A BRIEF INTRODUCTION TO KNOT THEORY AND THE JONES POLYNOMIAL

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Abstract. This paper introduces knot theory, a branch of topology. We begin with the definition of knots in mathematics, knot equivalence, and common types of knots. Then, we discuss the Jones Polynomial as an invariant of knots, with an example of calculating the Jones Polynomial provided. Two results about reduced alternating links are then given, with their applications discussed, as these results can be proven with the knot polynomials. Eventually, we briefly mention the applications of the Jones Polynomials in other fields, including those outside of mathematics.

Contents

1. Introduction

Informally speaking, a mathematical *knot* is a knotted piece of string with the two loose ends connected. The formal definition is given below.

Definition 1.1. A *knot* is a continuous injective map $f : S^1 \to \mathbb{R}^3$ that embeds the circle S^1 into three-dimensional Euclidean space $\mathbb{R}^3$.

An example of a knot is shown in figure 1.

Definition 1.2. A *link* is a disjoint union of knots. A link with one component is a knot.

The *Hopf link* is the simplest link with two components. It is shown in figure 2.

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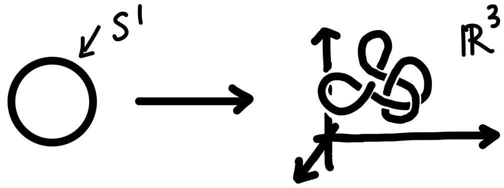


FIGURE 1. Example of a Knot

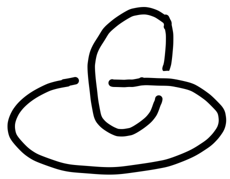


FIGURE 2. Hopf Link

(A) Multiple crossing (B) Parallel crossing

FIGURE 3. Crossings that are not allowed in a diagram

If you have a knot $f : S^1 \to \mathbb{R}^3$, then you can take a projection operator $p : \mathbb{R}^3 \to \mathbb{R}^2$, such that $p \circ f : S^1 \to \mathbb{R}^2$ is injective except at crossing points, where it is two to one. A *knot diagram* is a projection of a knot in $\mathbb{R}^2$ that has no multiple crossings (fig.3a) or parallel crossings (fig.3b), because then you cannot tell which strand is which. A legitimate knot diagram is like the one of Hopf Link in figure 2.

1.1. **Knot Equivalence.** A central problem in knot theory is concerned with distinguishing whether two knots are equivalent. To achieve this, it is necessary to mathematically define knot equivalence. While the standard definition of knot equivalence involves homeomorphism and ambient isotopy [7], in practice, two knots can be identified as equivalent if one can be derived from the other by a series of basic moves that do not cut and then re-attach any segments.

1.2. **Reidemeister Moves.** Reidemeister moves are a few basic moves on a knot diagram. They can be used to identify equivalent knots. There are three types of Reidemeister moves in total, shown in figure 4.

The first Reidemeister move adds or removes one crossing by doing a clock-wise or anti-clockwise partial rotation. The second Reidemeister move shifts a strand either toward or away from another strand. The third Reidemeister move shifts a strand from one side of a crossing to another. All Reidemeister moves change a knot's projection and the relationship between the crossings [1].

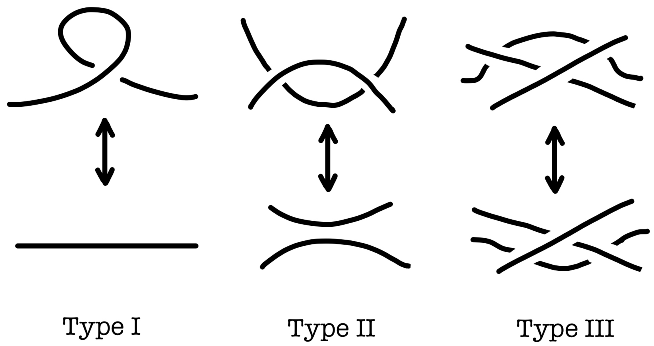


FIGURE 4. Reidemeister Moves

Theorem 1.3. [4] *Reidemeister Theorem: two links can be continuously deformed into each other if and only if any diagram of one can be transformed into a diagram of the other by a sequence of Reidemeister moves.*

The Reidemeister theorem above is a powerful tool in identifying equivalent knots, as it uses an "if and only if" condition.

2. Types of Knots

The *unknot* is the simplest knot. It has the shape of a ring with no knots on it, which can be drawn as ◯.

The *torus knot* is a knot that lies on the surface of an unknotted torus. Figure 5 shows three examples of torus knots. The leftmost knot in the figure is also known as the trefoil.

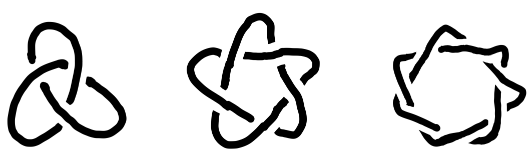


FIGURE 5. Torus knots

The other knots can be classified as either *satellite knots* or *hyperbolic knots* [8].

3. Tabulating Knots

The original interest in classifying knots was inspired by chemistry in the early nineteenth century. Lord Kelvin hypothesized that atoms were knots in ether, such that different knots corresponded to different elements [1, p. 111]. This led another physicist, Peter Guthrie Tait, to attempt to make a list of all knots, because he believed that classifying knots was equivalent to tabulating elements. Early attempts like these often involved drawing knots by hand and sometimes contained mistakes, while modern methods can be efficiently automated by computers [5].

Definition 3.1. [6] The *crossing number* of a knot is the smallest number of crossings of any diagram of the knot.

For example, a standard *unknot* is just an empty circle with zero crossings. While an *unknot* can be drawn with different diagrams as in figure 6, it always has the crossing number 0.

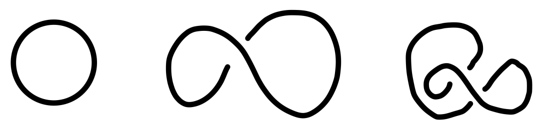


FIGURE 6. Crossing Number = 0

The table of knots consists of drawings of different knots' diagrams and their labels. A sample tabulation is shown in figure 7. Under each knot diagram, the number with larger size denotes the crossing number, and the subscript shows the index within that category of crossing number. The knot 7_5 , for example, is the fifth knot with crossing number seven. The sub-ordering of subscripts does not follow an explicit rule, except that some types of knots, such as the torus knots and twist knots, are listed first [6].

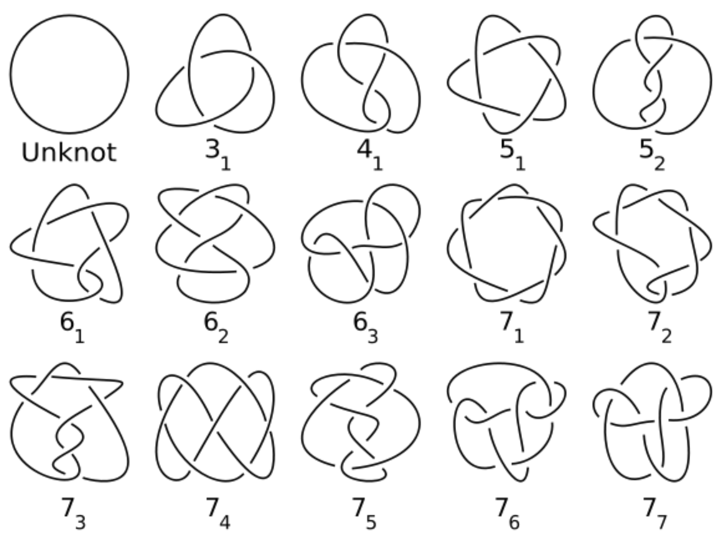


FIGURE 7. Tabulation of Knots [5]

4. Polynomials of Knots and Links

Definition 4.1. A knot invariant is a quantity that is the same for equivalent knots.

Thus, two knots are not equivalent if they have different invariants, but the opposite direction might not be true.

Example 4.2. Crossing number is a knot invariant.

Proof. Assume for the sake of contradiction that crossing number is not a knot invariant, then there exist two knots k_1 and k_2 that are equivalent but have different crossing numbers c_1 and c_2. If k_1's crossing number c_1 is smaller, then the corresponding diagram d_{k1} cannot be a diagram of k_2 since it has fewer than c_2 crossings. But this is a contradiction because k_1 and k_2 are equivalent knots that can be deformed into each other, and the diagram of k_1 must be a diagram of k_2. □

Research on invariants is important, as it not only helps with distinguishing knots from each other but also with understanding fundamental properties of knots and applications to other fields of mathematics. [10]. In this section we will discuss a knot invariant which takes the form of a polynomial. Its applications to other fields will be briefly mentioned in the next section.

A quantity derived from a knot's diagram is an invariant of a knot as long as it remains unchanged after Reidemeister moves, because Reidemeister moves never change the underlying knot to a different one.

The Jones polynomial is constructed in a way that makes it unchanged under Reidemeister moves. In order to define the Jones polynomial we first need to introduce the bracket polynomial (also known as the Kauffman bracket).

4.1. **Bracket Polynomial.** The bracket polynomial of a link L is a Laurent polynomial in the symbol A, denoted by $\langle L \rangle \in \mathbb{Z}[A, A^{-1}]$. A Laurent polynomial is a polynomial where you are also allowed to have negative powers, so a Laurent polynomial with integer coefficients in the symbol A is the same thing as an element of $\mathbb{Z}[A, A^{-1}]$. There are three rules for calculating the bracket polynomial of a link [2, p.24]:

(1) $\langle \bigcirc \rangle = 1$
(2) $\langle L \cup \bigcirc \rangle = (-A^2 - A^{-2})\langle L \rangle$ for any link L
(3) $\langle \times \rangle = A \langle \asymp \rangle + A^{-1} \langle)(\rangle$

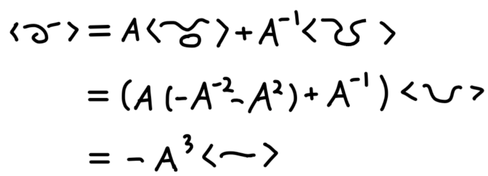
Rule (1) implies that the bracket polynomial for the unknot is 1. Rule (2) defines the resulting bracket polynomial when placing the unknot next to a link L. It implies, for example, that the bracket polynomial of n unknots placed next to each other is $(-A^2 - A^{-2})^{n-1}$. Rule (3) expresses a bracket polynomial as the combination of two bracket polynomials, both of which break the same pair of crossing segments into two pairs of non-crossing segments, in different directions. The notation of rule (3) might be confusing at first, so we have included the following example for clarity.

Performing Rule (3): 

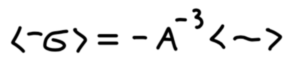
In the following two lemmas we see how the bracket polynomial changes under the three types of Reidemeister moves.

Lemma 4.3. [2, p.24] *The bracket polynomial is not an invariant under type I Reidemeister move. Applying a type I Reidemeister move multiplies the entire bracket polynomial by* $-A^3$ *or* $-A^{-3}$.

Proof. Here we follow the proof given in [2, p.24].



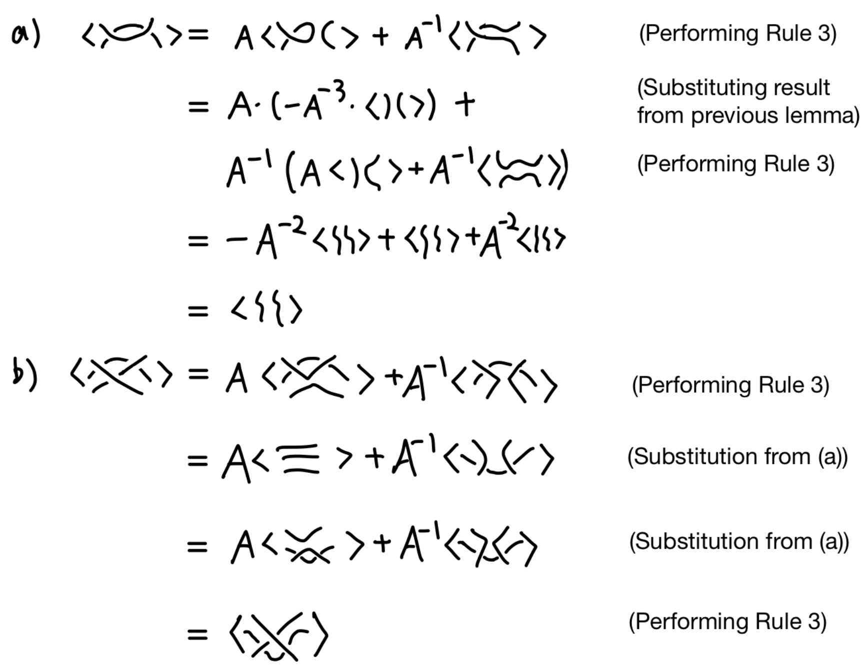
From this we see that the bracket polynomial is not an invariant under the type I Reidemeister move. Similarly,



□

Lemma 4.4. [2, p.25] *The bracket polynomial is invariant under type II and type III Reidemeister moves.*

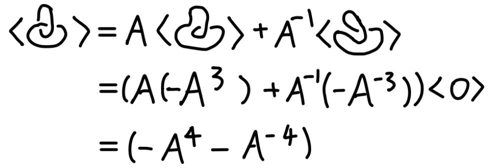
Proof. Here we follow the proof given in [2, p.25]



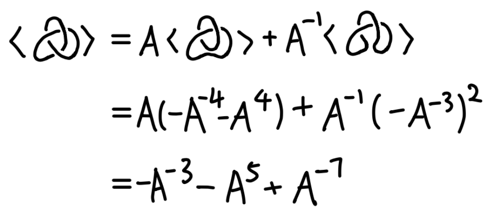
□

Example 4.5. As in the example provided in [2, p.25], the bracket polynomial of a trefoil can be calculated in the following way.

We first calculate the bracket polynomial of a Hopf Link:



Substituting this result, we can calculate the bracket polynomial of the trefoil:



4.2. **Writhe.** A knot's *orientation* is defined by choosing a direction to travel around the knot [1, p.10]. It can be denoted by placing directed arrows on the projection of a knot. There are two orientations for any given knot, as in figure 8.

Given a knot and an orientation, we assign the value of $+1$ to each crossing of the form and assign the value -1 to each crossing of the form . Then the writhe of a knot, denoted $w(L)$, equals the sum of these assigned numbers. For example, the knot in the left of figure 8 has writhe $w(L) = 4 - 1 = 3$, and the same knot with opposite orientation has $w(L) = -4 + 1 = -3$.

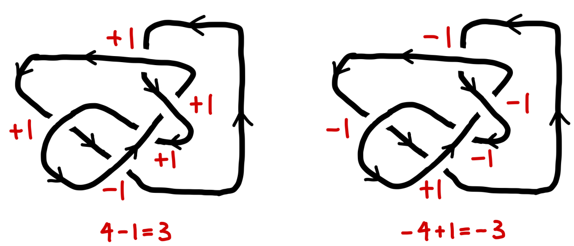
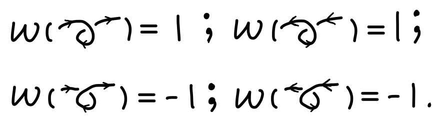


FIGURE 8. Writhe of Opposite Orientations

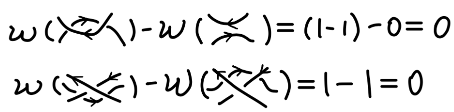
Lemma 4.6. [2, p.24-25] *Type II and III Reidemeister moves do not change the writhe, while type I Reidemeister moves always change the writhe by* ± 1.

Proof.

Type I:



Type II & III :



Other orientations can be derived similarly.

□

4.3. **Jones Polynomial.** To define the Jones polynomial of a knot, we need the knot to be oriented, so from here on, any given knot will be assumed to be oriented.

In addition, we can first define a polynomial $X(L) \in \mathbb{Z}[A, A^{-1}]$ by $X(L) = (-A^3)^{-w(L)} \langle L \rangle$. Essentially, it multiplies the bracket polynomial of a knot L by $(-A^3)$, so that the product does not change under all three types of Reidemeister moves.

For type I, the polynomial X remains unchanged because the term $(-A^3)^{-w(L)}$ negates the change in the bracket polynomial $\langle L \rangle$. Example 4.7 is given below to clarify this.

Example 4.7. This example shows one case in which the polynomial X remains unchanged under type I Reidemeister moves. The other cases can be proven in a similar way. First, after twisting an oriented segment into a crossing with a type I Reidemeister move, as shown below,

we can calculate the changes in the coefficient and the bracket polynomial respectively, and then see if they cancel each other out when multiplied together. Since the writhe increases by 1 due to an extra crossing, the coefficient term $(-A^3)^{-w(L)}$ is multiplied by $(-A^3)^{-1}$. At the same time, since a type I Reidemeister move is performed, the bracket polynomial $\langle L \rangle$ is multiplied by $-A^3$ (lemma 4.3), as the following equations show.

$$\Delta \text{ coefficient} : (-A^3)^{-1}$$
$$\Delta \text{ bracket} : -A^3$$

These changes will cancel each other out when the coefficient and the bracket polynomial are multiplied together to become $X(L)$:

$$\Rightarrow (-A^3)^{-1} \cdot (-A)^3 = 1$$

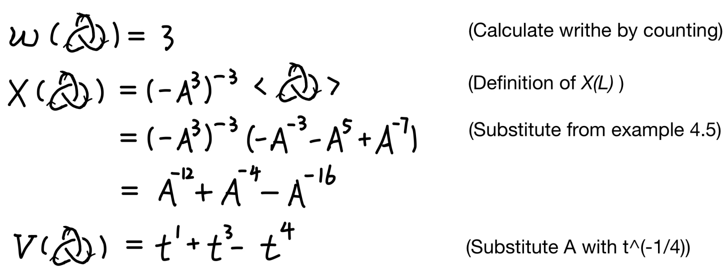
Therefore, we can see that the entire polynomial is multiplied by 1 after a type I Reidemeister move, which means it is unchanged. To obtain a complete proof, you have to consider the opposite orientation and the other direction of applying a type

I Reidemeister move. But the proofs of these other situations are quite similar and yield the same result [2, p.26]. This concludes the example.

For types II and III, the polynomial X remains unchanged because both the bracket polynomial and the writhe do not change, as shown in lemmas 4.4 and 4.6.

As a result, the polynomial X is invariant under all three Reidemeister moves. As a final step, substituting A with $t^{-1/4}$ results in the Jones polynomial, which is a Laurent polynomial with integer coefficients.

Example 4.8. The Jones Polynomial for an oriented trefoil can now be calculated. The resulting Jones polynomial is calculated to be $t^1 + t^3 - t^4$.



5. Reduced Alternating Link

When calculating the bracket polynomial of a link L, we can eliminate a crossing using rule (3), that is $\langle \times \rangle = A \langle \asymp \rangle + A^{-1} \langle)(\rangle$. This motivates two operations on links. The first sends L to L_A with one crossing $\times$ replaced with $\asymp$. The second sends L to L_B with the same crossing $\times$ replaced with $)($. Link L_A and L_B's bracket polynomials are then summed together to form $\langle L \rangle$.

The split that generates the link L_A is called an A-split, and the split that generates the link L_B is called a B-split [1, p.156]. Notice that A-splits and B-splits are not equivalent or interchangeable. An A-split removes the center of the crossing and connects the initial under-strand with the next over-strand, going counter-clockwise. A B-split, on the other hand, removes the center of the crossing and connects the initial over-strand with the next under-strand, also going counter-clockwise. This is shown in figure 9.

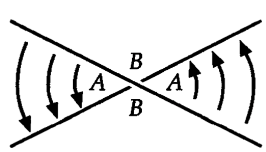


Figure 9. A-split and B-split [1, p.156]

Applying rule (3) recursively on the original link L with n crossings doubles the amount of links in each level of recursion. In the n'th level of recursion, we generate 2^n unknotted links, each of them having the bracket polynomial $(-A^2 - A^{-2})^{|s|-1}$, where $|s|$ can be found by counting the number of unknots in that link.

The choice of n splits that turns the original link L into one of the 2^n unknotted link, L', is called a *state* [1, p.157]. The bracket polynomial of the original link

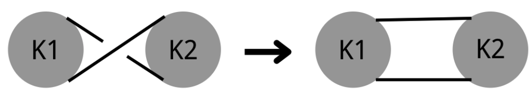
$\langle L \rangle$ can be calculated with the bracket polynomial of each state and the number of A-splits and B-splits using formula 5.1 [1, p.158]

$$\langle L \rangle = \sum_s A^{a(s)} A^{-b(s)} (-A^2 - A^{-2})^{|s|-1}. \tag{5.1}$$

For each state s, $a(s)$ is the number of A-splits, $b(s)$ is the number of B-splits, and $|s|$ is the number of unknots in s. For example, a state with three unknots has the polynomial $(-A^2 - A^{-2})^2$. If this state is generated through three A-splits and one B-split, then it is multiplied by $A^3 A^{-1}$. It contributes, in total, $A^3 A^{-1}(-A^2 - A^{-2})^2$ to the bracket polynomial of L.

Definition 5.2. A link's diagram is called *reduced* if it has no reducible crossings.

A reducible crossing separates the knot into two parts K_1 and K_2 and can be reduced by a twist, as illustrated in the graph below.



Definition 5.3. An *alternating link* is one that possesses an alternating diagram in which crossings alternate between over and under passing.

Figure 10 uses examples to clarify the definitions of *reduced* and *alternating*.

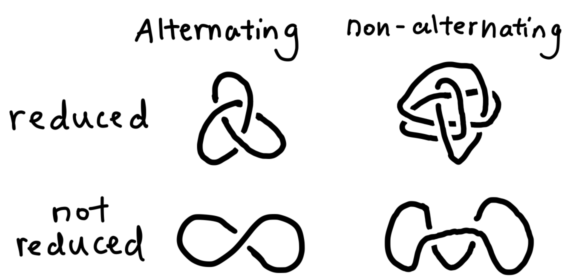


FIGURE 10. Table of Reduced and Alternating Links

There are two important conjectures about reduced alternating projections.

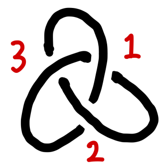
Conjecture 5.4. [1, p.159] *Two reduced alternating projections of the same knot have the same number of crossings.*

Conjecture 5.5. [1, p.159] *A reduced alternating projection of a knot has the least number of crossings for any projection of that knot.*

Both were proved to be true by Louis Kauffman, Morwen Thistlethwaite, and Kunio Murasugi in 1986, using the polynomials of knots [1, p.160]. The proof uses formula 5.1 to calculate the span of the polynomial, which is the highest power minus the lowest power, and builds the rest of the argument from there. The entire proof is slightly long and thus skipped in this section, but it can be found in The Knot Book [1, p.160]

With these two conjectures in mind, we know that whenever we see a reduced alternating diagram, we can tell its crossing number by just counting the number of crossings appearing in that diagram, as the following example shows.

Example 5.6. The trefoil diagram shown below is reduced and alternating, so we can tell its crossing number by just counting the number of crossings we see on the diagram, which is 3.



6. Other Applications of Jones Polynomial

There are many other applications of the Jones polynomial and the bracket polynomial. It has links to Chern–Simons theory, Khovanov homology, etc. [9]. Outside of mathematics, the polynomials of knots and links are applied to the study of quantum field theory [11] and DNA structure [3].

Acknowledgments

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