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ON FUZZY FUNCTIONAL COMPACTNESS AND FUZZY SEMINORMALITY IN FUZZY TOPOLOGICAL SPACES

G. Palani Chetty
Department of Mathematics
I. R. T. Polytechnic College
Krishnagiri - 635108
Tamilnadu.

G. Balasubramanian
Ramanujan Institute for Advanced
Study in Mathematics
University of Madras
Chepauk, Madras - 600005

Abstract:

In this paper the concept of fuzzy functional compactness and fuzzy seminormality is introduced in fuzzy topological spaces. Some interesting properties and characterizations of these spaces are obtained.

Key words and phrases: Fuzzy absolutely closed, fuzzy funcitonally compact, fuzzy semi-normal, fuzzy regular seminormal.

1. Introduction

Ever since C. L. Chang [2] introduced and developed the concept of fuzzy topological spaces based on the concept of fuzzy set introduced by L. A. Zadeh [14], the fuzzy concept has invaded many branches of Mathematics. Since then various important concepts in classical topology such as compactness have been extended to fuzzy topological spaces.

The concept of functional compactness and seminormality in general topology was introduced and studied in [1,3,5,6,10,11,13] and [4,8,9,12] respectively. They are focinating classes of spaces possessing many interesting properties. The purpose of this paper is to introduce fuzzy functional compactness and fuzzy seminormality in fuzzy topological spaces. We give various characterizations along with some interesting preliminary properties of these spaces.

2. Preliminaries

The interior and the closure of the fuzzy set λ in X will be denoted by *Int* λ and *cl* λ respectively. A fuzzy set λ in X is said to be fuzzy regularly open if *Int* (*cl* λ) = λ and fuzzy regularly closed if *cl* (*Int* λ) = λ. A fuzzy set is fuzzy regularly open if its complement is fuzzy regularly closed. For the concepts not defined in this paper we refer to [2,7].

3. Fuzzy Functionally Compactness in Fuzzy Topological Spaces

Definition 1: A fuzzy Hausdorff space (X, T_1) is fuzzy functionally compact ⇔ For every fuzzy closed set λ in X, satisfying the property that for every fuzzy continuous map f from (X, T_1) onto a fuzzy Hausdorff space (Y, T_2), $f(\lambda)$ is fuzzy closed.

Definition 2: A fuzzy Hausdorff space (X, T) is said to be fuzzy absolutely closed ⇔ For every fuzzy T - fuzzy open cover $G_n = \{ \mu_\alpha / \alpha \in \Delta \}$ of X there exists a finite subfamily $\mu_{\alpha_1}, \mu_{\alpha_2}, \ldots, \mu_{\alpha_n}$ such that $1_X \leq \bigvee_{i=1} cl\,(\mu_{\alpha_i})$.

Definition 3: A base for (X, T) is called fuzzy regular open, if it consists of fuzzy regular open sets.

Definition 4: Let p be a fuzzy point in X and (X, T) be a fuzzy topological space. $\lambda \in I^X$ is said to be a fuzzy regular open neighbourhood of p if there is some regular open fuzzy set μ in X such that $p \in \mu$ and $\mu \leq \lambda$.

Definition 5: A fuzzy topological space (X, T) is called fuzzy semi normal ⇔ given a fuzzy closed set λ and fuzzy open set μ such that $\lambda \leq \mu$, then there exists a fuzzy regular open set σ such that $\lambda \leq \sigma \leq \mu$.

Definition 6: A fuzzy topological space (X, T) is said to be fuzzy regular semi normal if given a fuzzy regular closed set λ of X and a fuzzy open set μ of X such that $\mu \geq \lambda$, there exist a fuzzy regular open set θ of X such that $\lambda \leq \theta \leq \mu$.

Proposition 1. A fuzzy topological space (X, T) is fuzzy semi normal $\Leftrightarrow$ Given any fuzzy closed set λ and a fuzzy closed set μ such that $\lambda + \mu \leq 1$, then there exists a fuzzy open set σ such that $\mu \leq (cl\ \sigma)$ and $(\text{cl}\ \sigma) + \lambda \leq 1$.

Proof: Assume (X, T) is fuzzy semi normal. Given that λ and μ are fuzzy closed sets such that $\lambda + \mu \leq 1$. Now $\lambda + \mu \leq 1 \Rightarrow \lambda \leq 1 - \mu$. Then λ is fuzzy closed and $1 - \mu$ is fuzzy open. Hence by assumption there exists a fuzzy regular open set ρ such that

$$\lambda \leq \rho \leq 1 - \mu \quad (1)$$

Put $\sigma = 1 - cl(\rho)$, then

$$cl(\sigma) = cl\,[\,1 - cl\,(\rho)]$$

$$\geq cl\,[\,1 - cl\,(1 - \mu)]$$

$$\geq cl\,[Int\ \mu] \geq \mu$$

Also

$$cl\,(\sigma) + \lambda = cl\,[\,1 - cl\,(\rho)] + \lambda$$

$$= cl\,[\,Int\,(1 - \rho)] + \lambda = (1 - \rho) + \lambda$$

Now from (1)

$$\rho \geq \lambda$$

$$\therefore 1 - \rho \leq 1 - \lambda$$

i.e., $1 - \rho + \lambda \leq 1$

i.e., $cl\,(\rho) + \lambda \leq 1.$

Conversely, suppose λ is fuzzy closed and μ is fuzzy open set such that $\lambda \leq \mu$.

Choose a fuzzy open set ρ such that

$$1 - \mu \leq cl\,(\rho),\ cl\,(\rho) + \lambda \leq \mu.$$

This means $\lambda \leq 1 - cl\,(\rho) \leq \mu$. Therefore it is sufficient if we show that $1 - cl\,(\rho)$ is a fuzzy regular open set.

$$Int\ cl\ [\ 1 - cl\ (\rho)] = Int\ cl\ [\ 1 - \{\ 1 - Int\ (\ 1 - \rho)\}]$$

$$= Int\ cl\ [\ Int\ (\ 1 - \rho)]$$

$$= Int\ [\ 1 - Int\ \{\ 1 - Int\ (1 - \rho\ )\}]$$

$$= Int\ [\ 1 - Int\ (cl\ (\rho))] \leq Int\ (\ 1 - \rho\ )$$

$$\leq 1 - cl\ (\rho)$$

Also,

$$1 - cl\ (\rho) \geq Int\ cl\ (\ 1 - cl\ (\rho))$$

$$\Rightarrow Int\ cl\ (\ 1 - cl\ (\rho)) = 1 - cl\ (\rho)$$

$\Rightarrow 1 - cl\ (\rho)$ is fuzzy regular open. □

Proposition 2: Let (X,T) be a fuzzy Hausdorff functional compact space. If $\mathcal{F} = \{\ \mu_\alpha\ /\ \alpha \in \Delta\ \}$ is a base on X such that the intersection λ of the elements of $\mathcal{F}$ is equal to the intersection of the closures of the elements of $\mathcal{F}$, then $\mathcal{F}$ is a base for the fuzzy neighbourhood of λ.

Proof: Given that every fuzzy continuous mapping f from X to a fuzzy Hausdorff space is fuzzy closed. Also given that $\mathcal{F}$ is a fuzzy base on X such that

$$\lambda = \underset{\alpha \in \Delta}{\wedge} \{\lambda_\alpha\ /\ \lambda_\alpha \in \mathcal{F}\}$$

$$= \underset{\alpha \in \Delta}{\wedge} \{cl\lambda_\alpha\ /\ \lambda_\alpha \in \mathcal{F}\}$$

We claim that $\mathcal{F}$ is a base for the neighbourhood of λ.

Suppose not, then there exist a fuzzy open set μ in X such that

(i) $\mu \geq \lambda$

(ii) $\forall \gamma \in \mathcal{F}, (1 - \mu) + \gamma \nleq 1$ i.e. $\gamma - \mu \nleq 0$ i.e. $\gamma \nleq \mu$ i.e., $\gamma \geq \mu$

Take $Y = X$ and define a base $\mathcal{B}$ for a fuzzy topology τ on Y as follows:

$$\rho \in \mathcal{B} \Leftrightarrow (1) f^{-1}(\rho)\ (1-\lambda) \in \mathcal{F}$$

or

$$(2) f^{-1}(\rho) \in \mathcal{F}$$

Thus (Y, τ) is a fuzzy Hausdorff space. Let $f : X \to Y$ be the identity map. Then f is fuzzy continuous and therefore by hypotheses f is fuzzy closed. But $f(1 - \mu)$ is not fuzzy closed. This is a contradiction. Hence the proposition. □

Proposition 3: In the following statements the following implications are valid.

(i) ⇔ (ii) and (iii) ⇔ (iv) ⇔ (v).

(i) (X, T) is fuzzy functionally compact.

(ii) Every fuzzy continuous function from X into any fuzzy Hausdorff space is fuzzy closed.

(iii) Given a fuzzy regular open set λ of X, a fuzzy open cover $\mathcal{B}$ of ($1 - \lambda$) and a fuzzy open neighbourhood μ of λ, then there exists $\rho_{\alpha_i} \in \mathcal{B}$, $1 \leq i \leq n$ such that

$$1_X = \mu \vee \left[cl_X \bigvee_{i=1}^{n} \rho_{\alpha_i} \right]$$

(iv) Given a fuzzy regular open cover $\mathcal{B}$ of any fuzzy closed set λ, then there exists $\rho_{\alpha_i} \in \mathcal{B}$, $1 \leq i \leq n$ such that

$$\lambda\ cl_X \bigvee_{i=1}^{n} \rho_{\alpha_i}$$

(v) X is fuzzy absolutely closed and fuzzy regular seminormal.

Proof: The equivalence of (i) and (ii) follows from the definition.

(iii) ⇒ (iv)

Let λ be any fuzzy closed set. Let $\mathcal{B} = \{\rho_\alpha / \alpha \in \Delta\}$ be any fuzzy regular open cover of λ.

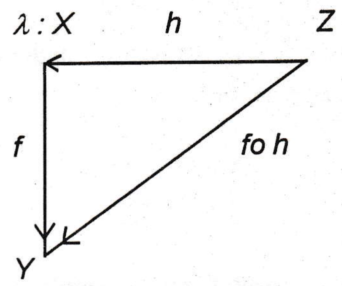
Then $\mu = 1 - \bigvee_{\alpha\in\Delta} \rho_\alpha$ is fuzzy regular closed and $\bigvee_{\alpha\in\Delta} \rho_\alpha = 1 - \mu \Rightarrow \mathcal{B}$ is a fuzzy

Proof: Let f be a fuzzy continuous mapping of X into a fuzzy Hausdorff space Y. Let λ be a fuzzy closed set in X. Then λ / X_i is fuzzy closed in X_i for i = 1 to n and f / X_i are also fuzzy continuous for i = 1 to n.

Then $f(\lambda) = \bigvee_{i=1}^{n} (f / X_i)(\lambda / X_i)$ is fuzzy closed in Y. Therefore X is fuzzy functionally compact.

Proposition 5: Let X be a fuzzy Hausdorff space and Z a fuzzy functionally compact, fuzzy Hausdorff space Y. Let h be a fuzzy continuous mapping of Z onto X. Then X is fuzzy functionally compact.

Proof: Let f be a fuzzy continuous mapping of X into a fuzzy Hausdorff space Y.



Let λ be any fuzzy closed set in X. Then $f(\lambda) = (f \circ h)(h^{-1}(\lambda))$ is fuzzy closed since Z is fuzzy functionally compact and $f \circ h$ is fuzzy continuous.

This proves that X is fuzzy functionally compact.

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