CHAPTER ONE

The Development of Mathematical Logic During the Middle Ages in Europe

1. Survey and Periodization

Despite the fact that mathematical logic is primarily an offspring of the nineteenth century, individual ideas in this scientific discipline were brought to light much earlier. In fact, the foundations of a number of concepts in mathematical logic were laid in the Western European Scholastic tracts on logics.

The achievements of the logicians of the Middle Ages have not received enough attention in the Soviet literature. However, the foreign literature includes a number of important works. Source material on medieval logic is presented in a number of books, notably References 17 and 370. The works of K. Prantl in Reference 17 still command admiration for his scrupulousness (although we must remember that Prantl, who was not well versed in mathematical logic and wrote at a time when the subject was just beginning to develop as an independent scientific discipline, naïvely believed that such general logical tracts as *de consequentis* and *de insolubilis* were conglomerations of foolishness).

The Middle Ages has an important influence on the field of logic. The scholars of that time excelled in subtle and abstract arguments; however, isolation from experiment and the necessity of making their arguments agree in every detail with the tenets of Catholicism, the religion of many of the prominent researchers, were burdens on their creative powers. The existing atmosphere of verbal battles led to an increased interest in the study of the laws of deductive logic. In addition, this was sometimes used as a convenient means for escaping from the ideological domination of clerical dogmatism. Even Abelard, in the twelfth century, actually proved that the "theses of revelation" were inconsistent with the logical studies of the Greeks and Arabs. The fact that specific examples and religious content as well as "secular" illustrations were contained in logic is, of course, of no fundamental importance. Thus, as an example of a fixed true statement, the Scholastics frequently used the phrase *deus est* while, as an example of an impossible proposition, they used the phrase *deus non est*. Incidentally, the reader may find it humorous that such statements as *deus est bonus* (God is good) and *asinus est animal* (the ass is an animal) were taken to be logically equivalent.

The Scholastic logic of the Middle Ages can be divided into three phases. The first phase — the *vetus logica*, which included Aristotle's *On Commentaries*, his *Categories* with Porphyry's commentary, and the logical tracts of Boethius — extended into the middle of the twelfth century. The second phase, the *logica nova*, was characterized by the introduction, between 1136 and 1141, of the *Analytics*, *Topics*, and *On Sophistic Refutation* by Aristotle. The third phase began with the tract *On the Properties of Terms*, which led to the formation of the so-called modern logic (*logica modernorum*) in the form of various summulae.

Attempts to separate the development of medieval logic into periods are found in References 370 and 21. The first period extends from the beginning of the Middle Ages through the time of Abelard (roughly, through the middle of the twelfth century). The principal results of this period are related primarily to the works of Byzantine logicians, especially Michael Psellus (1018–1096), *Synopsis Organi Aristotelici Michaele Psello autore*, edited by M. Elia, Ehingero, 1597. Psellus was a Byzantine who lived in Constantinople. A platonist in his philosophic views, he subscribed to the Neoplatonic interpretation of the philosophy of Plato and Proclus. Of the latter Psellus said nothing without appending the epithet "the divine." However, the elements of Neoplatonism in the philosophical views of Psellus had no great effect on his work in logic. Psellus' main contribution to the Byzantine Renaissance was that, long before Spinoza, he applied the numerical methods of mathematics and geometry to proving philosophical statements.

The late Middle Ages tract *de terminorum proprietatibus* (on the properties of terms) has, for its starting point, a commentary of Psellus on the last chapter of Aristotle's *Topics*, which considers the importance of the various parts of speech from the viewpoint of logic.

According to Prantl, Western European scholarship became familiar with the *Synopsis* of Psellus in the thirteenth century (see *Synopsis quinque vocum Porphyrii et Aristotelis Praedicamentum*, Paris, 1541, and *Synopsis Logicae Aristotelis*, Augusta Vindelicorum [Augsburg], 1597, 1600).

In particular, Psellus examined the problem of the equivalence of propositions (*aequipollentia propositionum*) and found logical conditions required for such equivalence. Much space in the *Synopsis* is occupied by a study of substitutions of terms for other terms that would apparently be scarcely possible without explicit distinction between logical constants and logical variables. However, a large part of his logical research is nonetheless devoted to the problems that occupied the first commentators on Aristotelian logic during the time of Boethius and is associated primarily with a scrupulous study of various forms of syllogisms.

One of the most colorful figures in the first period of development of medieval logic was Peter Abelard (1079–1142). Among his works, published at various times, we should note *Introductio ad theologiam*, *Dialectica* (Reference 73), *De eodem ad diverso*, *Quaestiones naturales*, *Sic et non*, *A History of My Calamities* (St. Petersburg, 1902), *Die Glossen zu Porphyrius*, Beiträge zur Geschichte der Philosophie im Mittelalter (Münster, 1895).

In his commentaries on Porphyry, apparently written during his youth, Abelard went far beyond a simple interpretation of the Greek text. He defined logic as the science of evaluating and distinguishing arguments on the basis of their truth. Logic, said Abelard, teaches us neither how to use arguments nor how to construct them. Since arguments consist of propositions, and propositions consist of expressions, logic must begin with a study of simple statements and then more complex ones. Abelard devoted special attention to the copula and subjected it to logical analysis.

The problem of universals occupies an important place in the logical works of Abelard (this is, the problem of the nature of general concepts, which must be solved if a satisfactory theory of abstraction is to be constructed). The nature of the general, he said, depends not on words (*voces*) but on their meanings (*sermones*). His conception, which differed from that of more extreme nominalism, is sometimes called sermonism. Abelard attempted, first of all, to emphasize the semantic aspect of universals, that is, the relation between the symbols used for universals and the objects they represent.

Abelard's *Dialectics* contains an early form of a modally implicative treatment of conditional statements. For example, consider the following rules, which concern the truth of an implication:

1. If the antecedent is true, so is the consequent;
2. If the antecedent is possible, so is the consequent;
3. If the consequent is false, so is the antecedent;
4. If the consequent is impossible, so is the antecedent.

Abelard sometimes stated the first of these rules in the following form (Reference 73, p. 287):

1*. Nothing that is true ever implies something that is false.

According to Reference 50, p. 221, in one place Abelard used 1* as a general principle for construction of proper arguments.

The logical problematics of Boethius has a rather strong influence in a number of places in *Dialectics*. This is evident particularly in Abelard's discussion of equivalent expression of certain logical constants in terms of others. Thus, following Boethius, Abelard used the following equivalents:

5. $$(A \oplus B) \equiv \overline{(A \rightarrow B) \mathbin{\&} (B \rightarrow A)},$$

where $\oplus$ denotes the exclusive "or" (*Aut est A aut est B*). Abelard was also familiar with the use of disjunction in the ordinary (as we would now say, non-Boolean) sense (Reference 50, p. 223; *A vel B*). In particular he used the conjunction *vel* to combine predicates into categorical propositions. Incidentally, Abelard never chose definitely between the "exclusive" (Boethius) and "inclusive" (Chrysippus) use of the word "or." In the last analysis, he was inclined to treat disjunctive propositions as necessary disjunctions of the form $N(A \vee B)$, interpreting them as an alternative form of necessary implications of the form $N(\bar{A} \rightarrow B)$, where N is the necessity functor, and the antecedent of the implication is therefore negative (Reference 73, p. 489–499). Thus, Abelard placed the necessity operator before the weakened disjunction as a whole. For him, modal disjunction was in some sense more elementary than implication and other logical unions.

Another important logician of the twelfth century was English-born Adam of Balsham (*Parvipontanus* or Adam du Petit Pont), who taught the "trivium"* in Paris in his own school not far from Petit-Pont. Methodologically, he was opposed to the noted representative of early French Scholasticism Gilbert de La Porrée, (*ca.* 1070–1158/9). As a logician, Adam was known primarily for his adaptation of the "First Analytics" of Aristotle. The French historian Cousin refers to one of Adam of Balsham's manuscripts under the heading *Anno MCXXXII ab incarnatione Domini editus liber Adam de arte dialectica.*†

The Italian historian L. Minio-Paluello has analyzed the logical works of Adam in detail (Reference 74). He believes that Adam took a big step forward in his logic, in comparison with Aristotelian logic. For example, he conducted a detailed analysis of the liar's paradox in the form of the statement of a man claiming to be lying (*qui se mentiri dicit*).

As noted in Reference 79, Adam admitted the possibility of the existence of a set of objects with a proper subset having as many objects as the set itself. In other words, he anticipated Cantor's definition of an infinite class. Thomas Ivo (Reference 79) was therefore absolutely right in saying that one of the paradoxes of the infinite had already been stated in the twelfth century. In another place in his *Ars Disserendi*, the criterion Adam used to distinguish between finite classes and infinite classes was that the former do not have the property he noted as characteristic of the latter. But this actually anticipates C. S. Peirce's (nineteenth century) idea that a finite set cannot be mapped one-to-one onto a proper subset.

* The trivium was composed of grammar, rhetoric, and dialectics.

† Cousin, *Fragment d. philos. du moyen-âge*, Paris, 1885, p. 335. Concerning Adam of Balsham, see also Reference 17, vol. 2, pp. 104, 212, 213.

From the standpoint of mathematical logic, the period we have been considering is distinguished primarily by the attempts of Psellus to develop general methods for finding the "middle term" in connection with problems on finding propositions (*inventio propositionum*) and by his interest in logical symbolism, mnemonics, and finally a number of problems he considered on the logic of modality.

The beginning of the second period dates to approximately the middle of the twelfth century and continues until the last decade of the thirteenth. The most important logicians of this period were William Sherwood* (William of Shyreswood, died 1249), Albert von Bollstädt (Albertus Magnus or Albert the Great, 1193–1280), and their students Peter of Spain (1210–1277), John Duns Scotus (1270–1308), and Raymond Lully (1234–1315).

William was born in Durham in the last decade of the twelfth century and taught first at Oxford and later in Paris. He translated the *Synopsis* of Psellus long before Peter of Spain did. The *Synopsis* was the basis of the handbook of logic that Sherwood wrote in Latin; this book remained in manuscript form until it was rediscovered during the nineteenth century (*Codex Sorbonnicus*, 1797).

Sherwood was concerned primarily with a series of problems in modal logic. His study of the "liar's antinomy" became the basis for studies associated with analysis of undecidable propositions. This group of problems subsequently led to Middle Ages tracts on the theme *de insolubulis*. In addition, William devoted much attention to further development of the mnemonic methods of Psellus, among which we should note, for example, the well-known "logical square."

Peter of Spain's *Summulae Logicales* enjoyed great popularity as a text for teaching logic. Peter expended much effort in developing mnemonics to fit the needs of logic. His name is also associated with the revival of the traditions of the Peripatetic school in the Middle Ages. The logical works of Peter of Spain may be found in the following publications: *Textus omnium tractatum Petri Hispani*

* It appears that the source of William's surname was the same Sherwood Forest associated with the legendary Robin Hood.

(Venice, 1487, 1489, 1503; Cologne, 1489, 1494, 1496); *Summulae Logicales of Peter of Spain* (Notre Dame, Indiana, 1945).*

The *Summulae* consist of six sections: studies on suppositions, propagation, designation, distribution, limitation, and statements requiring additional interpretation. The *Summulae* present various mnemonic methods for remembering the rules of logic. Consider, for example, the following mnemonic verse for remembering the rules for proving equivalence of statements:

Non omnis, quidam non. Omnis non quasi nullus.
Nonnullus, quidam: sed nullus non valet omnis.
Non aliquis nullus. Non quidam non valet omnis.
Non alter, neuter. Neuter non praestat uterque.

It is easy to see, for example, that the first phrase of the first line (*Non omnis, quidam non*: "Not for all true, therefore false for some") corresponds to the formula $\overline{\forall} x\mathfrak{A}(x) \equiv \exists x\overline{\mathfrak{A}}(x)$, while the first phrase of the second line (*Nonnullus, quidam*: "Not for all false, so true for some") corresponds, in modern symbolic notation, to the relation $\overline{\forall} x\overline{\mathfrak{A}}(x) \equiv \exists x\mathfrak{A}(x)$, where the bar indicates the logical operation of negation, $\forall$ is the universal quantifier, $\exists$ is the existential quantifier, and $\equiv$ denotes equivalence. In the *Summulae* much attention is devoted to the tract *de terminorum proprietatibus* (on the properties of terms), which, in particular, analyzes words that in ordinary language have the same function as quantifiers have in formal logic, that is, the words every (*omnis*)†, no (*nullus*), both (*uterque*), neither (*neuter*), all (*totus*), and so on. There is also a special study of the construction of statements made by placing the word "only" before the subject of a sentence and of the elimination of statements containing the word "except" or containing suppositions, etc.

This last concept deserves further comment. The very word "supposition" literally means "proposition." The theory of suppositions is a theory of notation (for terms). A Scholastic would

* The literature on Peter of Spain includes References 25 and 61.

† Later Duns Scotus used the term *omnis* in the sense of the universal quantifier for predicates and the term *unusquisque* for the universal quantifier for individuals.

say that a term denoting a really existing object is being used in the mode of formal supposition (*suppositio formalis*). For example, in the proposition "the Earth is round" the term "Earth" denotes an actually existing object, the terrestrial sphere. On the other hand, a Scholastic would say that a term signifying a word that is the name of an existing object is used in the mode of material supposition (*suppositio materialis*). For example, in the proposition "'earth' consists of five letters," the term "earth" signifies a word (and not an object), and so is used in the mode of *suppositio materialis*. Although it is impossible to establish a direct connection between the suppositions of logicians of the Middle Ages and the precise logical notions of the present, there is, for example, reason to attempt to find semantic analogs for the use of terms in the mode of *suppositio materialis*.* The great popularity of the *Summulae* is indicated by the fact that it underwent forty-eight printings in the half century following the invention of the printing press.

Albert the Great (Albertus Magnus), a contemporary of Peter of Spain's, was born in Lauingen in Swabia, studied at Padua, and taught in Cologne and Paris. In the theory of abstractions, he leaned toward the compromise solution of the problem of universals, which was due to Avicenna. His writings were collected in *Opera Omnia*, vols. 1–38, Paris, 1890–1899.

Albert, who was strongly influenced by the Arabian Averroës (Reference 17, vol. 2, p. 105), was, in particular, responsible for development of Psellus' theory on methods of finding propositions; his primary goal was to solve the type of problem that appears when corollaries of given premises are sought. The problem he solved could be applied only to the syllogistic apparatus developed by Aristotle; hence his results were of a very special nature.

Particularly unique logical views were held by John Duns Scotus, who was also an outstanding political figure in the Middle Ages: he raised the problem of "apostolic poverty" to the hated papal curia, attacked the richness of the church, and exposed papal robbery.

Early in his career, Duns Scotus showed a great interest in

* Partially anticipating Peter of Spain, William Sherwood had already distinguished between formal and material supposition.

mathematical disciplines. At 23, he became a teacher at Oxford. Since he was a member of the Franciscan order, Duns Scotus was in outright opposition to the prevailing Catholic dogmatism of his day and, in particular, to the theological determinism of Thomas Aquinas. Duns Scotus' output is striking in scope. For example, the Lyon edition of his works (1639) consists of twelve volumes. Detailed analyses of the logical views of Duns Scotus may be found in Reference 56 and in E. Gilson, *Jean Duns Scotus*, 1952.

Following al-Farabi, Duns Scotus distinguished between the logic of the theoretical (*docens*) (as the science of necessary conclusions from necessary premises) and the logic of the applied (*utens*). He provided great impetus to the development of Peripatetic logic, basing his work primarily on the compendium of Peter of Spain. Duns Scotus, in particular, is responsible for the logical law which is formally expressed as: $p \supset (\bar{p} \supset q)$, where the bar indicates negation.

To a considerable extent, his ideas are responsible for intensive development of an important aspect of medieval logic, namely treatises on the theme *de obligatoris*. As Vladislavlev (Reference 19) notes, the formal impetus for these treatises is a single statement of Aristotle's (which occurs in his work twice — in the first book of the second *Analytics* and in the ninth chapter of *Metaphysics*): A premise accepted as really possible cannot entail an impossible conclusion. The Scholastics noted that if certain propositions are related (for example, one proposition implies another), then other propositions, on the contrary, can be treated as independent of one another.

One of the aims of the *de obligatoris* treatises consisted in finding logical conditions under which the assertion or the negation of some proposition belongs to this class of necessary implication. The notion of *obligatio* is defined as a defensible system of assertions such that none imply anything impossible, that is, in modern terms, the system of assertions satisfies the criterion of consistency.*

It appears that tracts on the theme *de obligatoris* can be treated, although far from absolutely so, as predecessors of axiomatic-

* "*Obligatio est praefixio alicjus enunciabilis ne sequitur impossibile*" (Reference 19).

deductive studies. Analysis of the logical legacy of Duns Scotus shows that it can be treated as a predecessor of not only the calculus of propositions in contemporary mathematical logic but also studies of Husserl type. It is scarcely necessary to add that the first tendencies in the logic and methodology of this prominent Scottish thinker were progressive.

Another bright figure in medieval logic was Raymond Lully. Although a number of authors have attempted to place him off the "main line" in the development of logic, it seems natural to say that the achievements of Lully (part of which, we should note in passing, belong to *de consequentis*) are a natural consequence of the raised level of logical formalization that was characteristic of the period after the *Summulae* appeared. It should be noted that Vladislavlev (Reference 18) attempted to overcome the nihilistic view of Lully and his school. According to Vladislavlev, Lully's "art" is a natural result of the evolution of logic in the Middle Ages.

Raymund Lully was born in 1234 in Palma, on the island of Majorca. Until he was nearly thirty, he was a courtier at the court of Jacob, King of Aragon, and he obtained some measure of acclaim as a poet. Lully's courtly life was conducted in very grand style. Once, in courting a married woman, he rode a horse at top speed into a church during a mass. As the legend has it (Reference 63), the "lady of his heart" said to her suitor: "Would you see my breasts, which your sonnets have so lavishly adorned? Well then, I can afford you that pleasure!" With these words, she lifted her mantle and exposed her breasts, which were covered with bleeding ulcers. This sight stunned Lully, and on the spot, he decided to devote himself to the church. The shock experienced by Lully is comparable to that experienced by Prosper Merimée's Don Juan in *A Soul in Hell* when, learning from a funeral procession that the "object of his affections" has died, Don Juan rapidly converts from a thoroughgoing hedonist to a hermit who confines himself to a monastic cell. The decision so quickly made by Lully abruptly changed his life and he removed himself from worldly vanity. Fortunately for science, Lully decided to devote his time to much more useful pursuits than the Don Juan of Prosper Merimée.

Exchanging the costume of a courtier for the habit of a monk, Lully devoted himself almost completely to the study of logic. He defined logic as the "art and science by means of which truth and untruth can be recognized by reason and separated from each other — the science of finding truth and eliminating falseness" (*Logica est ars et scientia, quae verum et falsum ratiocinando cognoscuantur et unum ab altero discernitur, verum eligendo et falsum dimittendo*); cited in Reference 17, vol. 3, p. 150. Lully's works are collected in *Raymundi Lulli Opera*, Argentorati, 1617. Of material on the nature and extent of Lully's logical ideas, the following sources should be emphasized: References 14, 15, 60, and 62.

After he developed the "great art," Lully threw himself into an ascetic life, attempting to find methods for practical application of his studies. He traveled to all the principal cities of Western Europe, acquainting the academic world with his discovery and simultaneously igniting a zeal among the clergy to eradicate Mohammedanism. He traveled to Africa three times, attempting to lead the "unfaithful" to the "way of truth." Lully spent the remainder of his life attempting, first, to build his own school of logic and, second, to "persuade" Mohammedans to change their relation to Christianity. He was successful in neither of his aims: a Lullist school of logic appeared only long after his death, and the Mohammedans held to their own convictions.

The device invented by Lully for mechanization of syllogistic processes was rather primitive and limited in its capabilities. Lully began by constructing four special figures. The *first figure* included nine absolute predicates ("is a quantity," "is perfect," etc.), nine subjects ("quantity," "perfection," etc.), and nine letters: *B*, *C*, *D*, *E*, *F*, *G*, *H*, *J*, and *K*. The outer ring of the associated circle was comprised of letters; then came subjects and, finally, predicates. The inner ring consisted of a circumscribed disk with a star-shaped figure drawn on it; the lines of this figure indicated the directions of all possible combinations of the initial concepts.

On the *second figure* nine relative predicates were written ("is different from," "is in the relation of agreement with," "is in the relation of mutual exclusion to," etc.).

Absolute predicates and relative predicates were combined in the *third figure*; combinations of the form *BB* were eliminated, and a combination of the form *XY* was assumed to be identical to the combination *YX*. The third figure generated two-term combinations of terms, that is, sentences.

The *fourth figure* was a system of three disks of which only the outermost was stationary. This one, according to Lully, could be used to simulate the syllogistic process, by which he meant the procedure of combining (and substituting) terms. Here the conjunction *est* was taken by Lully to express the relation between the whole and a part. It is interesting that, although Lully sometimes used the operation or quantification (refinement of volume) in an implicit form, he was, on the whole, far from a systematic analysis of this operation. In effect, the only operation in his logic of terms is the operation of intersection. This naturally greatly restricted Lully's combinatorial techniques and made it impossible to construct algorithms for his individual methods. For the structural details of Lully's figures, see Reference 19, pp. 100–107.

Among the members of Lully's school, we should note the following: the pantheist Giordano Bruno, Agrippa von Nettesheim (1487–1535), Athanasius Kircher (died 1680), J. H. Alsted (1588–1638), and the materialist and Epicurean Pierre Gassendi (1592–1655).

In his philosophical works, Lully fought Averroism and, in particular, the study of the dual nature of truth. Unfortunately, his arguments, as we would now say, were a "critique of the right."

In addition to his attempts to mechanically simulate logical operations, Lully also busied himself with analysis of the relations between the logical constants "and" and "or" as well as with a study of the logical essence of interrogative propositions. For example, in Lully's *Introduction to Dialectics*, statements are initially divided into true and false (Reference 19, p. 111). On pp. 151–152 of the same work, Lully presents conditions under which conjunctive and disjunctive statements may be true or false. These conditions correspond exactly to the modern rules for truth of conjunctions and disjunctions. And Lully's rule that "from the universal we can proceed to the corresponding particular, indefinite, and individual"

(*Introduction to Dialectics*, F8 rB) corresponds to the following three formulas of the calculus of predicates:

$$\forall x \mathfrak{A}(x) \vdash \mathfrak{A}(x_p) \text{ (to the particular!)}, \quad (1)$$

$$\forall x \mathfrak{A}(x) \vdash \mathfrak{A}(y) \text{ (to the indefinite!)}, \quad (2)$$

$$\forall x \mathfrak{A}(x) \vdash \mathfrak{A}(a) \text{ (to the individual!)}, \quad (3)$$

where $\forall$ is the universal quantifier and $\vdash$ denotes derivability.

The bridge between Lully's logic and that of the Jansenists of Port Royale was Lully's analysis of so-called general points of view (aspects) for study of various objects. The *loci communes* of the scholars of Port Royale was nothing more than a further development of Lully's "general points of view," directed toward a known standardization of problems on a certain theme (see Lully's work *Tractatus de venatione medii inter subjectum et praedicatum*). To the section on statements, Lully attempted to append the Scholastic theory of suppositions, which he greatly simplified: he separated suppositions into formal (simple and complex) and material.*

A particularly important achievement of the second period was the appearance of the *tractatus syncategorematibus*, which contains an analysis of simple and complex terms pertaining to the formal structure of propositions, for example, "not," "and," "if..., then," "each," etc. In the fourteenth century, the Scholastics augmented this treatise with a section entitled *Sophismata*, which

* Lully's simplification of the theory of suppositions was a direct consequence of well-known weaknesses of this viewpoint (for example, in the form used by Peter of Spain).

The theory of supposition was later criticized and even parodied. Francois Rabelais, for one, parodied it in the following manner in *Gargantua and Pantagruel* (Chapter XX, Book I):

As a sign of gratitude, Gargantua offers the Scholastic Janotus a bolt of cloth. Then there is an argument between Janotus and his colleagues, who are experts in logic, about who should get the cloth. "What is the supposition of this cloth?" (*Pannus pro quo supponit?*) asks Janotus. "Mixed and distributive" (*confuse et distributive*), says Bandouille "I am not asking you, blockhead, about the nature of this supposition (*quomodo supponit*) but about its object (*pro quo*). Answer: This piece of cloth was intended for my legs (*pro tibiis meis*), as the supposition carries the supplementaries (*sicut suppositum portat adpositum*)."

considers the most typical errors that appear in arguments, as well as an entire class of problematical statements.

The culmination of the achievements of the medieval logicians occurs in the third period in the development of Scholastic logic, which covers the epoch between William of Ockham (1300–1347) and the end of the Middle Ages (Ockham studied logic intensively between 1328 and 1329; see Reference 370). During this period a significant shift in Scholastic methodology occurred. In the first place, we should note the studies of intensification and remission of forms. Study of the variability of the intensity symbols other than the uniquely defined symbols of the Scholastics was a step toward the mathematical theory of variables (Reference 29).

This study is closely related to the attempt at symbolization in philosophy, which Maier characterizes as an "innovation of the fourteenth century"; she also considers the "inclination to mathematization of philosophical conceptions and proofs" (Reference 29, p. 79). Nonetheless, this tendency cannot be said to be fundamentally "new"; in essence, it is only a development of the older methodological ideas of the Chartres school.

The fourteenth century exhibits a distinctly new approach to the problem of the infinite; this approach is associated with abandonment of the "physicalization" of this notion by Aristotle. At this time, a considerably more abstract (than Aristotle's) base led to animated discussions of the relations between "categorematic" and "syncategorematic" infinity, which, to some extent, correspond to the modern problem on the relations between actual and potential infinity. Naturally, this controversy had an important influence on the process of refining the terms in logical works on the theme *tractatus syncategorematibus*. In these works, in addition to purely logical constants, there begin to appear analyses of such nonlogical notions as "infinitely many" and "infinitely small."

In view of the new tendencies of the fourteenth century in methodology, a symptomatic and unique (in his generation) attempt was made by the nominalist and skeptic Nicholas of Autrecourt (born about 1300, died about 1350) to revive the atomism of Democritus. His opposition to the current of orthodox Catholic dogmatism is eloquently indicated by the papal curia's censure of Nicholas' works in 1346 and by the subsequent burning of his "heretical" books.

The most prominent philosopher and logician of the Middle Ages, William of Ockham, was born at the very end of the thirteenth century. He studied at Oxford, where he later taught until 1324. As the result of a charge of heresy, Ockham spent four years in prison at Avignon. Rescued from his ideological adversaries, he found refuge at the court of Louis IV of Bavaria. Decidedly at odds with the papacy in the controversy over separation of church and state, Ockham supported the view that church and state should be completely separated. He admitted the possibility that the Pope and ecumenical councils might deviate into heresy, that is, he was actually a distant predecessor of the Reformation.* His conception of the duality of truth was sociologically reflected in his battle for secularization of the state. Ockham died of plague in Munich in 1347.

Many historians have given the name "terminism" to Ockhamist nominalism in the field of logic since Ockham believed that the point of logic is analysis of signs. In addition, he believed that the scientific methods applicable to logic, rhetoric, and grammar were closely related. "Logic, rhetoric, and grammar," he said, "are really cognitive studies, and not speculative disciplines, since they actually control the intellect in its activity."† Literature on Ockham can be found in References 1–10. In Boehner's opinion (Reference 3), Ockham considered three truth values: "true," "false," and "indefinite." We will denote truth by the numeral 1, falseness by numeral 0, and indefiniteness by N. Consider the function $p \supset q$ (if p, then q), defined on the set $\{0, 1, N\}$. For $p \supset q$ we construct the truth table shown in Figure 1, which, according to Boehner, is the form it took in Ockham's studies.

* In the dispute between Pope Boniface VIII and Philip the Fair, Ockham took the side of the temporal sovereign. He sharply criticized the Pope and his retainers in his *Defensorium* (a pamphlet). It is interesting to note that Ockham believed that it is impossible to prove the existence of God.

† Ockham's logical works are *Summa totius logicae*, Oxford, 1675; *Summa logicae*. Part 1, published by P. Boehner, Louvain, 1951; *Quodlibeta septem*, Paris, 1487.

In our opinion, we are not actually dealing, in this case, with a multivalued logic in the strict sense of the word. Rather, because "indefiniteness" is considered in addition to "true" and "false," the situation reflects a well-known occurrence in the logical square of elementary logic (for example, in this square the proposition that a partially affirmative statement is "true" implies that the

p	q	$p \supset q$
1	1	1
0	1	1
N	1	1
1	0	0
0	0	1
N	0	N
1	N	N
0	N	1
N	N	N

Figure 1

corresponding generally affirmative statement is only "indeterminate"). As a result, Boehner's assertion of the existence of a trivalent logic in Ockham's work seems to be a modernization of the problem.

Signs, according to Ockham, can be applied in two ways: artificially (words) and naturally (thoughts). Knowledge is composed of signs that, instead of reflecting reality, denote it. Signs become terms when they enter into propositions, and terms interpreted by the intellect form concepts; the value of spoken or written terms is, to a great degree, conventional, while concepts, according to Ockham, have a natural significance. Universals, in his opinion, are only terms that are used in logic to denote a group of objects and relations, but in no case do they actually exist as divine essences. Ockham believed that universals are also related to the mental act of understanding, that is, obliquely, with the external world. The essential pattern of abstraction, he asserted, depends on the so-called similarity of real substances.

Like all nominalists of the Middle Ages, Ockham carefully distinguished so-called connotative names (from the grammatical viewpoint, adjectives and participles). Thus, for example, the adjective "triangular" not only applies to a concrete triangle but also indicates its property of "triangularity." In other words, the general notion of "triangular" contains not only "concrete objects" (such as equilateral triangles) but "abstract" objects (such as "triangularity"). On the other hand, Ockham also discussed so-called absolute signs (such as "Plato," "virtue") each of which denote either only concrete objects (in a particular case, material objects) or abstract objects.

Ockham classified the sciences as rational or real. He defined the distinction between these two groups as follows: "Real sciences are distinguished from rational sciences by the fact that the terms, that is, the part of known propositions, in real science indicate and denote real objects, while rational sciences deal with terms that indicate and denote other terms" (Ockham, *Expositio aurea super totam artem veterem*).

The fact that Ockham's gnosiology is absolutely consistent not only with deductive methodology but also with inductive methodology was not, of course, accidental. Inductive empiricism gradually developed despite the triumph of deductive habits and concepts. The Scholastic proponents of the inductive theory of abstraction and the inductive method were Roger Bacon (1214–1294) and his teacher Robert Grosseteste (1175–1253).

Robert Grosseteste was born in Strandbrook, England (Suffolk). He made his reputation as a master at Oxford and a commentator on the physical and logical works of Aristotle. Of his writings, we should note *Roberti Grosseteste epistolae* ed. H. R. Luard (1861), vol. 25, *Rerum Britannicarum medii aevi scriptores*. Of his original works, we should note *Summa in octo physicorum Aristotelis libros*, 1498. He is discussed in References 77 and 78. According to Grosseteste, physics studies form as filled by matter, while mathematics abstracts form from matter. As a concession to the spirit of his time, he begins with the thesis that knowledge of universals requires intuitive comprehension. However, recognizing the role of intuition in gnosiology and ontology, Robert was not alien to rationalistic tendencies when he passed to formulation of problems in scientific methodology proper. Using the Aristotelian concept of mathematics as a more accurate physics (*Metaphysics* 12. 3. 1077b–1078a), Grosseteste went considerably further than the Stagirite, asserting that mathematics must be the basis of all physical disciplines.

"Logical and metaphysical truths, as a result of their distance from the senses and as a result of their precision," he said, "escape the intellect which must view them, as it were, from a distance; hence, the fine distinctions are not recognized. It is here that consideration from a distance, together with blurring of fine distinctions, proves to be the source of certain errors. Similarly, the reliability of physical conclusions is reduced because of the changeability of natural substances. It is these three — that is, logic, metaphysics, and physics — that Aristotle calls rational (*rationales*) since the unreliability of our comprehension of them requires us to make greater use of argument and probability than in the sciences; these latter, of course, contain both science and proof, but not in the most rigorous sense of the words. However, mathematics is at once a science and a proof in the most rigorous and proper sense (*maxime et particulariter dicta*)" (cited in Reference 76, p. 111).

Elsewhere, Grosseteste asserts that "all sources of natural actions must be given directly by lines, angles, and figures" (Reference 79, p. 293), which explicitly contradicts Aristotle's physicalism. On the whole, the methodology of Grosseteste can be characterized as mathematical atomism.

Thus, according to Grosseteste, mathematics is the most exact and exemplary scientific discipline. However, he took only the first steps along the path of practical application of mathematical methods to analysis of empirical data. The ideas that he developed concerning the necessity of experimental investigation of nature were later adopted and made concrete by the Franciscan monk Roger Bacon.

Roger Bacon was born into rich nobility near Ilchester in Dorset. In 1225 he entered Oxford and, after finishing his studies, remained there to teach. In Paris, Bacon joined the Franciscan order. Upon his return to Oxford in 1252, he lectured on mathematics, physics, and foreign languages. In 1257 he was banished from the university as a "wizard and sorcerer," and between 1278 and 1292 Bacon was imprisoned by the ignorant monks as a "dangerous heretic." The major works of Bacon are *Opus majus* (published 1733); *Opus secundum; Opus minus; Opus tertium* (*ca.* 1260); *Essays*, ed. Little (Oxford, 1914); *Summulae Dialectices* (Oxford, 1940). Bacon's work in logic is discussed in References 43 and 80–82.

According to Bacon, there are "two methods of knowing: by proof and by experiment. Proof yields a solution to a problem, but does not provide us with certitude, since the correctness of the solution need not be confirmed by experiment" (Reference 80, p. 18).

Bacon decisively separated theology from science and philosophy. In contrast to the qualitative estimates used by Aristotle in discussions of the world, he took important steps in the direction of rigorous quantitative evaluations of being.

In the fourth part of his *Opus majus*, he defines mathematics as the "alphabet of philosophy." "Mathematics has had the misfortune to be unknown to the church fathers," said this pioneer of Western European empiricism with biting irony. Mathematics, he believed, must be treated as an important goal in any branch of scientific endeavor.

"Mathematics provides," said Bacon, "universal techniques . . . that can be applied to all sciences . . . (and) no science can be known without mathematics" (translated by O. V. Trakhtenberg, Reference 43, p. 164).

"The problem of logic," Bacon said, "is to construct arguments that will move the practical intellect toward goals of love, courage, and felicitous being" (*Opus majus*, 59). No doubt this definition of the problems of logic is, to some extent, a flight of poetic rhetoric.

He includes logic, which he treats as an auxiliary scientific discipline like grammar, in the study of method.

In addition to Ockham and Bacon, among the outstanding logicians in the third period of the development of Scholastic dialectics we should note John Buridan (1300–1358), Albert of Saxony (1316–1390), John of Cornubia (second half of the fourteenth century), Ralph Strode (second half of the fourteenth century, peak in 1370), Marsilius von Inghen (1330–1396), Nicollette Paulus of Venice (Paul of Venice) (died 1429), and Peter of Mantua (second half of the fourteenth century). The third period witnessed the apogee of nominalism (and its variant, terminism), which occurred with the establishment of a methodological basis for logical investigations that is actually close to materialism. Note that the Paris physicist and Ockhamist John Buridan was the teacher of Albert of Saxony; the most extreme position of terminism was taken by Peter of Mantua; among the students of Buridan was Marsilius von Inghen; among students of the Scotists were Ockham and Strode.

The achievements of this era are highly respected in the classical works of Marxism-Leninism. In his article "Debate on the Free Press" Marx said, "The twenty tremendous volumes of Duns Scotus are as stunning . . . as a Gothic edifice," and from them we obtain a "real sense of value."* In another place, the founder of scientific communism said "Materialism is the unmistakable offspring of Great Britain. Even her great Duns Scotus wondered whether 'matter can think.'"†

Let us briefly summarize the problems dealt with by the nominalist logicians. The main direction taken by their research appears in the treatises *de consequentiis* and *de insolubiliis*, considered to be important supplements not only to Aristotelian logic but also to the compendium of Peter of Spain, which provided a direction for work on syllogisms and the theory of suppositions. The source of work in *de consequentiis* in the Middle Ages was the third book of the "Topics," where Aristotle focused attention on the existence of

* K. Marx and F. Engels, Collected Works (Russian translation) (Moscow–Leningrad, 1928), vol. 1, p. 140.

† K. Marx and F. Engels, *op. cit.*, vol. 2, p. 142.

contingent conclusions in which the assertions about one subject make it possible to make assertions about other subjects of the same type (Reference 19, p. 39). While the Scholastics were able to draw the idea of formal implication from Aristotle, for the elements of the theory of material implication they turned to the works of the Arabian logicians Avicenna, al-Farabi, al-Ghazzali (Algazel), and Averroës (Reference 17, vol 3, p. 138).

The tract *de insolubiliis* was brought to the nominalists via William Sherwood. The subject was highly developed by Thomas Bradwardine* (1290–1349), Walter Burleigh (1273–1357), Buridan, Albert of Saxony, and others. Many examples of semantic antinomies can be found, for instance in the sixth part of *Perutilis Logica Magistri Alberti de Saxonia* (Venice, 1592).

The liar's paradox was analyzed by Ockham in the third part of the fifth tract of his *Summa Logicae*. A theory of formal implication was constructed by Strode† in his work *Consequentiae Strodi* (Venice, 1493). The exceptionally broad range of logical research and the tremendous dimensions (even on the scale of contemporary works) of the logical compendia are astounding. Bochenski notes in Reference 370 that, for example, Paul of Venice's‡ *Logica Magna*

* This British Scholastic is noteworthy, in particular, for his attempts to apply mathematical methods of research to justification of a rationalistic theological system.

† Ralph Strode was a logician and pedagogue who taught at Oxford in the second half of the fourteenth century. He traveled widely, visiting France, Germany, Italy, Syria, and Palestine. The logical works of Strode (among them, in particular, *Consequentiarum formulae*) were placed in the *Consequentiae* and *Obligationes* classifications at the end of the fifteenth and sixteenth centuries.

‡ Paul of Venice, the Averroist from Udino, began teaching at Oxford in 1390 and wrote extensive commentaries on all of the fundamental philosophical and logical works of Aristotle. Paul's logical works comprise perhaps the highest state of Scholastic logic. He died in Padua in 1429. In addition to his *Logica Magna*, among his best known works we should note *Dubia circa philosophiam* (1493), published in 1498 under the title *Quadratura*. This work presents, in particular, the following classification of semantic antinomies:

1. The consequent is simultaneously correct and incorrect;
2. the statement is simultaneously true and false;
3. contradictory notions about the same object;
4. statements of mutual exclusion (Reference 11, p. 636).

(fourteenth century) contained about 3,650,000 words, that is, (according to the most conservative calculations), about the same amount as is contained in five or six volumes of a modern encyclopedia.

Aristotle's *Organon* provided great impetus to further evolution in logical studies. Leibniz considered the invention of syllogistics a major accomplishment of the human intellect. Thus far, Aristotelian logic had preserved its scientific value. Oriented basically toward gnosiological, linguistic, and grammatical factors, it nevertheless influenced the Greek mathematicians contemporary with Aristotle. A number of prominent men (among them Weyl) believe that the Aristotelian theory of proof has definite points in common with the structure of Euclidean geometry (Reference 23, p. 35). The usual characterization of Aristotelian logic as a logic of monadic predicates (properties) is, in general, correct.

However, discussions of the relationship between Aristotelian formalism and other logical studies must not neglect the fact that even Leibniz believed it possible (in principle) to reduce all relations to properties, and here he proceeded as a follower of the Stagirite. The consensus of many contemporary logicians is that Aristotelian logic includes the theory of formal implication and the beginnings of the calculus of modalities. It is not an accident, therefore, that the elements of mathematical logic appeared in the Scholastic tracts at the end of the twelfth century, when the Scholastics began to work directly from Aristotle's *Organon* and not from the investigations of Porphyry of Tyre (232–304) or A. M. T. S. Boethius (480–526), as had been the case before.

We should note that the scholars of the Middle Ages who dealt with logic constantly emphasized that they were the students of Aristotle and carefully cited all the places in his works that stimulated them to research in new directions.* The study of

* Some logicians have erroneously believed that Scholastic logic led away from Aristotle. In particular, this is the belief of the French logician Charles Serrius, who berated the nominalist logicians for illegitimately substituting words for ideas. This argument makes absolutely no sense: we must realize that no one has yet succeeded in operating with ideas outside his verbal–material skeletons (Reference 24, pp. 58–59).

mathematical logic in Scholastic treatises was stimulated by two fundamental circumstances: first of all, the influence of direct acquaintance with the original logical research of Aristotle and, second, the strengthening of sensualistic tendencies and the attraction toward problems in mathematics and natural science that were characteristic of the latter days of orthodox Scholasticism.

We will limit the analysis primarily to the characteristics of the two principal treatises (*de consequentiis* and *de insolubiliis*) that are of the most interest from the viewpoint of the history of logic (especially mathematical logic). We will first briefly describe the medieval logic of modalities and the *tractatus exponibilium* of Peter of Spain. Reference will not be made formally to the original texts except when it proves necessary for clarification of important points or when the meaning of the text is not clear. For literature on medieval logic, see References 1–63.

2. The Modal Logic of the Middle Ages

No special treatises on modalities were written during the Middle Ages, although discussions of modality appeared in various, more general works. Sherwood had already stated six forms of modal "modes": true, false, possible, impossible, contingent, and necessary. Later, Scholastics tended to limit themselves to three modes which, in their opinion, were of the most value: necessity, possibility, and impossibility. As the result of research on semantic antinomies, the logicians of the fifteenth century dealt primarily with the modes of truth, falseness, and undecidability (*insolubile*).

Sherwood was the first to apply the method of the so-called logical square (introduced by Psellus, see Reference 25) to analysis of the relations between statements that were related "materially" (i.e., in content) but were of different modalities. Sherwood's square of modal statements was later reproduced by his student Peter of Spain.

We denote the necessity functor by N, the possibility functor by M, the impossibility functor by U, and the contingent functor by T. Then Sherwood's square takes the form shown in Figure 2.

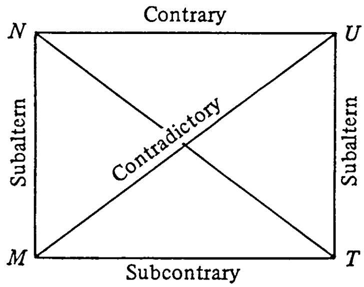

Figure 2

If, for example, N is true, then M is also true, T is false (N and T are in contradictory relation to each other), and U is false (N and U are in the contrary relation to each other). The same method of logical squares was applied to the study of direct modal conclusions by Ockham (Reference 17, vol. 3, pp. 317–318). We will consider, for example, one of his squares of modalities. Let N and M, respectively, be the modal necessity and possibility functors, where the symbols a, e, i, and o are respectively used to denote universal affirmation (XaY), universal negation (XeY), partial affirmation (XiY), and partial negation (XoY) in Aristotelian syllogistics. Ockham's square took the form shown in Figure 3.

The diagram in Figure 3 can easily be justified if we recall the relationship between the functors N and M, that is, $N(t) = \overline{M(\bar{t})}$, where the bar denotes logical negation and the symbol t denotes an

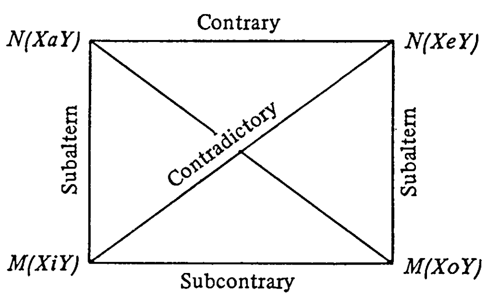

Figure 3

arbitrary statement. Then, if we substitute the expression XaY for t, we find that

$$N(XaY) = \overline{M(\overline{XaY})} = \overline{M(XoY)},$$

and, by setting t equal to XeY, we find that

$$N(XeY) = \overline{M(\overline{XeY})} = \overline{M(XiY)}.$$

In other squares, Ockham placed modal functors not only before a sentence but also before its components (that is, before both the subject of the sentence and its predicate). It should be emphasized that modal logic of propositions was, as a rule, still not differentiated from modal syllogistics, and the latter developed much more rapidly than the former. A special place was always occupied by the study of direct modal implications, which achieved its apogee in the work of Buridan.* Desiring mnemonic and graphical solidifications of the results he obtained, he was forced to abandon Psellus' square and to adopt a star-shaped figure (see Fig. 4). He considered modal functors (necessary, impossible, and possible) and the syllogistic functors a, e, i, and o. The subject–predicate structure of sentences was preserved, although modal functors as well as the symbol for logical negation could be placed before a statement as a whole or before any of the statement's individual components.

In Figure 4 we have substituted m for a and t for e to prevent confusion with the corresponding syllogistic functors. At each of

* John Buridan (1300–1358) studied logic, ethics, and physics. He taught at the University of Paris. His philosophical viewpoint comes under the heading of Ockhamism. From some time during the fifth decade through some time during the sixth decade of the fourteenth century, Buridan remained at the university where he was twice elected rector. Of his works, we should note the following: *Sophismata* (Paris, 1493); *Quaestiones super libros quattuor de caelo e mundo* (Cambridge, Mass., 1942); *Summa de dialectica* (Paris, 1487); *Perutile compendium totius logicae, cum jo. Dorp. expositione* (Venice, 1490). Buridan was known as a student of Ockham and was an intellectual determinist: In choosing one of several possible decisions, the "will" is controlled by reason. It acts only when the reason decides that one of the possibilities is the best (the will also chooses this possibility). But if reason decides that several possibilities are equivalent, the will does not act. This is the source of the well-known example of "Buridan's ass," which died while standing between two identical handfuls of grass.

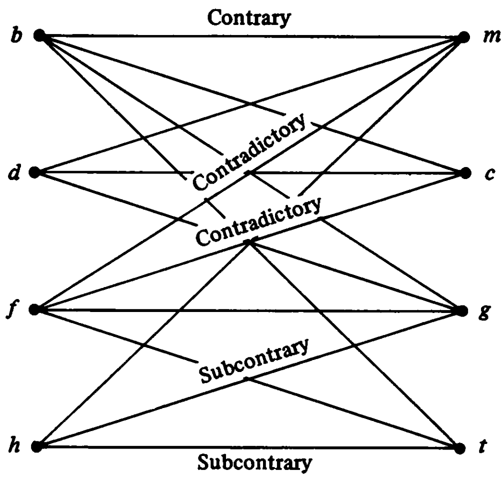

Figure 4

the vertices of the star-shaped figure, labeled *m*, *b*, *c*, *d*, *g*, *f*, *h*, and *t*, there is one of nine equivalent statements. Thus, it is possible to construct eight sets of statements (with nine in each set, where all statements in a given set are equivalent).

The following types of relations hold between these sets: contradiction, contrariness, inclusion (subordination), etc. We should note, for example, the contradictions. They constitute the following combinations: *gb*, *hm*, *tb*, *fm*, *gd*, and *fc*. Thus, for example, the sentence $Xa(N(Y))$, where N is the necessity functor and X and Y are the subject and predicate, respectively, is true at the vertex *m* if, let us say, the statement $Xo(N(Y))$ at the vertex *h* is false; conversely, $Xa(N(Y))$ is false if $Xo(N(Y))$ is true.

We will now present an example of equivalence of sentences of the set at the vertex *m*. The sentences $Xa(N(Y))$ and $Xe(M(\overline{Y}))$ of this set are equivalent, or we have that

$$Xe(M(\overline{Y})) \equiv Xe(\overline{N(\overline{\overline{\overline{Y}}}})) \equiv Xe\overline{(N(Y))} \equiv Xa\overline{\overline{(N(Y))}} \equiv Xa(N(Y)),$$

where $\equiv$ denotes logical equivalence.

The above examples give some idea of medieval research on modal logic. Other, more interesting modal concepts of the Middle Ages are closely related to the contents of treatises on the theme *de consequentiis*, which we will consider in Section 4.

3. Analysis of Separative and Exclusive Sentences

Medieval logical treatises on the theme of exponibilia (exponibilia are propositions that require interpretation) are of interest because the study of exponibilia led the Scholastics close to the so-called De Morgan's laws. These rules were not explicitly stated by the Scholastics, although they were definitely used implicitly.

For separative exponibilia (that is, sentences of the form "only X has the property Y"), Peter of Spain presented the following four rules in his *Summulae* (Reference 19):

1. If a separative sentence is affirmative, it is equivalent to some conjunctive sentence whose first term is some affirmative statement (coinciding in content with the initial exponibilium without the word "only"), while the second statement is negative and such that its predicate is negated relative to all that is not the subject of the initial exponibilium.
2. "Only S is P" implies "every P is S."
3. If the symbol of negation is placed before the exponibilium, that is, if the exponibilium is of the form "it is not true that only S is P," then it is equivalent to a sentence of the form "it is not true that all S are P or (*vel*) something other than S is P."
4. The statement "only S is not P" is equivalent to an affirmative conjunctive sentence of the form "S is not P and all other than S are P."

Rules 1 through 4 admit the following symbolic representations (the expression X_Δ denotes "only X"); (1′ corresponds to 1, 2′ corresponds to 2, etc.):

1′. $(X_\Delta aY) \equiv [(XaY) \,\&\, (Xe\bar{Y})]$, where "&" denotes "and";
2′. $(X_\Delta aY) \supset (YaX)$, where $\supset$ denotes "if . . ., then";
3′. $\overline{(X_\Delta aY)} \equiv [\overline{(XaY)} \vee (\bar{X}iY)] \equiv [XoY \vee \bar{X}iY]$, where $\vee$ denotes inclusive "or" (*vel*);
4′. $(X_\Delta eY) \equiv [(XeY) \,\&\, (\bar{X}aY)]$.

In Expression 3, De Morgan's law $x_1 \& x_2 \equiv \bar{x}_1 \vee \bar{x}_2$ is used implicitly. Similarly, De Morgan's laws are used for analysis of exclusive statements (that is, sentences of the form "any S_1 except S_2 has the predicate P").

The mnemonic aids of the logical square have been widely used for study of the relations between exponibilia that distinguish quantity and quality (that is, on the basis of the volume of the

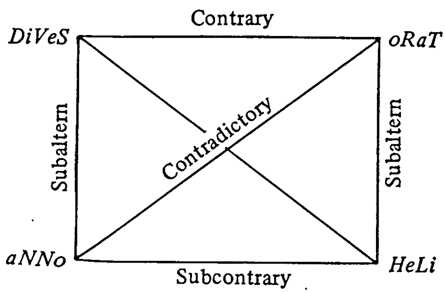

Figure 5

subject or the affirmative or negative nature of a copula) or that distinguish between the subject and predicate of a statement (in the corresponding sentence).

Consider, for example, the square constructed by Peter of Tartaret* (see Reference 17, vol. 4, p. 208). This square (or more properly, "metasquare" since it contains several logical squares) took the form shown in Figure 5.

As usual, *i*, *a*, *o*, and *e* are the syllogistic functors, while the letters *D*, *S*, *R*, *T*, *N*, *H*, *V*, and *L* have no semantic meaning — they are simply parts of the mnemonic signs.

Assume that we are dealing with exponibilia of type *i*, that is,

* Peter of Tartaret was a fifteenth-century Scotist and an active commentator on the logical, physical, and ethical works of Aristotle. A Franciscan who is representative of the later Scholastics, he became rector of the University of Paris in 1490. Of his works, we note *Expositio in Summulas* (1501); *In Summulas Petri Hispani, in Isagogen Porphyrii et Aristotelis Logicam* (Venice, 1592).

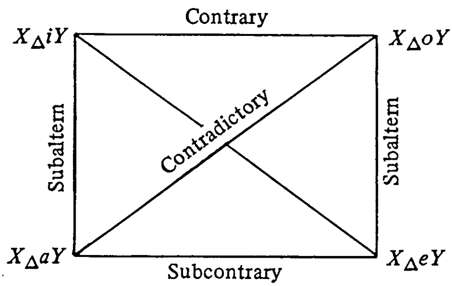

Figure 6

with expressions of the form $X_\Delta iY$. For this case the "metasquare" of exponibilia takes the form shown in Figure 6:

$$X_\Delta iY \equiv \overline{X_\Delta eY} \equiv \overline{XeY \,\&\, \bar{X}aY} \equiv (XiY) \vee (\bar{X}oY)$$

$$X_\Delta oY \equiv \overline{X_\Delta aY} \equiv \overline{XaY \,\&\, \bar{X}eY} \equiv (XoY) \vee (\bar{X}iY)$$

If $X_\Delta aY$ is true, then $X_\Delta oY$ is false.

In conclusion, we note some of the fundamental rules of negation for syllogistic propositions. These rules are explained in many places, so we will present them immediately in their symbolic form:

1. $\overline{SaP} \equiv SoP$; 2. $\overline{SeP} \equiv SiP$; 3. $\overline{SoP} \equiv SaP$;
4. $\overline{SiP} \equiv SeP$; 5. $Sa\bar{P} \equiv SeP$; 6. $Se\bar{P} \equiv SaP$;
7. $Si\bar{P} \equiv SoP$; 8. $So\bar{P} \equiv SiP$; 9. $\overline{Se\bar{P}} \equiv SoP$;
10. $\overline{Sa\bar{P}} \equiv SiP$; 11. $\overline{Si\bar{P}} \equiv SaP$; 12. $\overline{So\bar{P}} \equiv SeP$.

Despite the obvious use of De Morgan's laws in certain places in the *tractatus exponibilium*, it took a long time to find medieval works that would make it clear whether the Scholatics were aware of the meaning of De Morgan's laws in their general form (that is, without special conditions on the formal structure of the variables treated by the laws). Finally Boehner (Reference 26) succeeded in demonstrating that Ockham used De Morgan's laws. A number of logical historians (such as J. Lukasiewicz, Reference 27) believe that one could say Ockham's laws instead of De Morgan's laws. For the history of the application of De Morgan's laws during the Scholastic period, the interested reader is referred to the valuable material of Reference 26.

4. The Theory of Logical Implication

Here we will present a detailed analysis of the division *de consequentiis*. Fundamental to this section is the notion of implication, which is rather broad.

First of all, this notion includes the modern view stated by the expression "A implies B" (B can be derived from A by way of definite rules; symbolically, $A \vdash B$, where implication is meant in the sense of some relation between A and B).

Secondly, this notion includes the idea corresponding to interpretation of implication as an operation or function. In modern "language," this approach corresponds, for example, to the symbolic relation $A \supset B$, which, without stretching the point, can be read "If A, then B." Interpretation of implication as an operation has been traced to the Megarian-Stoic school (Philo). This approach, however, was comparatively rare in the logic of the Middle Ages, and, as we shall shortly see, it was unknown to Duns Scotus, who treated implication in the sense of derivability.

Duns Scotus classified the forms of implication as follows: "Implications may be separated into proper and improper. Proper implications are enthymematic or syllogistic. Enthymematic implications can be separated into formal and material, the latter being simple or factual."*

According to Duns Scotus, proper implication (*bona consequentia*) is concerned with a hypothetical relation in which the antecedent and the consequent are related by a contingent or causal relation such that it is impossible for the antecedent to be true and the consequent to be (simultaneously) false.† Duns Scotus also claimed

* Duns Scotus, *Qu. Sup. Anal. Pr,* 1, 20, p. 302 B; see Reference 17, vol. 3, p. 139.
† *Ibid,* 10, p, 287 B: "*Consequentia est propositio hypothetica composita ex antecedente et consequente mediante conjunctione conditionali vel rationali, quae denotat, quod impossibile est ipsis, quod antecedens sit verum et consequens falsum.*" See Reference 17, vol. 3, p. 139, fn. 614.

that any proper implication that is not an Aristotelian syllogism is an enthymeme. He characterized "formal implication" as follows:

"A formal implication occurs when . . . both terms can be transformed (*uterque terminus est convertibilis*)" (Reference 17, vol. 3, p. 229, fn. 199).

Although it might seem from the above quotation that Duns Scotus included material implication in the modern sense in the notion of formal implication in a similarly modern sense, it is clear from elsewhere in his work that he used the terms *consequentia materialis* and *consequentia formalis* in other than the modern sense.

He associated the first term with conclusions that cannot be obtained in a purely formal manner and, if the conclusion is to be logically justifiable, must be supplemented by additional information. (The simplest case of such conclusions is the common enthymeme of elementary logic.) On the other hand, he associated *consequentia formalis* with conclusions of a purely formal nature. However, *consequentia materialis* can be reduced to *consequentia formalis* by reconstitution of omitted premises, after which the content of the implication plays absolutely no part in determining the logical correctness of the conclusion. Material implication can be reduced to formal implication in two different ways.

Duns Scotus called material implications that could be reduced to formal implications by appending a statement of necessity to the antecedent "simple material conclusions" (*consequentia materialis simpliciter*). For example, we can reduce the implication "Man walks. Therefore, animals walk," to a formal implication (in the Duns Scotus sense) by adding the statement "Every man is an animal."

On the other hand, Duns Scotus called a material implication that he reduced to a formal implication by appending a contingent statement to the antecedent "a factual (*ut nunc*) material implication." Thus, for example, the proposition "Socrates goes. Therefore, what is white goes," can be reduced by constructing the contingent statement "Socrates is white."

Although Duns Scotus treated the relation of logical implication as dealing with content, he and his closest student, William of

Ockham, treated implication itself in a manner very close to that of formalization provided by material implication. Thus, among Ockham's rules for implication, we might note, for example, the following: (1) An impossible statement implies any conclusion. (2) A necessary statement follows from whatever is convenient.*

These rules correspond to the following identically true expressions in the propositional calculus: $A \supset (\bar{A} \supset B)$ and $A \supset (B \supset A)$, where the bar denotes logical negation.

Elsewhere, Ockham states the following rules: (3) A true statement never implies a false statement. (4) A possible statement never implies an impossible statement (Reference 17, vol. 3, p. 129).

These rules correspond to the following formulas in propositional calculus and modal logic:

$$A \;\&\; \bar{B} \supset \overline{(A \supset B)} \quad \text{and} \quad ((A \equiv p) \;\&\; (B \equiv imp)) \supset \overline{(A \supset B)},$$

where p denotes some fixed statement of possibility (from the Latin *possible*) and *imp* denotes a fixed statement of impossibility (from the word *impossibile*).

We should also note certain other of Ockham's rules of implication (see Reference 17, vol. 3, pp. 390, 392, 396, 411–415, 417–419): (5) A false statement may imply a true statement (*ex falsis potest sequi verum*). (6) What follows from the consequent also follows from the antecedent (*Quidquid sequitur ad consequens, sequitur ad antecedens*). This is a statement of the factual transitivity of implication. (7) Necessity does not imply contingency (*Ex necessario non sequitur contigens*). (8) What contradicts the consequent contradicts the antecedent (*Quidquid repugnat consequenti, repugnat antecedenti*). We can formalize Statement 7 as follows:

$$(x \equiv N(x)) \;\&\; (y \equiv (M(y) \;\&\; M(\bar{y}))) \supset \overline{(x \supset y)},$$

where $N(x)$ denotes the statement "x must occur," and the conjunction $M(y)$ & $M(\bar{y})$ states that y is contingent (here contingency is defined as the possibility of M: M may or may not be true).

We should keep in mind that Ockham took necessity to be the

* Ockham, *Summa Logicae*, c. 70A, cited in Reference 17, vol. 3, pp. 129–130.

negation of contingency, so his seventh rule is actually only a restatement of his third (it is sufficient to replace the term "true" in the latter with the term "necessary" and the word "false" with the word "contingent"). Of course, it should be noted that here there is no assertion of metaphysical identification of the categories of truth and necessity on the one hand and the notions of falseness and contingency on the other; there is merely a statement that necessity bears the same relation to contingency as truth to falseness.

Duns Scotus' successor Ralph Strode developed what is actually a theory of formal implication (in the sense of Duns Scotus), for which he stated twenty-four rules (Reference 17, vol. 4, pp. 46–48, 50–52). Some of these rules are interesting because Strode introduced a further gradation in the modes of modal logic. He included the mode "questionable" between "true" and "false." Some of his rules read as follows: (9) If the consequent is questionable, the antecedent is either questionable or known to be false (*Si consequens est dubium, et antecedens est dubium vel scitum esse falsum*). (10) If the antecedent is questionable, the consequent is not necessarily negative (*Si antecedens est dubitandum, ergo consequens non est ab codem negandum*). (11) If the consequent is negated, the antecedent is not necessarily questionable (*Si consequens est negandum, antecedens non est ab codem dubitandum*). (12) If the antecedent is known, the consequent is known (*Si antecedens est scitum, et consequens est scitum*). This rule states perfectly the essentially analytic nature of implication in the sense of deductive implication. (13) If the consequent is impossible, the antecedent is also impossible (*Si consequens est impossibile, igitur et antecedens est impossibile*). (14) If the consequent is contingent, the antecedent is either contingent or impossible (*Si consequens est contigens, et antecedens est contingens vel impossibile*).

Analysis of other rules stated by Strode shows that, together with material implication* in the sense of the Stoics and Avicenna, and

* Strode defines material implication as follows: "Material implication is governed by two rules: (1) An impossible statement implies anything; (2) A necessary statement follows from anything" (*Pro consequentia materialis sunt duae regulae*: (1) *ex impossibile sequitur quodlibet*; (2) *Necessarium sequitur ad quodlibet*).

formal implication in the sense of Duns Scotus, he permitted a type of implication that was not considered by either the Stoics or Duns Scotus. This is made clear by Strode's remark, "a conditional relationship may be false, even if the antecedent and consequent are simultaneously true" (. . . *Conditionalem esse falsum, cuis tam antecedens, quam consequens esset verum*). Strode provides the following example of such a false conditional relationship: *Si tu est homo, ergo sum homo.*

Thus, Strode moved away from the Stoic notion of implication (which he did not reject entirely, for he reserved it for particular types of sentences) toward the notion of a type of implication which might be called semantic and which assumes that the consequent must be part of the antecedent if the corresponding conditional relationship is to be true. The material available does not permit us to determine which of the contemporary notions of implication — the Lewisonian or the Ackermannian (Reference 28) — Strode was moving toward; we can only be sure that he abandoned the treatment of implication as strictly a function of truth, an approach taken by the logical school of the Stoics.

The rules of implication stated by Albert of Saxony are also of interest (Reference 17, vol. 4, pp. 73–75). In particular, he stated the rule that, if A, together with some statement of necessity, implies B, then B follows from A alone.* Symbolically, we have—

$$(15)\quad \frac{A, N \vdash B}{A \vdash B},$$

where the N is associated with the phrase "statement of necessity" (from the word *necesse*) and the symbol $\vdash$ denotes derivability (following from the preceding).

Marsilius von Inghen (Reference 17, vol. 4, pp. 101–102) was responsible for an important contribution to the development of *de*

* Reference 17, vol. 4, p. 73: *Logica Alberticuii*, C2, F24rB: *Si ad A cum aliqua necessaria sibi apposita sequitur B, tunc B sequitur ad A solum.* In the literature, Albert of Saxony is also called Albert von Riegensdorf. He is known as a mathematician and commentator on the works of Aristotle and Ockham. After syllogistics, he considered the *Topics*, which he treated as part of a study on implication in the form of a theory of so-called maximal propositions (*propositione maxima*) (Reference 17, vol. 4, p. 78).

*consequentis.** Among the rules for which he is responsible we should note the following:

(16) The entire disjunction follows from each of its terms.†

(17) A universal statement makes it possible to conclude arbitrary individual terms.‡

It is easy to see that the sixteenth rule corresponds to the modern rules

$$A \vdash A \vee B, \quad B \vdash A \vee B,$$

for introducing the sign of disjunction, while the seventeenth rule corresponds to the rule

$$\forall x \mathfrak{A}(x) \vdash \mathfrak{A}(y)$$

of the restricted calculus of predicates, where $\forall$ denotes the universal quantifier and can be read "for all x." The letter $\mathfrak{A}$ denotes some property (predicate) of the subject x. The expression "the subject x has the property" can be symbolically written in the form $\mathfrak{A}(x)$.

In his work *Logica magistri Petri Mantuani* (published in 1483 in Pavia and in 1492 in Venice), the terminist Peter of Mantua considered the problem of *de consequentiis* in close connection with problems of modal logic. However, he is important not because of his results as a whole but because of one of the rules pertaining to the problem of the modality of rules, namely, the rule that leads to the conclusion *ad nullam de necessario*, that is, in contemporary terms, a necessary statement follows from the empty set of premises (*ad*

* Marsilius von Inghen was a German successor of John Buridan. He received his education in Paris, reached the level of magister, and became rector of Heidelberg. His *Dialectics* was appended in 1512 to an edition of Peter of Spain's *Summulae.*

Doctrinally Marsilius was a Thomist, but in the theory of material implication he leaned toward the views of Duns Scotus; in logic he was at one with Ockham. In the theory of abstraction he adopted a compromise position between realism and nominalism. Marsilius wrote glosses for Aristotle's *Categories* and for the well-known *Introduction* of Porphyry. Among his works we should note *Quaestiones super quator libros Sententiarum* (*Petri Lombardi*) (1947), which was published in a second edition in 1501.

† Reference 17, vol. 4, pp. 101–102.

‡ *Ibid.* Raymond Lully stated this rule in a somewhat more complicated form: "From a universal statement we can conclude the corresponding particular, indeterminate, and individual statements."

nullam). In addition to this rule, Peter stated sixty-three other rules of the same type.

Gradually a tendency arose to consider the general notion of logical implication. As far as we can judge from Prantl's classification (Reference 17, vol. 4, p. 181) of views on the nature of implication in medieval logic, there were three fundamental approaches to the definition of logical implication. According to the first view, implication is the relation of derivability of a consequent from an antecedent (Duns Scotus, Ferabrikh); according to the second, implication consists in establishing a situation where the consequent is part of the antecedent (Strode, Ockham); according to the third, implication is nothing more than an aggregate consisting of an antecedent, a consequent, and a conjunction (the extreme terminists). The first conception is close to the theory of formal implication, the second is very nearly the same as what is meant by implication in the semantic sense, and the third treats implication in a manner close to that admitting formalization in terms of the apparatus of material implication.

5. The Theory of Semantic Paradoxes

Paradoxes of the "liar's" type had been dealt with as early as the time of Adam of Balsham (twelfth century). Basically, however, rapid progress concerning *de insolubiliis* occurred during the fourteenth and fifteenth centuries. Important contributions to these studies were made by Albert of Saxony and John Buridan.

Buridan, in particular, considered such *insolubilia** (that is, paradoxical statements) as the statement "everything written in this volume is false," where this statement itself is the only thing written in the entire volume.

We denote the proposition in quotes by p. It is required to determine whether p is true or false. Assume that p is true. But then, as the statement says, it is false, because it is written in the given book. We now assume that p is false. But then the book should contain at

* Buridan's statement of this *insolubilium* reads as follows: *Propositio scripta in illo folio est falsa*. We have presented it in a form somewhat more convenient for analysis.

least one true proposition, and this can be only p since it is the only proposition in the given book. As a result, p must be true. Thus, if we assume that p is true, we must conclude that it is false, and conversely, if we assume that p is false, we must conclude that it is true.

As a result, we have before us two statements of the form $p \supset \bar{p}$ and $\bar{p} \supset p$, demonstrating the presence of a paradox since it follows from them that p is equivalent to $\bar{p}$, that is, either both statements are true (we have p and $\bar{p}$) or both are false (we have $\bar{p}$ and $\bar{\bar{p}}$). In both cases, we have an explicit formal contradiction since both the proposition and its negation are true (in the second case both are false: we have $\bar{p}$ and $\bar{\bar{p}}$).

We will now present several examples of the semantic antinomies encountered in the work of Albert of Saxony that permit, without changes in form, changes in some part of the content of the paradoxical statements within them. We state the first of these antinomies in the following form:

"If $2 \times 2 = 4$, then some contingent statement made by a liar in a given time interval t is false," and, except for the phrase in quotes, over the time interval t the liar N makes no contingent statements.* It is required to determine whether the liar N has made a true or false statement.

To rigorously analyze this paradoxical situation, we initially introduce two predicates: (1) during the time interval t someone has asserted x," which (predicate) we denote by "ass x," and (2) the identity predicate, which has the property that, if x and y are identical (that is, if we have $x = y$), then all that is true concerning x is also true concerning y. We can now state the problem in the form

1′. $\text{ass}(2 \times 2 = 4) \supset (\exists X \exists Y((\text{ass}(X \supset Y)) \;\&\; \overline{(X \supset Y)}))$,

which is a symbolic representation of the sentence "someone says, 'if $2 \times 2 = 4$, then some stated contingent proposition is false.'"

2′. $\forall t \forall v(\text{ass}(t \supset v)) \supset (t = (2 \times 2 = 4))$
$\;\;\;\& \; (v = \exists X \exists Y(\text{ass}(X \supset Y) \;\&\; \overline{(X \supset Y)}))$,

* *Si deus est, aliqua conditionalis est falsa, et sit nulla alia conditionalis* — Albert, Perutilis Logica Magistri Albert de Saxonia, VI (Venice, 1522). Albert set out to determine whether the proposition in italics is true or false (assuming that it is unique).

which states that for N any contingent statement made by N in the time interval t coincides with that of 1′. We now assume something that N said, that is,

$$2 \times 2 = 4 \supset \exists X \exists Y(\text{ass}(X \supset Y) \ \& \ (\overline{X \supset Y})) \qquad (1)$$

is true. Since the premise of the implication is true, in view of our assumption the conclusion is also true, that is, the expression

$$\exists X \exists Y(\text{ass}(X \supset Y) \ \& \ (\overline{X \supset Y})) \qquad (2)$$

is true.

Applying a common procedure of introducing symbols U_0 and V_0 to be treated as constants (that is, for which nothing can be substituted and to which we cannot apply the rule of generalization), we write Implication 2 in the form

$$\text{ass}(U_0 \supset V_0) \ \& \ (\overline{U_0 \supset V_0}). \qquad (3)$$

Applying the rule for elimination of the universal quantifier (*dictum de omni*)* to Condition 2′, we find also that

$$\text{ass}(U_0 \supset V_0) \supset (U_0 = (2 \times 2 = 4)) \ \& \ (V_0 = \exists x \exists y(\text{ass}(X \supset Y) \ \& \ (\overline{X \supset Y}))). \qquad (4)$$

Since, in view of Implication 3, the premise $\text{ass}(U_0 \supset V_0)$ in Implication 4 is true, the consequent of Implication 4 is also true; that is, U_0 coincides with $2 \times 2 = 4$, and V_0 coincides with

$$\exists x \exists y(\text{ass}(X \supset Y) \ \& \ (\overline{X \supset Y})).$$

But this means that $U_0 \supset V_0$ is precisely our Implication 1. Nonetheless, Implication 3 implies that $\overline{U_0 \supset V_0}$. Assuming that Implication 1 is true (that is, that $U_0 \supset V_0$), we thus find that Implication 1 is false.

We now assume that Implication 1 is not true. Since its antecedent is true, the conclusion must be false; that is, we have

$$\overline{\exists x \exists y(\text{ass}(X \supset Y) \ \& \ (\overline{X \supset Y}))} \quad \text{or} \quad \forall x \forall y((\text{ass}(X \supset Y)) \supset (X \supset Y))$$

* We do this by substituting U_0 for t and V_0 for V, which this rule permits us to do.

from which it follows, by virtue of the rule for elimination of the universal quantifier by replacing X by $2 \times 2 = 4$ and Y by $\exists x \exists Y(\text{ass}(X \supset Y) \,\&\, (\overline{X \supset Y}))$, that

$$\text{ass}(2 \times 2 = 4) \supset \exists x \exists y(\text{ass}(X \supset Y) \,\&\, (\overline{X \supset Y})) \supset (2 \times 2 = 4) \supset \exists x \exists y(\text{ass}(X \supset Y) \,\&\, (\overline{X \supset Y})). \quad (5)$$

Since by Condition 1′ the antecedent of Implication 5 is true, the conclusion is also true. But this conclusion is also the consequent of Implication 1, from which it follows that all of Implication 1 is true, even though we assumed it to be false. Thus, if Implication 1 is true, it is simultaneously false. Similarly, if it is false, it is also true. We are therefore dealing with an unsolvable proposition, a paradox.

We can analyze the following examples of Albert of Saxony (which have been changed in content but not in form) in the same way:

a. "$2 \times 2 = 5$ or some disjunction is false," where the universe of discourse contains no more than one disjunctive proposition;*

b. "$2 \times 2 = 4$ or some conjunction is false," where the conclusion in quotes is the only conjunction in the universe of discourse;†

c. "Socrates says, 'Man is an animal,' while Plato asserts that 'only Socrates speaks the truth,' and there are no other assertions. It is required to determine whether or not Plato has lied."‡

We will analyze Example c. First of all, it is quite easy to obtain a paradox contensively. Assume that Plato is telling the truth. In this case only the assertion "Man is an animal" is true, and all others are false.

But Plato's statement differs from Socrates'. Thus, Plato has lied. We now assume that Plato has lied. Then we must find at least one true statement other than that of Socrates. But the only other possible statement is Plato's (recall that, except for his statement,

* Albert de Saxonia, *Logica . . .*, VI; '*Homo est asinus vel alique disjunctiva est falsa,' et sit nulla alia disjunctiva in mundo.*

† *Ibid.*; '*Deus est et aliqua copulativa est falsa,' et sit sic, quod nulla alia copulative sit in mundo, haec ipsa. Tunc quaeritur, utrum sit vera.*

‡ *Ibid.*; *Dicat Socrates: "Deus est," et Plato dicat: "Solus Socrates dicit verum," et non simul alii loquentes in mundo. Tunc quaeritur, utrum Plato dicit verum.*

there is no other). As a result, Plato is not lying; so we have a paradox.

To formalize this paradox, we denote the statements of Socrates and Plato by p_1 and p_2, respectively; we introduce the identity predicate,* and we introduce the predicates SS (Socrates said) and PS (Plato said). The antinomy is formalized by the following system of premises:

1′. p_1 (in other words, p_1 is a fixed true statement).
2′. $SS(p_1)$.
3′. $p_2 \equiv SS(p_1) \,\&\, \forall z((z \neq p_1) \supset \bar{z})$.
4′. $PS(p_2)$.
5′. $\forall w((w = p_1) \vee (w = p_2))$, which means that except for p_1 and p_2, no other statement is made in the time interval t.
6′. $p_1 \neq p_2$.

We assume that what Plato said is true, that is, the following formula is true:

$$SS(p_1) \,\&\, \forall z((z \neq p_1) \supset \bar{z}). \tag{1}$$

But in Formula 1 the first conjunctive term is true (Premise 2′), it can therefore be eliminated. Formula 1 is equivalent to the expression

$$\forall z((z \neq p_1) \supset \bar{z}). \tag{2}$$

By *dictum de omni*, it follows from 2 that

$$(p_2 \neq p_1) \supset \bar{p}_2. \tag{3}$$

But because the antecedent of Implication 3 corresponds to Premise 6′, it follows that 3 is equivalent to

$$\bar{p}_2. \tag{4}$$

Thus, if we assume that p_2 is true, we are led to the conclusion that it is false.

Assume that p_2 is false, that is, the formula

$$\overline{SS(p_1) \,\&\, \forall z((z \neq p_1) \supset \bar{z})} \tag{5}$$

* It is unnecessary to introduce a special distinction predicate, because distinction is nothing more than simple negation of identity.

is true; this formula is equivalent to

$$SS(p_1) \vee \exists z((z \neq p_1) \; \& \; \bar{\bar{z}}), \tag{6}$$

from which it follows, by Premise 2′, that

$$\exists z(z \neq p_1) \; \& \; \bar{\bar{z}}. \tag{7}$$

We now introduce the symbol z_0 to replace z, using the same technique we already used in formalizing the preceding paradox. We have

$$(z_0 \neq p_1) \; \& \; \bar{\bar{z}}_0. \tag{8}$$

Applying the rule for elimination of the universal quantifier to Premise 5′, we find that

$$(z_0 = p_1) \vee (z_0 = p_2). \tag{9}$$

From 8, 9, and the fact that $\bar{\bar{z}}_0$ is equivalent to z_0, it follows that z_0 coincides with $\bar{p}_2$. By substituting p_2 for z_0 we obtain:

$$(p_2 \neq p_1) \; \& \; p_2. \tag{10}$$

But the first term of 10 can be eliminated since it is Premise 6′. Thus, the final statement is:

$$p_2. \tag{11}$$

As a result, if we assume that p_2 is false, we find that p_2 is true, a fact which in conjunction with Formula 4 leads to an antinomy. We leave it to the interested reader to obtain antinomies from the following two examples of Albert of Saxony, which have been slightly changed in content but not in form:

d. We are given three statements p_1, p_2, and p_3, where p_1 states that p_2 is false, p_2 states that p_3 is false, and p_3 states that p_1 is false. Determine whether or not p_1 is false.

e. We are given only the two statements p_1 and p_2, where p_1 states that p_2 is false and p_2 states that p_1 is false. Determine whether p_1 is true or false.

f. $2 \times 2 = 4$; therefore a given consequence does not hold. Determine whether or not Example f is true or false (under the assumption that Example f is unique).

g. We are given two premises:

g_1. "man is an animal"

and

g_2. any statement except g_1 is true.

Determine whether or not Premise g_2 is true.

h. We are given the following premises:

h_1. man is an animal,

h_2. the earth is round,

and

h_3. every statement except h_3 is true.

Is Premise h_3 true?

i. We are given the following premises:

i_1. twice two is four,

i_2. unicorns exist,

and

i_3. every statement other than i_1 and i_2 is false.

Determine whether Premise i_3 is true or false.

We will obtain paradoxes for Examples f through i, beginning with Example f. Assume that Example f is true; in this case the antecedent of Example f can be eliminated as a fixed true statement. But the assertion of the truth of the consequent of f is equivalent to asserting that all of Proposition f is false. As a result, f is false. We now assume that Example f is false, a condition that can occur only if the antecedent is true (and this is indeed so) and the consequent of f is false. But the assumption that the consequent is false is, in this case, equivalent to asserting that the entire proposition f is false. As a result, Example f is true. Paradox.

Examples g and i are of the same type in the sense that they include separative exponibilia (with the word "except"). They can be reduced to paradoxes in exactly the same way, so we will limit our analysis to Example h. Under the assumption that Premise h_3 is true, we can easily show that h_3 is false. If, however, we assume that Premise h_3 is false, that is, at least one statement in addition to h_3 is false, we can easily show that this is impossible since Premises h_1 and h_2 are both true, and, except for them and h_3, we have no other statements at our disposal. As a result, we must assert that Premise h_3 is true (reasoning from the general principle that, if an assumption leads to a contradiction, the given assumption is false).

Albert of Saxony presents a number of examples that, instead of containing insolubilia, contain only some system of premises composed of inconsistent statements. For example, we are given two premises *A* and *B*, where *B* denotes some fixed false statement and *A* states that "everything but *A* is different from *B*." It is required to determine whether *A* is true or false. Initially, assume that *A* is true, that is, the statement "only *A* does not differ from *B*" is true. We must also conclude that *B* differs from *A* (since the universe of discourse consists of only *A* and *B*), and this is impossible (it violates the law of identities). As a result, the assumption that *A* is true is untenable; thus, *A* is false. If we assume that *A* is false, we can only conclude that *B* is not different from *B*; that is, the statement "*A* is true," required to obtain a paradox, does not appear. Thus, the system of premises consisting of *A* and *B* is inconsistent, and analysis of it proves that *A* is false.

Buridan is responsible for a curious system of statements in which no statement is true, the system is not paradoxical, and each of them is defined in terms of another by means of the same method. Namely, he calls attention to the case of two premises, *p* and *g*, where *p* says: "*g* is false" and *g* says "*p* is false." It is easy to see that the assumptions

I. *p* is true and *g* is false and
II. *p* is false and *g* is true

are consistent. There is no way of deciding whether Assumption I is true or Assumption II is true.

Thus far, our discussion of antinomies has remained within the framework of two-valued formalism. The problem of how the Scholastics formalized modal antinomies (i.e., paradoxes whose statements contain modal terms) remains unanswered. Albert of Saxony, for example, presents an antinomy whose literal expression contains such expressions as "Socrates doubts that...," and "Plato proposes that...," (Reference 17, vol. 4, p. 80; Anmerkung, Reference 77). These can scarcely qualify as modal antinomies in the narrow sense since they can apparently be expressed completely within the framework of two-valued formalism. Albert himself offered the opinion that the nature of such antinomies is linguistic and psychological rather than logical.

Discussions of paradoxes in Scholastic logic were very lively. A classification of the viewpoints on the nature of insolubilia is contained in the work of Johann Mayoris Scott (1478–1540), who lived and taught in Paris (Reference 17, vol. 4, p. 250).

A close examination of the opinions (of which Scott counted eight and Bochenski (Reference 370) counted thirteen) shows that there were essentially three approaches: (a) "rejection" (*cassatio*), (b) "restriction" (*restrictio*), and (c) "resolution" (*solutio*).

According to Approach (a), an *insolubilium* is not a proposition (Paul of Venice) and, because it cannot be said to be true or false, it is simply meaningless. We will illustrate Approach (b) with an example taken from the work of Ockham. According to Ockham, the source of an antinomy lies in the fact that the terms required for notation of propositions are sometimes used for notation of the same propositions in which they are used as constituents. More plainly, Ockham meant that part of a proposition (the predicate "is false") must not (in order to eliminate an antinomy) refer to the entire assumption in which it appears.*

Ockham's view therefore reduces to an interdiction of circular definitions. In other words, it is not permissible to require linguistic constructions in which, for example, a given proposition appeals directly to falseness proper (or unprovability). By eliminating circular arguments in his studies of paradoxes, Ockham was led to his famous "razor" ("essentials should not be multiplied more than is necessary"). He believed that his solution to the problem was the most general possible (in the sense that if his proscription is followed, antinomies do not appear). However, Buridan found an example in which Ockham's proscription is not violated (there is no direct appeal to falseness), but an antinomy nonetheless appears.

* The problem of this type of restriction did not appear in connection with the predicate "is true," because, in defining truth, Ockham, following Aristotle, assumed that the propositions (1) "*p*" and (2) "'*p*' is true" are equivalent.

The system Σ consists of the following statements:

C_1. "man is an animal,"
C_2. "only C_1 is true,"
C_3. "C_1 and C_2 are the only available statements."

(See Example c in the discussion of the antinomies of Albert of Saxony; in that example, C_2 is an insolubilium, although it does not appeal directly to falseness.)

The system Σ therefore contains a paradoxical statement, C_2. The special property of C_2 is that it does not contain a direct appeal to the falseness of C_2. Rather, it refers only to Premise C_1. Without knowing anything about C_1, it would still be impossible to draw definite conclusions about the truth or falseness of C_2. Indeed, if C_1 stated that "the meaning of C_1 is identical to that of C_2," no antinomy would appear.

In view of this type of expression, Buridan distinguished two types of paradoxical statements: The first type contains "direct reference" (relative to the truth value ascribed to a proposition in accordance with the Aristotelian definition of truth), while the second type contains "indirect reference." For instance, in the liar's paradox, there is "direct reference," while in the above system of propositions, C_2 contains an indirect reference.

The existence of "indirect reference" provided Buridan with the occasion to declare Ockham's approach to elimination of semantic antinomies ineffective and to attempt to find other methods for elimination of paradoxes. In modern terms, Ockham's approach leaned toward Russell's theory of logical types, while Buridan's approach is somewhat reminiscent of Tarski's viewpoint in *The Concept of Truth in Formalized Languages* (Lvov, 1935).

Of the attempts made during the Middle Ages to eliminate antinomies, we will discuss in more detail only Buridan's theory since this approach is apparently the one of most value. It is presented primarily in his works *Sophismata* and *Johannes Buridani Quaestiones in Methaphysicam Aristotelis* (Paris, 1518).

From the formal viewpoint, Buridan's considerations on a method of eliminating semantic paradoxes in *Johannes Buridani Quaestiones in Methaphysicam Aristotelis* (VI, Qu. 11) can be stated as follows. We assume that p is some unsolvable proposition such as the one we considered above: "all that is written in this book is false." To eliminate the antinomy appearing in this case, Buridan proposes to append to p (the symbol for the given paradoxical proposition) an auxiliary premise Δp, where Δp states that the proposition p is actually spoken by someone, say Socrates. From the conjunction $p \,\&\, \Delta p$, we can derive the proposition p. But from p, in view of the value of p, that is, the actual words in the proposition, we can derive the statement $\bar{p}$.

Thus, from $p \,\&\, \Delta p$ we can obtain the two propositions p and $\bar{p}$; that is, we can obtain an explicit formal contradiction. But, if a given statement leads to a contradiction, it is false. That is, the expression $\overline{p \,\&\, \Delta p}$ is true, but $\overline{p \,\&\, \Delta p}$ implies that $p \supset \overline{\Delta p}$. In other words, the proposition that p is true implies only that Socrates cannot be responsible for it. We should note that $\overline{p \,\&\, \Delta p}$ implies $\Delta p \supset \bar{p}$; that is, the premise that Socrates has spoken p implies that p is false. Thus, there is no paradox.

Buridan's ideas can also be formalized on the basis of an analysis of texts that are not in Prantl's compendium but are presented in Reference 20 (which gives several excerpts from Buridan's works). This latter reference initially attempts to find a concrete form of the Aristotelian meaning of the predicate "to be true" and presents the following method of formalizing Buridan's ideas.

Buridan criticized the viewpoint according to which all (including, therefore, paradoxical) propositions imply another proposition in which the subject is the name of the initial proposition itself and the predicate is "is true." If $\forall$ is the universal quantifier, x is some proposition, x_n is the name of this proposition, and T is the predicate "is true," then the proposition Buridan had in mind takes the form

$$\forall x(x \supset T(x_n)). \tag{1}$$

From the viewpoint of material implication, Formula 1 is, of course, irrefutable. While x may be false in Formula 1, the entire implication is nonetheless true. However, Buridan used the following argument to avoid the thesis that Expression 1 is always true. Information may actually be contained in x, but it need not be fixed in a particular statement. Hence x is only the name of a statement ("the consequent [that is, $T(x_n)$ in Formula 1], because it is affirmative, can denote nothing," said Buridan). He added an additional premise, asserting the existence of the proposition x. Thus, the expression "x_n is the name of x and x is" is equivalent to the phrase "x_n is true." Symbolically, we have

$$T(x_n) = x_n x \& x, \tag{2}$$

where the expression $x_n x$ is taken to be a single, inseparable symbol denoting the statement "x_n is the name of x" ("x_n denotes x"). Definition 2 can easily be used to eliminate the liar's paradox. Let some paradoxical statement (we call it m) mean (after it is appropriately interpreted) "m is false." Symbolically, we have $m = \overline{T(m_n)}$, where m_n is the name of m.

If we accept Thesis 1, a paradox is obtained as follows:

$$m = T(m_n) \qquad \text{(by 1)}, \tag{3}$$

$$m = \overline{T(m_n)} \qquad \text{(premise)} \tag{4}$$

It follows from (3) and (4) that $(Tm_n) = \overline{T(m_n)}$; that is, we have an antinomy that is easily eliminated by use of Premise 2. Assume that m is true. Then we obtain

$$T(m_n) = m_n m \& m. \tag{5}$$

By (4), we have that

$$T(m_n) = m_n \overline{T(m_n)} \& \overline{T(m_n)} \supset \overline{T(m_n)}$$

or

$$T(m_n) \supset \overline{T(m_n)}.$$

We now assume that m is false, i.e., that we have

$$\overline{T(m_n)} = \overline{m_n m \& m} \tag{6}$$

or

$$\overline{T(m_n)} = \overline{m_n T(m_n) \& (\overline{T(m_n)})} \tag{7}$$

from which we obtain

$$\overline{T(m_n)} \supset \overline{m_n T(m_n)} \vee \overline{\overline{T(m_n)}}. \tag{8}$$

It is now impossible to derive $T(m_n)$ from $\overline{T(m_n)}$, so a paradox does not appear.

Thus, the assumption that m is false leads only to the statement "either m is true or m_n is not the name of a true proposition m," but not to the categorical statement (required for an antinomy) "m is true."

In addition to Buridan's theory of elimination of paradoxes, we should also note a similar study by Paul of Venice; we will present a discussion of these ideas, somewhat modifying the formalization presented in Reference 370, pp. 291–292. Paul assumed that, because an insolubilium is a proposition, there is no basis for assuming that the Aristotelian criteria of truth do not extend to it. To eliminate antinomies, however, it is necessary to verify the difference between the "ordinary" and "precise" value of a paradoxical proposition. We will see that this requirement by Paul is equivalent to a refinement of the Aristotelian notion of truth and, in connection with this, that Paul's ideas did not extend past the basic superstructure constructed by Buridan.

Before proceeding to what Paul actually meant, we introduce the following conventions. By f we denote the falseness functor, and by t we denote the truth functor. We will write an insolubilium in the form

$$A \equiv (A \equiv f). \tag{1}$$

Adding Paul's refinement of Aristotle's notion of truth, we obtain the following system of premises:

$$(A \equiv p) \rightarrow ((A \equiv t) \equiv ((A \equiv t) \,\&\, p)), \tag{2}$$

$$(A \equiv p) \rightarrow ((A \equiv f) \equiv \overline{((A \equiv t) \,\&\, p)}), \tag{3}$$

to which we must add the usual definition of falseness,

$$(A \equiv f) \equiv \overline{(A \equiv t)}. \tag{4}$$

We leave it to the interested reader to verify that Premises 1 through 4 and the assumption that A is true imply that A is false. On the other hand, let us consider the consequences of Premises 1 through 4 under the assumption that A is false. First of all, substituting $A \equiv f$ for p in Premise 3, we obtain

$$(A \equiv (A \equiv f)) \rightarrow ((A \equiv f) \equiv \overline{((A \equiv t) \,\&\, (A \equiv f))}). \quad (5)$$

Application of *modus ponens* to 1 and 5 yields

$$(A \equiv f) \equiv \overline{(A \equiv t) \,\&\, (A \equiv f)}. \quad (6)$$

Since we have assumed that A is false, 6 reduces to

$$\overline{(A \equiv t) \,\&\, (A \equiv f)}, \quad (7)$$

which is equivalent to the expression

$$\overline{A \equiv t} \vee \overline{A \equiv f}. \quad (8)$$

But, by Premise 4, $\overline{A \equiv t}$ is equivalent to $A \equiv f$. Therefore, finally we obtain

$$(A \equiv f) \vee (\overline{A \equiv f}), \quad (9)$$

that is, a variation of the principle of *tertium non datur*, from which naturally we cannot derive the necessary conclusion (for a paradox) that A is true. Thus, the paradox has been eliminated, and, according to Paul, the assumption that the insolubilium is false leads only to a statement about the validity of the law of the excluded middle.

It should be noted that many of the viewpoints of medieval scholars on the nature of insolubilia anticipate in one way or another some of the contemporary approaches to elimination of antinomies. For example, some Scholastics insisted on strict prohibitions against the "vicious circle" (*circulus vitiosus*), which they saw as an immediate source of paradoxes. In the opinion of others, an "insolubilium is neither true nor false but lies somewhere in between these notions, different from both,"* an opinion, in modern terms, equivalent to the statement that paradoxes must be resolved in trivalent logic.

* For the text of Paul of Venice on the opinion of his opponents on the subject of insolubilia, see Reference 370, p. 281.

By the end of the sixteenth century, the emphasis on logical research had diminished. And, about the same time, it appears that the differentiation between the logical schools became complete. At the end of the Middle Ages, there were basically three logical schools: (1) the school of peripatetics (which had its origins in the *Summulae* of Peter of Spain); (2) the Ramists, the followers of Peter Ramus (1515–1572), who persistently advanced proposals for reforming Aristotelian logic; and (3) the Lullists, who based their doctrine on the studies of Raymond Lully.

The Frenchman Peter Ramus was not satisfied with the level of Aristotelian logic. Following Lorenzo della Valle and other Stoics, he sharply censured the gap between logic and rhetoric. A number of like-minded scholars in Germany, for example, Johannes Sturm (1507–1589), were drawn to Ramus' criticism.

Ramus criticized the Aristotelian system of categories, which, in his opinion, did not permit classification of logical methods. He had some influence on the logic of Leibniz (in particular, on the problem of using certain figures of syllogisms for proving particular rules concerning the conversion of sentences). Mainly, Ramus proposed using syllogisms of the second and third figures for this purpose. For instance, the modus *Datisi* of the third figure and the principle of identity can be used to prove the reversal rule that SiP implies PiS (that is, the statement "some S are P" implies that "some P are S"). Indeed, if we substitute the term P for M in the statement $(MaP.\,MiS) \rightarrow (SiP)$, we obtain $(PaP.\,PiS) \rightarrow (SiP)$, from which, by appealing to the identically true statement PaP, we obtain $PiS \rightarrow SiP$.

An important place in the logic of Ramus was occupied by problems of methodology and, in particular, the theory of loci, (*loci communes*), that is, points of view that make it possible to construct proofs. He distinguished five "original" points of view and nine "derived" points of view.

Of the works of Ramus, we should note the following: *Animadversiones in dialecticam Aristotelis* (1543); *Dialecticae partitiones* (1543); *Institutiones dialecticae* (Paris, 1543, 1547); *Scholae in liberales artes* (Basel, 1569, 1578, 1582).

The logical ideas of Lully were completely adopted by the well-known Spanish pedagogue and humanist Luis Vives (1492–1540) and the great thinker Giordano Bruno (1548–1600).

Vives insisted that scientific research be based not on the authority of Aristotle but on experiment. In his works *On the Means of the Probable* and *Against Pseudodialectics* (1519) he argued against exaggerating the value of the rhetorical applications of logic, preparing the way for the empirical methodology of Bacon. Vives was responsible also for an ideographic representation of the relations between syllogistic notions. The diagram in Figure 7 is

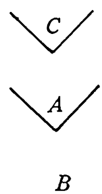

Figure 7

reproduced from his *De censura veri* (1555), where *C* denotes the minor premise of a syllogism, *A* denotes the middle, and *B* is the major term; the inclusion relation is indicated by the V-shaped sign.

Vives' work can be treated as a precursor of ideographic methods later developed by Hamilton. In 1555 his dialogues were translated into German by Breringen in Oldenburg. The collected works of Vives were published that same year in Basel and in 1782–1790 in an eight-volume edition in Valencia. In the Russian language we have *A Guide to Knowledge* (St. Petersburg, 1768), which was translated from the Latin. Further information about Vives can be found in: A. Lange, *Ludvig Vives* and *Schmidt'schen Encyclopädie des Erziehungs- und Unterrichtswesens*, vol. 9 (1869).

A number of ideas in the propositional calculus were slowly formulated by the Lullists in the special study called "axiomatica" that was used as an introduction to syllogistics. In particular, this is the way J. H. Alsted proceeded in his handbook *A New Form of General Logic* (1652). Because he strictly adhered to certain logical ideas of Ramus, Alsted attempted to reconcile the dialectics of

Ramus and Lully with Aristotelian logic. Born in 1588 in Nassau, he was appointed professor of philosophy at Strassburg in 1610. Alsted had commented on Lully's work in *Sources of Lully's Art and True Logic* in 1609. In 1624 he made an attempt at further development of the ideographic methods of Vives. In one of his Frankfurt works, written about 1641, Alsted attempted to state the axioms basic to a number of the philosophical systems of his contemporaries.

Bruno was an active commentator on Lully, as we can see from his works *De compendioso architectura et complemento artis Raymundi Lullii* (Paris, 1582); *De Lulliano specierum scrutinio* (Prague, 1588); and *De Lampade combinatoria Lulliana.*

In keeping with his fondness for convenient mnemonic devices, Bruno attempted to justify the logical technique of the *Ars Magna* by gnosiological apparatus. In *De Lampade combinatoria Lulliana* he considered three forms of activities of the human mind: perception of elementary objects, identification and distinction of objects, and argumentation. Accordingly, in *Ars Magna* a theory for a method of combining terms and argumentation based on the laws of combining terms is derived from a list of initial, undefined terms in a given alphabet.

Bruno was also responsible for a detailed critique of the Scholastic Peripatetic school of logic from the viewpoint of his somewhat improved version of Lully's principles. In his *On the Shadows of Ideas* (1582), he raised the more general problem of improving memory.

A final important achievement of the Lullists was their simplification of the cumbersome studies of Peter of Spain on the forms of suppositions.

The foregoing discussion, of course, raises the puzzling problem of the extent to which the considerable logical achievements of the Middle Ages were forgotten during the Renaissance. Before we attempt to answer this question, we should note the fact that there was an objective limit to further evolution of the Scholastic form of logic. Just as Greek geometry could not advance beyond Apollonius since literal algebra had not yet been developed, so the Scholastic logic of the Middle Ages could not advance beyond its state in the *Logica Magna* of Paul of Venice since no formalism could be found that would make possible consolidation of prior achievements, thus making possible further progress.*

Another source of the well-known *cul de sac* of Scholastic logic can be seen in the fact that the mathematics and technology of the Middle Ages could not sufficiently stimulate its development, not even to the point of developing effective symbolism for the formalisms. The decisive factor, however, is that the technology of the Middle Ages could not stimulate logic the way the industrial practice of rising capitalism could. By virtue of these circumstances, the individual great logical achievements of the Middle Ages, which were not based on an artificial "language," were sometimes formalized by extremely unreliable mnemonic devices and they thus necessarily remained deeply hidden.

Mathematical logic began to flower only in the nineteenth century under the influence of the exact sciences, which made complex computational problems the order of the day. Even the form of the new logic differed from Scholastic logic. Syntactical problems of logic were relegated to the background of semantic problems, which, for a time, were of absolutely no interest to scholars. For example, the Scholastic theory of suppositions was treated as being of doubtful logical value. To some extent, there has been a return to the syntactical form of ancient logic, although the modern form is no trivial reproduction since natural science and technology have advanced greatly, with consequent major influences on the evolution of logical thought. By no means can we assume that the logic of the Middle Ages has been completely analyzed. There are many "gaps," such as the problem of determining the major medieval conceptions of semantic implication and what modern forms of implication can be used to formalize them. In addition, we do not know whether it was only semantic

* It is quite reasonable to argue that the source of this circumstance is the underdeveloped technology and the social structure of Greece at the time of Apollonius (and later) or of the European Middle Ages. Certainly, industrial technology and the nature of the social structure are related to the development of science, which led to both algebra and formalisms not available to the Middle Ages. However, the immediate reason that Greek geometers and the logicians of the Middle Ages were unable to progress was their lack of the necessary research techniques.

paradoxes (and not other types of antinomies) that were analyzed. It can also be asked whether inductive logic developed during the Middle Ages.* This list can be greatly extended; it is only meant to indicate the necessity of further research on the logical heritage of the Middle Ages.

It is beyond the scope of this book to consider the immense subject of the fate of our logical inheritance from the Middle Ages. However, we should note, for example, that the problem of the extent to which Western European logic affected the development of logical thought in Russia and in other Eastern European countries has received little attention. The Russian scholars M. V. Bezobrazova (Reference 14) and N. A. Sokolov (Reference 15) have discovered Russian manuscripts of the seventeenth and eighteenth centuries that contain extensive critical commentaries on Lully's *Ars Magna*. One variant of this manuscript was found by Sokolov in the Kazan library of the Solovetsky monastery.

It is interesting to note that initially the manuscript belonged to a peasant, Semen Ivanov, who lived on the holdings of the Solovetsky monastery close to Archangel; it was sold to a servant, Grigory Titov of the Solovetsky monastery (Reference 15, p. 332). Thus, it would seem that there were many copies of the Russian interpretation of Lully's *Ars Magna*. One of the copyists has called it a "fragrantly productive garden."

It was demonstrated in References 14 and 15 that the author of the Russian interpretation of *Ars Magna* was a translator of the "ambassadorial office," Andrei Khristoforovich Belobodsky (Reference 15, p. 337). According to Savitskiy (*Trudy Kievskoy Dukhovnoy Akademii*, November 1902, p. 444), Belobodsky "discovered the logical principles of Lully completely independently." As Trakhtenberg has properly remarked, Belobodsky stated as his aim an attempt to find a universal method for obtaining knowledge, using graphical methods for this purpose (a table is a "grid"

* Of course, individual remarks on inductive logic can be observed in the works of the scholars of the Middle Ages. For example, Lully considered induction (*inductio incompleta*), and Duns Scotus asserted that a defect in Aristotelian induction existed in Aristotle's thesis that, in the process of perception, the individual precedes the general.

classifying "all reality," the "chain of being," etc.). See O. V. Trakhtenberg, "Social and Political Thought in Russia During the fifteenth through seventeenth centuries," in *On the History of Russian Philosophy* (Moscow, 1951), p. 87.

The logical ideas of Lully were being introduced into the Secondary schools of Russia at that time, and they penetrated even into rhetoric. However, the influence of these ideas was not strong and waned rapidly. The same must be said of the influence of Scholastic methodology on Russian academic philosophy in the seventeenth century.