a Dean of the Faculty of Science at the university, now renamed the Ivan Franko University. Banach continued his research, his textbooks writing, lecturing and sessions in the cafes. He also attended conferences in the Soviet Union. He was staying in Kiev (Polish: Kijów) when Germany invaded the Soviet Union and he returned immediately to Lwów to his family - his wife Łucja and his son Stefan.

The Nazi occupation of Lwów in June 1941 caused that Banach's conditions of living totaly changed. He was arrested under suspicion of trafficking in German currency but released after a few weeks. He survived a period when Polish academics were murdered, his doctoral supervisor Łomnicki dying on the tragic night of 3 July 1941 when many massacres occurred. [8] From July 1941 up to February 1944 he worked feeding lice with his blood in the Typhus Research Institute of German Profesor Rudolf Weigl. This job gave him a document which prevented him from Nazi occupants and helped him survive the brutal German occupation.

[8] See more in [10] and [11]

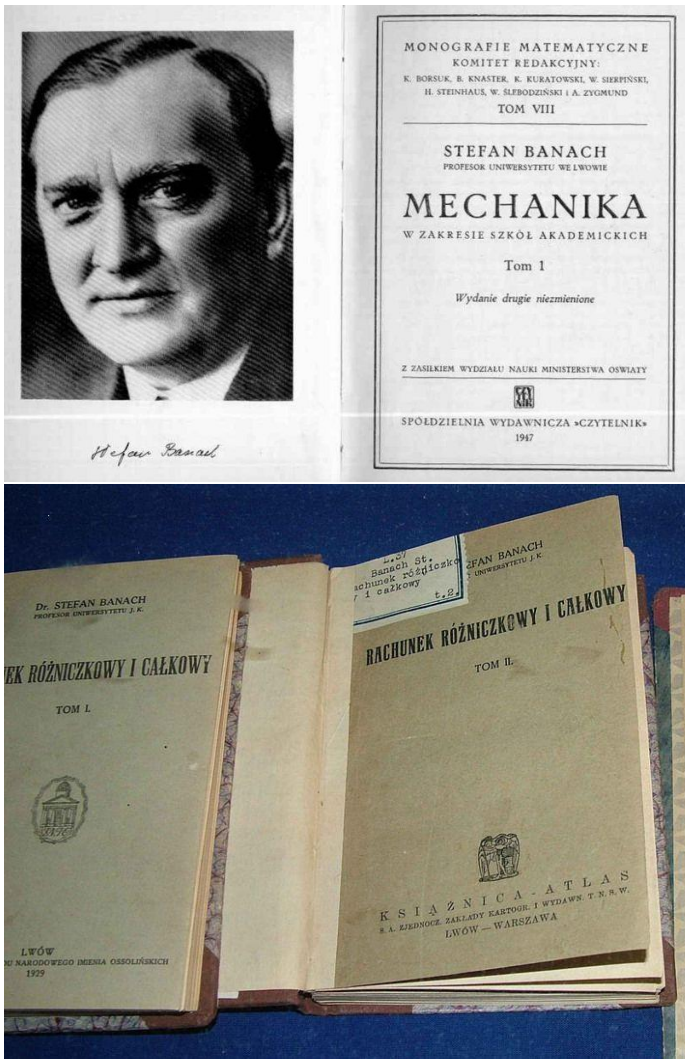
As soon as the Soviet troops retook Lwów Banach renewed his contacts. He continued his work at Lwów University as Dean of the Faculty of Science. He planned to go to Kraków after the war to take over the chair of mathematics at the Jagiellonian University but he died in Lwów in 1945 of lung cancer. He is buried at the Łyczakowski Cemetery in Lwów (figure 6).

FIGURE 6. Grave in which Stefan Banach is buried at the Łyczakowski Cemetery in Lwów

2. Work

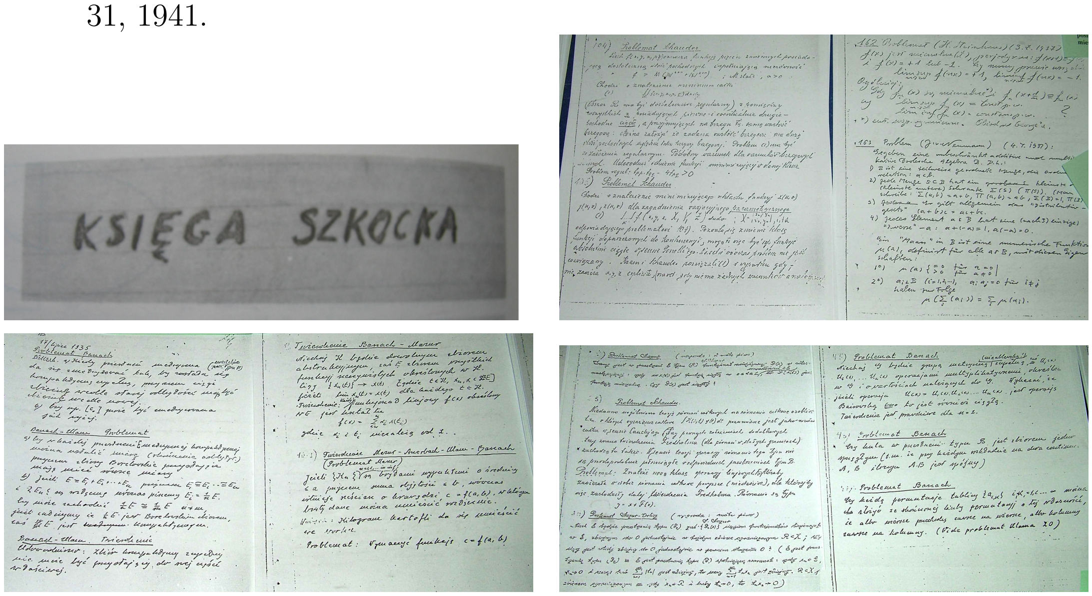
Banach was a great academic teacher and author of many mathematics books, including books for secondary and high schools (figure 7). Full list of Banach's publications contains 58 positions.

FIGURE 7. Stefan Banach's books for high schools



The way that Banach worked was unconventional. He liked to do mathematics with his colleagues in the Scottish Cafe (Polish: Kawiarnia Szkocka) in Lwów (figure 3). As his friends related it in their memories: *We discussed problems proposed right there, often with no solution evident even after several hours of thinking. The next day Banach was likely to appear with several small sheets of* *paper containing outlines of proofs he had completed. Banach would spend most of his days in cafes, not only in the company of others but also by himself. He liked the noise and the music. They did not prevent him from concentrating and thinking. There were cases when, after the cafes closed for the night, he would walk over to the railway station where the cafeteria was open around the clock. There, over a glass of beer, he would think about his problems.* The tables of the cafe had marble tops, so mathematicians could write in pencil, directly on the table, during their discussions. To avoid the results being lost Stefan Banach's wife provided the mathematicians with a large notebook, which was used for writing the problems and answers that eventually became known as the Scottish Book (figure 8). The book, a collection of solved, unsolved and even unsolvable dilemmas, could be borrowed by any of the guests of the cafe. Solving any of the mathematical paradoxes was prized with often absurd awards. [9].

FIGURE 8. Few pages from Scottish Book which was used for writing mathematical problems and their solutions during meetings in Scottish Cafe. The first problem was written by Stefan Banach July 17, 1935. The last one was written by Hugo Steinhaus May 31, 1941.



[9]See [12] and [13]

Theory of Linear Operations is regarded as Banach's most influential work. In it he formulated the concept, now known as *Banach space*, and proved many fundamental theorems of functional analysis. He founded modern *functional analysis* and made major contributions to the theory of topological vector spaces. *Banach algebra* was also named after him.

Besides being one of the founders of functional analysis, Banach also made important contributions to the measure theory, integration, the theory of sets, orthogonal series and other branches of mathematics. He proved a number of fundamental results on normed linear spaces and many important theorems are today named after him. There is the *Hahn-Banach theorem* on the extension of continuous linear functionals, the *Banach-Steinhaus theorem* on bounded families of mappings, the *Banach fixed point theorem* and the *Banach-Tarski paradoxical decomposition of a ball.*

2.1. **Banach space and Banach algebra.** Banach spaces are one of the central objects of study in functional analysis. Many of the infinite-dimensional function spaces studied in functional analysis are examples of Banach spaces.

Banach space is defined as complete normed vector space. This means that a Banach space is a vector space V over the real or complex numbers with a norm $\|\cdot\|$ such that every Cauchy sequence (with respect to the metric $d(x,y) = \|x-y\|$) in V has a limit in V. Since the norm induces a topology on the vector space, a Banach space provides an example of a topological vector space.

Examples. Throughout, let K stand for one of the fields $\mathbb{R}$ or $\mathbb{C}$.

(i) The familiar Euclidean spaces K^n where the Euclidean norm of $x = (x_1, ..., x_n)$ is given by

$$\|x\| = (|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2)^{\frac{1}{2}}$$

are Banach spaces.

(ii) The space of all continuous functions $f : [a,b] \to K$ defined on a closed interval $[a,b]$ becomes a Banach space if we define the norm of such a function as $\|f\| = \sup\{|f(x)| : x \in [a,b]\}$

(iii) If $p \geq 1$ is a real number, we can consider the space of all infinite sequences $(x_1, x_2, x_3, \ldots)$ of elements in K such that the infinite series $\sum |x_i|^p$ is finite. The p-th root of this series' value is then defined to be the p-norm of the sequence. The space, together with this norm, is a Banach space and it is denoted by l^p.

(iv) If X and Y are two Banach spaces, then we can form their direct sum $X \cup Y$, which is again a Banach space.
(v) If M is a closed subspace of the Banach space X, then the quotient space $\frac{X}{M}$ is again a Banach space.
(vi) Every Hilbert space is a Banach space by definition.

If V is a Banach space and K is the underlying field (either the real or the complex numbers), then K is itself a Banach space (using the absolute value as norm) and we can define the *dual space* V^* as $V^* = L(V, K)$, the space of continuous linear maps into K. This is again a Banach space (with the operator norm). It can be used to define a new topology on V - the *weak topology.*

Banach algebra is an associative algebra A over the real or complex numbers which at the same time is also a Banach space. The algebra multiplication and the Banach space norm are required to be related by the following inequality

$$\bigwedge_{x,y \in A} \|xy\| \leq \|x\| \cdot \|y\|.$$

This ensures that the multiplication operation is continuous.

2.2. **The Banach-Steinhaus theorem.** The Banach-Steinhaus theorem (the uniform boundedness principle) is one of the fundamental results in functional analysis and together with the Hahn-Banach theorem and the open mapping theorem, considered one of the cornerstones of the field. In its basic form, it asserts that for a family of continuous linear operators whose domain is a Banach space, pointwise boundedness is equivalent to boundedness.

More precisely, let X be a Banach space and N be a normed vector space. Suppose that Φ is a collection of continuous linear operators from X to N. The uniform boundedness principle states that

$$\text{if} \quad \bigwedge_{x \in X} \sup_{\phi \in \Phi} |\phi(x)| < \infty \quad \text{then} \quad \sup_{\phi \in \Phi} \|\phi\| < \infty.$$

2.3. **The Hahn-Banach theorem.** The Hahn-Banach theorem [10] is a central tool in functional analysis. It allows one to extend linear operators defined on a subspace of some vector space to the whole space and it also shows that there are "enough" continuous linear functionals defined on every normed vector space to make the study of the dual space interesting.

[10]Hans Hahn - an Austrian mathematician, proved this theorem independently.

Let X be a vector space over the scalar field K (which is either the real or the complex numbers), let M be a subspace of X ($M \subset X$) and let a function $\varphi : M \to K$ be sublinear such that

$$\bigwedge_{x \in M} |\varphi(x)| \leq c \cdot \|x\| \quad , \quad c \text{ - constant.}$$

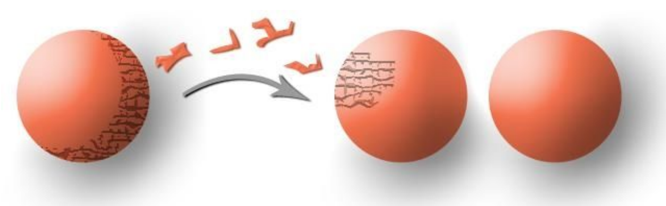
Then exist a sublinear function $\Phi : X \to K$ such that

$$|\Phi(x)| \leq c \cdot \|x\| \quad \text{and} \quad \Phi|_M = \varphi.$$

If X is a normed vector space with subspace M and if $\varphi : M \to K$ is continuous and linear, then there exists an extension $\Phi : X \to K$ which is also continuous and linear and which has the same norm as φ ($\|\varphi\| = \|\Phi\|$).

2.4. **The Banach-Tarski paradox.** The Banach-Tarski paradox [11] appeared in a joint paper of the two mathematicians in 1926 in *Fundamenta Mathematicae* entitled *Sur la decomposition des ensembles de points en partiens respectivement congruent*. This paradox is called also *doubling the ball paradox*. It shows that a ball can be divided up into subsets which can be fitted together to make two balls each identical to the first. The axiom of choice is needed to define the decomposition and the fact that it is able to give such a non-intuitive result has made some mathematicians question the use of the axiom. The Banach-Tarski paradox was a major contribution to the work being done on axiomatic set theory around this period.

FIGURE 9. The BanachTarski "paradox": A ball can be decomposed and reassembled into two balls the same size as the original



[11]Alfred Tarski was a logician and mathematician of considerable philosophical importance. A brilliant member of the interwar Warsaw School of Mathematics and active in the USA after 1939, he wrote on topology, geometry, measure theory, mathematical logic, set theory, metamathematics, and most of all, on model theory, abstract algebra and algebraic logic. Tarski's first paper, published when he was only 19 years old, was on set theory, a subject to which he returned throughout his life.

Formal treatment. Let A and B be two subsets of Euclidean space. We call them equi-decomposable if they can be represented as finite unions of disjoint subsets

$$A = \bigcup_{i=1}^{n} A_i \quad \text{and} \quad B = \bigcup_{i=1}^{n} B_i$$

such that for any i the subset A_i is congruent to B_i. Then, the paradox can be reformulated as follows: *the ball is equi-decomposable with two copies of itself.*

For the ball, five pieces are sufficient to do this; it cannot be done with fewer than five. There is an even stronger version of the paradox: *any two bounded subsets of* 3*-dimensional Euclidean space with non-empty interior are equi-decomposable.*

2.5. **Banach fixed point theorem.** The Banach fixed point theorem is an important tool in the theory of metric spaces; it guarantees the existence and uniqueness of fixed points of certain self maps of metric spaces and provides a constructive method to find those fixed points.

Let (X, d) be a non-empty complete metric space. Let $T : X \to X$ be a contraction mapping on X, i.e: there is a real number $q < 1$ such that

$$d(Tx, Ty) \leq qd(x, y) \quad \text{for all} \quad x, y \in X.$$

Then the map T admits one and only one fixed point $x* \in X$ (this means $Tx^* = x^*$). Furthermore, this fixed point can be found as follows: start with an arbitrary element $x_0 \in X$ and define an iterative sequence by $x_n = Tx_{n-1}$ for $n = 1, 2, 3, \ldots$. This sequence converges and its limit is x^*. The following inequality describes the speed of convergence

$$d(x^*, x_n) \leq \frac{q^n}{1-q} d(x_1, x_0).$$

Equivalently,

$$d(x^*, x_{n+1}) \leq \frac{q}{1-q} d(x_{n+1}, x_n) \quad \text{and} \quad d(x^*, x_{n+1}) \leq qd(x_n, x^*).$$

References

1. *Encyklopedia szkolna. Matematyka*, WSiP, Warszawa 1989.
2. Kordos M., *Wyklady z historii matematyki*, SCRIPT, Warszawa 2005.
3. Musielak J., *Wstep do analizy funkcjonalnej*, PWN, Warszawa 1989.
4. Skowski T., *Czowiek i matematyka*, Polska Oficyna Wydawnicza BGW, Warszawa 1996.
5. http://pl.wikipedia.org/wiki/Stefan_Banach
6. http://www.lwow.home.pl/naszdziennik/banach.html
7. http://www.lwow.com.pl/m.htm
8. http://www.matematyka.org/main2326520210,2,yisvp.htm
9. http://www-history.mcs.st-andrews.ac.uk/Mathematicians/Banach.html
10. http://www.lwow.com.pl/lwow_profs.html
11. http://en.wikipedia.org/wiki/Massacre_of_Lw%C3%B3w_professors
12. http://www-history.mcs.st-and.ac.uk/history/Miscellaneous/Scottish_Cafe.html
13. http://www-groups.dcs.st-and.ac.uk/ history/HistTopics/Scottish_Book.html#s5
14. http://www.emis.de/newsletter/54/page24.pdf
15. http://www.gap-system.org/ history/Societies/Polish.html
16. http://en.wikipedia.org/wiki/Warsaw_School_of_Mathematics