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J. Kevorkian

Partial Differential Equations

Analytical Solution Techniques

Second Edition

With 128 Illustrations

Springer

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Mathematics Subject Classification (1991): 00A06, 35-01

Library of Congress Cataloging-in-Publication Data
Kevorkian, J.
Partial differential equations: analytical solution techniques/
J. Kevorkian. — [2nd ed.]
p. cm. — (Texts in applied mathematics; 35)
Includes bibliographical references and index.
ISBN 978-1-4419-3139-9 ISBN 978-1-4757-3266-5 (eBook)
DOI 10.1007/978-1-4757-3266-5
1. Differential equations, Partial—Numerical solutions.
I. Title. II. Series.
QA77.K48 2000
515'.353—dc21 99-16552

Printed on acid-free paper.

9 8 7 6 5 4 3 2

ISBN 978-1-4419-3139-9 SPIN 11005797

springeronline.com

Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series: *Texts in Applied Mathematics (TAM).*

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and encourage the teaching of new courses.

TAM will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the *Applied Mathematical Sciences (AMS)* series, which will focus on advanced textbooks and research level monographs.

Preface

This is a text for a two-semester or three-quarter sequence of courses in partial differential equations. It is assumed that the student has a good background in vector calculus and ordinary differential equations and has been introduced to such elementary aspects of partial differential equations as separation of variables, and eigenfunction expansions. Some familiarity is also assumed with the application of complex variable techniques, including conformal mapping, integration in the complex plane, and the use of integral transforms. In this second edition, much of the needed background is reviewed in the Appendix. In addition, new material has been added to all the chapters, and some of the derivations and discussions have been streamlined.

Linear theory is developed in the first half of the book and quasilinear and nonlinear problems are covered in the second half, but the material is presented in a manner that allows flexibility in selecting and ordering topics. For example, it is possible to start with the scalar first-order equation in Chapter 5, to include or delete the nonlinear equation in Chapter 6, and then to move on to second-order equations selecting and omitting topics as dictated by the course. At the Univeristy of Washington, the material in Chapters 5, and 1-3 is covered during the third quarter of a three-quarter sequence that is part of the required program for first-year graduate students in Applied Mathematics. We offer the material in Chapters 4, and 6-8 to more advanced students in a two-quarter sequence.

The primary purpose of this book is to discuss the formulation and solution of representative problems that arise in the physical sciences and engineering and are modeled by partial differential equations. To achieve this goal, all the basic physical principles of a given subject are first considered in detail and then incorporated into the analysis. Although proofs are often omitted, the underlying mathematical concepts are carefully explained. The emphasis throughout is on deriving explicit analytical results, rather than on the abstract properties of solutions. Whenever a new idea is introduced, it is illustrated by an example from an appropriate area of application. The ideas are further explored through problems that range in difficulty from straightforward extensions of the textual material to rather challenging departures testing the student's skill at application. Several new problems have been included in this edition, and all problems are now grouped by sections rather than chapters.

The numerical solution of partial differential equations is a vast topic requiring a separate volume; here, the emphasis is on analytical techniques. Numerical solutions are mentioned only in connection with particular examples and, more generally, to illustrate the solution of hyperbolic problems in terms of characteristic variables. Certain analytical techniques coverd in specialized texts have also been left out. The notable omissions concern the asymptotic expansion of solutions obtained by integral transforms, integral equation methods, the Wiener–Hopf method, and inverse scattering theory.

Acknowledgments

I rededicate this second edition with gratitude and admiration to the memory of Paco A. Lagerstrom and Julian D. Cole. I want to thank my wife Seta again for her patience and support during this project. Frances Chen created the TeX files for this book from my rough handwritten notes and edited pages. Her skill and cheerful cooperation throughout this process are greatly appreciated.

Seattle, Washington J. Kevorkian
August 1999

Contents

The Diffusion Equation

In this chapter we study the diffusion equation

$$u_t - (u_{xx} + u_{yy} + u_{zz}) = p(x, y, z, t),$$

which describes a number of physical models, such as the conduction of heat in a solid or the spread of a contaminant in a stationary medium.

We shall use this equation to introduce many of the solution techniques that will be useful in subsequent chapters in our study of other types of linear partial differential equations. To begin with, it is important to have a physical understanding of how the diffusion equation arises in a particular application, and we consider the simple model of heat conduction in a solid.

1.1 Heat Conduction

Consider a thin axisymmetric rod of some heat-conducting material with variable density $\rho(x)$ (g/cm^3) (for example, a copper–silver alloy with a variable copper/silver ratio along the rod). Let $A(x)$ (cm^2) denote the cross-sectional area and assume that the surface of the rod is perfectly insulated so that no heat is lost or gained through this surface. (See Figure 1.1.) Thus, the problem is one-dimensional in the sense that all material properties depend on the distance x along the rod. We assume that at each spatial position x and time t there is one temperature θ that does not depend on the transverse coordinates y or z. Let x_1 and x_2 be two arbitrary fixed points on the axis.

In the basic law of conservation of heat energy for the rod segment $x_1 \leq x \leq x_2$, the rate of change of heat inside this segment is equal to the net flow of heat through the two boundaries at x_1 and x_2, plus the heat produced by a possible distribution of internal heat sources in the interval. Consider an infinitesimal section of length dx in the interval $x_1 \leq x \leq x_2$. Using elementary physics, we have dQ, the heat content in this section, proportional to the mass and the temperature:

$$dQ \equiv c(\rho A dx)\theta, \tag{1.1.1}$$

where the constant of proportionality c is the specific heat in cal/g°C. Thus, the total heat content in the interval $x_1 \le x \le x_2$ is*

$$Q(t) \equiv \int_{x_1}^{x_2} c(x)\rho(x)A(x)\theta(x,t)dx. \tag{1.1.2}$$

Next, we invoke Fourier's law for heat conduction, which states that the rate of heat flowing *into* a body through a small surface element on its boundary is proportional to the area of that element and to the *outward* normal derivative of the temperature at that location. The constant of proportionality here is $k \sim$ (cal/cm s°C), the *thermal conductivity*. Note that this sign convention implies the intuitively obvious fact that the direction of heat flow between two neighboring points is toward the relatively cooler point. For example, if the temperature increases as a boundary point is approached from inside a body, then the outward normal derivative of the temperature is positive, and this correctly implies that heat flows into the body.

For the present one-dimensional example, the net inflow of heat through the boundaries x_1 and x_2 is

$$R(t) \equiv A(x_2)k(x_2)\frac{\partial\theta}{\partial x}(x_2,t) - A(x_1)k(x_1)\frac{\partial\theta}{\partial x}(x_1,t). \tag{1.1.3}$$

Let $h(x,t)$ (cal/g s) denote the heat produced per unit mass and time by the sources. Thus, the total time rate of heat production by the sources is

$$H(t) \equiv \int_{x_1}^{x_2} h(x,t)\rho(x)A(x)dx. \tag{1.1.4}$$

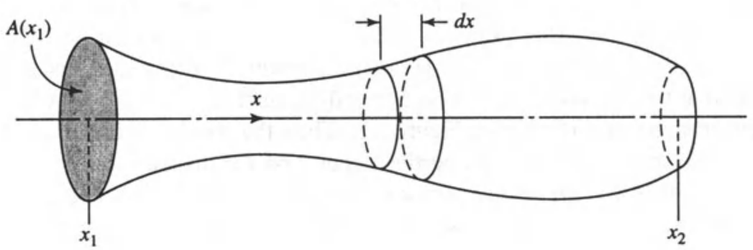

FIGURE 1.1. Thin axisymmetric heat conductor

* In this text we shall often use the notation $\equiv$ instead of $=$ when it is important to indicate that a new quantity is being defined, as in (1.1.1) and (1.1.2). As a special case of this notation, the statement $f(x,y) \equiv 0$ indicates that the function f of x and y vanishes identically; that is, it equals zero for all x and y by definition.

The conservation of heat then implies

$$\frac{dQ}{dt} = R(t) + H(t), \tag{1.1.5}$$

or

$$\frac{d}{dt}\int_{x_1}^{x_2} c(x)\rho(x)A(x)\theta(x,t)dx = A(x_2)k(x_2)\frac{\partial\theta}{\partial x}(x_2,t) - A(x_1)k(x_1)\frac{\partial\theta}{\partial x}(x_1,t) + \int_{x_1}^{x_2} h(x,t)\rho(x)A(x)dx. \tag{1.1.6}$$

Equation (1.1.6) is a typical integral *conservation law*, which has general applicability. For example, (1.1.6) remains true if material properties have a discontinuity at a given point $x = \xi$ inside the interval, as would be the case if we had a perfect thermal bond between two rods of different materials. We shall encounter other examples of such conservation laws later on in the book and shall study how discontinuities propagate in detail in Chapter 5.

For smooth material properties, that is, if c, ρ, A, and k are continuous and have a continuous first derivative, the solution $\theta(x,t)$ is also continuous with continuous first partial derivatives $\partial\theta/\partial x$ and $\partial\theta/\partial t$, and we may rewrite (1.1.6) in the following form after we express $R(t)$ as the integral of a derivative:

$$\int_{x_1}^{x_2}\left\{c(x)\rho(x)A(x)\frac{\partial\theta}{\partial t}(x,t) - \frac{\partial}{\partial x}\left[A(x)k(x)\frac{\partial\theta}{\partial x}(x,t)\right] - h(x,t)\rho(x)A(x)\right\}dx = 0. \tag{1.1.7}$$

Since (1.1.7) is true for any x_1 and x_2, it follows that the integrand must vanish:

$$c(x)\rho(x)A(x)\frac{\partial\theta}{\partial t} - \frac{\partial}{\partial x}\left[A(x)k(x)\frac{\partial\theta}{\partial x}\right] = h(x,t)\rho(x)A(x). \tag{1.1.8}$$

For constant area and material properties, this reduces to

$$\frac{\partial\theta}{\partial t} - \kappa^2\frac{\partial^2\theta}{\partial x^2} = \sigma(x,t), \tag{1.1.9}$$

where $\kappa^2 \equiv k/c\rho$ (cm^2/s) is the *thermal diffusivity* and $\sigma \equiv h/c$. The dimensionless form of (1.1.9) follows when characteristic constants with dimensions of temperature, length, and time are used to define nondimensional variables.

For example, let us study (1.1.9) for a rod of length L that is initially at a constant temperature θ_0 and has one end, $x = L$, held at $\theta = \theta_0$ while the other end, $x = 0$, has a prescribed temperature history $\theta(0,t) = \theta_0 f(t/T)$, where T is a characteristic time scale. For simplicity assume $\sigma \equiv 0$. We set

$$u \equiv \frac{\theta}{\theta_0}, \quad x^* \equiv \frac{x}{L}, \quad t^* \equiv \frac{t\kappa^2}{L^2},$$

and obtain the following dimensionless formulation:

$$\frac{\partial u}{\partial t^*} - \frac{\partial^2 u}{\partial x^{*2}} = 0, \quad 0 \le x^* \le 1, \tag{1.1.10a}$$

$$u(x^*, 0) = 1 \tag{1.1.10b}$$

$$u(0, t^*) = f(\lambda t^*), \quad t^* > 0, \tag{1.1.10c}$$

$$u(1, t^*) = 1, \tag{1.1.10d}$$

where λ is the dimensionless parameter $L^2/(\kappa^2 T)$. The original dimensional formulation of this problem involves the four constants κ, θ_0, L, and T. The dimensionless description is considerably simpler, as it involves only the one parameter λ. Once the dimensionless problem has been solved, say $u = U(x^*, t^*)$, the dimensional result is easily obtained in the form

$$\theta = \theta_0 U\left(\frac{x}{L}, \frac{t\kappa^2}{L^2}\right).$$

The corresponding derivation for three-dimensional heat conduction follows from similar steps. If a solid occupies the domain G with surface S and outward unit normal $\mathbf{n}$, as shown in Figure 1.2, the total heat content of the solid is given by

$$Q(t) \equiv \iiint_G c\rho\theta \, dV, \tag{1.1.11}$$

where dV is the volume element; for instance, $dV = dx\,dy\,dz$ in Cartesian variables. The net *inflow* of heat through the boundary S is

$$R(t) \equiv \iint_S k \operatorname{grad} \theta \cdot \mathbf{n} \, dA. \tag{1.1.12}$$

We can express $R(t)$ in terms of a volume integral over G using Gauss' theorem. This theorem states that if $\mathbf{F}$ is a one-valued vector field with continuous first partial

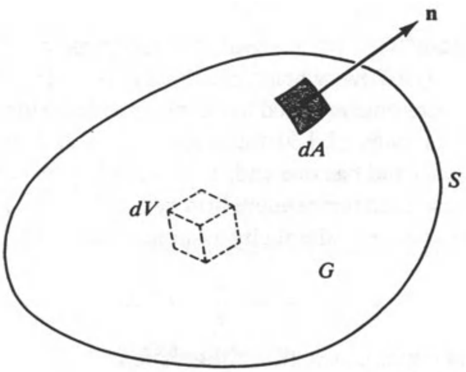

FIGURE 1.2. Three-dimensional heat conductor

derivatives in G, then*

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n} dA = \iiint\limits_G \operatorname{div} \mathbf{F} dV. \tag{1.1.13}$$

Therefore, for a medium where c, ρ, and k are smooth, we identify $\mathbf{F}$ in (1.1.13) with $k \operatorname{grad} \theta$, and (1.1.12) becomes

$$R(t) = \iiint\limits_G \operatorname{div}(k \operatorname{grad} \theta) dV. \tag{1.1.14}$$

Also, since G is fixed in space we have

$$\frac{dQ}{dt} \equiv \frac{d}{dt} \iiint\limits_G c\rho\theta dV = \iiint\limits_G c\rho\theta_t dV. \tag{1.1.15}$$

The conservation law of heat energy (1.1.5) becomes

$$\iiint\limits_G c\rho\theta_t dV = \iiint\limits_G \operatorname{div}(k \operatorname{grad} \theta) dV + \iiint\limits_G h\rho dV. \tag{1.1.16}$$

Therefore, assuming continuity of the integrands in (1.1.16), the three-dimensional version of (1.1.8) is

$$c\rho\theta_t - \operatorname{div}(k \operatorname{grad} \theta) = h\rho. \tag{1.1.17}$$

For constant k, this reduces to

$$\theta_t - \kappa^2 \Delta\theta = \sigma, \tag{1.1.18}$$

where $\kappa^2 \equiv k/(c\rho)$, $\sigma \equiv h/c$, and Δ is the Laplacian operator $\Delta \equiv \operatorname{div} \operatorname{grad}$, given by

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.1.19}$$

in Cartesian coordinates.

1.2 The Fundamental Solution

The *fundamental solution* of a second-order partial differential equation is just Green's function for that equation over the infinite domain with zero boundary conditions (if appropriate) at infinity. See Appendix A.1 for a review of the use of

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* Thus, in writing $R(t)$ in the form given in (1.1.7), we have used the "one-dimensional version" of Gauss' theorem relating the definite integral of the derivative of a function to values of the function at the endpoints.

Green's function in ordinary differential equations. For example, the fundamental solution of the one-dimensional diffusion equation obeys

$$u_t - u_{xx} = \delta(x - \xi)\delta(t - \tau) \tag{1.2.1}$$

on $-\infty < x < \infty, 0 \leq t < \infty$, where ξ and τ are fixed constants, $|\xi| < \infty$, $0 \leq \tau < \infty$, and δ denotes the Dirac delta function. We may interpret (1.2.1) physically as the equation governing the temperature in an infinite conductor that is subjected to a *concentrated unit source of heat* at the point $x = \xi$. This source of heat is turned on only for the "instant" $t = \tau$ and is absent for all other times; its location is also concentrated at the point $x = \xi$.

Prior to the application of the heat source, the conductor has a constant temperature that we normalize to equal zero. Thus, the boundary conditions are

$$u(x, t) \to 0 \quad \text{as } |x| \to \infty, \tag{1.2.2}$$

and the initial condition is

$$u(x, t) = 0; \quad 0 \leq t < \tau. \tag{1.2.3}$$

The solution of (1.2.1)–(1.2.3) is the fundamental solution, which is a function of $x - \xi$ and $t - \tau$,

$$u = F(x - \xi, t - \tau). \tag{1.2.4}$$

There is no loss of generality in taking the initial and boundary temperatures equal to zero in (1.2.2)–(1.2.3); any constant value u_0 can be used and then reduced to (1.2.2)–(1.2.3) by simply considering the new dependent variable $u - u_0$. This is a consequence of the absence of nondifferentiated terms in (1.2.1). Also, since the left-hand side of (1.2.1) does not involve x or t, we need only consider the simpler problem corresponding to $\xi = \tau = 0$

$$u_t - u_{xx} = \delta(x)\delta(t), \tag{1.2.5}$$

$$u(x, 0^-) = 0, \tag{1.2.6}$$

$$u(x, t) \to 0 \quad \text{as } |x| \to \infty. \tag{1.2.7}$$

Once the solution $u = F(x, t)$ of (1.2.5)–(1.2.7) is found, the general result $F(x - \xi, t - \tau)$ is obtained by translation.

In Section 1.3 we shall show that once the fundamental solution is known, we can solve the following *general* initial-value problem for the diffusion equation on the infinite domain

$$u_t - u_{xx} = p(x, t); \quad -\infty < x < \infty; \quad 0 \leq t < \infty, \tag{1.2.8}$$

$$u(x, 0) = f(x), \tag{1.2.9}$$

$$u(x, t) \to f(\pm\infty) \quad \text{as } x \to \pm\infty, \tag{1.2.10}$$

where p and f are prescribed functions and $p(x, t) \equiv 0$ if $t < 0$.

In the next three subsections we derive the fundamental solution F using different techniques that have a broad range of applicability in solving partial differential equations.

1.2.1 Similarity (Invariance)

In this very useful approach, we ask under what scalings of the dependent and independent variables the system (1.2.5)–(1.2.7) is invariant. If such scalings exist, we can reduce (1.2.5) to an ordinary differential equation in terms of a "similarity" variable using arguments that go as follows.

Assume that we have found the solution of (1.2.5)–(1.2.7) in the form $u = F(x,t)$. Is it possible to use this result to obtain a second solution $u = G(x,t)$ by setting $\bar{x} = \beta x$ and $\bar{t} = \gamma t$ and defining G by

$$G(x,t) \equiv \alpha F(\beta x, \gamma t) \tag{1.2.11}$$

for positive constants α, β, and γ?

We compute $G_t = \alpha\gamma F_{\bar{t}}$, $G_{xx} = \alpha\beta^2 F_{\bar{x}\bar{x}}$, and use of the fact that for any constant c, we may set (See (A.1.16))

$$\delta(cx) \to \frac{1}{|c|}\delta(x). \tag{1.2.12}$$

If $G(x,t)$ is to be a solution of (1.2.5)–(1.2.7), we must have

$$G_t - G_{xx} = \delta(x)\delta(t), \quad G(x,0^-) = 0, \quad G(x,t) \to 0 \quad \text{as } |x| \to \infty. \tag{1.2.13}$$

Expressing G_t and G_{xx} in terms of $F_{\bar{t}}$ and $F_{\bar{x}\bar{x}}$ and using $\delta(x)\delta(t) = \delta(\bar{x}/\beta)\delta(\bar{t}/\gamma) = \beta\gamma\delta(\bar{x})\delta(\bar{t})$ in (1.2.13) gives

$$\alpha\gamma F_{\bar{t}} - \alpha\beta^2 F_{\bar{x}\bar{x}} = \beta\gamma\delta(\bar{x})\delta(\bar{t}),$$
$$\alpha F(\bar{x},0^-) = 0, \quad \alpha F(\bar{x},\bar{t}) \to 0 \quad \text{as } |x| \to \infty,$$

or

$$F_{\bar{t}} - \left(\frac{\beta^2}{\gamma}\right) F_{\bar{x}\bar{x}} = \left(\frac{\beta}{\alpha}\right)\delta(\bar{x})\delta(\bar{t}),$$
$$F(\bar{x},0^-) = 0, \quad F(\bar{x},\bar{t}) \to 0 \quad \text{as } |x| \to \infty.$$

But we know that $F(\bar{x},\bar{t})$ must satisfy (1.2.5)–(1.2.7) in terms of the $\bar{x}$, $\bar{t}$ variables. Therefore, $G(x,t)$, as defined by (1.2.11), can be a solution only if $\beta^2/\gamma = 1$ and $\beta/\alpha = 1$; that is, if $\beta = \alpha$ and $\gamma = \alpha^2$. Thus, (1.2.11) must be of the form

$$G(x,t) = \alpha F(\alpha x, \alpha^2 t). \tag{1.2.14}$$

Have we discovered a new solution of (1.2.5)–(1.2.7)? Of course not; the solution for this problem is unique, $G = F$, as is physically obvious and can be proved. Therefore, (1.2.14) is just a statement of the *similarity structure* of the solution F, and (1.2.14) must read

$$\alpha F(\alpha x, \alpha^2 t) = F(x,t). \tag{1.2.15}$$

That is to say, if we replace x by αx and t by $\alpha^2 t$ in F and then multiply the result by α (for any $\alpha > 0$), *the resulting expression is identical to* $F(x,t)$. This property implies that $F(x, t)$ must be of the form

$$F(x,t) = \frac{1}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}\right), \quad \text{or } \frac{1}{\sqrt{t}} g\left(\frac{x^2}{t}\right), \quad \text{or } \frac{1}{x} h\left(\frac{x}{\sqrt{t}}\right), \ldots$$

for certain functions $f, g, h, \ldots$ of the indicated arguments.

Any one of an infinite number of possibilities that satisfy the similarity condition (1.2.15) may be used. Each choice will reduce (1.2.5) to an ordinary differential equation, which, when solved, will give the *same* result for F. Let us pick the form

$$F(x,t) = \frac{1}{\sqrt{t}} f(\zeta), \quad \zeta \equiv \frac{x}{\sqrt{t}}.$$

We compute

$$F_x = \frac{1}{t} f'; \quad F_{xx} = \frac{1}{t^{3/2}} f''; \quad F_t = -\frac{1}{2t^{3/2}} f - \frac{x}{2t^2} f',$$

where $' \equiv d/d\zeta$.

Since the delta function on the right-hand side of (1.2.5) is identically equal to zero for $t > 0$, we need to solve only the homogeneous diffusion equation for $t > 0$. However, the initial condition $u(x, 0^-) = 0$ in (1.2.6) does not remain valid for $t = 0^+$. (If it did, the result would be the trivial solution $u(x,t) \equiv 0$.) The effect of the delta function on the right-hand side is to generate impulsively a nonzero value for $u(x, 0^+)$ (see (1.2.22)), which is the appropriate initial condition to be used in solving the homogeneous equation (1.2.5) for $t > 0$.

Consider now the homogeneous version of (1.2.5). Using the results we computed for F and its derivatives gives

$$-\frac{1}{2t^{3/2}} f - \frac{x}{2t^2} f' - \frac{1}{t^{3/2}} f'' = 0,$$

which is the linear second-order ordinary differential equation

$$f'' + \frac{\zeta}{2} f' + \frac{1}{2} f = 0 \tag{1.2.16}$$

with the independent variable ζ.

Integrating once gives $f' + (\zeta/2)f = A = \text{constant}$, and the solution of this is

$$f = Ae^{-\zeta^2/4} \int^{\zeta} e^{s^2/4} ds + Be^{-\zeta^2/4}, \quad B = \text{constant}.$$

The constants A and B are determined by considering the total heat content $H(t)$ in the bar. In terms of our dimensionless units, the total heat is just the integral of the temperature:

$$\begin{aligned} H(t) &\equiv \int_{-\infty}^{\infty} F(x,t)dx \\ &= \frac{A}{\sqrt{t}} \int_{-\infty}^{\infty} f_1\left(\frac{x}{\sqrt{t}}\right) dx + \frac{B}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-x^2/4t} dx, \end{aligned} \tag{1.2.17}$$

where we have defined

$$f_1(\zeta) \equiv e^{-\zeta^2/4} \int^{\zeta} e^{s^2/4} ds = e^{-\zeta^2/4} \int^{\zeta^2/4} e^{\sigma} \sigma^{-1/2} d\sigma.$$

Integrating the second expression for f_1 by parts shows that (see Section A.3.5)

$$f_1(\zeta) = \frac{2}{|\zeta|} + O(\zeta^{-3}) \quad \text{as } |\zeta| \to \infty.$$

Therefore, $(1/\sqrt{t}) \int_{-\infty}^{\infty} f_1 \, dx$ in (1.2.17) is unbounded. Since the total heat must be finite, we set $A = 0$ and have

$$F = \frac{B}{\sqrt{t}} e^{-x^2/4t}, \quad t > 0. \tag{1.2.18}$$

The idea now is to pick B in order to satisfy (1.2.5) at $t = 0^+$. If we differentiate the integral defining $H(t)$ in (1.2.17) with respect to t and use (1.2.5), we obtain

$$\frac{dH}{dt} = \int_{-\infty}^{\infty} F_t(x,t)dx = \int_{-\infty}^{\infty} [F_{xx}(x,t) + \delta(x)\delta(t)]dx,$$

so that

$$\frac{dH}{dt} = F_x(\infty, t) - F_x(-\infty, t) + \delta(t) = \delta(t),$$

because the temperature gradient at $\pm\infty$ due to a unit source must be zero. Therefore, $H(t)$ is the Heaviside function (see (A.1.14)), and for $t > 0$, we have

$$1 = \int_{-\infty}^{\infty} \frac{B}{\sqrt{t}} e^{-x^2/4t} dx. \tag{1.2.19}$$

Thus, after switching on a unit source of heat for an instant at the origin, the total heat content in the rod remains constant, and this constant can be set equal to unity under an appropriate nondimensionalization.

We can rewrite (1.2.19) as

$$1 = 2B \int_{-\infty}^{\infty} \frac{e^{-x^2/4t}}{\sqrt{4t}} dx = 2B \int_{-\infty}^{\infty} e^{-\xi^2} d\xi = 2B\sqrt{\pi},$$

or

$$B = \frac{1}{2\sqrt{\pi}},$$

and the fundamental solution is

$$F(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t}. \tag{1.2.20}$$

More generally, the solution of (1.2.1)–(1.2.3) is

$$F(x-\xi, t-\tau) = \frac{1}{2\sqrt{\pi(t-\tau)}} e^{-(x-\xi)^2/4(t-\tau)}. \tag{1.2.21}$$

It is important to note that the use of similarity is not restricted to linear problems. For example, a classical use of similarity arguments is provided by the boundary-layer equations for viscous incompressible flow over an infinite wedge (or the special case of a semi-infinite flat plate if the wedge angle is zero). See Section B.14 of [31]. Here, the nonlinear partial differential equation for the flow stream function is reduced to a third-order nonlinear ordinary differential equation.

A crucial requirement for the applicability of similarity arguments is that both the governing equations *and* initial and/or boundary conditions be reducible to similarity form. In the preceding example, this was trivially true for the given initial condition $F = 0$, as this also immediately implied $G = 0$. For further reading on similarity methods, see [6] and [41].

The fundamental solution (1.2.20) can also be derived using Fourier or Laplace transforms. A review of these techniques appears in Appendix 2, where this problem is used as one of the illustrative examples.

1.2.2 Qualitative behavior; diffusion

Figure 1.3 shows three temperature profiles for $F(x, t)$ given by (1.2.20) taken at three successive times $0 < t_1 < t_2 < t_3$. In each case, the area under the curve is, according to (1.2.19), equal to unity. For t smaller and smaller, the contribution to this area becomes more and more concentrated at the origin. This is just one of the many possible representations of the delta function (for instance, see (A.1.11b) with $\Delta\xi = 4t$), and we may write

$$F(x, 0^+) = \delta(x). \tag{1.2.22}$$

Equation (1.2.22) also follows by integrating (1.2.5) with respect to t from $t = 0^-$ to $t = 0^+$ and noting that $\int_{0^-}^{0^+} u_{xx} dt = 0$.

The fundamental solution can be used to give a precise definition of *diffusion*. First, notice that if we regard the source at $x = 0$ as a disturbance introduced at time $t = 0$, the "signal speed" due to this disturbance is *infinite* because for any positive t, no matter how small, the value of u is nonzero for all x. Thus, the entire rod instantly "feels" the effect of the source. Of course, a real temperature gauge would fail to detect the very weak disturbance at large distances. Thus, the idea of a signal speed is not very useful in this case, and we would like to have a better characterization of how the rod "heats up" for $t > 0$. Suppose we ask instead where a given fraction of the *total heat* in the rod is to be found at any specified time. We know that at $t = 0^+$, all the heat is concentrated at the origin. For any $t > 0$, the heat is nonuniformly distributed over the entire rod with the maximum temperature at the origin, as shown in Figure 1.4.

Suppose that d is a fixed constant with $0 < d < 1$. At some time $t > 0$, the temperature distribution is the even function of x given by (1.2.20) and sketched in Figure 1.4. The shaded area represents the fraction d of the total area (which equals unity). Thus, as t increases, so does x_d. The question is, how does x_d depend

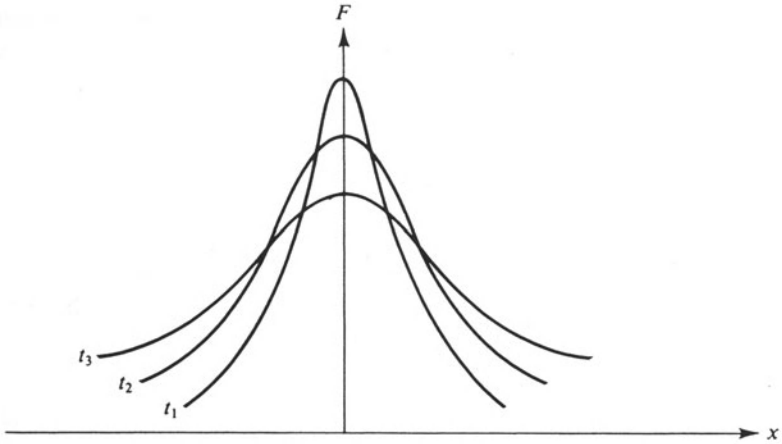

FIGURE 1.3. Fundamental solution for the temperature at three different times

on t? It follows from (1.2.20) and symmetry that

$$d = \frac{2}{2\sqrt{\pi t}} \int_0^{x_d} e^{-\sigma^2/4t} \, d\sigma,$$

or, changing variables, that

$$d = \frac{2}{\sqrt{\pi}} \int_0^{x_d/2\sqrt{t}} e^{-\eta^2} \, d\eta \equiv \operatorname{erf}\left(\frac{x_d}{2\sqrt{t}}\right), \tag{1.2.23}$$

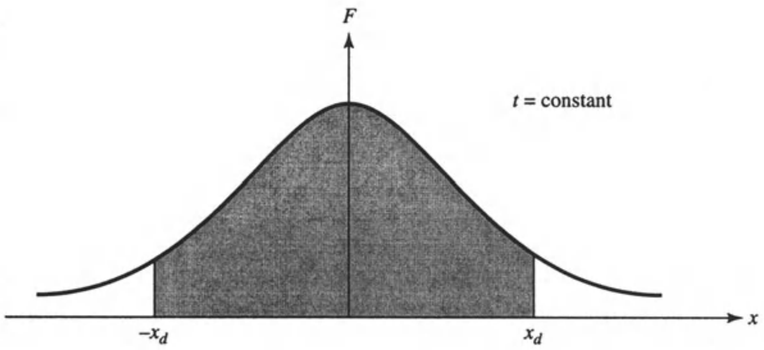

FIGURE 1.4. Interval $(-x_d, x_d)$ containing the fraction d of the total heat

where the error function erf is defined in (A.2.76). Since the left-hand side of (1.2.23) is a constant, we conclude that $x_d/2\sqrt{t}$ remains constant as t increases. Therefore, $x_d \sim \sqrt{t}$, and we say that heat due to a point source at $x = 0, t = 0$ *diffuses* according to $|x| \sim \sqrt{t}$.

Problems

1.2.1. Consider the diffusion equation with variable coefficient

$$2xu_t - u_{xx} = 0, \quad 0 \le x < \infty, \quad t \ge 0, \tag{1.2.24}$$

with boundary conditions

$$u(0,t) = C_1 = \text{constant} \qquad \text{if } t > 0, \tag{1.2.25a}$$

$$u(\infty,t) = C_2 = \text{constant} \qquad \text{if } t > 0, \tag{1.2.25b}$$

and initial condition

$$u(x,0) = C_3 = \text{constant}. \tag{1.2.26}$$

a. What is the most general choice for the constants C_1, C_2, and C_3 for which the solution of the above initial- and boundary-value problem can be obtained in similarity form?
b. For the choice of constants obtained in part (a), calculate the solution and evaluate all integration constants explicitly.

1.2.2. Use similarity to reduce the following initial- and boundary-value problem for a nonlinear diffusion equation to an ordinary differential equation and corresponding boundary conditions:

$$u_{xx} - uu_t = 0, \quad 0 \le x, \quad 0 \le t, \tag{1.2.27}$$

$$u(0,t) = 0, \tag{1.2.28b}$$

$$u(\infty,t) = 1, \tag{1.2.28b}$$

$$u(x,0) = 1. \tag{1.2.29}$$

Discuss the behavior of the solution.

1.2.3 A semi-infinite bar ($x \ge 0$) insulated everywhere except at $x = 0$ loses heat to the adjacent medium ($x < 0$) by blackbody radiation according to the boundary condition

$$\theta^4(0,t) - \theta_0^4 = \alpha\theta_x(0,t), \ t > 0, \tag{1.2.30}$$

where α is a constant (equal to the conductivity divided by the product of the emissivity and the Stefan-Boltzmann constant), θ_0 is the constant temperature of the medium, and $\theta(x,t)$ is the temperature at the point x and time t in the bar. Equation (1.1.9) with $\sigma \equiv 0$ governs the temperature distribution in $x \ge 0$, and we assume that the initial temperature is given in the form

$$\theta(x,0) = \theta_1 f\left(\frac{x}{L_0}\right), \tag{1.2.31}$$

where θ_1 is a characteristic temperature and L_0 is a characteristic length. The boundary condition at $x = \infty$ is

$$\theta(\infty, t) = \theta_1 f(\infty) < \infty. \tag{1.2.32}$$

a. Introduce appropriate dimensionless variables u, x^*, t^* to reduce (1.2.30)–(1.2.32) to the form

$$\frac{\partial u}{\partial t^*} - \frac{\partial^2 u}{\partial x^{*2}} = 0, \tag{1.2.33a}$$

$$u(x^*, 0) = \frac{1}{\epsilon} f(x^*), \tag{1.2.33b}$$

$$u^4(0, t^*) - 1 = \lambda \frac{\partial u}{\partial x^*}(0, t^*), \quad t^* > 0, \tag{1.2.33c}$$

$$u(\infty, t^*) = \frac{1}{\epsilon} f(\infty), \tag{1.2.33d}$$

where ϵ and λ are dimensionless constants.

b. What does the limiting case

$$\lambda \gg 1, \quad \epsilon \ll 1, \quad \lambda\epsilon^3 = \text{constant} = \overline{\lambda} = O(1), \tag{1.2.34}$$

describe physically? Since u is initially large, it is appropriate to consider the rescaled dependent variable $\overline{u} = u/\epsilon$, where $\overline{u}$ is $O(1)$. Thus, to leading order, $\overline{u}$ satisfies

$$\frac{\partial \overline{u}}{\partial t^*} - \frac{\partial^2 \overline{u}}{\partial x^{*2}} = 0, \tag{1.2.34a}$$

$$\overline{u}(x^*, 0) = f(x^*), \tag{1.2.34b}$$

$$\overline{u}^4(0, t^*) = \overline{\lambda} \frac{\partial \overline{u}}{\partial x^*}(0, t^*) + O(\epsilon^4), \quad t^* > 0, \tag{1.2.34c}$$

$$\overline{u}(\infty, t^*) = f(\infty). \tag{1.2.34d}$$

c. For what $f(x^*)$ (possibly singular) can (1.2.34) be solved by similarity? For this choice of f derive, but do not solve, the ordinary differential equation and boundary conditions governing the solution.

1.3 Initial-Value Problem in the Infinite Domain; Superposition

The general initial-value problem for the inhomogeneous diffusion equation in the infinite interval is

$$u_t - u_{xx} = p(x,t), \quad -\infty < x < \infty, \quad t \geq 0, \tag{1.3.1a}$$

$$u(x, 0^+) = f(x), \tag{1.3.1b}$$

where p and f are arbitrarily prescribed functions with $p \equiv 0$ if $t < 0$. For heat conduction, p represents a dimensionless heat-source distribution, and f an initial temperature distribution.

Because of linearity, the solution of (1.3.1) can be expressed as the sum of the following two problems:

$$u_t - u_{xx} = p(x,t), \quad -\infty < x < \infty, \quad t \geq 0, \tag{1.3.2a}$$

$$u(x, 0^-) = 0, \tag{1.3.2b}$$

$$u_t - u_{xx} = 0, \quad -\infty < x < \infty, \quad t \geq 0, \tag{1.3.3a}$$

$$u(x, 0^+) = f(x). \tag{1.3.3b}$$

We now show that knowing the fundamental solution $F(x - \xi, t - \tau)$ allows us to write the solution of the first problem immediately in terms of a "superposition integral." The derivation of this superposition integral is a straightforward generalization of the single-variable case discussed in Appendix 1 (see (A.1.23)–(A.1.28)). We consider the solution of (1.3.2) arising from the contribution of p coming from a small neighborhood of the fixed point $x = \xi, t = \tau$, with p set equal to zero everywhere outside this neighborhood. Let $R(\xi, \tau)$ denote the small neighborhood $\xi - \Delta\xi/2 \leq x \leq \xi + \Delta\xi/2, \tau - \Delta\tau/2 \leq t \leq \tau + \Delta\tau/2$, over which we may regard the value of p as the constant $p(\xi, \tau)$.

If $\tilde{p}$ denotes the incremental contribution to p from R, we have the following expression defining $\tilde{p}$:

$$\tilde{p} \equiv p(\xi,\tau)\left[H\left(t - \tau + \frac{\Delta\tau}{2}\right) - H\left(t - \tau - \frac{\Delta\tau}{2}\right)\right]\left[H\left(x - \xi + \frac{\Delta\xi}{2}\right) - H\left(x - \xi - \frac{\Delta\xi}{2}\right)\right], \tag{1.3.4}$$

where H is the Heaviside function, and the bracketed expressions ensure that the left-hand side vanishes outside R and equals $\tilde{p}$ in R. We now multiply and divide this expression for $\tilde{p}$ by $\Delta\tau\Delta\xi$ and observe that since $dH/ds = \delta(s)$, the first bracketed expression divided by $\Delta\tau$ represents $\delta(t - \tau)$, whereas the second bracketed expression divided by $\Delta\xi$ represents $\delta(x - \xi)$. Therefore, in the "limit" as $\Delta\tau \to 0$, $\Delta\xi \to 0$, we have

$$\tilde{p} = p(\xi,\tau)\delta(t - \tau)\delta(x - \xi)d\tau\, d\xi. \tag{1.3.5}$$

Since the solution of the diffusion equation with right-hand side $\delta(t - \tau)\delta(x - \xi)$ is the fundamental solution $F(x - \xi, t - \tau)$ defined in (1.2.21), linearity implies that the solution due to the right-hand side $\tilde{p}$ is just

$$\tilde{u} = p(\xi,\tau)F(x - \xi, t - \tau)d\tau\, d\xi. \tag{1.3.6}$$

Linearity also implies that we may superpose the $\tilde{u}$ contributions arising from each of the infinitesimal domains R that cover the half-space $-\infty < \xi < \infty$,

$0 \leq \tau < t$, and this leads to the desired superposition integral

$$u(x,t) = \int_{\xi=-\infty}^{\infty} \int_{\tau=0^-}^{t} F(x-\xi, t-\tau) p(\xi,\tau) d\tau\, d\xi$$
$$= \int_{\xi=-\infty}^{\infty} \int_{\tau=0^-}^{t} \frac{p(\xi,\tau)}{2\sqrt{\pi(t-\tau)}} e^{-(x-\xi)^2/4(t-\tau)} d\tau\, d\xi. \qquad (1.3.7)$$

To confirm this formal derivation, it is easy to verify explicitly that (1.3.7) solves (1.3.2); this is left as an exercise (Problem 1.3.1).

To solve (1.3.3), we note that it is equivalent to

$$u_t - u_{xx} = \delta(t) f(x), \quad -\infty < x < \infty, \quad t \geq 0, \qquad (1.3.8a)$$
$$u(x, 0^-) = 0, \qquad (1.3.8b)$$

as can be verified by noting that integrating the inhomogeneous diffusion equation (1.3.8a) with respect to t from $t = 0^-$ to $t = 0^+$ gives $u(x, 0^+) = f(x)$. Since the right-hand side of (1.3.8a) vanishes when $t > 0$, (1.3.3) and (1.3.8) are equivalent. To solve (1.3.8), we set $p(\xi, \tau)$ in (1.3.7) equal to $\delta(\tau) f(\xi)$ and obtain

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{\xi=-\infty}^{\infty} f(\xi) e^{-(x-\xi)^2/4t} d\xi. \qquad (1.3.9)$$

This result can also be derived using transforms (see (A.2.32)–(A.2.36) for a derivation using Fourier transforms). Therefore, the solution of (1.3.1) is the sum of the solutions (1.3.7) and (1.3.9).

Note that

$$u(x, 0^+) = \lim_{t \to 0^+} \int_{-\infty}^{\infty} f(\xi) \frac{e^{-(x-\xi)^2/4t}}{2\sqrt{\pi t}} d\xi, \qquad (1.3.10)$$

and according to (1.2.22), this is just

$$u(x, 0^+) = \int_{-\infty}^{\infty} f(\xi) \delta(x-\xi) d\xi = f(x), \qquad (1.3.11)$$

which is the correct initial condition.

We can also verify that the initial condition is satisfied by the following alternative approach that does not involve use of the delta function. We write (1.3.9), as the sum of three integrals over the intervals $(-\infty, x-\epsilon)$, $(x-\epsilon, x+\epsilon)$, and $(x+\epsilon, \infty)$, where ϵ is an arbitrarily small, fixed positive number. As $t \to 0^+$, the integrals tend to zero except over the interval $(x-\epsilon, x+\epsilon)$. Thus,

$$u(x, 0^+) = \lim_{t \to 0^+} \int_{x-\epsilon}^{x+\epsilon} f(\xi) \frac{e^{-(x-\xi)^2/4t}}{2\sqrt{\pi t}} d\xi. \qquad (1.3.12)$$

Changing the variable of integration from ξ to $\sigma = (x-\xi)/2t^{1/2}$ gives

$$u(x, 0^+) = \lim_{t \to 0^+} \frac{1}{\sqrt{\pi}} \int_{-\epsilon/\sqrt{4t}}^{\epsilon/\sqrt{4t}} f(x + \sigma\sqrt{4t}) e^{-\sigma^2} d\sigma$$
$$= \frac{f(x)}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\sigma^2} d\sigma = f(x). \qquad (1.3.13)$$

Problems

1.3.1. Verify by direct substitution that the sum of the expressions given by (1.3.7) and (1.3.9) solves the initial-value problem (1.3.1).

1.3.2a Specialize (1.3.9) to the case where

$$f(x) = \begin{cases} 1/2\epsilon, & -\epsilon < x < \epsilon, \\ 0, & |x| > \epsilon, \end{cases} \tag{1.3.14}$$

and show that the solution reduces to

$$u(x,t) = \frac{1}{4\epsilon}\left[\operatorname{erf}\left(\frac{x+\epsilon}{2\sqrt{t}}\right) - \operatorname{erf}\left(\frac{x-\epsilon}{2\sqrt{t}}\right)\right], \tag{1.3.15}$$

where the error function erf is defined in (A.2.76).

b. Show that as $\epsilon \to 0$ the result in (1.3.15) tends to the fundamental solution (1.2.20), as expected, since (1.3.14) is a representation of the delta function (see (A.1.3)), and the solution (1.3.9) with $f(x) = \delta(x)$ is just (1.2.20).

1.3.3. Specialize (1.3.7) to the case where $p(x, t)$ is a uniformly moving source

$$p(x,t) = \delta(x - vt), \quad v = \text{constant}. \tag{1.3.16}$$

1.4 Problems in the Semi-infinite Domain; Green's Functions

In studying the diffusion equation over the semi-infinite interval with a prescribed boundary condition at $x = 0$, it is useful first to consider the solution that results from a unit source somewhere in the domain and subject to a homogeneous (zero) boundary condition at the origin. This solution will be denoted by Green's function of the first kind, G_1, or second kind, G_2, depending on whether the boundary condition at $x = 0$ is $u = 0$ or $u_x = 0$.

1.4.1 Green's Function of the First Kind

Consider first the case where $u = 0$ at the origin; that is, we seek the solution for

$$u_t - u_{xx} = \delta(t)\delta(x - \xi) \tag{1.4.1a}$$

on $0 \leq x < \infty$, with ξ equal to a positive constant, and impose the boundary condition

$$u(0,t) = 0, \quad t > 0, \tag{1.4.1b}$$

and initial condition

$$u(x, 0^-) = 0. \tag{1.4.1c}$$

(Unless stated otherwise, we shall take the boundary condition for u at $x = \infty$ to be the same as the limit as $x \to \infty$ of the initial value. Thus, in the present case, we have $u(\infty, t) = u(\infty, 0) = 0$.)

Thus, we have introduced a concentrated unit source of heat at $x = \xi$ and $t = 0$. (Note that we can derive the solution for the case where (1.4.1) involves $\delta(t - \tau)$ by replacing t everywhere in the solution by $t - \tau$.) The rod is initially at zero temperature, and its left end is maintained at zero temperature for all time, for example, by attaching this end to an infinite solid of zero temperature.

The only difference between this problem and the fundamental solution is the fact that we require u to vanish at $x = 0$ and $x \to \infty$ instead of $x \to \pm\infty$. Thus, Green's function is the response to a source with a homogeneous boundary condition imposed at a finite point.

An intuitively appealing procedure invokes symmetry relative to the origin to construct the solution once the fundamental solution is known. (This is often called the *method of images*.)

Consider the temperature that results in the *infinite* domain if we turn on a positive source of unit strength at $x = \xi$ and $t = 0$, and *simultaneously* turn on a negative source of unit strength at $x = -\xi$, the image point.

At any time $t > 0$, the temperature in the rod will be the sum of the two temperatures $F(x - \xi, t)$ and $-F(x + \xi, t)$, corresponding to the positive and negative sources, respectively. These individual temperature profiles at some $t > 0$ are sketched in Figure 1.5. In particular, the combined temperature will always vanish at $x = 0$ for $t > 0$, by symmetry. Moreover, since the image source is located at $x = -\xi$, outside the domain of interest, the combined temperature satisfies (1.4.1a). Therefore, the solution of (1.4.1) is Green's function:

$$G_1(x, \xi, t) \equiv F(x - \xi, t) - F(x + \xi, t), \tag{1.4.2}$$

where F is defined by (1.2.20).

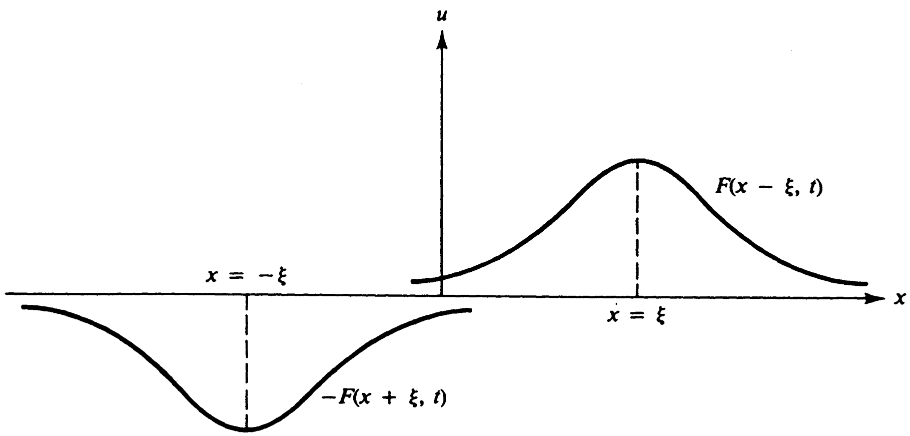

FIGURE 1.5. Temperature due to a unit positive source at $x = \xi$ and a unit negative source at $x = -\xi$

More generally, the solution of

$$u_t - u_{xx} = \delta(x-\xi)\delta(t-\tau), \quad \xi > 0, \quad \tau > 0, \tag{1.4.3}$$

with initial condition $u(x, \tau^-) = 0$ and boundary condition $u(0, t) = 0$ for $t > \tau$ and x on the semi-infinite interval $0 \le x < \infty$ is Green's function of the first kind for the semi-infinite domain and has the form

$$G_1(x, \xi, t-\tau) = \frac{1}{2\sqrt{\pi(t-\tau)}}[e^{-(x-\xi)^2/4(t-\tau)} - e^{-(x+\xi)^2/4(t-\tau)}]. \tag{1.4.4}$$

1.4.2 Homogeneous Boundary-Value Problems

Consider the following *inhomogeneous* diffusion equation with zero initial condition and *homogeneous* boundary condition:

$$u_t - u_{xx} = p(x, t), \quad 0 \le x, \quad 0 \le t, \tag{1.4.5a}$$

$$u(x, 0^-) = 0, \tag{1.4.5b}$$

$$u(0, t) = 0, \quad t > 0. \tag{1.4.5c}$$

The superposition idea leading to (1.3.7) also applies for this case, and we have

$$u(x, t) = \int_{0^-}^{t} d\tau \int_0^\infty p(\xi, \tau) G_1(x, \xi, t-\tau) d\xi. \tag{1.4.6}$$

It is important to bear in mind that *Green's function and the desired solution of (1.4.5) must both satisfy a zero boundary condition at the origin* in order for the superposition idea and the result (1.4.6) to make sense. For example, if $G_1(0, \xi, t-\tau) \neq 0$, then (1.4.6) does not satisfy (1.4.5c). Conversely, if we wish to solve the problem (1.4.5) with the right-hand side of (1.4.5c) replaced by some prescribed function $g(t)$, the representation (1.4.6) fails, since it automatically has $u(0, t) = 0$. We shall see in Section 1.4.3 that this case is easily handled once the problem is transformed to one with a zero boundary condition at the origin.

Consider now the case where the initial condition (1.4.5b) is prescribed arbitrarily. Since the homogeneous problem

$$u_t - u_{xx} = 0, \quad 0 \le x, \quad 0 \le t, \tag{1.4.7a}$$

with nonzero initial condition

$$u(x, 0^+) = f(x) \tag{1.4.7b}$$

and homogeneous boundary condition

$$u(0, t) = 0, \quad t > 0, \tag{1.4.7c}$$

is equivalent to

$$u_t - u_{xx} = \delta(t) f(x) \tag{1.4.8}$$

with $u(x, 0^-) = 0$ and $u(0, t) = 0$, we can express the solution of (1.4.7) using the result (1.4.6) with $p = \delta(\tau)f(\xi)$; that is,

$$u(x, t) = \int_0^\infty \int_{0^-}^t \delta(\tau)f(\xi)G_1(x, \xi, t-\tau)d\tau d\xi = \int_0^\infty f(\xi)G_1(x, \xi, t)d\xi. \tag{1.4.9}$$

For the special case where $f(\xi) = c$, a constant, (1.4.9) gives

$$u(x, t) = \frac{c}{2\sqrt{\pi t}}\left[\int_0^\infty e^{-(x-\xi)^2/4t}d\xi - \int_0^\infty e^{-(x+\xi)^2/4t}d\xi\right]. \tag{1.4.10}$$

Changing the variable of integration from ξ to $\eta = (x-\xi)/2t^{1/2}$ in the first integral and to $\eta = (x+\xi)/2t^{1/2}$ in the second integral results in

$$u(x, t) = \frac{c}{\sqrt{\pi}}\left\{-\int_{x/2\sqrt{t}}^{0} e^{-\eta^2}d\eta - \int_0^{-\infty} e^{-\eta^2}d\eta - \int_{x/2\sqrt{t}}^{\infty} e^{-\eta^2}d\eta\right\}. \tag{1.4.11a}$$

It is important to note that because $x - \xi$ vanishes for $\xi = x$, which is a point in $(0, \infty)$, the first integral in (1.4.11a) must be decomposed into two parts. Simplifying this expression gives

$$u(x, t) = \frac{2c}{\sqrt{\pi}}\int_0^{x/2\sqrt{t}} e^{-\eta^2}d\eta = c\,\mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right), \tag{1.4.11b}$$

where the error function erf is defined in (A.2.76).

The qualitative behavior of the solution (1.4.11) has u rising rapidly from its zero boundary value to the asymptotic value $u = c$. Temperature profiles at various times are sketched in Figure 1.6.

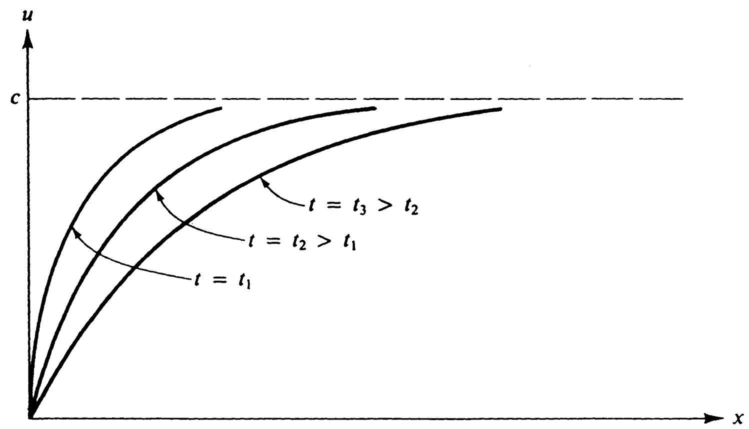

FIGURE 1.6. Temperature profiles at various values of t

Notice that $\lim_{\substack{t\to 0^+\\ x>0}} u(x,t) = c$, in agreement with (1.4.7b), and that $\lim_{\substack{x\to 0^+\\ t>0}} u(x,t) = 0$, in agreement with (1.4.7c). In particular, $u(0^+, 0^+)$ is undefined, as is to be expected from (1.4.7b) and (1.4.7c).

1.4.3 Inhomogeneous Boundary Condition $u(0,t) = g(t)$

As pointed out in Section 1.4.2, the crucial requirement for applying superposition is that the boundary condition at $x = 0$ be homogeneous. Does this mean that we cannot use Green's functions to solve an inhomogeneous boundary-value problem? We shall show next that if it is possible to transform the problem to one with a homogeneous boundary condition at $x = 0$ (as is often the case), a solution derived by superposition of Green's functions can still be used.

Consider the inhomogeneous boundary-value problem

$$u_t - u_{xx} = 0, \quad 0 \le x < \infty, \quad 0 \le t < \infty, \tag{1.4.12a}$$

with zero initial condition

$$u(x, 0^+) = 0, \tag{1.4.12b}$$

and a prescribed boundary condition at $x = 0$:

$$u(0,t) = g(t), \quad t > 0. \tag{1.4.12c}$$

Again, in view of (1.4.12b), it is understood that $u(\infty, t) = 0$.

The idea is to transform $u(x,t)$ to a new dependent variable $w(x,t)$, which obeys a homogeneous boundary condition at the origin. Clearly, the simple *homogenizing transformation*

$$w(x,t) \equiv u(x,t) - g(t) \tag{1.4.13}$$

works, since w obeys the inhomogeneous diffusion equation

$$w_t - w_{xx} = -\dot{g}(t), \quad t > 0, \tag{1.4.14}$$

with constant initial condition

$$w(x, 0^+) = -g(0^+), \tag{1.4.15a}$$

and zero boundary condition

$$w(0,t) = 0, \quad t > 0. \tag{1.4.15b}$$

Note that $w(\infty, t) = -g(t)$ if $t > 0$, but this does not preclude superposition. A problem equivalent to (1.4.14)–(1.4.15) is

$$w_t - w_{xx} = -\dot{g}(t) - g(0^+)\delta(t), \tag{1.4.16a}$$

$$w(x, 0^-) = 0, \tag{1.4.16b}$$

$$w(0,t) = 0, \quad t > 0, \tag{1.4.16c}$$

and the system (1.4.16) is a special case of (1.4.5), with $p(x,t) = -\dot{g}(t) - g(0^+)\delta(t)$. Writing out the solution (1.4.6) for this case gives

$$w(x,t) = \int_{0^+}^{t}\int_{0}^{\infty} \frac{-\dot{g}(\tau)}{2\sqrt{\pi(t-\tau)}}[e^{-(x-\xi)^2/4(t-\tau)} - e^{-(x+\xi)^2/4(t-\tau)}]d\xi\, d\tau$$
$$- \int_0^\infty \frac{g(0^+)}{2\sqrt{\pi t}}[e^{-(x-\xi)^2/4t} - e^{-(x+\xi)^2/4t}]d\xi. \tag{1.4.17}$$

The solution (1.4.17) involves the two integrals

$$I = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-(x-\xi)^2/4(t-\tau)}}{2\sqrt{t-\tau}}d\xi \tag{1.4.18a}$$

and

$$K = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-(x+\xi)^2/4(t-\tau)}}{2\sqrt{t-\tau}}d\xi. \tag{1.4.18b}$$

In preparation for evaluating I, we set the exponent in the integrand equal to $-\eta^2$, where η is a new variable of integration. Again, we must be careful to take into account the fact that this exponent vanishes at the point $\xi = x > 0$, which is inside the interval of integration. Thus, we first split (1.4.18a) into two integrals over $0 \le \xi \le x$ and $x \le \xi < \infty$; then we change variables $\xi \to \eta$ by setting $(x-\xi)/2\sqrt{t-\tau} = \eta$, $d\xi = -2\sqrt{t-\tau}\,d\eta$ to obtain

$$I = \frac{1}{\sqrt{\pi}}\left[\int_{x/2\sqrt{t-\tau}}^{0} e^{-\eta^2}(-d\eta) + \int_0^{-\infty} e^{-\eta^2}(-d\eta)\right]$$
$$= \frac{1}{\sqrt{\pi}}\left[\int_0^{x/2\sqrt{t-\tau}} e^{-\eta^2}d\eta + \int_0^\infty e^{-\eta^2}d\eta\right].$$

It then follows from the definition (A.2.76) of the error function that

$$I = \frac{1}{2}\,\mathrm{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right) + \frac{1}{2}. \tag{1.4.18c}$$

Since $(x+\xi)$ does not vanish for $x > 0$ if $0 \le \xi < \infty$, we evaluate K directly by setting $(x+\xi)/2\sqrt{t-\tau} = \eta$ to obtain

$$K = \frac{1}{\sqrt{\pi}}\left[\int_{x/2\sqrt{t-\tau}}^{\infty} e^{-\eta^2}d\eta\right] = \frac{1}{2}\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t-\tau}}\right), \tag{1.4.18d}$$

where erfc denotes the complementary error function, $\mathrm{erfc}(y) = 1 - \mathrm{erf}(y)$. See (A.2.77).

Thus, I may also be written as

$$I = 1 - \frac{1}{2}\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t-\tau}}\right), \tag{1.4.19}$$

and (1.4.17) becomes

$$w(x,t) = \int_{0^+}^{t} \dot{g}(\tau)\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t-\tau}}\right)d\tau + g(0^+)\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t}}\right) - g(t). \tag{1.4.20}$$

Therefore, $u(x,t) = w(x,t) + g(t)$ is given by

$$u(x,t) = \int_{0^+}^{t} \dot{g}(\tau)\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t-\tau}}\right)d\tau + g(0^+)\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t}}\right). \tag{1.4.21}$$

Note that $w(\infty, t) = -g(t)$ as required. For the special case $g(t) = d =$ constant, $\dot{g} = 0$, and we have $u(x,t) = d\,\mathrm{erfc}(x/2\sqrt{t})$. Here, again, as for (1.4.11), $u(0^+, 0^+)$ is undefined. However, $\lim_{\substack{t\to 0^+\\ x>0}} u(x,t) = 0$, in agreement with (1.4.12b), and $\lim_{\substack{x\to 0^+\\ t>0}} u(x,t) = d$, in agreement with (1.4.12c).

Integrating the first term by parts in (1.4.21) gives the alternative form

$$u(x,t) = \frac{x}{2\sqrt{\pi}} \int_0^t \frac{g(\tau)e^{-x^2/4(t-\tau)}}{[t-\tau]^{3/2}}\,d\tau = \frac{x}{2\sqrt{\pi}} \int_0^t \frac{g(t-\tau)e^{-x^2/4\tau}}{\tau^{3/2}}\,d\tau. \tag{1.4.22}$$

The solution (1.4.22) is derived in Appendix A.2 using Laplace transforms. See (A.2.73). In Problem 1.4.4a this result is obtained as the solution of a related integral equation. Problem 1.4.6 explores the application of the preceding ideas to the case of discontinuous material properties. Problem 1.4.7 concerns the effect of moving boundaries.

Next, we consider problems on the semi-infinite domain subject to the homogeneous boundary condition $u_x = 0$ at $x = 0$ and see how Green's function may also be used to solve the problem where u_x is specified at $x = 0$.

1.4.4. Green's Function of the Second Kind

We can also use a symmetry argument to solve

$$u_t - u_{xx} = \delta(x-\xi)\delta(t) \tag{1.4.23a}$$

on $0 \le x < \infty$, with $\xi > 0$ subject to the boundary condition

$$u_x(0,t) = 0, \quad t > 0, \tag{1.4.23b}$$

and initial condition

$$u(x, 0^-) = 0. \tag{1.4.23c}$$

Here again, we assume that as $x \to \infty$, u remains equal to the value it has at infinity initially.

We might interpret the solution of (1.4.23) as the temperature in a semi-infinite rod in response to a unit source of heat at $x = \xi$, $t = 0$ for the case where the rod is insulated (that is, there is no heat flow) at the left end.

In order to ensure that condition (1.4.23b) holds for all $t > 0$ at the origin, we need to introduce an *image*, or *reflected*, source of unit *positive* strength at the image point $x = -\xi$. The situation corresponding to Figure 1.5 now has the two bell-shaped profiles above the x-axis and centered at the points $x = \pm\xi$. Therefore, the slope of the combined profile vanishes at $x = 0$, since the contributions to u_x from the source at $x = \xi$ and $x = -\xi$ cancel out exactly for all $t > 0$.

Thus, the solution of (1.4.23) is

$$G_2(x, \xi, t) \equiv F(x - \xi, t) + F(x + \xi, t), \tag{1.4.24}$$

where F is the fundamental solution defined by (1.2.20).

More generally, if the source is turned on at $t = \tau > 0$, we have

$$G_2(x, \xi, t - \tau) = \frac{1}{2\sqrt{\pi(t - \tau)}}[e^{-(x-\xi)^2/4(t-\tau)} + e^{-(x+\xi)^2/4(t-\tau)}]. \tag{1.4.25}$$

1.4.5 Homogeneous Boundary-Value Problems

As in Section 1.4.2 we can use superposition to express the solution of

$$u_t - u_{xx} = p(x, t), \quad 0 \le x, \ 0 \le t, \tag{1.4.26a}$$

$$u(x, 0^-) = 0, \tag{1.4.26b}$$

$$u_x(0, t) = 0, \quad t > 0, \tag{1.4.26c}$$

in the form

$$u(x, t) = \int_0^t d\tau \int_0^\infty [p(\xi, \tau)G_2(x, \xi, t - \tau)]d\xi. \tag{1.4.27}$$

Also, we can accommodate a nonzero initial condition

$$u(x, 0^+) = f(x)$$

by adding to (1.4.27) the contribution

$$u(x, t) = \int_0^\infty f(\xi)G_2(x, \xi, t)d\xi. \tag{1.4.28}$$

For the case $f(\xi) = c =$ constant, it is easily seen by changing the sign of the second term in (1.4.10) that (1.4.28) reduces to $u = c$, as expected.

1.4.6 Inhomogeneous Boundary Condition, $u_x(0, t) = h(t)$

To solve the problem

$$u_t - u_{xx} = 0, \quad 0 \le x < \infty, \quad 0 \le t < \infty, \tag{1.4.29a}$$

$$u(x, 0^+) = 0, \tag{1.4.29b}$$

$$u_x(0, t) = h(t), \quad t > 0, \tag{1.4.29c}$$

we introduce the homogenizing transformation

$$w(x, t) \equiv u(x, t) - xh(t). \tag{1.4.30}$$

It then follows that if u solves (1.4.29), w solves

$$w_t - w_{xx} = -x\dot{h}(t), \tag{1.4.31a}$$

$$w(x, 0^+) = -xh(0^+), \tag{1.4.31b}$$

$$w_x(0, t) = 0. \tag{1.4.31c}$$

Using the results in (1.4.27) and (1.4.28), we have

$$u(x,t) - xh(t) = -\int_0^t d\tau \int_0^\infty \xi\dot{h}(\tau)G_2(x,\xi,t-\tau)d\xi - h(0^+)\int_0^\infty \xi G_2(x,\xi,t)d\xi. \tag{1.4.32a}$$

This can be simplified to the form

$$u(x,t) = -\frac{1}{\sqrt{\pi}}\int_0^t h(\tau)(t-\tau)^{-1/2}e^{-x^2/4(t-\tau)}d\tau. \tag{1.4.32b}$$

In Problem 1.4.8 you are asked to derive this result and to reconcile it with the result obtained by Laplace transforms.

1.4.7 The General Linear Boundary-Value Problem

The general linear boundary-value problem over the semi-infinite domain is

$$u_t - u_{xx} = p(x, t), \tag{1.4.33a}$$

$$u(x, 0^+) = 0, \tag{1.4.33b}$$

$$a(t)u(0,t) + b(t)u_x(0,t) = c(t), \quad t > 0, \tag{1.4.33c}$$

as we have the most general linear boundary condition (1.4.33c) at the left end with arbitrarily prescribed nonvanishing functions a, b, and c. In our previous discussion, we have solved the two special cases $a = 0$ and $b = 0$. There is no loss of generality in setting $u(x, 0^+) = 0$ in (1.4.33b), since for a general initial condition $u(x, 0^+) = f(x)$, we can transform the problem to the form (1.4.33) by considering $u - f$ as a new dependent variable.

A Green's function approach is not feasible if a, b, and c are all nonzero, and we study two approaches next for solving (1.4.33).

(i) $au(0,t) + bu_x(0,t) = c(t)$, *a and b constant*

If a and b are constant, (1.4.33c) may be interpreted as Newton's law of cooling for a semi-infinite heat conductor with its left end ($x = 0$) in contact with a heat reservoir with prescribed time-dependent temperature. We write (1.4.33c) in the form

$$bu_x(0,t) = a[u_R(t) - u(0,t)], \tag{1.4.34}$$

and regard $-b$ as the thermal conductivity, $a > 0$ as the heat transfer coefficient, and $u_R(t) = c(t)/a$ as the reservoir temperature. Thus, for example, if $u_R(t) >$

$u(0, t)$, we expect heat to flow from the reservoir into the conductor, making the left end $x = 0$ hotter than the interior, i.e., $u_x(0, t) < 0$. This follows from (1.4.34), since $b < 0$ in this interpretation.

One approach for solving (1.4.33) is to introduce a new dependent variable $v(x, t)$ defined by

$$v(x, t) \equiv au(x, t) + bu_x(x, t). \tag{1.4.35}$$

If we compute $v_t - v_{xx}$ using (1.4.35) we obtain

$$v_t - v_{xx} = a(u_t - u_{xx}) + b(u_t - u_{xx})_x.$$

Thus, if u satisfies (1.4.33a), v satisfies

$$v_t - v_{xx} = ap(x, t) + bp_x(x, t) \equiv q(x, t), \tag{1.4.36a}$$

the same diffusion equation with a different, but known, right-hand side. Note that if a and b depend on t, this approach *does not* lead to the same diffusion equation; we pick up additional terms involving time-dependent coefficients.

The initial and boundary conditions for v are obtained in the form

$$v(x, 0) = 0, \tag{1.4.36b}$$

$$v(0, t) = c(t). \tag{1.4.36c}$$

Therefore, using (1.4.6) and (1.4.9), we have

$$\begin{aligned} v(x, t) = {} & \frac{x}{2\sqrt{\pi}} \int_0^t \tau^{-3/2} c(t - \tau) e^{-x^2/4\tau} d\tau \\ & + \int_0^t d\tau \int_0^\infty q(\xi, \tau) G_1(x, \xi, t - \tau) d\xi, \end{aligned} \tag{1.4.37}$$

where G_1 is defined in (1.4.4).

Knowing $v(x, t)$, we compute $u(x, t)$ by solving the linear inhomogeneous ordinary differential equation (1.4.35). This gives

$$u(x, t) = \phi(t) e^{-ax/b} + \frac{e^{-ax/b}}{b} \int_0^x v(\xi, t) e^{a\xi/b} d\xi, \tag{1.4.38}$$

where $\phi(t)$ is as yet unspecified. The initial condition $u(x, 0) = 0$ and the fact that $v(x, 0) = 0$ imply that $\phi(0) = 0$. It is easy to verify by direct substitution that (1.4.38) satisfies the boundary condition (1.4.33c) identically. To determine $\phi(t)$ we substitute (1.4.38) into the governing equation (1.4.33a). We have

$$u_t = \dot{\phi}(t) e^{-ax/b} + \frac{1}{b} e^{-ax/b} \int_0^x v_t(\xi, t) e^{a\xi/b} d\xi, \tag{1.4.39a}$$

$$u_x = -\frac{a}{b} \phi(t) e^{-ax/b} - \frac{a}{b^2} e^{-ax/b} \int_0^x v(\xi, t) e^{a\xi/b} d\xi + \frac{1}{b} v(x, t), \tag{1.4.39b}$$

$$\begin{aligned} u_{xx} = {} & \frac{a^2}{b^2} \phi(t) e^{-ax/b} + \frac{a^2}{b^3} e^{-ax/b} \int_0^x v(\xi, t) e^{a\xi/b} d\xi - \frac{a}{b^2} v(x, t) \\ & + \frac{1}{b} v_x(x, t). \end{aligned} \tag{1.4.39c}$$

The integral in the expression defining u_{xx} can be developed by integration by parts twice to give

$$u_{xx} = \frac{a^2}{b^2}\phi(t)e^{-ax/b} - \frac{a}{b^2}v(0,t)e^{-ax/b} + \frac{1}{b}v_x(0,t)e^{-ax/b} + \frac{1}{b}e^{-ax/b}\int_0^x v_{xx}(\xi,t)e^{a\xi/b}d\xi. \tag{1.4.39d}$$

Substituting (1.4.39a) for u_t and (1.4.39d) for u_{xx} into (1.4.33a) and using the boundary condition (1.4.33c) gives

$$p(x,t) = \left[\dot{\phi}(t) - \frac{a^2}{b^2}\phi(t)\right]e^{-ax/b} + \frac{1}{b}e^{-ax/b}\int_0^x [ap(\xi,t) + bp_x(\xi,t)]e^{a\xi/b}d\xi + \left[\frac{a}{b^2}c(t) - \frac{1}{b}v_x(0,t)\right]e^{-ax/b}. \tag{1.4.40}$$

Now, when we integrate by parts the integral of $pe^{a\xi/b}$, the integrals involving p_x cancel. We also pick up a $p(x,t)$ on the right-hand side of (1.4.40) that cancels the $p(x,t)$ on the left-hand side. Finally, we multiply though by $e^{ax/b}$ to obtain the following first-order linear inhomogeneous ordinary differential equation governing $\phi(t)$:

$$\dot{\phi}(t) - \frac{a^2}{b^2}\phi(t) = -\frac{a}{b^2}c(t) + p(0,t) + \frac{1}{b}v_x(0,t). \tag{1.4.41}$$

Since v is given by (1.4.37), the right-hand side of (1.4.41) is a known function of t. The solution of (1.4.41) subject to $\phi(0) = 0$ defines $\phi(t)$ uniquely. When this result is used in (1.4.38), we have the solution of (1.4.33).

We work out the details next for the special case $p(x,t) \equiv 0$, $c =$ constant. Thus, according to (1.4.21) (with $\dot{g}(0)$, $g(0^+) = c$) we have

$$v(x,t) = c\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t}}\right). \tag{1.4.42}$$

Using the definition (A.2.77) of the complementary error function, we have

$$\frac{d}{dz}\mathrm{erfc}(z) = -\frac{2}{\sqrt{\pi}}e^{-z^2}. \tag{1.4.43}$$

Therefore, $v_x(0,t) = -c/\sqrt{\pi t}$, and (1.4.41) reduces to

$$\dot{\phi} - \frac{a^2}{b^2}\phi = -\frac{ac}{b^2} - \frac{c}{b\sqrt{\pi t}}. \tag{1.4.44}$$

The solution of (1.4.44) subject to $\phi(0) = 0$ is

$$\phi(t) = \frac{c}{a}\left\{1 - e^{a^2t/b^2}\left[1 + \mathrm{erf}\left(\frac{a\sqrt{t}}{b}\right)\right]\right\}. \tag{1.4.45}$$

Thus, $u(x, t)$ is given by (1.4.38) in the form

$$u(x,t) = \frac{c}{a} e^{-ax/b} \left\{ 1 - e^{-a^2 t/b^2} \left[1 + \operatorname{erf}\left(\frac{a\sqrt{t}}{b} \right) \right] \right\} + \frac{c}{b} e^{-ax/b} \int_0^x e^{a\xi/b} \operatorname{erfc}\left(\frac{\xi}{2\sqrt{t}} \right) d\xi. \qquad (1.4.46)$$

This result can be further simplified by evaluating the integral on the right-hand side. We outline the calculations next, although the final result may be obtained directly using Mathematica or Maple. We have

$$\begin{aligned} I_0 &\equiv \int_0^x e^{a\xi/b} \operatorname{erfc}\left(\frac{\xi}{2\sqrt{t}} \right) d\xi \\ &= \frac{b}{a} \left[e^{ax/b} \operatorname{erfc}\left(\frac{x}{2\sqrt{t}} \right) - 1 + \frac{1}{\sqrt{\pi t}} \int_0^x \exp\left(\frac{a\xi}{b} - \frac{\xi^2}{4t} \right) d\xi \right]. \end{aligned}$$

Denoting

$$I_1 \equiv \int_0^x \exp\left(\frac{a\xi}{b} - \frac{\xi^2}{4t} \right) d\xi,$$

we find, upon completing the square in the exponential, that

$$I_1 = e^{a^2 t/b^2} \int_0^x \exp\left(\frac{\xi - 2at/b}{2\sqrt{t}} \right)^2 d\xi.$$

The above is a useful trick for integrals with quadratic exponents as in I_1. Now we evaluate I_1 by splitting the integral into two parts as in (1.4.18c) to obtain

$$I_1 = \sqrt{\pi t} e^{a^2 t/b^2} \left[\operatorname{erf}\left(\frac{a\sqrt{t}}{b} \right) + \operatorname{erf}\left(\frac{x - 2at/b}{2\sqrt{t}} \right) \right].$$

Therefore, using this result in the expression for I_0 gives

$$I_0 = \frac{b}{a} \left\{ e^{ax/b} \operatorname{erfc}\left(\frac{x}{2\sqrt{t}} \right) - 1 + e^{a^2 t/b^2} \left[\operatorname{erf}\left(\frac{a\sqrt{t}}{b} \right) + \operatorname{erf}\left(\frac{x - 2at/b}{2\sqrt{t}} \right) \right] \right\}.$$

Now we substitute this expression for I_0 into (1.4.46) to obtain the solution

$$u(x,t) = \frac{c}{a} \left[\operatorname{erfc}\left(\frac{x}{2\sqrt{t}} \right) - \exp\left(\frac{a^2 t}{b^2} - \frac{ax}{b} \right) \operatorname{erfc}\left(\frac{x - 2at/b}{2\sqrt{t}} \right) \right]. \qquad (1.4.47)$$

It is a straightforward matter using Mathematica or Maple to verify that (1.4.47) satisfies (1.4.33).

(ii) $a(t)u(0,t) + b(t)u_x(0,t) = c(t)$, *a and b depend on t*

As pointed out earlier, the transformation of dependent variable (1.4.35) is not helpful in this case. Instead, we assume an unknown boundary value for u at $x = 0$,

$$u(0,t) = k(t), \quad t > 0, \tag{1.4.48}$$

where $k(t)$ is as yet unspecified. The solution of the problem consisting of (1.4.33a), (1.4.33b), and (1.4.48) was worked out in (1.4.21). We have

$$u(x,t) = \int_0^t \dot{k}(\tau)\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t-\tau}}\right) d\tau + k(0^+)\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t}}\right). \tag{1.4.49}$$

We now compute

$$u_x(x,t) = -\frac{1}{\sqrt{\pi}} \int_0^t \frac{\dot{k}(\tau)}{\sqrt{t-\tau}} e^{-x^2/4(t-\tau)} d\tau - \frac{k(0^+)}{\sqrt{\pi t}} e^{-x^2/4t}, \tag{1.4.50}$$

where we have used (1.4.43) to calculate the derivative of the complementary error function. Evaluating (1.4.50) at $x = 0$ and using the result together with (1.4.48) in the boundary condition (1.4.33c) gives the following *integral equation* for $k(t)$:

$$c(t) = a(t)k(t) - \frac{b(t)}{\sqrt{\pi}} \left[\int_0^t \frac{\dot{k}(\tau)}{\sqrt{t-\tau}} d\tau - \frac{k(0^+)}{\sqrt{t}} \right]. \tag{1.4.51}$$

In the first term on the right-hand side of (1.4.50), note the occurrence of the integrable singularity proportional to $(t-\tau)^{-1/2}$ at $\tau = t$. Had we used the form (1.4.22) for the solution u, the corresponding singularity would have been proportional to $x^2(t-\tau)^{-5/2}$, requiring further manipulations to derive a well-behaved result at $x = 0$.

A discussion of techniques for solving the integral equation (1.4.51) is beyond our scope. Once $k(t)$ has been determined, the solution for $u(x,t)$ is given by (1.4.49).

Problems

1.4.1. Verify by direct substitution that (1.4.6) solves (1.4.5), and that (1.4.9) solves (1.4.7).

1.4.2. Verify by direct substitution that (1.4.22) solves (1.4.12).

1.4.3. Consider the linear equation

$$u_t - u_{xx} = 0, \quad 0 \le x, \quad 0 \le t, \tag{1.4.52a}$$

with initial condition

$$u(x,0) = 0, \tag{1.4.52b}$$

and the following boundary condition at $x = 0$:

$$u(0,t) = Ct^n, \quad t > 0, \tag{1.4.52c}$$

where n is a nonnegative constant and C is a positive constant.

As usual, (1.4.52b) implies the boundary condition at infinity

$$u(\infty, t) = 0, \quad t \geq 0. \tag{1.4.52d}$$

a. Use the result (1.4.22) to express the solution in the form

$$u(x, t) = Ct^n f(\theta), \tag{1.4.53}$$

where

$$\theta \equiv \frac{x}{2t^{1/2}} \tag{1.4.54a}$$

and

$$f(\theta) = \frac{2}{\sqrt{\pi}} \int_\theta^\infty \left(1 - \frac{\theta^2}{s^2}\right)^n e^{-s^2} ds. \tag{1.4.54b}$$

b. Show that the similarity form (1.4.53) satisfies (1.4.52), and derive the following differential equation and boundary conditions for $f(\theta)$:

$$f'' + 2\theta f' - 4nf = 0, \tag{1.4.55a}$$

$$f(0) = 1, \tag{1.4.55b}$$

$$f(\infty) = 0. \tag{1.4.55c}$$

Show that the solution of (1.4.55) gives (1.4.54b).

c. Now consider the nonlinear diffusion equation

$$u_t - [k(u)u_x]_x = 0, \; 0 \leq x, \;\; 0 \leq t, \tag{1.4.56}$$

where $k(u)$ is a prescribed function of u.

The initial condition is (1.4.52b), and the boundary condition at $x = \infty$ is (1.4.52d), whereas at $x = 0$ we have

$$u(0, t) = g(t), \quad t > 0, \tag{1.4.57}$$

for some prescribed function $g(t)$. This problem is discussed in [6].

i. If $k(u) = \lambda u^\nu$, where λ and v are positive constants, show that the most general $g(t)$ for which a similarity solution exists is

$$g(t) = Ct^n, \tag{1.4.58}$$

where C and n are constants as in (1.4.52c). In this case, the similarity form is

$$u(x, t) = t^n \phi(\zeta), \;\; \zeta \equiv \frac{x}{t^{(\nu n+1)/2}}, \tag{1.4.59}$$

and ϕ obeys

$$\lambda \frac{d}{d\zeta}\left(\phi^\nu \frac{d\phi}{d\zeta}\right) + \frac{\nu n + 1}{2} \zeta \frac{d\phi}{d\zeta} - n\phi = 0, \tag{1.4.60a}$$

subject to the boundary conditions

$$\phi(0) = C, \;\; \phi(\infty) = 0. \tag{1.4.60b}$$

ii. If $k(u)$ is prescribed arbitrarily, show that the most general $g(t)$ for which a similarity solution exists is $g(t) = C =$ constant. In this case the similarity form is

$$u(x,t) = \phi(\theta), \ \theta = \frac{x}{t^{1/2}}, \tag{1.4.61}$$

and ϕ obeys

$$\frac{d}{d\theta}\left[k(\phi)\frac{d\phi}{d\theta}\right] + \frac{\theta}{2}\frac{d\phi}{d\theta} = 0, \tag{1.4.62a}$$

with boundary conditions

$$\phi(0) = C, \ \ \phi(\infty) = 0. \tag{1.4.62b}$$

1.4.4a. Assume that the solution of (1.4.12) on the positive axis may be regarded as the response due to a source of unknown strength $q(t)$ at the origin for an infinite conductor. Therefore, $u(x,t)$ may be expressed in the form (1.3.7) with $p = \delta(x)q(t)$. In this case, (1.3.7) reduces to

$$u(x,t) = \frac{1}{2\sqrt{\pi}}\int_0^t q(\tau)\frac{e^{-x^2/4(t-\tau)}}{\sqrt{t-\tau}}d\tau. \tag{1.4.63}$$

But in order to satisfy the boundary condition (1.4.12c), we must have

$$g(t) = \frac{1}{2\sqrt{\pi}}\int_0^t \frac{q(\tau)d\tau}{\sqrt{t-\tau}}. \tag{1.4.64}$$

This is an integral equation (solved by Abel) for the unknown $q(t)$ in terms of the known $g(t)$.

Use Laplace transforms and the convolution integral to show that

$$q(t) = \frac{2}{\sqrt{\pi}}\frac{d}{dt}\int_0^t \frac{g(\tau)d\tau}{\sqrt{t-\tau}}. \tag{1.4.65}$$

Therefore, the solution of (1.4.12) may also be expressed in the form

$$u(x,t) = \frac{1}{\pi}\int_0^t \left[\frac{e^{-x^2/4(t-\tau)}}{\sqrt{t-\tau}}\frac{d}{d\tau}\int_0^\tau \frac{g(s)ds}{\sqrt{\tau-s}}\right]d\tau. \tag{1.4.66}$$

Show that (1.4.66) reduces to (1.4.22).

b. For the case $g(t) = 1$, we have shown that (1.4.66) reduces to $u(x,t) = \operatorname{erfc}(x/2t^{1/2})$. Suppose that we wish to regard this solution in $0 \leq x < \infty$, $0 \leq t < \infty$ as being produced by an *unknown initial specification* of u of the form

$$u(x,0) = \begin{cases} 0 & \text{if } x \geq 0, \\ f(x) & \text{if } x < 0, \end{cases} \tag{1.4.67}$$

for the same diffusion equation (1.4.12a) over the infinite interval $-\infty < x < \infty$. With $\tilde{f}(x) = f(-x)$ show that $\tilde{f}$ obeys the integral equation

$$\int_0^\infty \tilde{f}(\xi)e^{-\xi^2/4t}d\xi = 2\sqrt{\pi t}. \tag{1.4.68}$$

Use Laplace transforms to show that $\tilde{f}(\xi) = 2$.

1.4.5a. Modify the calculations leading to (1.4.22) so that you obtain the solution of (1.4.12) with (1.4.12b) replaced by the arbitrary initial condition

$$u(x, 0^+) = f(x). \tag{1.4.69}$$

b. Specialize the results in (a) to the case f = constant = u_1, and express the solution in a form such that $u_x(0^+, t)$ is free of singularities. (*Note*: (1.4.22) has an apparent singularity at $x = 0$, whereas (1.4.21) does not.)

1.4.6a. Consider two semi-infinite rods with initial temperatures $u = u_1$ = constant and $u = u_2$ = constant, thermal diffusivities (see (1.1.9)) κ_1^2 = constant and κ_2^2 = constant, and thermal conductivities k_1 = constant and k_2 = constant. Suddenly, at $t = 0$, the two conductors are brought into perfect contact at $x = 0$. Let the first conductor lie on $0 \leq x < \infty$ and let the second conductor lie on $-\infty < x \leq 0$.

It follows from the integral conservation law (1.1.6) with A = constant that the interface conditions for $t > 0$ are $u(0^+, t) = u(0^-, t)$ and $k_1 u_x(0^+, t) = k_2 u_x(0^-, t)$. Show this. Use the result in Problem 1.4.5b to show that the heat flow $k_1 u_x(0^+, t)$ (or $k_2 u_x(0^-, t)$) at the point of contact and $t > 0$ is given by

$$F(t) = \frac{1}{\sqrt{\pi t}} \frac{k_1}{\kappa_1}(u_1 - c), \tag{1.4.70}$$

where c is the constant temperature at $x = 0$:

$$c = \frac{u_2 - \alpha u_1}{1 - \alpha}, \quad \alpha = -\frac{k_1 \kappa_2}{k_2 \kappa_1}. \tag{1.4.71}$$

b. Now consider the situation where these two rods are initially at zero temperature and in perfect thermal contact. Use the method of images to calculate the fundamental solution; that is, solve

$$u_t - \kappa^2 u_{xx} = \delta(t)\delta(x - \xi), \quad 0 < \xi, \tag{1.4.72}$$

on $-\infty < x < \infty$ with $u(x, 0^-) = 0$, where $\kappa = \kappa_1$ if $x > 0$ and $\kappa = \kappa_2$ if $x < 0$. Use the interface conditions $u(0^+, t) = u(0^-, t)$ and $k_1 u_x(0^+, t) = k_2 u_x(0^-, t)$. *Hint*: Assume that in the domain $x < 0$, the solution $u_2(x, t)$ may be regarded as the response to a source of unknown strength B and unknown location ($\xi_1 > 0$) in an infinite medium with the uniform properties κ_2, k_2 *throughout*. Thus, $u_2(x, t)$ corresponds to a "transmitted" temperature due to the primary source at $x = \xi$ and $t = 0$. For the solution $u_1(x, t)$ in the domain $x > 0$, assume that in addition to the response due to the primary source, there is a "reflected" contribution, which may be regarded as the response to an image source of unknown strength A located at the unknown point $x = \xi_2 < 0$ in an infinite rod with properties k_1 and κ_1 throughout. Use the interface conditions to determine A, B, ξ_1, and ξ_2. Verify that in the limits $(k_2/k_1) \to 1$ and $(\kappa_2/\kappa_1) \to 1$, $A \to 0$ and $B \to 1$.

1.4.7. Consider the diffusion equation

$$u_t - u_{xx} = 0, \quad 0 \leq t < \infty, \tag{1.4.73}$$

on the *time-dependent* domain $at \leq x < \infty$, where a is a constant. We wish to solve the initial- and boundary-value problem having

$$u(x, 0^+) = 0, \tag{1.4.74}$$

$$u(at, t) = g(t), \tag{1.4.75}$$

for $t > 0$ and a prescribed $g(t)$. Thus, u is prescribed as a function of time on the left boundary that moves at a constant speed a.

a. Introduce the transformation of variables $\bar{x} = x - at, \bar{t} = t$ and solve the resulting problem by Laplace transforms.

b. Calculate the appropriate Green's function for the problem in x, t variables and rederive the solution using this.

1.4.8. Use the expression (1.4.25) for G_2 to simplify the solution in (1.4.32a) to the form given by (1.4.32b). Rederive the same result using Laplace transforms.

1.5 Problems in the Finite Domain; Green's Functions

The next step in our development involves problems on the finite domain, which may be taken as the unit interval $0 \leq x \leq 1$ with no loss of generality (that is, we choose the length L of the domain as the scale to normalize (1.1.9)). As in Section 1.4, we distinguish problems that have $u = 0$ or $u_x = 0$ at either end. Thus, we need to study *four* different Green's functions, and we start with the simplest case.

1.5.1 Green's Function of the First Kind

We refer to the solution satisfying the boundary condition $u = 0$ at both ends as Green's function of the first kind, G_1. More precisely, define the solution of

$$u_t - u_{xx} = \delta(x - \xi)\delta(t - \tau), \quad 0 \leq x \leq 1, \quad \tau \leq t, \tag{1.5.1a}$$

$$u(x, \tau^-) = 0, \tag{1.5.1b}$$

$$u(0, t) = u(1, t) = 0, \quad t > \tau, \tag{1.5.1c}$$

as Green's function $G_1(x, \xi, t - \tau)$. Here, ξ and τ are constants with $0 < \xi < 1$, $0 < \tau$.

Let us construct G_1 using symmetry arguments in terms of appropriate fundamental solutions. Consider the "primary" source $\delta(x - \xi)\delta(t - \tau)$ sketched as $\uparrow$ at the point $x = \xi$, $0 < \xi < 1$, on the unit interval in Figure 1.7.

In order to cancel the contribution of the primary source at the left boundary $x = 0$, we need to introduce a reflected (or image) source of negative unit strength (sketched as $\downarrow$) at the image point $x = -\xi$. This image source must also be turned

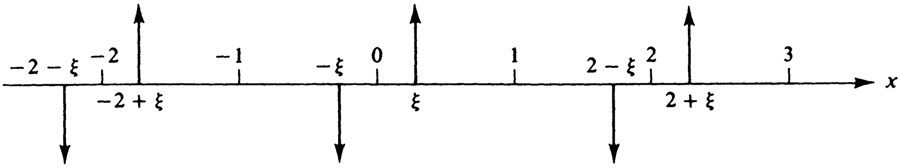

FIGURE 1.7. Primary and reflected sources to give $u = 0$ at $x = 0$ and $x = 1$

on at $t = \tau$. Similarly, to take care of the boundary contribution of the primary source at $x = 2$, we introduce another image source at $x = 2 - \xi$, also turned on at $t = \tau$. But now, the image source at $x = -\xi$ contributes to the boundary value at $x = 1$, and the image source at $x = 2 - \xi$ contributes to the boundary value at $x = 0$. To take care of the first, we introduce the $\uparrow$ unit source at $x = 2 + \xi$. To take care of the second, we introduce the $\uparrow$ unit source at $-2 + \xi$, and so on. The pattern that emerges has positive unit sources at $x = 2n + \xi, n = 0, \pm 1, \pm 2, \ldots$, and negative unit sources are at $x = 2n - \xi, n = 0, \pm 1, \pm 2, \ldots$. The sum of all these source contributions is a representation for Green's function G_1 in the following series form:

$$G_1(x, \xi, t-\tau) \equiv \sum_{n=-\infty}^{\infty} \{F[x-(2n+\xi), t-\tau] - F[x-(2n-\xi), t-\tau]\}, \quad (1.5.2)$$

where F is defined in (1.2.20).

Green's function G_1 has the interesting symmetry property

$$G_1(x, \xi, t - \tau) \equiv G_1(\xi, x, t - \tau). \quad (1.5.3)$$

The corresponding steady-state result is noted in Appendix A.1.3. To demonstrate this symmetry property, we note that the right-hand side of (1.5.3) is by definition given by

$$G_1(\xi, x, t-\tau) = \sum_{n=-\infty}^{\infty} [F(\xi - 2n - x, t-\tau) - F(\xi - 2n + x, t-\tau)]. \quad (1.5.4)$$

Since F is an even function of its first argument, we can rewrite the first term in the summation as $F(-\xi + 2n + x, t - \tau)$. Furthermore, since the summation ranges over $-\infty < n < \infty$, the infinite sum of these terms remains the same if we replace n by $-n$. Therefore, we may write

$$G_1(\xi, x, t-\tau) = \sum_{n=-\infty}^{\infty} [F(-\xi - 2n + x, t-\tau) - F(\xi - 2n + x, t-\tau)], \quad (1.5.5)$$

which is just $G_1(x, \xi, t - \tau)$.

In terms of heat conduction, the result (1.5.3) is intuitively obvious and physically consistent. Suppose we consider a conductor with uniform properties and with its two endpoints maintained at the same temperature, here normalized to be zero. Fix any two distinct locations x and ξ on the conductor and carry out the following two experiments. In the first experiment we turn on a unit source of heat at time τ at the point ξ and measure the temperature at the point x and time $t > \tau$. This gives the result $G_1(x, \xi, t - \tau)$ for the measured temperature. In the second experiment, we reverse the locations of the source and observer without changing the values of τ or t and find that the temperature at ξ, given by $G_1(\xi, x, t - \tau)$, is the same as that measured in the first experiment.

Using G_1 and superposition, we can now solve the inhomogeneous problem (see (1.4.5))

$$u_t - u_{xx} = p(x, t), \quad 0 \le x \le 1, \quad 0 \le t, \tag{1.5.6a}$$

with zero initial condition

$$u(x, 0^-) = 0 \tag{1.5.6b}$$

and zero boundary conditions at both ends,

$$u(0, t) = u(1, t) = 0 \quad \text{for} \quad t > 0, \tag{1.5.6c}$$

in the form

$$u(x, t) = \int_0^t d\tau \int_0^1 [p(\xi, \tau)G_1(x, \xi, t - \tau)]d\xi. \tag{1.5.7}$$

Similarly, as in (1.4.7)–(1.4.8), we solve the problem with $p(x, t) = 0$ and nonzero initial condition

$$u(x, 0^+) = f(x), \tag{1.5.8}$$

instead of (1.5.6b), in the form

$$u(x, t) = \int_0^1 f(\xi)G_1(x, \xi, t)d\xi. \tag{1.5.9}$$

Green's functions for the remaining three homogeneous boundary-value problems are listed in Problem 1.5.2.

1.5.2 Connection with Separation of Variables

You may be wondering how the result in (1.5.9) is related to the solution we obtain by the more conventional separation of variables approach that is usually discussed in a first course in partial differential equations. We explore this question next. (Problem 1.5.6 gives a review of the basic ideas of separation of variables.)

To solve

$$u_t - u_{xx} = 0, \tag{1.5.10a}$$

$$u(x, 0^+) = f(x), \tag{1.5.10b}$$

$$u(0, t) = u(1, t) = 0, \quad t > 0, \tag{1.5.10c}$$

we assume that u can be expressed in the "separated" form:

$$u(x, t) = X(x)T(t). \tag{1.5.11}$$

Substituting (1.5.11) into (1.5.10a) gives

$$X\dot{T} - X''T = 0, \quad \text{or } \frac{\dot{T}}{T} = \frac{X''}{X}, \tag{1.5.12}$$

where the dot indicates d/dt and the double prime indicates d^2/dx^2. The second part of (1.5.12) can hold only if it equals a constant, and we quickly convince ourselves that this constant must be negative, say $-\lambda^2$. (Why?)

So, we obtain the *eigenvalue problem*

$$X'' + \lambda^2 X = 0, \quad X(0) = X(1) = 0, \tag{1.5.13}$$

associated with (1.5.10). The solution is the *eigenfunction*

$$X_n = b_n \sin \lambda_n x, \quad \lambda_n = n\pi,$$

where b_n is arbitrary and n is an integer. Thus, the solution of (1.5.10) in a series of eigenfunctions is just the *Fourier sine series*

$$u(x, t) = \sum_{n=1}^{\infty} B_n(t) \sin n\pi x. \tag{1.5.14}$$

Substituting (1.5.14) into (1.5.10a), or using $\dot{T}_n + \lambda_n^2 T_n = 0$, gives $B_n = c_n e^{-n^2\pi^2 t}$, where $c_n =$ constant.

To determine the c_n, we impose the initial condition (1.5.10b) and make use of orthogonality to obtain

$$c_n = 2 \int_0^1 f(\xi) \sin n\pi\xi \, d\xi. \tag{1.5.15}$$

Thus, the solution of (1.5.10) may be written in series form as

$$u(x, t) = \sum_{n=1}^{\infty} \left[2 \int_0^1 f(\xi) \sin n\pi\xi \, d\xi \right] e^{-n^2\pi^2 t} \sin n\pi x.$$

If we interchange summation and integration (a step that is nearly never questioned in a course in applied mathematics!), we obtain

$$u(x, t) = \int_0^1 f(\xi) H(x, \xi, t) d\xi, \tag{1.5.16a}$$

where

$$H(x, \xi, t) \equiv 2 \sum_{n=1}^{\infty} (\sin n\pi\xi) e^{-n^2\pi^2 t} \sin n\pi x. \tag{1.5.16b}$$

Comparing (1.5.16) with (1.5.9) shows that these two results do not look alike. In fact, in order for the two results to agree, we must be able to show that $G_1 = H$. This is indeed the case, and is a consequence of a certain identity for the *theta* *function*. For example, see page 75 of [12]. It is instructive to work out this identity in detail next.

We may use trigonometric identities to rewrite H in the form

$$H(x, \xi, t) = \frac{1}{2}\sum_{n=-\infty}^{\infty} e^{-n^2\pi^2 t}\cos n\pi(x-\xi) - \frac{1}{2}\sum_{n=-\infty}^{\infty} e^{-n^2\pi^2 t}\cos n\pi(x+\xi). \tag{1.5.17}$$

Now, the expression for G_1 in (1.5.2) agrees with (1.5.17) if we can show that

$$\sum_{n=-\infty}^{\infty} F(x+\xi-2n, t) = \frac{1}{2}\sum_{n=-\infty}^{\infty} e^{-n^2\pi^2 t}\cos n\pi(x+\xi) \tag{1.5.18a}$$

and

$$\sum_{n=-\infty}^{\infty} F(x-\xi-2n, t) = \frac{1}{2}\sum_{n=-\infty}^{\infty} e^{-n^2\pi^2 t}\cos n\pi(x-\xi). \tag{1.5.18b}$$

These two conditions are equivalent and reduce to the simple condition

$$\frac{1}{\sqrt{\eta}}\sum_{n=-\infty}^{\infty} e^{-\pi(z-n)^2/\eta} = \sum_{n=-\infty}^{\infty} e^{-n^2\pi\eta}\cos 2n\pi z, \tag{1.5.19}$$

if we write $(x+\xi)$ or $(x-\xi)$ as $2z$, set $\eta = \pi t$, and use the expression (1.2.20) defining F.

Denote $\sqrt{\eta}$ times the expression on the left-hand side of (1.5.19) by ϕ; that is,

$$\phi(\eta, z) \equiv \sum_{n=-\infty}^{\infty} e^{-\pi(z-n)^2/\eta}. \tag{1.5.20}$$

Clearly, ϕ is an even function of z (that is, $\phi(\eta, -z) = \phi(\eta, z)$). Also, it is periodic in z with unit period: $\phi(\eta, z+1) = \phi(\eta, z)$. Therefore, we may expand ϕ in a Fourier cosine series:

$$\phi(\eta, z) = \sum_{\nu=-\infty}^{\infty} \alpha_\nu(\eta)\cos 2\pi\nu z, \tag{1.5.21a}$$

where

$$\alpha_\nu(\eta) = \int_0^1 \sum_{n=-\infty}^{\infty} e^{-\pi(\zeta-n)^2/\eta}\cos 2\pi\nu\zeta\, d\zeta. \tag{1.5.21b}$$

Interchanging integration and summation in (1.5.21b) gives

$$\alpha_\nu(\eta) = \sum_{n=-\infty}^{\infty}\int_0^1 e^{-\pi(\zeta-n)^2/\eta}\cos 2\pi\nu\zeta\, d\zeta. \tag{1.5.22}$$

Now change the integration variable and let $s = n - \zeta$ to obtain

$$\alpha_\nu(\eta) = \sum_{n=-\infty}^{\infty}\int_{n-1}^{n} e^{-\pi s^2/\eta}\cos 2\pi\nu s ds$$

$$= \int_{-\infty}^{\infty} e^{-\pi s^2/\eta} \cos 2\pi \nu s \, ds = \sqrt{\eta} e^{-\pi \nu^2 \eta}. \tag{1.5.23}$$

Thus, we have proven the identity

$$\sum_{n=-\infty}^{\infty} e^{-\pi(z-n)^2/\eta} = \sum_{n=-\infty}^{\infty} \sqrt{\eta} e^{-\pi n^2 \eta} \cos 2\pi n z, \tag{1.5.24}$$

which is (1.5.19) when we divide by $\sqrt{\eta}$.

In conclusion, the series representation for G_1 converges to the same result as the series for H, even though these series *do not agree term by term*. This latter observation means that if we truncate the series for G_1, the resulting approximation will be valid in a different sense than the approximation obtained by truncating the Fourier series H. Let us pursue this idea further, as it will provide a useful characterization of the two approaches we have used.

Consider first what happens if we truncate the series (1.5.2) at $n = N$ for G_1. Clearly, we are neglecting all the heat sources located at distances greater than $2N + \xi$ on the positive axis and greater than $2N - \xi$ on the negative axis. For short times, the response due to these sources is very small over the unit interval (because we are ignoring only the weak exponential tails of the corresponding F functions). Thus, the Green's function representation (1.5.9), when G_1 is truncated for some $n = N$, *should be valid for short times*. In particular, the boundary conditions at $x = 0$ and $x = 1$ are only approximately satisfied with the truncated series, and this approximation deteriorates as t gets large. On the other hand, if we truncate the Fourier series representation (1.5.16), the boundary conditions are *exactly* satisfied for all times, but the initial condition will be described only approximately. Thus, the truncated series (1.5.16) should provide a *good approximation for t large*. A more careful analysis of the convergence properties of the G_1 and H series confirms the above intuitive conclusions.

We reiterate that both expressions converge to the same solution if the infinite series are summed. We shall see in Chapter 3 in examples for the wave equation that this property of Green's functions versus eigenfunction expansions is also true there. It is a useful result, as we are able to have an approximation involving a finite number of terms for both t small and t large.

1.5.3 Connection with Solution by Laplace Transforms

A third approach for solving the problem in (1.5.10) is to use Laplace transforms with respect to t. For simplicity, consider the special case $f = 1$. Using the notation $U(x, s)$ for the Laplace transform of $u(x, t)$ (see Section A.2.6), we obtain

$$U_{xx} - sU = -1, \quad U(0, s) = U(1, s) = 0.$$

The solution is easily obtained in the form

$$U(x, s) = \frac{1}{s(e^{\sqrt{s}} - e^{-\sqrt{s}})} [e^{\sqrt{s}} - e^{-\sqrt{s}} + (e^{-\sqrt{s}} - 1)e^{\sqrt{s}x} - (e^{\sqrt{s}} - 1)e^{-\sqrt{s}x}]. \tag{1.5.25}$$

The solution for $u(x, t)$ is then given by the inversion integral (A.2.41b); that is,

$$u(x,t) = \frac{1}{2\pi i}\int_{0^+-i\infty}^{0^++i\infty} e^{st}U(x,s)ds. \tag{1.5.26}$$

Note the branch points at $s = 0$ and $s = \infty$. Since we must choose the branch of $\sqrt{s}$ that is positive when s is along the positive real axis, it is convenient to cut the s-plane along the negative real axis; hence $c = 0^+$ for the vertical contour.

The expression (1.5.26) cannot be evaluated in terms of a finite number of elementary functions. One standard approximation for a Laplace transform inversion is the "large s" approximation, which consists of expanding (1.5.25) in series form for s large and then integrating the result, term by term, in (1.5.26). As discussed in texts on complex variables (for example, see page 279 of [8]), this gives an approximation for $u(x, t)$ valid for t *small.*

To see this, just change the variables in (1.5.26), setting $s = \sigma/t$ and consider the limit $|s| \to \infty$, $|\sigma|$ *fixed.* Clearly, this implies that we need to take $t \to 0$, and in effect, the substitution σ/t for s in $U(x, s)$ accomplishes this.

If we expand the denominator of (1.5.25) and take the product of this series with the numerator, we find that U equals the particular solution $1/s$ plus four series in the form

$$\begin{aligned}
U(x,s) = &\frac{1}{s} + \frac{1}{s}\left[e^{-\sqrt{s}(2-x)} + e^{-\sqrt{s}(4-x)} + e^{-\sqrt{s}(6-x)} + \dots\right] \\
&- \frac{1}{s}\left[e^{-\sqrt{s}x} + e^{-\sqrt{s}(2+x)} + e^{-\sqrt{s}(4+x)} + \dots\right] \\
&- \frac{1}{s}\left[e^{-\sqrt{s}(1-x)} + e^{-\sqrt{s}(3-x)} + e^{-\sqrt{s}(5-x)} + \dots\right] \\
&+ \frac{1}{s}\left[e^{-\sqrt{s}(1+x)} + e^{-\sqrt{s}(3+x)} + e^{-\sqrt{s}(5+x)} + \dots\right].
\end{aligned}$$

These series can be rearranged in the form

$$U(x,s) = \frac{1}{s} + \frac{1}{s}\sum_{n=1}^{\infty}(-1)^n e^{-\sqrt{s}(n-x)} - \frac{1}{s}\sum_{n=0}^{\infty}(-1)^n e^{-\sqrt{s}(n+x)}. \tag{1.5.27}$$

Using (1.5.26) or tables of Laplace transforms, we find that the transform of

$$f(t) = \operatorname{erfc}\left(\frac{\lambda}{2\sqrt{t}}\right), \tag{1.5.28a}$$

with $t > 0$ and λ real, is

$$F(s) = \frac{1}{s}e^{-\lambda\sqrt{s}}. \tag{1.5.28b}$$

Therefore, the termwise inversion of (1.5.27) gives the series

$$u(x,t) = 1 + \sum_{n=1}^{\infty}(-1)^n \operatorname{erfc}\left(\frac{n-x}{2\sqrt{t}}\right) - \sum_{n=0}^{\infty}(-1)^n \operatorname{erfc}\left(\frac{n+x}{2\sqrt{t}}\right). \tag{1.5.29}$$

It is left as an exercise (Problem 1.5.4) to show that this series is the same as the one resulting from the Green's function representation (1.5.9) when we take $f = 1$ and integrate the series for G_1 term by term. This gives a confirmation of our earlier intuitive argument that the truncated Green's function representation of the solution is valid for t small.

At any rate, the *exact* expressions (1.5.9), (1.5.16a) with $f = 1$, and (1.5.26) define the *same* function $u(x, t)$. The advantage of (1.5.9) and (1.5.16a) over (1.5.26) is that these are in terms of real quadratures, whereas (1.5.26) is a complex integral. Another example of the use of Laplace transforms to calculate the solution of the diffusion equation in a bounded domain is given in Problem 1.5.5.

1.5.4 Uniqueness of Solutions

In this section we show that solutions of the initial- and boundary-value problem for the diffusion equation are unique. We consider solutions of

$$u_t - u_{xx} = 0, \quad 0 \leq x \leq 1, \quad 0 \leq t < \infty, \tag{1.5.30a}$$

with initial condition

$$u(x, 0^+) = f(x), \tag{1.5.30b}$$

and one of the following four boundary conditions:

$$u(0, t) = g(t), \quad u(1, t) = h(t), \tag{1.5.31a}$$

$$u(0, t) = g(t), \quad u_x(1, t) = h(t), \tag{1.5.31b}$$

$$u_x(0, t) = g(t), \quad u(1, t) = h(t), \tag{1.5.31c}$$

$$u_x(0, t) = g(t), \quad u_x(1, t) = h(t). \tag{1.5.31d}$$

Here g and h are arbitrarily prescribed in each case.

In preparation for this proof, we first derive an integral identity for solutions of (1.5.30a). Multiply (1.5.30a) by $u(x, t)$ and integrate the result with respect to x on the unit interval to obtain

$$\int_0^1 uu_t dx = \int_0^1 uu_{xx} dx.$$

Since the interval is independent of t, we may write the left-hand side of this expression as $(d/dt) \int_0^1 (u^2/2) dx$, and integrating the right-hand side by parts, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_0^1 u^2(x, t) dx = uu_x \Big|_0^1 - \int_0^1 u_x^2(x, t) dx. \tag{1.5.32}$$

The identity (1.5.32) is true for any solution of (1.5.30a). Suppose that u_1 and u_2 are two solutions of (1.5.30a), each of which satisfies the initial condition (1.5.30b) and one of the four pairs of boundary conditions (1.5.31). If we denote the difference by $u_1 - u_2 \equiv v(x, t)$, then $v(x, t)$ satisfies the problem

$$\begin{aligned} v_t - v_{xx} &= 0, \\ v(x, 0) &= 0, \\ v v_x &= 0 \quad \text{at } x = 0 \quad \text{and} \quad x = 1. \end{aligned}$$

Therefore, the identity (1.5.32) for v becomes

$$\frac{1}{2}\frac{d}{dt}\int_0^1 v^2(x, t)dx = -\int_0^1 v_x^2(x, t)dx \leq 0.$$

Or if we let

$$I(t) \equiv \frac{1}{2}\int_0^1 v^2(x, t)dx \geq 0$$

and

$$G(t) \equiv -\int_0^1 v_x^2(x, t)dx \leq 0,$$

we have

$$\frac{dI}{dt} = G(t), \quad \text{that is,} \quad I(t) - I(0) = \int_0^t G(\tau)d\tau \leq 0. \tag{1.5.33}$$

Thus, $I(t) - I(0) \leq 0$. But $I(0) = 0$; hence, $I(t) \leq 0$. According to its definition, $I(t) \geq 0$. So, we must have $I(t) \equiv 0$, and the integral of a nonnegative quantity, such as v^2, can vanish only if $v(x, t) = 0$. Thus, we have proven that $u_1(x, t) = u_2(x, t)$.

1.5.5 Inhomogeneous Boundary Conditions

As discussed in Section 1.4, we can transform a homogeneous equation with inhomogeneous boundary conditions to an inhomogeneous equation with homogeneous boundary conditions. To illustrate the idea, consider (1.5.30a) with initial condition (1.5.30b) and boundary conditions (1.5.31a).

To homogenize the boundary conditions, assume a transformation of dependent variable $u \rightarrow w$ in the following form that is linear in x,

$$u(x, t) \equiv w(x, t) + \alpha(t)x + \beta(t), \tag{1.5.34}$$

with as yet unspecified functions α and β of the time, to be chosen such that the boundary conditions for the resulting problem for w are homogeneous.

Using (1.5.34), we compute

$$\begin{aligned} u_t &= w_t + \dot{\alpha}x + \dot{\beta}, \\ u_x &= w_x + \alpha, \quad u_{xx} = w_{xx}. \end{aligned}$$

Therefore,

$$u_t - u_{xx} = w_t + \dot{\alpha}x + \dot{\beta} - w_{xx} = 0,$$

that is, w obeys the inhomogeneous problem

$$w_t - w_{xx} = -\dot{\alpha}x - \dot{\beta}.$$

In order to have $w(0, t) = 0$, we find from (1.5.34) that we must set $\beta(t) = g(t)$. Similarly, in order to have $w(1, t) = 0$, we must set $\alpha(t) = h(t) - g(t)$. Thus, the transformation relation is

$$u(x, t) \equiv w(x, t) + x[h(t) - g(t)] + g(t), \tag{1.5.35}$$

and w obeys the inhomogeneous equation

$$w_t - w_{xx} = [\dot{g}(t) - \dot{h}(t)]x - \dot{g}(t) \equiv p(x, t) \tag{1.5.36a}$$

subject to the initial condition

$$w(x, 0) = f(x) - x[h(0^+) - g(0^+)] - g(0^+) \equiv q(x) \tag{1.5.36b}$$

and homogeneous boundary conditions

$$w(0, t) = w(1, t) = 0. \tag{1.5.36c}$$

The solution of the problem (1.5.36) is just the sum of the solutions (1.5.7) and (1.5.9) with $f = q$; that is,

$$w(x, t) = \int_0^t d\tau \int_0^1 p(\xi, \tau)G_1(x, \xi, t - \tau)d\xi + \int_0^1 q(\xi)G_1(x, \xi, t)d\xi. \tag{1.5.37}$$

Having found $w(x, t)$, we obtain $u(x, t)$ from (1.5.35). Note that the form (1.5.36) is also appropriate for a solution using Fourier series, as homogeneous boundary conditions are also crucial in being able to superpose eigensolutions. Problem 1.5.7 concerns the solution for the case (1.5.31b).

Problems

1.5.1a. Show that Green's function for the following general homogeneous boundary-value problem for the steady-state diffusion equation

$$-u'' = \delta(x - \xi); \quad 0 \le x \le 1, \quad 0 < \xi < 1, \tag{1.5.38}$$

$$u(0) + a_0 u'(0) = 0; \quad a_0 = \text{constant}, \tag{1.5.39a}$$

$$u(1) + a_1 u'(1) = 0; \quad a_1 = \text{constant}, \tag{1.5.39b}$$

is given by

$$G(x, \xi) = \begin{cases} \frac{(1-\xi+a_1)(x-a_0)}{1+a_1-a_0}; & x < \xi, \\ \frac{(1-x+a_1)(\xi-a_0)}{1+a_1-a_0}; & x > \xi. \end{cases} \tag{1.5.40}$$

Give a physical reason why G becomes infinite if $a_0 - a_1 = 1$.

b. Give a physical reason why Green's function for (1.5.38) with the homogeneous boundary conditions $u'(0) = u'(1) = 0$ does not exist.

1.5.2. Use symmetry arguments to show that Green's function for the diffusion equation

$$u_t - u_{xx} = \delta(t-\tau)\delta(x-\xi) \tag{1.5.41}$$

with zero initial condition and each of the following three types of homogeneous boundary conditions is given in the specified form.

a. $u(0,t) = u_x(1,t) = 0$ has

$$G_2(x,\xi,t-\tau) = \sum_{n=-\infty}^{\infty} (-1)^n \{F[x-(2n+\xi), t-\tau] - F[x-(2n-\xi), t-\tau]\}. \tag{1.5.42a}$$

b. $u_x(0,t) = u(1,t) = 0$ has

$$G_3(x,\xi,t-\tau) = \sum_{n=-\infty}^{\infty} (-1)^n \{F[x-(2n+\xi), t-\tau] + F[x-(2n-\xi), t-\tau]\}. \tag{1.5.42b}$$

c. $u_x(0,t) = u_x(1,t) = 0$ has

$$G_4(x,\xi,t-\tau) = \sum_{n=-\infty}^{\infty} \{F[x-(2n+\xi), t-\tau] + F[x-(2n-\xi), t-\tau]\}. \tag{1.5.42c}$$

What symmetry properties, if any, can you uncover for G_2, G_3, and G_4 if $x \to \xi, \xi \to x$?

d. Use the results of parts (a)–(c) to solve the general initial/boundary value problem for

$$u_t - u_{xx} = p(x,t), \tag{1.5.43a}$$

$$u(x,0) = f(x), \tag{1.5.43b}$$

and each of the following pairs of boundary conditions for $t > 0$ after introducing an appropriate homogenizing transformation as in Section 1.5.5

$$u(0,t) = g_1(t); \quad u_x(1,t) = g_2(t), \tag{1.5.44a}$$

$$u_x(0,t) = h_1(t); \quad u(1,t) = h_2(t), \tag{1.5.44b}$$

$$u_x(0,t) = h_1(t); \quad u_x(1,t) = g_2(t). \tag{1.5.44c}$$

1.5.3. Evaluate (1.5.7) for the special case where $p = \delta(x-\zeta)$, where ζ is a fixed constant on $0 < \zeta < 1$. Show that as $t \to \infty$, your result reduces to Green's function for the steady-state problem derived in Appendix A.1 (see (A.1.40)).

1.5.4. Evaluate (1.5.9) for $f = 1$ and show that the resulting series is the same as (1.5.29).

1.5.5a. Show that the Laplace transform $U(x, s)$ of the solution of

$$u_t - u_{xx} = 0; \quad 0 \le x \le 1; \quad 0 \le t, \tag{1.5.45a}$$

$$u(0, t) = 1; \quad u(1, t) = 0, \tag{1.5.45b}$$

$$u(x, 0) = 0, \tag{1.5.45c}$$

is

$$U(x, s) = \frac{1}{s} \frac{\sinh \sqrt{s}(1 - x)}{\sinh \sqrt{s}}. \tag{1.5.46}$$

b. Rewrite (1.5.46) in the form

$$U(x, s) = \frac{1}{s} \cdot \frac{1}{(1 - e^{-2\sqrt{s}})} (e^{-\sqrt{s}x} - e^{-\sqrt{s}(2-x)}), \tag{1.5.47}$$

and expand the factor

$$\frac{1}{1 - e^{-2\sqrt{s}}} = \sum_{n=0}^{\infty} e^{-2n\sqrt{s}} \tag{1.5.48}$$

for large s to obtain the series

$$U(x, s) = \frac{1}{s} \sum_{n=0}^{\infty} (e^{-\sqrt{s}(2n+x)} - e^{-\sqrt{s}(2n+2-x)}). \tag{1.5.49}$$

Now use (1.5.28) to show that the solution $u(x, t)$ has the series form

$$u(x, t) = \sum_{n=0}^{\infty} \left[\operatorname{erfc} \left(\frac{2n + x}{2\sqrt{t}} \right) - \operatorname{erfc} \left(\frac{2n + 2 - x}{2\sqrt{t}} \right) \right]. \tag{1.5.50}$$

c. Calculate the solution of (1.5.45) using Green's function and superposition after homogenizing the boundary condition at $x = 0$. Show that this result agrees with (1.5.50).

1.5.6. This is a review problem to illustrate separation of variables and Fourier series. Consider

$$u_t - u_{xx} = x \sin t, \quad 0 \le x \le 1, \quad 0 \le t, \tag{1.5.51a}$$

$$u(x, 0) = x(1 - x), \tag{1.5.51b}$$

$$u(0, t) = u_x(1, t) = 0 \quad \text{if } t > 0. \tag{1.5.51c}$$

a. Look for a solution of the homogeneous equation (1.5.51a) in the separated form $u(x, t) = X(x)T(t)$, and show that X is given by any one of the eigenfunctions

$$X_n(x) = \alpha_n \sin \lambda_n x, \tag{1.5.52}$$

where the eigenvalues are $\lambda_n = (2n + 1)\pi/2$ for $n = 0, 1, 2, \ldots$ and $\alpha_n = \text{constant}$.

b. Based on this result assume a solution of (1.5.51) in the form of a series of eigenfunctions:

$$u(x, t) = \sum_{n=0}^{\infty} A_n(t) \sin \lambda_n x, \tag{1.5.53}$$

where the $A_n(t)$ are functions of t to be specified. Also, expand the right-hand side of (1.5.51a) in a series of the eigenfunctions X_n,

$$x \sin t = \left(\sum_{n=0}^{\infty} b_n \sin \lambda_n x \right) \sin t. \tag{1.5.54}$$

Use orthogonality to show that $b_n = 8(-1)^n/\pi^2(2n+1)^2$.

Now substitute (1.5.52) into (1.5.51a) with (1.5.54) for its right-hand side to show that the $A_n(t)$ satisfy

$$\frac{dA_n}{dt} + \lambda_n^2 A_n = b_n \sin t. \tag{1.5.55}$$

c. Solve (1.5.55) to obtain

$$A_n(t) = A_n(0)e^{-\lambda_n^2 t} + \frac{b_n}{\lambda_n^2 + 1}(\lambda_n^2 \sin t - \cos t + e^{-\lambda_n^2 t}). \tag{1.5.56}$$

d. Use (1.5.53) with $A_n(t)$ given by (1.5.56) in the initial condition (1.5.51b) to obtain

$$A_n(0) = \frac{32 - 8\pi(-1)^n(2n+1)}{\pi^3(2n+1)^3}. \tag{1.5.57}$$

1.5.7. Solve

$$u_t - u_{xx} = p(x, t), \quad 0 \le x \le 1, \quad 0 \le t, \tag{1.5.58a}$$

$$u(x, 0^+) = f(x), \tag{1.5.58b}$$

$$u(0, t) = g(t), \quad u_x(1, t) = h(t), \tag{1.5.58c}$$

using Green's function as well as separation of variables after having transformed to a homogeneous boundary-value problem.

1.6 Higher-Dimensional Problems

The diffusion equation in two or more space dimensions is given by the following dimensionless form of (1.1.18):

$$u_t - \Delta u = p, \tag{1.6.1}$$

where Δ is the Laplace operator and p is a prescribed function of the spatial variables and the time. For certain domains where one or more of the coordinates are bounded, solutions may be calculated using separation of variables. This technique may also be combined with Fourier transforms with respect to one or more coordinates that have an unbounded range. An example is outlined in Problem 1.6.4b.

For a complete discussion of separation of variables see [22]. Another approach for solving (1.6.1) is to take its Laplace transform with respect to t. The result is a Helmholtz equation for the transformed variable, and this equation is discussed in Chapter 2. See Section 2.3.2 and Problems 2.3.4, 2.6.1, and 2.6.2. Here we will only consider solutions using Green's functions, and we begin our discussion with a derivation of the fundamental solution.

1.6.1 The Fundamental Solution

Consider the n-dimensional diffusion equation with a unit source turned on at the origin at time $t = 0$:

$$u_t - \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} = \delta(t)\delta(x_1)\delta(x_2)\ldots\delta(x_n), \tag{1.6.2}$$

where n is a positive integer. As in (1.2.6)–(1.2.7), we have the zero initial condition

$$u(x_1, \ldots, x_n, 0^-) = 0, \tag{1.6.3}$$

and require u to vanish at infinity:

$$u(x_1, \ldots, u_n, t) \to 0 \quad \text{as} \quad r = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} \to \infty. \tag{1.6.4}$$

In view of the fact that the source term on the right-hand side of (1.6.2) produces a spherically symmetric solution, we need only consider the spherically symmetric Laplacian, and (1.6.2) has the form

$$u_t - u_{rr} - \frac{(n-1)}{r} u_r = \delta(t)\delta_n(r). \tag{1.6.5}$$

We have used the notation $\delta_n(r)$ to denote the n-dimensional delta function

$$\delta_n(r) = \delta(x_1)\delta(x_2)\ldots\delta(x_n). \tag{1.6.6}$$

Consider the volume integral in terms of the Cartesian coordinates $x_1, \ldots, x_n$ of the n-dimensional delta function over some domain D in this n-dimensional space. By simply applying the properties of the one-dimensional delta function to each of the n integrals defining the volume integral, we have the following generalization of the definition for the one-dimensional case

$$\int \cdots \int_D \delta(x_1)\delta(x_2)\ldots\delta(x_n)dx_1dx_2\ldots dx_n = \begin{cases} 1 & \text{if the origin is in } D, \\ 0 & \text{otherwise.} \end{cases} \tag{1.6.7}$$

For $n \geq 2$, we may also write (1.6.7) in terms of the n-dimensional delta function $\delta_n(r)$ and the appropriate volume element dV

$$\int \cdots \int_D \delta_n(r)dV = \begin{cases} 1 & \text{if } r = 0 \text{ is in } D, \\ 0 & \text{otherwise.} \end{cases} \tag{1.6.8}$$

For example, if $n = 2$ and D is the interior of a circle of radius ϵ centered at the origin, then $dV = r\,dr\,d\theta$, and we have

$$\int_{r=0}^{\epsilon}\int_{\theta=0}^{2\pi}\delta_2(r)r\,d\theta\,dr = 2\pi\int_0^{\epsilon} r\delta_2(r)dr = 1, \qquad (1.6.9a)$$

where r and θ are polar coordinates in the plane: $x = r\cos\theta$, $y = r\sin\theta$. If $n = 3$ and D is the interior of a sphere of radius ϵ centered at the origin, the corresponding result is

$$\int_{r=0}^{\epsilon}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\delta_3(r)r^2\sin\phi\,d\phi\,d\theta\,dr = 4\pi\int_0^{\epsilon} r^2\delta_3(r)dr = 1, \qquad (1.6.9b)$$

where r, θ, and ϕ are the spherical polar coordinates: $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, $z = r\sin\phi$. More generally, for an n-dimensional sphere of radius ϵ centered at the origin, (1.6.8) reduces to

$$\omega_n\int_{r=0}^{\epsilon} r^{n-1}\delta_n(r)dr = 1, \qquad (1.6.9c)$$

where ω_n is the "area" of the n-dimensional *unit* sphere.

To calculate ω_n consider the following identity:

$$\int_{-\infty}^{\infty}\cdots\int e^{-x_1^2-x_2^2-\dots x_n^2}dx_1dx_2\dots dx_n = \int_0^{\infty} e^{-r^2}r^{n-1}\omega_n dr, \qquad (1.6.10)$$

where $r^2 = x_1^2+\dots+x_n^2$. The left-hand side of (1.6.10) is just $\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^n = \pi^{n/2}$. The right-hand side is

$$\omega_n\int_0^{\infty} e^{-r^2}r^{n-1}dr = \frac{\omega_n}{2}\int_0^{\infty} e^{-\sigma}\sigma^{\frac{n}{2}-1}d\sigma = \frac{\omega_n}{2}\Gamma\left(\frac{n}{2}\right), \qquad (1.6.11)$$

where $\Gamma(z)$ is the *gamma function* defined by

$$\Gamma(z) \equiv \int_0^{\infty} e^{-\sigma}\sigma^{z-1}d\sigma;\ z > 0. \qquad (1.6.12)$$

Therefore, the area ω_n of the n-dimensional unit sphere is

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \qquad (1.6.13)$$

Coming back to (1.6.5), we solve the homogeneous problem using similarity. Proceeding as in Section 1.2.1, we find that the fundamental solution $F(r, t)$ has the following similarity structure (cf. (1.2.15))

$$\alpha^n F(\alpha r, \alpha^2 t) = F(r, t) \qquad (1.6.14)$$

for any positive constant α. Setting

$$F(r, t) = t^{-n/2}f(\theta),\ \theta = rt^{-1/2}, \qquad (1.6.15a)$$

we see that (1.6.15a) satisfies (1.6.14) and gives the following ordinary differential equation for f:

$$f'' + \left(\frac{\theta}{2} + \frac{n-1}{\theta}\right) f' + \frac{n}{2} f = 0.$$

It is easily seen that

$$f = Ce^{-\theta^2/4}, \quad C = \text{constant},$$

is a solution that upon substitution into (1.6.15) gives

$$F(r,t) = \frac{C}{t^{n/2}} e^{-r^2/4t}. \tag{1.6.15b}$$

This solution has the appropriate singularity at $r = 0, t = 0$ and decays as $r \to \infty$, $t > 0$ or $t \to \infty$, $r > 0$. The other solution of (1.6.15) gives an unbounded contribution to the total heat content in the domain as in the one-dimensional case.

To evaluate C, we integrate (1.6.5) over the entire n-dimensional space D_∞ to obtain

$$\int_{D_\infty}\!\!\cdots\!\int F_t dV = \int_{D_\infty}\!\!\cdots\!\int \Delta F dV + \int_{D_\infty}\!\!\cdots\!\int \delta(t)\delta_n(r) dV. \tag{1.6.16}$$

Using Cartesian coordinates we have

$$\int_{D_\infty}\!\!\cdots\!\int \Delta F dV \equiv \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\frac{\partial^2 F}{\partial x_1^2} + \ldots + \frac{\partial^2 F}{\partial x_n^2}\right] dx_1 \ldots dx_n = 0, \tag{1.6.17}$$

because for each $i = 1, \ldots, n$, $\partial F/\partial x_i$ vanishes if any one of its arguments x_j equals $\pm\infty$. The second integral on the right-hand side of (1.6.16) gives $\delta(t)$, and we obtain

$$\int_{D_\infty}\!\!\cdots\!\int F_t dV = \frac{d}{dt}\int_{D_\infty}\!\!\cdots\!\int F dV = \delta(t).$$

Therefore,

$$\int_{D_\infty}\!\!\cdots\!\int F dV = 1 \quad \text{if} \quad t > 0, \tag{1.6.18}$$

as in the one-dimensional case (cf. (1.2.19)). Using the result (1.6.15b) for F in (1.6.18) gives

$$\frac{C}{t^{n/2}} \int_0^\infty e^{-r^2/4t} \omega_n r^{n-1} dr = 1. \tag{1.6.19a}$$

Changing the integration variable from r to $s = r^2/4t$ gives

$$2^{n-1}\omega_n C \int_0^\infty e^{-s} s^{\frac{n}{2}-1} ds = 2^{n-1}\omega_n C\Gamma\left(\frac{n}{2}\right) = 1. \tag{1.6.19b}$$

Therefore,

$$C = \frac{1}{2^{n-1}\omega_n \Gamma(n/2)} = \frac{1}{2^n \pi^{n/2}}, \tag{1.6.20}$$

and the fundamental solution is

$$F(r, t) = \frac{1}{2^n \pi^{n/2} t^{n/2}} e^{-r^2/4t}. \tag{1.6.21}$$

More generally, the fundamental solution at time t at a point P with coordinates $x_1, \ldots, x_n$ due to a source located at the point Q with coordinates $\xi_1, \ldots, \xi_n$ and turned on at time τ is

$$F(r_{PQ}, t - \tau) = \frac{e^{-r_{PQ}^2/4(t-\tau)}}{2^n \pi^{n/2}(t - \tau)^{n/2}}, \tag{1.6.22}$$

where we have introduced the notation

$$r_{PQ} = \sqrt{\sum_{i=1}^{n}(x_i - \xi_i)^2} \tag{1.6.23}$$

for the distance between the observer at P and the source point Q.

1.6.2 Initial-Value Problem in the Infinite Domain

Consider the general initial-value problem for the three-dimensional diffusion equation in the infinite domain:

$$u_t - u_{xx} - u_{yy} - u_{zz} = p(x, y, z, t), \tag{1.6.24a}$$

$$u(x, y, z, 0) = f(x, y, z). \tag{1.6.24b}$$

Here p and f are prescribed, and $p = 0$ if $t < 0$. The corresponding one-dimensional problem was discussed in Section 1.3. The basic ideas are the same; we split (1.6.24) into two problems as in (1.3.2) and solve each using the fundamental solution. The result is

$$u(x, y, z, t) = \int_{\tau=0}^{\infty}\int_{\xi=-\infty}^{\infty}\int_{\eta=-\infty}^{\infty}\int_{\zeta=-\infty}^{\infty} F(r_{PQ}, t - \tau)p(\xi, \eta, \zeta, \tau)d\zeta\, d\eta\, d\xi\, d\tau$$

$$+ \int_{\xi=-\infty}^{\infty}\int_{\eta=-\infty}^{\infty}\int_{\zeta=-\infty}^{\infty} f(\xi, \eta, \zeta)F(r_{PQ}, t)d\zeta\, d\eta\, d\xi, \tag{1.6.25}$$

where $r_{PQ}^2 = (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2$, and F is given by (1.6.22) with $n = 3$. Similar formulas can be written down for the solution for any n.

(i) *Example, axisymmetric problem in two dimensions*

Consider the axisymmetric problem in two dimensions

$$u_t - \left(u_{rr} + \frac{1}{r}u_r\right) = 0, \tag{1.6.26a}$$

$$u(r, 0) = f(r). \tag{1.6.26b}$$

The fundamental solution is ($n = 2$, $r_{PQ}^2 = (x-\xi)^2 + (y-\eta)^2$)

$$F(r_{PQ}, t-\tau) = \frac{1}{4\pi(t-\tau)} \exp\left(-\frac{(x-\xi)^2 + (y-\eta)^2}{4(t-\tau)}\right). \tag{1.6.27}$$

We introduce the polar coordinates r, θ for P defined by $x = r\cos\theta$, $y = r\sin\theta$ and set $\xi = \rho\cos\phi$, $\eta = \rho\sin\phi$. Then $r_{PQ}^2 = r^2 - \rho^2 - 2r\rho\cos(\theta - \phi)$, and (1.6.27) becomes

$$F(r_{PQ}, t-\tau) = \frac{1}{4\pi(t-\tau)} \exp\left(-\frac{r^2 + \rho^2 - 2r\rho\cos(\theta-\phi)}{4(t-\tau)}\right). \tag{1.6.28}$$

In the superposition integral corresponding to (1.6.25) only the second term contributes, and we use polar coordinates to obtain

$$u(r,t) = \frac{e^{-r^2/4t}}{4\pi t} \int_0^\infty e^{-\rho^2/4t} \rho f(\rho) \left[\int_0^{2\pi} e^{(r\rho\cos\phi)/2t} d\phi\right] d\rho. \tag{1.6.29}$$

The definite integral with respect to ϕ can be evaluated explicitly. For any positive constant α, we have

$$\int_0^{2\pi} e^{\alpha\cos\phi} d\phi = 2\pi I_0(\alpha), \tag{1.6.30}$$

where I_0 is the modified Bessel function of order zero. Therefore, (1.6.29) simplifies to

$$u(r,t) = \frac{e^{-r^2/4t}}{2t} \int_0^\infty f(\rho)\rho e^{-\rho^2/4t} I_0\left(\frac{r\rho}{2t}\right) d\rho. \tag{1.6.31}$$

1.6.3 Green's Function for Various Simple Domains

The use of image sources to satisfy boundary conditions also generalizes to higher-dimensional problems for certain simple geometries. Three planar examples are discussed next to illustrate ideas.

(i) *The half-plane $y \geq 0$ with $u(x, 0, t) = 0$*

Consider

$$u_t - (u_{xx} + u_{yy}) = \delta(x-\xi)\delta(y-\eta)\delta(t-\tau) \tag{1.6.32a}$$

in the upper half-plane: $-\infty < x < \infty$, $0 \leq y \leq \infty$ for positive constants ξ, η, τ. The initial condition is

$$u(x, y, \tau^-) = 0, \tag{1.6.32b}$$

and boundary conditions are

$$u(x, 0, t) = u(x, \infty, t) = 0. \tag{1.6.32c}$$

Using a negative image source at $x = \xi$, $y = -\eta$, $t = \tau$ we obtain Green's function using (1.6.22) with $n = 2$

$$G(x - \xi, y, \eta, t - \tau) = \frac{1}{4\pi(t - \tau)} \left[e^{-r_{PQ}^2/4(t-\tau)} - e^{-r_{P\overline{Q}}^2/4(t-\tau)} \right], \quad (1.6.33)$$

where

$$r_{PQ}^2 = (x - \xi)^2 + (y - \eta)^2 \quad (1.6.34a)$$

$$r_{P\overline{Q}} = (x - \xi)^2 + (y + \eta)^2. \quad (1.6.34b)$$

Knowing G, we can solve the general initial- and boundary-value problem in the upper half-plane,

$$u_t - u_x - u_{yy} = p(x, y, t), \quad (1.6.35a)$$

$$u(x, y, 0) = f(x, y), \quad (1.6.35b)$$

$$u(x, 0, t) = g(x, t), \ t > 0, \quad (1.6.35c)$$

for prescribed functions p, f, and g using Green's function and superposition after the boundary condition (1.6.35c) is homogenized. The details are entirely analogous to the one-dimensional case discussed in Section 1.4 and are left as an exercise (Problem 1.6.1).

The same ideas can be used to compute Green's function in the half-space in three dimensions and to construct Green's function of the second kind where the normal derivative of u vanishes along the boundary.

(ii) *The quarter-plane* $x \geq 0$, $y \geq 0$ *with* $u(x, 0, t) = u(0, y, t) = 0$

Green's function satisfies

$$u_t - (u_{xx} + u_{yy}) = \delta(x - \xi)\delta(y - \eta)\delta(t - \tau), \quad (1.6.36a)$$

$$u(x, y, \tau^-) = 0, \quad (1.6.36b)$$

$$u(x, 0, t) = 0, \ x > 0, \quad (1.6.36c)$$

$$u(0, y, t) = 0, \ y > 0. \quad (1.6.36d)$$

For this domain, the positive primary source of unit strength is located at $x = \xi > 0$, $y = \eta > 0$ and turned on at time $t = \tau$. In order to have $u = 0$ on both the positive x- and y-axes we need to introudce negative image sources of unit strength at the points $x = \xi$, $y = -\eta$ and $x = -\xi$, $y = \eta$. We also need a positive image source of unit strength at $x = -\xi$, $y = -\eta$. This maintains the symmetry relative to the two coordinate axes.

Therefore, the solution is given by

$$G(x, \xi, y, \eta, t - \tau) = \frac{1}{4\pi(t - \tau)} \left[e^{-r_1^2/4(t-\tau)} - e^{-r_2^2/4(t-\tau)} - e^{-r_3^2/4(t-\tau)} + e^{-r_4^2/4(t-\tau)} \right], \quad (1.6.37)$$

where

$$r_1^2 = r_{PQ} = (x-\xi)^2 + (y-\eta)^2, \tag{1.6.38a}$$

$$r_2^2 = (x-\xi)^2 + (y+\eta)^2, \tag{1.6.38b}$$

$$r_3^2 = (x+\xi)^2 + (y-\eta)^2, \tag{1.6.38c}$$

$$r_4^2 = (x+\xi)^2 + (y+\eta)^2. \tag{1.6.38d}$$

Using (1.6.37) we can solve the general initial- and boundary-value problem in the quarter-plane. See Problem 1.6.2. Problem 1.6.3 concerns the solution in the quarter plane with $u_y(x,0,t)$ prescribed.

The symmetry idea also generalizes to corner domains in higher dimensions, e.g., $x \geq 0$, $y \geq 0$, $z \geq 0$ in three dimensions.

(iii) *The infinite strip* $0 \leq y \leq 1$, $-\infty < x < \infty$ *with* $u(x,0,t) = u(x,1,t) = 0$

Green's function of the first kind for this domain satisfies

$$u_t - u_{xx} - u_{yy} = \delta(x-\xi)\delta(y-\eta)\delta(t-\tau), \tag{1.6.39a}$$

$$u(x,y,\tau^-) = 0, \tag{1.6.39b}$$

$$u(x,0,t) = u(x,1,t) = 0, \ t > 0. \tag{1.6.39c}$$

The solution of (1.6.39) is entirely analogous to the one- dimensional version (1.5.2), and we have

$$G(x-\xi, y, \eta, t-\tau) = \sum_{n=-\infty}^{\infty} \{F(r_n, t-\tau) - F(\bar{r}_n, t-\tau)\}, \tag{1.6.40}$$

where

$$F(r,t) = \frac{e^{-r^2/4t}}{2\pi t}, \tag{1.6.41}$$

$$r_n^2 = (x-\xi)^2 + [y-(2n+\eta)]^2, \tag{1.6.42a}$$

$$\bar{r}_n^2 = (x-\xi)^2 + [y-(2n-\eta)]^2. \tag{1.6.42b}$$

We can now use (1.6.40) to solve the general initial- and boundary -value problem in the infinite strip. See Problem 1.6.4a. This problem can also be solved by Fourier transforms with respect to x followed by separation of variables as discussed in Problem 1.6.4b.

Boundary-value problems where u_y is specified on $y = 0$ or $y = 1$ or both can also be solved using the appropriate Green's function as in Problem 1.5.2.

Problems

1.6.1. Use the homogenizing transformation $w(x,y,t) = u(x,y,t) - g(x,t)$ to show that if u solves (1.6.35), then w is the solution of

$$w_t - w_{xx} - w_{yy} = h(x,t) + \delta(t)k(x,y), \tag{1.6.43a}$$

$$w(x, y, 0^-) = 0, \tag{1.6.43b}$$

$$w(x, 0, t) = 0, \quad t > 0, \tag{1.6.43c}$$

where

$$h(x, t) \equiv p(x, y, t) - g_t(x, t) + g_{xx}(x, t), \tag{1.6.44a}$$

$$k(x, y) \equiv f(x, y) - g(x, 0). \tag{1.6.44b}$$

Using Green's function (1.6.33), the solution of (1.6.35) then becomes

$$\begin{aligned} u(x, y, t) = &\int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} h(\xi, \tau) G(x - \xi, y, \eta, t - \tau) d\eta\, d\xi\, d\tau \\ &+ \int_{-\infty}^{\infty} \int_0^{\infty} k(\xi, \eta) G(x - \xi, y, \eta, t) d\eta\, d\xi \\ &+ g(x, t). \end{aligned} \tag{1.6.45}$$

Develop the result in (1.6.45) using (1.4.18) to obtain (see (1.4.21))

$$\begin{aligned} u(x, y, t) = &\frac{1}{2\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t - \tau}} \operatorname{erf}\left(\frac{y}{2\sqrt{t - \tau}}\right) \\ &\times \left[\int_{-\infty}^{\infty} h(\xi, \tau) e^{-(x-\xi)^2/4(t-\tau)} d\xi\right] d\tau \\ &- \frac{1}{2\sqrt{\pi t}} \operatorname{erf}\left(\frac{y}{2\sqrt{t}}\right) \int_{-\infty}^{\infty} g(\xi, 0) e^{-(x-\xi)^2/4t} d\xi \\ &+ \frac{1}{4\pi t} \int_{-\infty}^{\infty} \int_0^{\infty} f(\xi, \eta) \left[e^{-[(x-\xi)^2+(y-\eta)^2]/4t}\right. \\ &\left.- e^{-[(x-\xi)^2+(y+\eta)^2]/4t}\right] d\eta\, d\xi. \end{aligned} \tag{1.6.46}$$

1.6.2. Consider the general initial- and boundary-value problem in the quarter plane $x \geq 0, y \geq 0$:

$$u_t - u_{xx} - u_{yy} = p(x, y, t), \tag{1.6.47a}$$

$$u(x, y, 0) = f(x, y), \tag{1.6.47b}$$

$$u(x, 0, t) = g_1(x, t), \tag{1.6.47c}$$

$$u(0, y, t) = g_2(y, t). \tag{1.6.47d}$$

Introduce the homogenizing transformation

$$w(x, y, t) = u(x, y, t) - \alpha(x, y, t), \tag{1.6.48}$$

where α is a function that satisfies $\alpha(x, 0, t) = g_1(x, t)$, $\alpha(0, y, t) = g_2(y, t)$. For example, we may choose

$$\alpha(x, y, t) = \frac{g_1(x, t)x}{\sqrt{x^2 + y^2}} + \frac{g_2(y, t)y}{\sqrt{x^2 + y^2}}. \tag{1.6.49}$$

Show that if u satisfies (1.6.47), then w is governed by

$$w_t - w_{xx} - w_{yy} = h(x, y, t) + \delta(t)k(x, y), \quad (1.6.50a)$$

$$w(x, y, 0^-) = 0, \quad (1.6.50b)$$

$$w(x, 0, t) = w(0, y, t) = 0, \ \ t > 0, \quad (1.6.50c)$$

where

$$h(x, y, t) \equiv p(x, y, t) - \alpha_t(x, y, t) + \alpha_{xx}(x, y, t) + \alpha_{yy}(x, y, t), \quad (1.6.51a)$$

$$k(x, y) \equiv f(x, y) - \alpha(x, y, 0). \quad (1.6.51b)$$

Solve (1.6.50) using Green's function (1.6.37).

1.6.3. What is Green's function for the corner domain $0 \leq x < \infty, 0 \leq y < \infty$ with boundary conditions $u(0, y, t) = 0$, $u_y(x, 0, t) = 0$? Use this result to calculate the solution of (1.6.47), where we replace (1.6.47c) by $u_y(x, 0, t) = g_3(x, t)$.

1.6.4a. Consider the diffusion equation in two dimensions in the infinite strip $-\infty < x < \infty, 0 \leq y \leq 1$ with prescribed source distribution, and initial and boundary values for u given by

$$u_t - u_{xx} - u_{yy} = p(x, y, t), \quad (1.6.52a)$$

$$u(x, y, 0) = f(x, y), \quad (1.6.52b)$$

$$u(x, 0, t) = g_1(x, t), \ \ t > 0, \quad (1.6.52c)$$

$$u(x, 1, t) = g_2(x, t), \ \ t > 0. \quad (1.6.52d)$$

Introduce the homogenizing transformation

$$w(x, y, t) = u(x, y, t) + (y - 1)g_1(x, t) - yg_2(x, t) \quad (1.6.53)$$

to show that w satisfies

$$w_t - w_{xx} - w_{yy} = h(x, y, t) + \delta(t)k(x, y), \quad (1.6.54a)$$

$$w(x, y, 0^-) = 0, \quad (1.6.54b)$$

$$w(x, 0, t) = w(x, 1, t) = 0, \ \ t > 0, \quad (1.6.54c)$$

where

$$h(x, y, t) \equiv p(x, y, t) + (1 - y)(g_{1_t} - g_{1_{xx}}) - y(g_{2_t} - g_{2_{xx}}), \quad (1.6.55)$$

$$k(x, y) \equiv f(x, y) + (y - 1)g_1(x, 0) - yg_2(x, 0). \quad (1.6.56)$$

Calculate the solution of (1.6.54) using Green's function (1.6.40).

b. An alternative approach for solving (1.6.52) is to take Fourier transforms with respect to x. Show that the transformed variable $\overline{u}(k, y, t)$ satisfies (overbars indicate the Fourier transform, see (A.2.9a))

$$\overline{u}_t - \overline{u}_{yy} + k^2\overline{u} = \overline{p}(k, y, t), \tag{1.6.57a}$$

$$\overline{u}(k, y, 0) = \overline{f}(k, y), \tag{1.6.57b}$$

$$\overline{u}(k, 0, t) = \overline{g}_1(k, t), \ t > 0, \tag{1.6.57c}$$

$$\overline{u}(k, 1, t) = \overline{g}_2(k, t), \ t > 0. \tag{1.6.57d}$$

In preparation for solving (1.6.57) by separation of variables, introduce the homogenizing transformation (1.6.53),

$$\overline{w}(k, y, t) = \overline{u}(k, y, t) + (y - 1)\overline{g}_1(k, t) - y\overline{g}_2(k, t), \tag{1.6.58}$$

and show that $\overline{w}$ satisfies

$$\overline{w}_t - \overline{w}_{yy} + k^2\overline{w} = \overline{p} + (y-1)(\overline{g}_{1_t} + k^2\overline{g}_1) - y(\overline{g}_{2_t} + k^2\overline{g}_2) \equiv q(k, y, t), \tag{1.6.59a}$$

$$\overline{w}(k, y, 0) = \overline{f}(k, y) + (y - 1)\overline{g}_1(k, 0) - y\overline{g}_2(k, 0) \equiv r(k, y), \tag{1.6.59b}$$

$$\overline{w}(k, 0, t) = \overline{w}(k, 1, t) = 0, \ t > 0. \tag{1.6.59c}$$

Solve (1.6.59) by separation of variables in the form

$$\overline{w}(k, y, t) = \sum_{n=1}^{\infty} B_n(k, t) \sin n\pi y, \tag{1.6.60}$$

where

$$B_n(k, t) = \left[B_n(k, 0) + \int_0^t q_n(k, \tau) e^{(n^2\pi^2+k^2)\tau} d\tau \right] e^{-(n^2\pi^2+k^2)t}, \tag{1.6.61}$$

and $B_n(k, 0)$, $q_n(k, t)$ are the Fourier coefficients

$$B_n(k, 0) = 2 \int_0^1 r(k, y) \sin n\pi y \, dy, \tag{1.6.62a}$$

$$q_n(k, t) = 2 \int_0^1 q(k, y, t) \sin n\pi y \, dy. \tag{1.6.62b}$$

1.7 Burgers' Equation

The quasilinear diffusion equation

$$u_t + uu_x - \epsilon u_{xx} = 0, \ \epsilon > 0, \tag{1.7.1}$$

is attributed to Burgers, who in 1948 proposed it as a mathematical model for turbulence [7]. Actually, (1.7.1) was first studied by Bateman in 1915 in modeling the motion of a fluid with small viscosity ϵ [5]. Although (1.7.1) may be obtained as a limiting form of the x-component of the momentum equation for viscous flows, as first shown in [32], it does not model turbulence. Nevertheless, (1.7.1) is a fundamental *evolution equation* that arises in a number of unrelated applications where viscous and nonlinear effects are equally important. Examples are discussed in [16] and in Section 6.2.5 of [26]. This equation also models traffic flow and is derived in Section 5.1.2.

Hopf [24], and Cole [9] independently showed that (1.7.1) may be transformed to the linear diffusion equation of this chapter. We now work out this transformation and discuss how it may be used to solve initial- and boundary-value problems for (1.7.1).

1.7.1 The Cole–Hopf Transformation

This transformation of dependent variable $u \to v$ is defined by

$$u \equiv -2\epsilon \frac{v_x}{v}. \tag{1.7.2}$$

We then calculate

$$\begin{aligned} u_t &= -2\epsilon \frac{v_{xt}}{v} + 2\epsilon \frac{v_x v_t}{v^2}, \\ u_x &= -2\epsilon \frac{v_{xx}}{v} + 2\epsilon \frac{v_x^2}{v^2}, \\ u_{xx} &= -2\epsilon \frac{v_{xxx}}{v} + 6\epsilon \frac{v_x v_{xx}}{v^2} - \frac{4\epsilon v_x^3}{v^3}. \end{aligned}$$

Substituting these expressions into (1.7.1) gives

$$\frac{v_x}{v}(\epsilon v_{xx} - v_t) - (\epsilon v_{xx} - v_t)_x = 0. \tag{1.7.3}$$

Thus, *any* solution $v(x, t)$ of (1.7.3), when used in (1.7.2), gives an expression $u(x, t)$, that satisfies (1.7.1).

In particular, if v satisfies the diffusion equation

$$\epsilon v_{xx} - v_t = 0, \tag{1.7.4}$$

it also solves (1.7.3) trivially, and the resulting $u(x, t)$ satisfies (1.7.1).

Although the parameter ϵ may be transformed out of (1.7.1) (and hence also out of (1.7.4)) by an appropriate scaling of the x and t variables, it is more instructive to retain it in the solution because we can then study how the results behave in the limit $\epsilon \to 0$. This is a singular perturbation problem that we will discuss in Section 8.2.3.

1.7.2 Initial-Value Problem on $-\infty < x < \infty$

Let us study how we can use the preceding result to solve the initial-value problem for Burgers' equation:

$$u_t + uu_x - \epsilon u_{xx} = 0, \quad -\infty < x < \infty, \tag{1.7.5a}$$

$$u(x, 0) = f(x). \tag{1.7.5b}$$

According to (1.7.2), the new variable $v(x, t)$ must initially satisfy

$$f(x) = -\frac{2\epsilon v_x(x,0)}{v(x,0)}. \tag{1.7.6}$$

This is a linear first-order ordinary differential equation for $v(x, 0)$ and has the general solution

$$v(x,0) = \alpha e^{(-1/2\epsilon)\int_0^x f(s)ds} \equiv \alpha g(x), \ \ \alpha = \text{constant}. \tag{1.7.7}$$

Thus, for a given $f(x)$, we compute $g(x)$ by quadrature. Of course, it is understood that the integral $\int_0^x f(s)ds$ exists. So, we need to solve the following *linear problem* for $v(x, t)$:

$$v_t - \epsilon v_{xx} = 0, \ \ -\infty < x < \infty, \tag{1.7.8a}$$

$$v(x,0) = \alpha g(x). \tag{1.7.8b}$$

This is essentially (1.3.3) and has the solution (1.3.9) after replacing $u \to v$, $f \to \alpha g, t \to \epsilon t$:

$$v(x,t) = \frac{\alpha}{\sqrt{4\pi\epsilon t}} \int_{-\infty}^{\infty} g(\xi) e^{-(x-\xi)^2/4\epsilon t} d\xi. \tag{1.7.9}$$

It then follows that

$$v_x(x,t) = \frac{-\alpha}{\sqrt{4\pi\epsilon t}} \int_{-\infty}^{\infty} \frac{g(\xi)(x-\xi)}{2\epsilon t} e^{-(x-\xi)^2/4\epsilon t} d\xi. \tag{1.7.10}$$

Therefore, using (1.7.2) to compute $u(x, t)$ gives

$$u(x,t) = \frac{\int_{-\infty}^{\infty} g(\xi) \frac{(x-\xi)}{t} e^{-(x-\xi)^2/4\epsilon t} d\xi}{\int_{-\infty}^{\infty} g(\xi) e^{-(x-\xi)^2/4\epsilon t} d\xi}, \tag{1.7.11}$$

in which the constant α cancels out.

We shall use these results in discussing discontinuous solutions of the first-order equation

$$u_t + uu_x = 0 \tag{1.7.12}$$

in Chapter 5. We compute (1.7.11) explicitly for the case where $f(x)$ is piecewise constant in Problem 1.7.1.

1.7.3 Boundary-Value Problems

The solution of Burgers' equation on the semi-infinite or bounded interval in x is more complicated than the solution we have derived in (1.7.11). We now consider some special cases.

(i) *Semi-infinite interval:* $0 \le x < \infty$

The problem is

$$u_t + uu_x - \epsilon u_{xx} = 0, \ \ 0 \le x < \infty, \tag{1.7.13a}$$

$$u(x,0) = f(x), \tag{1.7.13b}$$

$$u(0,t) = h(t),\ t > 0. \tag{1.7.13c}$$

Using (1.7.2) we obtain the following problem for the new dependent variable $v(x,t)$

$$v_t - \epsilon v_{xx} = 0, \tag{1.7.14a}$$

$$v(x,0) = \alpha \exp\left(-\frac{1}{2\epsilon}\int_0^x f(s)ds\right) \equiv \alpha g(x), \tag{1.7.14b}$$

$$h(t)v(0,t) + 2\epsilon v_x(0,t) = 0. \tag{1.7.14c}$$

If h = constant and $f = 0$, (1.7.14b) and (1.7.14c) reduce to

$$v(x,0) = \alpha = \text{constant}, \tag{1.7.15a}$$

$$hv(x,0) + 2\epsilon v_x(x,0) = 0,\ h = \text{constant}. \tag{1.7.15b}$$

In (1.7.14b) and (1.7.15a), the constant α is arbitrary.

To use previously calculated results, we set

$$\overline{v} = v - \alpha,\ \overline{t} = \epsilon t,\ \overline{x} = x$$

to obtain

$$\overline{v}_{\overline{t}} - \overline{v}_{\overline{x}\overline{x}} = 0, \tag{1.7.16a}$$

$$\overline{v}(\overline{x},0) = 0, \tag{1.7.16b}$$

$$h\overline{v}(\overline{x},0) + 2\epsilon\overline{v}_{\overline{x}}(\overline{x},0) = -h\alpha. \tag{1.7.16c}$$

The solution is given by (1.4.47) with $u \to \overline{v}$, $a \to h$, $b \to 2\epsilon$, $c = -h\alpha$, $x \to \overline{x}$, $t \to \overline{t}$,

$$\overline{v}(\overline{x},t) = -\alpha\left[\operatorname{erfc}\left(\frac{\overline{x}}{2\sqrt{\overline{t}}}\right) - \exp\left(\frac{h^2\overline{t}}{4\epsilon^2} - \frac{h\overline{x}}{2\epsilon}\right)\operatorname{erfc}\left(\frac{\overline{x} - 2h\overline{t}/2\epsilon}{2\sqrt{\overline{t}}}\right)\right] \tag{1.7.17a}$$

or

$$v(x,t) = \alpha\left[1 - \operatorname{erfc}\left(\frac{x}{2\sqrt{\epsilon t}}\right) + \exp\left(\frac{h^2 t}{4\epsilon} - \frac{hx}{2\epsilon}\right)\operatorname{erfc}\left(\frac{x - ht}{2\sqrt{\epsilon t}}\right)\right]. \tag{1.7.17b}$$

We now use this result to evaluate (1.7.2) for $u(x,t)$ and obtain

$$u(x,t) = h\frac{\operatorname{erfc}\left(\frac{x-ht}{2\sqrt{\epsilon t}}\right)}{\exp\left(\frac{hx}{2\epsilon} - \frac{h^2 t}{4}\right)\operatorname{erf}\left(\frac{x}{2\sqrt{\epsilon t}}\right) + \operatorname{erfc}\left(\frac{x-t}{2\sqrt{\epsilon t}}\right)}. \tag{1.7.18}$$

In [31], this result is attributed to J.D. Cole.

If h is not constant, the above approach does not apply, but we may use the idea discussed in Section 1.4.7 of replacing (1.7.14c) by an unknown boundary value

$v(0, t) = k(t)$, then deriving an integral equation for $k(t)$. The details are entirely analogous to those discussed in Section 1.4.7. See Problem 1.7.2.

(ii) *Finite interval* $0 \leq x \leq \pi$

The following initial- and boundary-value problem for Burgers' equation is discussed in [9]:

$$u_t + uu_x = \epsilon u_{xx}, \quad 0 \leq x \leq \pi, \tag{1.7.19a}$$

$$u(x, 0) = f(x), \tag{1.7.19b}$$

$$u(0, t) = u(\pi, t) = 0, \quad t > 0. \tag{1.7.19c}$$

The problem for $v(x, t)$ defined by (1.7.2) satisfies

$$v_t - \epsilon v_{xx} = 0, \tag{1.7.20a}$$

$$v(x, 0) = \alpha g(x), \quad \alpha = \text{constant}, \tag{1.7.20b}$$

$$v_x(0, t) = v_x(\pi, t) = 0, \quad t > 0, \tag{1.7.20c}$$

where α is arbitrary and $g(x)$ is defined in (1.7.14b).

The solution for $v(x, t)$ is easily derived using separation of variables in the form

$$v(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2 \epsilon t} \cos nx, \tag{1.7.21}$$

where

$$a_n = \frac{2\alpha}{\pi} \int_0^{\pi} g(x) \cos nx \, dx. \tag{1.7.22}$$

The transformation relation (1.7.2) gives the solution of (1.7.19) in the form

$$u(x, t) = 2\epsilon \frac{\sum_{n=1}^{\infty} n a_n e^{-n^2 \epsilon t} \sin nx}{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2 \epsilon t} \cos nx}. \tag{1.7.23}$$

A discussion of the qualitative features of the solution is given in [9]. The problem where $f(x)$ has a discontinuity in the interval $0 \leq x \leq 1$ is discussed in [29]. This problem is of interest in understanding the long-term behavior of a shock layer for Burgers' equation. We still study shock layers in Chapter 5.

Problems

1.7.1. Consider Burgers' equation on $-\infty < x < \infty$,

$$u_t + uu_x = \epsilon u_{xx}. \tag{1.7.24}$$

a. For the piecewise constant initial condition

$$u(x, 0) = \begin{cases} 1 & \text{if } x < 0, \\ -1 & \text{if } x > 0, \end{cases} \tag{1.7.25}$$

derive the solution in the form

$$u(x,t) = \frac{e^{-x/\epsilon}\,\mathrm{erfc}\left(\frac{x-t}{2\sqrt{\epsilon t}}\right) - \mathrm{erfc}\left(-\frac{x+t}{2\sqrt{\epsilon t}}\right)}{e^{-x/\epsilon}\,\mathrm{erfc}\left(\frac{x-t}{2\sqrt{\epsilon t}}\right) + \mathrm{erfc}\left(-\frac{x+t}{2\sqrt{\epsilon t}}\right)}. \tag{1.7.26}$$

b. For the piecewise constant initial condition

$$u(x,0) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x > 0, \end{cases} \tag{1.7.27}$$

derive the solution in the form

$$u(x,t) = \frac{-\,\mathrm{erfc}\left(\frac{x+t}{2\sqrt{\epsilon t}}\right) + e^{-x/\epsilon}\,\mathrm{erfc}\left(\frac{t-x}{2\sqrt{\epsilon t}}\right)}{\mathrm{erfc}\left(\frac{x+t}{2\sqrt{\epsilon t}}\right) + e^{-x/\epsilon}\,\mathrm{erfc}\left(\frac{t-x}{2\sqrt{\epsilon t}}\right)}. \tag{1.7.28}$$

1.7.2. To study the problem (1.7.13) for Burgers' equation, replace (1.7.14c) with the boundary condition

$$v(0,t) = k(t), \tag{1.7.29}$$

where $k(t)$ is as yet unspecified. Use the results in Sections 1.4.2–1.4.3 to write the solution for $v(x,t)$ in terms of the unknown $k(t)$ in the form

$$\begin{aligned} v(x,t) =& \frac{\alpha}{2\sqrt{\pi\epsilon t}} \int_0^\infty g(\xi)\left[e^{-(x-\xi)^2/4\epsilon t} - e^{-(x+\xi)^2/4\epsilon t}\right] d\xi \\ &+ \int_0^t \dot{k}(\tau)\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t-\tau}}\right) d\tau \\ &+ k(0^+)\,\mathrm{erfc}\left(\frac{x}{2\sqrt{\epsilon t}}\right). \end{aligned} \tag{1.7.30}$$

Use the condition (1.7.14c) to derive the following integral equation for $k(t)$ (Assume $\alpha = k(0^+)$)

$$h(t)k(\tau) = \phi(t) + 2\sqrt{\frac{\epsilon}{\pi}} \int_0^t \frac{\dot{k}(\tau)}{\sqrt{t-\tau}}\, d\tau, \tag{1.7.31}$$

where

$$\phi(t) = 2k(0^+)\sqrt{\frac{\epsilon}{\pi t}}\left[1 - \int_0^\infty g(\xi)e^{-\xi^2/4\epsilon t}\, d(\xi^2/4\epsilon t)\right]. \tag{1.7.32}$$

Note that $\phi(t) = 0$ if $f(x) = 0$.

2

Laplace's Equation

2.1 Applications

There are numerous physical applications that are modeled by the inhomogeneous Laplace equation (Poisson equation)

$$\Delta u \equiv u_{xx} + u_{yy} + u_{zz} = Q(x, y, z).$$

Some of the standard examples are given in the following table.

Henceforth, to standardize terminology, we shall refer to u as the *potential*, even though in some applications it is not a potential and one is interested in the value of u itself rather than in its gradient. Also, we shall refer to a real function that satisfies Laplace's equation in some domain G as being harmonic in G.

We have already shown that for the steady-state problem of heat conduction in a material with constant properties and no heat sources, the temperature field satisfies Laplace's equation [see (1.1.18)].

A derivation of Laplace's equation for the deflection of a membrane (in the limit of small amplitudes) may be found on pp. 214–215 of [21].

In electrostatics, the potential due to a stationary distribution of charges follows directly from Maxwell's equation. For example, see p. 100 of [33]. A derivation of the gravitational potential for an arbitrary solid is given in Section 2.4.1, and a discussion of applications for incompressible flow follows next.

2.1.1 Incompressible Irrotational Flow

Consider the flow of a fluid having density $\rho(x, y, z, t)$ (g/cm^3) and defined by the vector velocity field $\mathbf{q}(x, y, z, t)$ (cm/s). As in Section 1.1, we can derive an integral law of mass conservation by equating the rate of change of mass inside a given fixed domain G to the net *inflow* of mass. If we also have an arbitrary distribution of mass sources of strength/unit volume equal to $Q(x, y, z, t)$ (g/cm^{3}s), the integral law of mass conservation analogous to (1.1.16) becomes

$$\frac{d}{dt} \iiint_G \rho \, dV = - \iint_S \rho \mathbf{q} \cdot \mathbf{n} \, dA + \iiint_G Q \, dV, \qquad (2.1.1)$$

TABLE 2.1. Some Applications of Poisson's Equation

Application	u	Q
Steady-state temperature in a solid	u = temperature	−Heat source strength/unit volume
Static deflection of a thin membrane in two dimensions	u = deflection	Pressure
Electrostatics	u = electrostatic potential, electric field = $\mathbf{E}$ = grad u	Charge/unit volume
Incompressible irrotational flow in two or three dimensions	u = velocity potential, velocity = $\mathbf{q}$ = grad u	Mass source strength/unit volume
Two-dimensional incompressible steady flow	u = stream function = ψ, velocity = $\mathbf{q} = \psi_y \mathbf{i} - \psi_x \mathbf{j}$	−Vorticity
Newtonian gravitation	u = gravitational potential, force of gravity = $\mathbf{F}$ = − grad u	Mass density

where again, $\mathbf{n}$ is the outward unit normal on the surface S bounding G. For smooth flows, (2.1.1) gives

$$\rho_t + \operatorname{div}(\rho \mathbf{q}) = Q. \tag{2.1.2}$$

Now, if the density is a constant (incompressible flow), (2.1.2) reduces to

$$\operatorname{div} \mathbf{q} = \tilde{Q} \equiv Q/\rho. \tag{2.1.3}$$

If in addition one assumes that the flow is irrotational—that is, curl $\mathbf{q} = 0$—it follows from vector calculus that $\mathbf{q}$ is the gradient of a scalar potential: $u(x, y, z, t)$ (cm^2/s); that is,

$$\mathbf{q} = \operatorname{grad} u. \tag{2.1.4}$$

Combining (2.1.3) and (2.1.4), we obtain the Poisson equation

$$\operatorname{div} \operatorname{grad} u \equiv \Delta u = \tilde{Q}. \tag{2.1.5}$$

2.1.2 Two-Dimensional Incompressible Flow

A flow is two-dimensional if the velocity field is independent of z, for instance. Consider such a flow and assume also that it is steady (independent of time), is source free, has a constant density, but is not necessarily irrotational. Mass conservation, (2.1.3), reduces in this case to

$$q_{1_x} + q_{2_y} = 0, \tag{2.1.6}$$

where $\mathbf{q} = q_1\mathbf{i} + q_2\mathbf{j}$ and $\mathbf{i}, \mathbf{j}$ are Cartesian unit vectors along x and y, respectively.

Consider an arbitrary simple curve C joining the origin to the point P as shown in Figure 2.1. The flow rate per unit depth across a given element $ds \equiv (dx^2+dy^2)^{1/2}$ is $dM \equiv \rho(q_1 dy - q_2 dx)$ (g/cm s). Therefore, the total flow/unit depth across the arc C is the line integral

$$\frac{M}{\rho} \equiv \int_C (q_1 dy - q_2 dx)(\text{cm}^2/\text{s}). \tag{2.1.7}$$

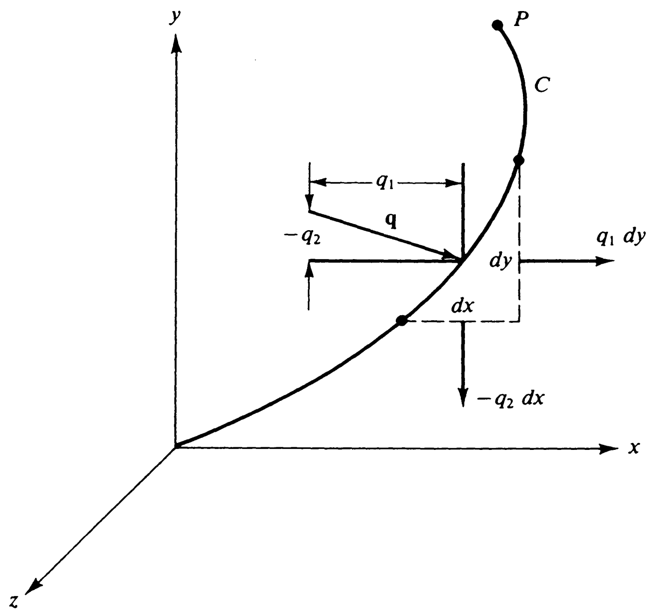

FIGURE 2.1. Two-dimensional flow across a curve

Introduce the vector $\mathbf{F} \equiv -q_2\mathbf{i} + q_1\mathbf{j}$, and note that curl $\mathbf{F} = (q_{1_x} + q_{2_y})\mathbf{k} = \mathbf{0}$ because of (2.1.6). Here $\mathbf{k}$ is the Cartesian unit vector in the z direction. Therefore, the line integral (2.1.7), which may also be written as

$$\frac{M}{\rho} = \int_C \mathbf{F} \cdot d\mathbf{r}, \quad d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}, \tag{2.1.8}$$

is a function only of the endpoint P and does not depend on the path C. Definee $M/\rho = \psi(x, y)$, where ψ is called the *stream function* for the flow. Thus, at any point P, ψ measures the total mass flow between P and the origin.

It follows from the fact that curl $\mathbf{F} = \mathbf{0}$ that

$$\mathbf{F} = \text{grad}\, \psi \equiv \psi_x\mathbf{i} + \psi_y\mathbf{j}. \tag{2.1.9a}$$

Therefore,

$$q_1 = \psi_y, \quad q_2 = -\psi_x, \tag{2.1.9b}$$

that is, the velocity vector $\mathbf{q}$ at any point is tangent to the curve ψ = constant passing through that point. The curves ψ = constant are called *streamlines* (see Figure 2.2) and measure loci of constant mass flow relative to a reference point in the sense just discussed. In particular, the mass flow between any two curves $\psi = c_1$ and $\psi = c_2$ remains constant. Thus, if the distance between these two curves narrows down, the velocity must increase to conserve mass flow.

For the velocity field defined by $\mathbf{q}$, let us define

$$\text{curl}\, \mathbf{q} \equiv \boldsymbol{\Omega} = \Omega(x, y)\mathbf{k}. \tag{2.1.10}$$

The vector $\boldsymbol{\Omega}$ is called the vorticity and corresponds to twice the average angular velocity of a fluid element. (For example, see p. 158 of [21].)

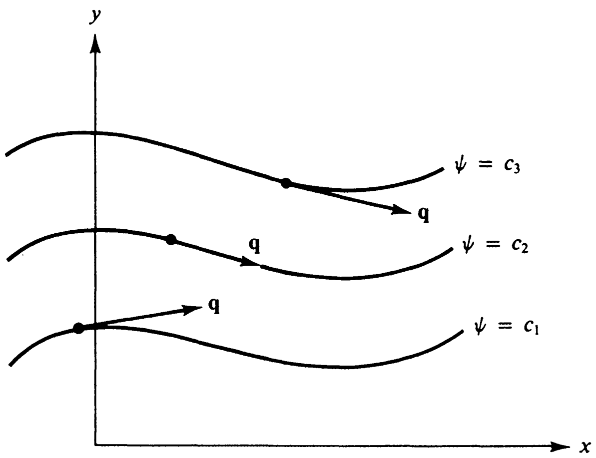

FIGURE 2.2. Streamlines

Now, by definition,

$$\operatorname{curl} \mathbf{q} = (q_{2_x} - q_{1_y})\mathbf{k},$$

and using (2.1.9b) and (2.1.10), we obtain

$$\operatorname{curl} \mathbf{q} = -\Delta\psi\mathbf{k} = \Omega(x, y)\mathbf{k}. \tag{2.1.11}$$

Thus, for a steady, incompressible, sourceless, two-dimensional, possibly rotational flow, the stream function $\psi(x, y)$ obeys

$$\Delta\psi = -\Omega(x, y) = -\text{vorticity}. \tag{2.1.12a}$$

If the flow is irrotational, we have

$$\Delta\psi = 0. \tag{2.1.12b}$$

2.2 The Two-Dimensional Problem; Conformal Mapping

The solution of Laplace's equation in two-dimensional domains is intimately related to the theory of analytic functions of a complex variable. In fact, this topic occupies a significant portion of texts on complex variables and will therefore only be outlined in this section.

2.2.1 Mapping of Harmonic Functions

We restrict our discussion to simply connected domains—that is, domains D for which every simple closed curve within D encloses only points of D.

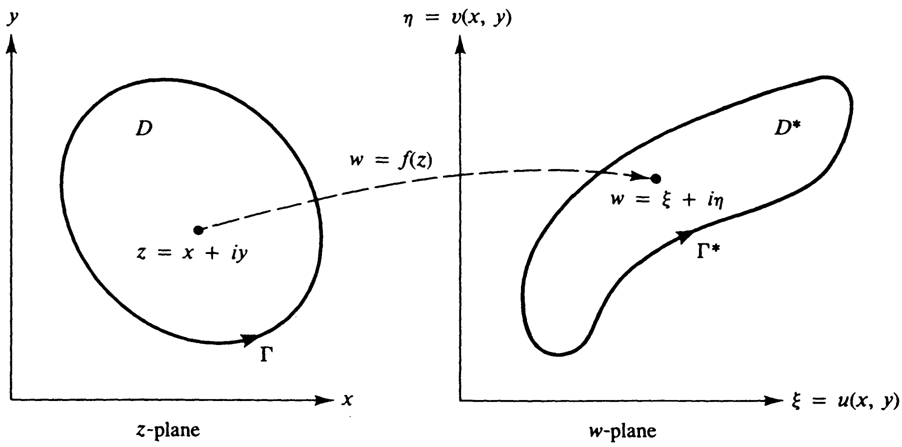

FIGURE 2.3. Mapping of a simply connected domain

The first theorem we invoke asserts that the real and imaginary parts of an analytic function are harmonic; that is, if

$$F(z) = \phi(x, y) + i\psi(x, y), \ z = x + iy, \tag{2.2.1}$$

is analytic, then

$$\phi_{xx} + \phi_{yy} = 0, \tag{2.2.2a}$$

$$\psi_{xx} + \psi_{yy} = 0. \tag{2.2.2b}$$

A second result asserts that if $w = f(z)$ is analytic in some domain D of the z-plane (and hence defines a conformal mapping of D to the domain D^* in the w-plane), then a harmonic function $\phi(x, y)$ defined on D maps to a harmonic function $\Phi(\xi, \eta)$ on D^*, and vice versa (see Figure 2.3). More precisely, consider the mapping $z \to w$ defined by

$$w = u(x, y) + iv(x, y) = f(z), \tag{2.2.3}$$

where $f(z)$ is analytic in D. With $w = \xi + i\eta$, regard the pair of equations $\xi = u(x, y)$, $\eta = v(x, y)$, as a coordinate transformation $(x, y) \leftrightarrow (\xi, \eta)$. If $\Phi(\xi, \eta)$ satisfies, $\Phi_{\xi\xi} + \Phi_{\eta\eta} = 0$ in D^*, then

$$\phi(x, y) \equiv \Phi(u(x, y), v(x, y)) \tag{2.2.4}$$

satisfies $\phi_{xx} + \phi_{yy} = 0$ in D, and vice versa. (See Problem 2.2.1.)

2.2.2 *Transformation of Boundary Conditions*

An important question associated with this mapping concerns the transformation of a boundary condition on Γ, the boundary of D, to Γ^*, the boundary of D^*. To fix ideas, let us parametrize Γ in the form

$$x = \alpha(s), \ y = \beta(s), \tag{2.2.5}$$

where s, $0 \leq s \leq s_0$, is a parameter that varies monotonically along Γ and $\alpha(s_0) = \alpha(0)$; $\beta(s_0) = \beta(0)$ if D is bounded. Moreover, let D lie to the left as Γ is traversed in the direction of increasing s. Now, Γ^*, the boundary of D^*, is defined by

$$\begin{aligned} \xi &= u(\alpha(s), \beta(s)) \equiv \lambda(s), \\ \eta &= v(\alpha(s), \beta(s)) \equiv \mu(s), \end{aligned} \tag{2.2.6}$$

and is also traversed in the same sense.

Consider now a general linear boundary condition on Γ of the form

$$A(s)\phi(\alpha, \beta) + B(s)\phi_n(\alpha, \beta) = C(s), \tag{2.2.7}$$

where $\alpha(s)$ and $\beta(s)$ are given in (2.2.5), the functions A, B, C are prescribed, and ϕ_n denotes the outward normal derivative of ϕ on Γ; that is,

$$\phi_n(\alpha, \beta) \equiv \phi_x(\alpha, \beta)n_1(\dot{\alpha}, \dot{\beta}) + \phi_y(\alpha, \beta)n_2(\dot{\alpha}, \dot{\beta}), \tag{2.2.8a}$$

with n_1 equal to the x-component and n_2 equal to the y-component of the unit outward normal; that is,

$$n_1 \equiv \frac{\dot{\beta}}{(\dot{\alpha}^2 + \dot{\beta}^2)^{1/2}}, \quad n_2 \equiv -\frac{\dot{\alpha}}{(\dot{\alpha}^2 + \dot{\beta}^2)^{1/2}}. \tag{2.2.8b}$$

It is easy to show that the boundary condition (2.2.7) transforms to

$$A(s)\Phi(\lambda, \mu) + B(s)\left(\frac{\dot{\lambda}^2 + \dot{\mu}^2}{\dot{\alpha}^2 + \dot{\beta}^2}\right)^{1/2} \Phi_N(\lambda, \mu) = C(s), \tag{2.2.9}$$

where again Φ_N is the outward normal derivative of Φ on Γ^*; that is,

$$\Phi_N(\lambda, \mu) \equiv \Phi_\xi(\lambda, \mu)N_1(\dot{\lambda}, \dot{\mu}) + \Phi_\eta(\lambda, \mu)N_2(\dot{\lambda}, \dot{\mu}), \tag{2.7.10a}$$

with

$$N_1 \equiv \frac{\dot{\mu}}{(\dot{\lambda}^2 + \dot{\mu}^2)^{1/2}}; \quad N_2 \equiv -\frac{\dot{\lambda}}{(\dot{\lambda}^2 + \dot{\mu}^2)^{1/2}}. \tag{2.2.10b}$$

In particular, if $B \equiv 0$ (Dirichlet's problem), the boundary values *of ϕ map unchanged to boundary values of Φ at corresponding points*. If $A \equiv 0$ (Neumann's problem), boundary values of the normal derivative at corresponding points are scaled by the factor $(\dot{\lambda}^2 + \dot{\mu}^2)^{1/2}/(\dot{\alpha}^2 + \dot{\beta}^2)^{1/2}$. A boundary condition $\phi_n \equiv 0$ is mapped unchanged to a boundary condition $\Phi_N \equiv 0$.

2.2.3 *Example, Solution in a Simpler Transformed Domain*

To illustrate an application of the preceding ideas, consider the Dirichlet problem for Laplace's equation in the corner domain sketched in Figure 2.4.

The boundary values for ϕ are specified in terms of the two functions $a(y)$ and $b(y)$.

For the case $a(y) \equiv a = \text{constant}$, $b(y) \equiv b = \text{constant} \neq a$, it is convenient to map D onto the strip domain D^* using $w = \log z$. Here, we use the principal branch of $\log z$, that is, $-\pi < \arg z < \pi$, and cut the z-plane along the negative real axis. It then follows that

$$\xi = \log(x^2 + y^2)^{1/2}, \quad \eta = \tan^{-1}\left(\frac{y}{x}\right). \tag{2.2.11}$$

Now Γ_1 maps to Γ_1^*, the horizontal line $\eta = \pi/2$, and Γ_2 maps to Γ_2^*, the horizontal line $\eta = -\pi/4$. Moreover, the boundary condition on Φ becomes

$$\Phi\left(\xi, \frac{\pi}{2}\right) = a, \quad \Phi\left(\xi, -\frac{\pi}{4}\right) = b. \tag{2.2.12}$$

Thus, we need to solve $\Delta\Phi = 0$ in the strip domain D^* subject to boundary conditions *that do not depend on ξ*. It then follows that Φ does not depend on ξ,

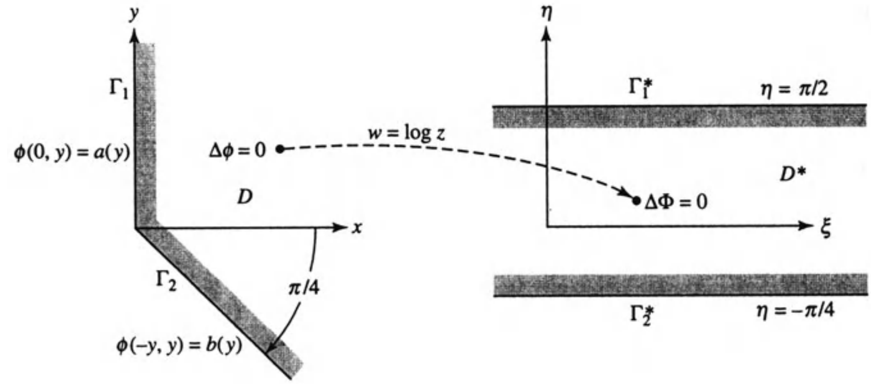

FIGURE 2.4. Mapping of corner domain

and we need to solve only $\partial^2\Phi/\partial\eta^2 = 0$, that is,

$$\Phi = C_1 + C_2\eta, \tag{2.2.13}$$

where C_1 and C_2 are constants. Imposing the boundary condition (2.2.12) determines C_1 and C_2, and we obtain

$$\Phi = \frac{2b + a}{3} + \frac{4(a - b)}{3\pi}\eta. \tag{2.2.14a}$$

Therefore, the solution in D is

$$\phi(x, y) = \frac{2b + a}{3} + \frac{4(a - b)}{3\pi}\tan^{-1}\left(\frac{y}{x}\right). \tag{2.2.14b}$$

The success of this approach is directly due to the fact that we were able to write the solution for Φ in D^* by inspection. This, in turn, is a direct consequence of the simple boundary conditions. In fact, if a and b are given functions of y, the transformation (2.1.11) to D^* certainly still holds, but (2.2.13) cannot, in general, satisfy the boundary conditions because Φ will depend on ξ along the two boundaries Γ_1^* and Γ_2^*. If we attempt to accommodate the variable boundary conditions by assuming $C_1(\xi)$ and $C_2(\xi)$, we immediately discover that this is a solution only if $C_1'' \equiv 0, C_2'' \equiv 0$. Thus, we can handle only the case for which a and b transform to linear functions on Γ_1^* and Γ_2^*.

To make any progress for the case where $a(y)$, $b(y)$ are arbitrarily prescribed, we need to transform D into a domain D^* *for which we can solve the Dirichlet problem for arbitrary boundary conditions*. Recall from your study of complex variables that we can solve Laplace's equation in the interior of the unit circle with arbitrary boundary values for Φ on the circumference of the unit circle. This is Poisson's formula, given next for the geometry sketched in Figure 2.5 (see, for example, p.47 of [8] for a derivation of this result using Cauchy's integral formula

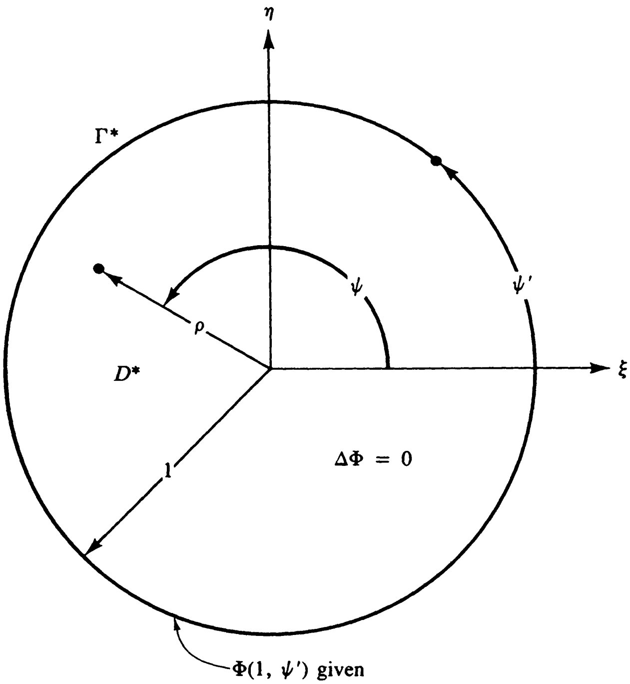

FIGURE 2.5. Dirichlet problem for the interior of a unit circle

or (2.6.36) for a derivation using Green's function):

$$\Phi(\rho, \psi) = \frac{1-\rho^2}{2\pi} \int_0^{2\pi} \frac{\Phi(1, \psi')d\psi'}{1+\rho^2 - 2\rho\cos(\psi - \psi')}. \tag{2.2.15}$$

Therefore, if we can transform D to the interior of the unit circle, we will have solved the problem. There is a famous theorem due to Riemann that asserts that any simply connected domain whose boundary consists of more than one point can be mapped conformally onto the interior of the unit circle. It is also possible to make an arbitrary point in D and a direction through this point correspond, respectively, to the origin and the positive real axis. If this is done, the mapping is unique. For a proof of this theorem, see Chapter 5 of [40].

Although this result is reassuring, it does not provide us the mapping itself. We shall see in Section 2.6.6 that *finding the mapping is equivalent to finding Green's function* for the domain.

Let us use our knowledge of the mapping properties of simple functions to map D into the unit circle via the sequence of simple mappings sketched in Figure 2.6.

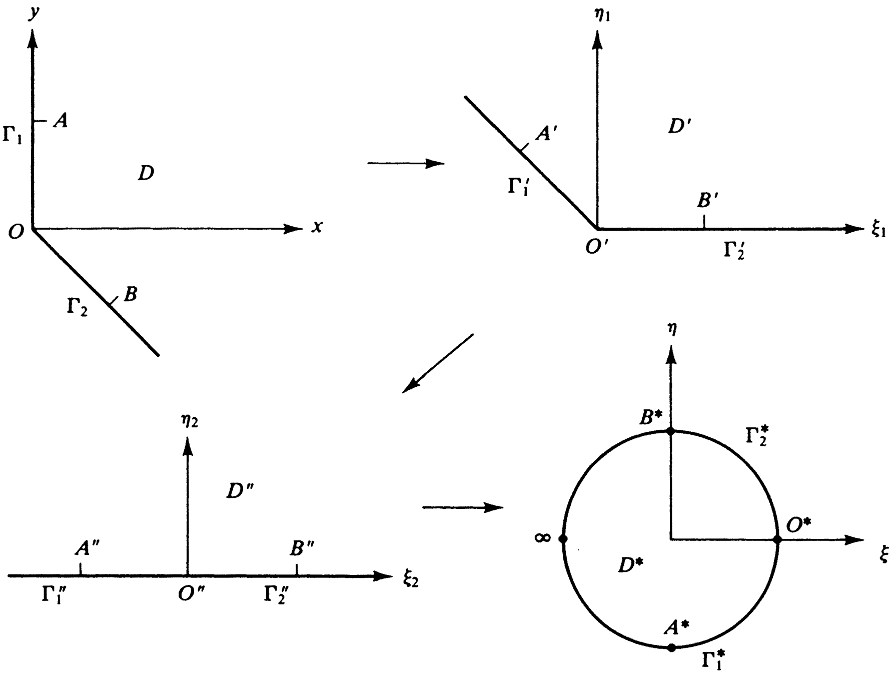

FIGURE 2.6. Sequence of mappings of a corner domain to the unit circle

The first mapping, $z \to w_1$, is a rotation by $\pi/4$ and therefore obeys

$$w_1 = ze^{i\pi/4}. \qquad (2.2.16a)$$

The second mapping, $w_1 \to w_2$, rotates Γ_1', the map of Γ_1 (which is now given by $\arg w_1 = 3\pi/4$) to the negative real axis of w_2 but leaves Γ_2', the map of Γ_2, unchanged. Therefore,

$$w_2 = w_1^{4/3}. \qquad (2.2.16b)$$

The third mapping must be a linear fractional transformation (sometimes also called a bilinear transformation), and we derive it in the form

$$w = (i - w_2)/(i + w_2). \qquad (2.2.16c)$$

See Problem 2.2.2 for a review of the properties of linear fractional transformations.

Combining these and simplifying gives

$$w = \frac{1 - e^{-i\pi/6}z^{4/3}}{1 + e^{-i\pi/6}z^{4/3}}, \qquad (2.2.17a)$$

or using the polar form $z = re^{i\theta}$, we obtain

$$w = \frac{1 - r^{8/3} - 2ir^{4/3}\sin\left(\frac{4\theta}{3} - \frac{\pi}{6}\right)}{1 + r^{8/3} + 2r^{4/3}\cos\left(\frac{4\theta}{3} - \frac{\pi}{6}\right)}. \tag{2.2.17b}$$

Therefore, the ξ and η components are defined by

$$\xi = \frac{(1 - r^{8/3})}{D(r, \theta)} \equiv u(r, \theta), \tag{2.2.18a}$$

$$\eta = \frac{2r^{4/3}\sin\left(\frac{4\theta}{3} - \frac{\pi}{6}\right)}{D(r, \theta)} \equiv v(r, \theta), \tag{2.2.18b}$$

where $D(r, \theta)$ is the denominator of (2.2.17b):

$$D(r, \theta) \equiv 1 + r^{8/3} + 2r^{4/3}\cos\left(\frac{4\theta}{3} - \frac{\pi}{6}\right). \tag{2.2.18c}$$

In order to specify the boundary conditions on Γ^*, we must transform the given $a(y)$ and $b(y)$ to obtain $\Phi(1, \psi)$. The details are straightforward but laborious and are therefore omitted. Knowing $\Phi(1, \psi)$ defines $\Phi(\rho, \psi)$ according to (2.2.15), and $\phi(r, \theta)$ is then given by

$$\phi(r, \theta) = \Phi\left(\sqrt{u^2 + v^2}, \tan^{-1}\frac{v}{u}\right), \tag{2.2.19}$$

where, $u(r, \theta)$ and $v(r, \theta)$ are defined in (2.2.18).

Actually, the transformation (2.2.17a) is not needed for this problem and was worked out for purposes of illustration only; the intermediate transformation to the upper half-plane defined by (2.2.16a) and (2.2.16b) suffices because the solution of the Dirichlet problem there is also available for arbitrary boundary values on the real axis (see Problems 2.2.3 and 2.6.6). Usually, the transformation to the interior of the unit circle is appropriate for bounded domains D.

The reader will find a number of interesting and challenging examples in Chapter 4 of [8]. The material in the present chapter does not dwell on techniques that are appropriate only for the two-dimensional Laplacian; rather, whenever the opportunity arises, we point out how general results specialize to two dimensions and relate to ideas from complex variables.

Problems

2.2.1. This problem illustrates the fact that under an arbitrary coordinate transformation the Cartesian Laplacian form $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$ is not necessarily preserved. However, if the mapping is defined by an analytic function of the complex variable $z = x + iy$, then a harmonic function ϕ of x, y maps to a function $\Phi(\xi, \eta)$ that satisfies $\Phi_{\xi\xi} + \Phi_{\eta\eta} = 0$.

Consider Laplace's equation

$$\phi_{xx} + \phi_{yy} = 0 \tag{2.2.20}$$

in some domain D of the xy-plane. Let ξ and η be general curvilinear coordinates in D defined by

$$\xi = u(x, y), \tag{2.2.21a}$$

$$\eta = v(x, y), \tag{2.2.21b}$$

for prescribed functions $u(x, y)$, $v(x, y)$. Assume that u, v, u_x, v_x, u_y, v_y, u_{xx}, u_{xy}, u_{yy}, v_{xx}, v_{xy}, v_{yy} are all continuous in D and that the mapping is one-to-one in D.

a. Define $\Phi(\xi, \eta)$, the image of $\phi(x, y)$ under mapping (2.2.21), by (2.2.4). Show that

$$\begin{aligned}\phi_{xx} + \phi_{yy} &= (u_x^2 + u_y^2)\Phi_{\xi\xi} + 2(u_x v_x + u_y v_y)\Phi_{\xi\eta} + (v_x^2 + v_y^2)\Phi_{\eta\eta} \\ &\quad + (u_{xx} + u_{yy})\Phi_\xi + (v_{xx} + v_{yy})\Phi_\eta. \end{aligned} \tag{2.2.22}$$

Thus, if (2.2.20) holds, $\Phi(\xi, \eta)$ *does not necessarily satisfy* the Cartesian form $\Phi_{\xi\xi} + \Phi_{\eta\eta} = 0$.

b. Take the special case where (2.2.21) defines polar coordinates in the plane, that is,

$$\xi = r \equiv (x^2 + y^2)^{1/2}, \quad \eta = \theta \equiv \tan^{-1} \frac{y}{x}. \tag{2.2.23}$$

Verify that the mapping is one-to-one except at $r = 0$. Show that (2.2.20) and (2.2.22) imply

$$\phi_{xx} + \phi_{yy} = \Phi_{rr} + \frac{1}{r}\Phi_r + \frac{1}{r^2}\Phi_{\theta\theta} = 0. \tag{2.2.24}$$

Thus, the Cartesian form of the Laplacian is not preserved.

c. Now assume that u and v in (2.2.21) are the real and imaginary parts, respectively, of an analytic function, as in (2.2.3). Use the Cauchy-Riemann conditions

$$u_x = v_y, \quad v_x = -u_y, \tag{2.2.25}$$

which must hold in this case, to conclude that (2.2.20) and (2.2.22) imply

$$\Phi_{\xi\xi} + \Phi_{\eta\eta} = 0. \tag{2.2.26}$$

Specialize your results for the case where $f(z)$ in (2.2.3) is

$$f(z) = z^2. \tag{2.2.27}$$

Verify that the Cauchy-Riemann conditions (2.2.25) do not hold for the pair of functions $u(x, y)$, $v(x, y)$ defined by (2.2.23).

2.2.2 In this problem we review some properties of linear fractional transformations. A linear fractional transformation is defined by

$$w = \frac{az + b}{cz + d} \tag{2.2.28}$$

for complex constants a, b, c, d. If $c = 0$, (2.2.28) reduces to a linear transformation. Factoring a and c from the numerator and denominator, respectively, shows that the right-hand side of (2.2.28) reduces to the constant a/c if $ad - bc = 0$. We exclude the cases $c = 0$ and $ad = bc$.

a. Show that (2.2.28) is the composition of the following five sequential mappings:

$w^{(1)} = z + \frac{d}{c}$,	translation,	(2.2.29*a*)
$w^{(2)} = cw^{(1)}$,	rotation and dilation,	(2.2.29*b*)
$w^{(3)} = \frac{1}{w^{(2)}}$,	inversion,	(2.2.29*c*)
$w^{(4)} = \frac{(bc-ad)}{c} w^{(3)}$,	rotation and dilation,	(2.2.29*d*)
$w = \frac{a}{c} + w^{(4)}$,	translation.	(2.2.29*e*)

b. Show that each of these mappings takes circles and straight lines into circles and straight lines. Therefore, (2.2.28) has this property.

c. Show that if z, z_1, z_2, z_3 are four points in the finite z-plane with images under (2.2.28) given by w, w_1, w_2, w_3, respectively, then the following identity holds:

$$\frac{(w_1 - w_2)(w_3 - w)}{(w_1 - w)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z)}{(z_1 - z)(z_3 - z_2)}. \tag{2.2.30}$$

Also, (2.2.30) is a linear fractional transformation. If any point w_i or z_i is the point at infinity, we suppress the factor containing that point from both numerator and denominator; for example, if $w_3 = \infty$, the left-hand side of (2.2.30) becomes $(w_1 - w_2)/(w_1 - w)$. Explain why the linear fractional transformation that takes the circle or straight line passing through three given points in the z-plane to the circle or straight line passing through three given points in the w-plane is unique. Thus, (2.2.30) may be regarded as a generating formula for linear fractional transformations.

d. Show that the linear fractional transformation taking:

i. $z_1 = -1, z_2 = 0, z_3 = 1$ to $w_1 = 0, w_2 = i, w_3 = 3i$ is

$$w = -3i\,\frac{z + 1}{z - 3}. \tag{2.2.31a}$$

ii. $z_1 = -1, z_2 = 0, z_3 = 1$ to $w_1 = -i, w_2 = 1, w_3 = i$ is [see (2.2.16c)]

$$w = \frac{i - z}{i + z}. \tag{2.2.31b}$$

iii. $z_1 = 0, z_2 = 1, z_3 = \infty$ to $w_1 = -1, w_2 = -i, w_3 = 1$ is

$$w = \frac{(1-i)z - (1+i)}{(1-i)z + (1+i)}. \tag{2.2.31c}$$

2.2.3 We wish to solve

$$u_{xx} + u_{yy} = 0 \tag{2.2.32}$$

in the domain $y \geq 0$, subject to

$$u(x,0) = f(x); \; f \to 0 \quad \text{as } |x| \to \infty, \tag{2.2.33}$$

$$u(x,\infty) = 0. \tag{2.2.34}$$

a. Use the mapping (2.2.16c) and Poisson's formula (2.2.15) to show that $u(x,y)$ is given in the form

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(s)ds}{(s-x)^2 + y^2}. \tag{2.2.35}$$

This result may also be observed directly (see Problem 2.6.6).

b. Use (2.2.35) and the mapping defined by combining (2.2.16a) with (2.2.16b) to solve (2.2.32) in the corner domain of Figure 2.4 with $a = e^{-y}$ on $y \geq 0$ and $b = (\sin y)/y$ on $y \leq 0$. Show that the solution is given by

$$\begin{aligned} u(x,y) = {} & \frac{\eta}{\pi}\int_{-\infty}^{0} \frac{e^{-(-s)^{3/4}}ds}{s^2 - 2s\xi + (x^2+y^2)^{4/3}} \\ & + \frac{2^{1/2}\eta}{\pi}\int_{0}^{\infty} \frac{s^{-3/4}\sin(s^{3/4}/\sqrt{2})ds}{s^2 - 2s\xi + (x^2+y^2)^{4/3}}, \end{aligned} \tag{2.2.36}$$

where ξ and η are the following functions of x and y:

$$\xi = (x^2+y^2)^{2/3}\cos\left[\frac{\pi + 4\tan^{-1}(y/x)}{3}\right], \tag{2.2.37a}$$

$$\eta = (x^2+y^2)^{2/3}\sin\left[\frac{\pi + 4\tan^{-1}(y/x)}{3}\right]. \tag{2.2.37b}$$

2.3 Fundamental Solution; Dipole Potential

As in Chapter 1, we begin our discussion of solution techniques by studying the influence of a unit source for Laplace's equation in the infinite domain. This is the fundamental solution that is the basic building block for constructing the solution of more general boundary-value problems.

2.3.1 Point Source in Three Dimensions

Consider the potential at a point $P = (x, y, z)$ due to a unit positive source located at the origin; that is, we seek the solution of

$$\Delta u = \delta(x)\delta(y)\delta(z). \tag{2.3.1}$$

Because of symmetry, we need consider only the spherically symmetric Laplacian and solve

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} = \delta_3(r), \tag{2.3.2}$$

where $\delta_3(r)$ is the three-dimensional delta function (see (1.6.6)–(1.6.9)). In particular, we have

$$\iiint_G \delta_3(r)dV = \begin{cases} 1 & \text{if } G \text{ contains the origin,} \\ 0 & \text{otherwise,} \end{cases} \tag{2.3.3}$$

where G is a given three-dimensional domain, and dV is the element of volume in G.

Solving (2.3.2) for $r > 0$ gives

$$u = \frac{C_1}{r} + C_2, \tag{2.3.4}$$

and we discard C_2 by normalizing $u(\infty) = 0$.

To evaluate C_1, we integrate (2.3.1) over G_ϵ, the interior of a sphere of radius ϵ centered at the origin. Hence,

$$\iiint_{G_\epsilon} \Delta u\, dV = 1. \tag{2.3.5}$$

The left-hand side of (2.3.5) can be computed using Gauss' theorem; that is,

$$\iiint_G \operatorname{div} \mathbf{F}\, dV = \iint_\Gamma \mathbf{F} \cdot \mathbf{n}\, dA, \tag{2.3.6}$$

where $\mathbf{F}$ is a prescribed vector field in the domain G with boundary Γ, outward unit normal $\mathbf{n}$, element of volume dV, and element of area dA. Let G in (2.3.6) be the interior of the ϵ-sphere and set $\mathbf{F} = \operatorname{grad} u$, where $u = C_1/r$. Therefore, $\mathbf{F} \cdot \mathbf{n} = du/dr$, and (2.3.6) gives

$$\iiint_{G_\epsilon} \Delta u\, dV = \iint_{\Gamma_\epsilon} \frac{du}{dr} dA = \int_{\theta=0}^{\pi} \int_{\psi=0}^{2\pi} \left(\frac{-C_1}{\epsilon^2}\right) \epsilon^2 \sin\theta\, d\psi\, d\theta = -4\pi C_1. \tag{2.3.7}$$

Thus, $C_1 = -1/4\pi$ and the solution of (2.3.1) with $u(\infty) = 0$ is $u = -1/4\pi r$. More generally, if $P = (x, y, z)$, and if the source is located at the point $Q = (\xi, \eta, \zeta)$, and we define

$$r_{PQ}^2 \equiv (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2, \tag{2.3.8a}$$

$$\delta_3(P, Q) \equiv \delta(x-\xi)\delta(y-\eta)\delta(z-\zeta), \tag{2.3.8b}$$

then the solution of

$$\Delta u = \delta_3(P, Q) \tag{2.3.9}$$

is

$$u = -\frac{1}{4\pi r_{PQ}}. \tag{2.3.10}$$

The fundamental solution for the n-dimensional Laplacian is discussed in Problem 2.3.2.

2.3.2 Fundamental Solution in Two Dimensions; Descent

The case $n = 2$ is special [because we cannot set $u(\infty) = 0$ (see problem 2.3.2)] and is discussed next. The axisymmetric equivalent of (2.3.1) with $r^2 \equiv x^2 + y^2$ is

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} = \delta_2(r), \tag{2.3.11}$$

where

$$\delta_2(r) \equiv \delta(x)\delta(y),$$

and

$$\iint_D \delta_2(r)dS = \begin{cases} 1, & \text{if } D \text{ contains the origin,} \\ 0, & \text{otherwise.} \end{cases} \tag{2.3.12}$$

Here, D is a given planar domain and dS is the area element.

The solution of (2.3.11) for $r > 0$ is

$$u = C_1 \log r + C_2, \tag{2.3.13}$$

and it is no longer possible to normalize $u(\infty) = 0$. However, set $C_2 = 0$ by requiring $u(1) = 0$. Now, we can evaluate C_1 by applying the two-dimensional Gauss theorem,

$$\iint_D \operatorname{div} \mathbf{F}\, dS = \int_\Gamma \mathbf{F}\cdot\mathbf{n}\, ds, \tag{2.3.14}$$

where $\mathbf{F}$ is a two-dimensional vector field in the planar domain D with element of area dS, boundary Γ with element of arc ds, and outward unit normal $\mathbf{n}$. Again, integrating (2.3.11) over D_ϵ, the interior of a circle of radius ϵ centered at the origin, gives

$$\iint_{D_\epsilon} \Delta u\, dS = 1. \tag{2.3.15}$$

Now we use $u = C_1 \log r$ and set $\mathbf{F} = \operatorname{grad} u$ in (2.3.14) to obtain $C_1 = 1/2\pi$. Thus,

$$u = \frac{1}{2\pi} \log r. \tag{2.3.16}$$

More generally, the fundamental solution of

$$u_{xx} + u_{yy} = \delta(x - \xi)\delta(y - \eta) \tag{2.3.17}$$

is

$$u = \frac{1}{2\pi} \log \sqrt{(x - \xi)^2 + (y - \eta)^2}. \tag{2.3.18}$$

It is interesting to derive (2.3.16) by "descent" from the three-dimensional result. Since a two-dimensional unit source is just a distribution of sources of unit strength along an infinite straight line, it must be possible to obtain the result (2.3.16) as a solution of the following three-dimensional problem:

$$u_{xx} + u_{yy} + u_{zz} = \delta(x)\delta(y). \tag{2.3.19}$$

By superposition of the fundamental solution (2.3.10), the potential at the point $r = \sqrt{x^2 + y^2}$, $z = 0$ is simply

$$\begin{aligned} u(r) &= -\frac{1}{4\pi} \iiint_{-\infty}^{\infty} \frac{\delta(\xi)\delta(\eta)d\xi\, d\eta\, d\zeta}{\sqrt{(x - \xi)^2 + (y - \eta)^2 + \zeta^2}} \\ &= -\frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{d\zeta}{\sqrt{r^2 + \zeta^2}}. \end{aligned} \tag{2.3.20}$$

(*Note*: There is no loss of generality in setting $z = 0$.)

The integral (2.3.20) diverges, which is not surprising because the value of u itself need not be bounded. Thus, if u in (2.3.19) is interpreted as the steady temperature due to the continuous distribution of point sources of heat along an infinite straight line, the temperature at any finite radius r is indeed infinite.

However, if u is interpreted as a potential, it is only its gradient that is of interest, and in computing this gradient, the contribution due to a large additive constant may be ignored. Therefore, we interpret (2.3.20) in the sense of Hadamard's definition of the "finite part" of a divergent integral, denoted by the notation $FP\left(\int\right)$. For more details concerning divergent integrals, see Section D, 14.1, of [23]. Thus, we regard u to be

$$\begin{aligned} u &= -\frac{1}{4\pi} FP\left(\int_{-\infty}^{\infty} \frac{d\zeta}{\sqrt{r^2 + \zeta^2}}\right) \\ &\equiv -\frac{1}{4\pi} \int_1^r \left[\int_{-\infty}^{\infty} \frac{\partial}{\partial\rho}\left(\frac{1}{\sqrt{\rho^2 + \zeta^2}}\right) d\zeta\right] d\rho. \end{aligned} \tag{2.3.21}$$

Since the integral in the square brackets converges, in fact,

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial \rho} \left[\frac{1}{\sqrt{\rho^2 + \zeta^2}} \right] d\zeta = - \int_{-\infty}^{\infty} \frac{\rho d\zeta}{(\rho^2 + \zeta^2)^{3/2}} = -\frac{2}{\rho},$$

we obtain $u = (1/2\pi) \log r$, as before. The definition (2.3.21) for the finite part of u involves two steps. First, we filter out the infinite constant in (2.3.20) by evaluating the derivative of u by the *convergent* integral

$$u_r = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{r d\zeta}{(r^2 + \zeta^2)^{3/2}} = \frac{1}{2\pi r}. \tag{2.3.22}$$

We then integrate this expression for u_r using the normalization $u(1) = 0$.

2.3.3 Effect of Lower-Derivative Terms

If we consider the n-dimensional Laplacian, to which are added arbitrary linear terms in the $\partial u / \partial x_i$ and u, we obtain the following general equation:

$$\sum_{i=1}^{n} \left(\frac{\partial^2 u}{\partial x_i^2} + a_i \frac{\partial u}{\partial x_i} \right) + au = 0. \tag{2.3.23}$$

Here, the a_i and a are constants.

The first-derivative terms in (2.3.23) can be removed by the transformation of the dependent variable $u \rightarrow w$ defined by

$$u(x_1, \ldots, x_n) \equiv w(x_1, \ldots, x_n) \exp \left\{ -\frac{1}{2} \sum_{i=1}^{n} a_i x_i \right\}. \tag{2.3.24}$$

An easy calculation shows that if u satisfies (2.3.23), then w obeys

$$\sum_{i=1}^{n} \frac{\partial^2 w}{\partial x_i^2} + \lambda w = 0, \tag{2.3.25a}$$

where λ is the constant

$$\lambda = a - \frac{1}{4} \sum_{i=1}^{n} a_i^2. \tag{2.3.25b}$$

With $\lambda > 0$, (2.3.25a) is the Helmholtz equation (and if $\lambda < 0$, it is called the "modified" Helmholtz equation), which also arises when the time derivative is eliminated by taking the Laplace transform of a diffusion or wave equation.

To calculate the fundamental solution of (2.3.25a) with $\lambda > 0$, we need to solve [compare with (2.3.74)]

$$\frac{d^2 w}{dr^2} + \frac{n-1}{r} \frac{dw}{dr} + \lambda w = \delta_n(r), \tag{2.3.26}$$

where $r \equiv (\sum_{i=1}^{n} x_i^2)^{1/2}$ and δ_n is the n-dimensional delta function [defined in (1.6.8)].

A second transformation of the dependent variable $w \to v$ is indicated. Set

$$w(r) = r^{-(n-2)/2} v(\theta); \;\; \theta \equiv \lambda^{1/2} r \tag{2.3.27}$$

to obtain

$$\frac{d^2 v}{d\theta^2} + \frac{1}{\theta}\frac{dv}{d\theta} + \left[1 - \left(\frac{n-2}{2\theta}\right)^2\right] v = 0, \tag{2.3.28}$$

for $r > 0$.

This is Bessel's equation of order $(n-2)/2$. We note that if n is even, $(n-2)/2$ is an integer, and the two linearly independent solutions of (2.3.28) are $J_{(n-2)/2}(\theta)$ and $Y_{(n-2)/2}(\theta)$, where J_p and Y_p are the Bessel functions (of order p) of the first and second kind, respectively. Conversely, if n is odd, $(n-2)/2$ is not an integer, and the two solutions $J_{(n-2)/2}(\theta)$ and $J_{-(n-2)/2}(\theta)$ are also independent and convenient to use.

We recall that J_p, J_{-p}, Y_p, and Y_0 have the following behavior as $\theta \to 0$ (for example, see [3]):

$$J_p(\theta) \sim \frac{\theta^p}{2^p \Gamma(1+p)}, \tag{2.3.29a}$$

$$J_{-p}(\theta) \sim \frac{2^p}{\theta^p \Gamma(1-p)}, \tag{2.3.29b}$$

$$Y_p(\theta) \sim -\frac{2^p (p-1)!}{\pi \theta^p}, \;\; p = \text{integer} \neq 0, \tag{2.3.29c}$$

$$Y_0(\theta) \sim \frac{2}{\pi} \log \theta. \tag{2.3.29d}$$

We discard the solution of (2.3.28), which, upon multiplication by $r^{-(n-2)/2}$, is finite at $r = 0$, and obtain

$$w(r) = B_n r^{-(n-2)/2} Y_{(n-2)/2}(\lambda^{1/2} r), \;\; n \text{ even}, \tag{2.3.30a}$$

$$w(r) = C_n r^{-(n-2)/2} J_{-(n-2)/2}(\lambda^{1/2} r), \;\; n \text{ odd}, \tag{2.3.30b}$$

where B_n and C_n are constants to be determined from evaluating the volume integral of (2.3.26) over the interior of an n-dimensional sphere of radius ϵ centered at the origin. For ϵ small, the term λw in (2.3.26) does not contribute anything to this integral, and using the n-dimensional Gauss theorem we obtain

$$\int \underset{\epsilon}{\cdots} \int \frac{dw}{dr} dA = 1, \tag{2.3.31}$$

where the integral is evaluated on the surface of the n-dimensional sphere of radius ϵ.

For example, with n equal to an odd integer and using the expression in (2.3.29b) for $J_{-(n-2)/2}$, (2.3.31) gives

$$C_n \left(\frac{2}{\lambda^{1/2}}\right)^{(n-2)/2} \frac{(2-n)}{\Gamma\left(\frac{4-n}{2}\right)} \omega_n = 1, \tag{2.3.32}$$

where ω_n is the surface area of the unit n-sphere, as defined in (1.6.13), and Γ is the gamma function defined in (1.6.12). Using this expression for ω_n and solving for C_n gives

$$C_n = \frac{\Gamma\left(\frac{4-n}{2}\right)\Gamma\left(\frac{n}{2}\right)}{2(2-n)\pi^{n/2}}\left(\frac{\lambda^{1/2}}{2}\right)^{(n-2)/2}. \tag{2.3.33}$$

It is easily verified that for $\lambda \to 0$, (2.3.30b) reduces to

$$w = \frac{1}{(2-n)r^{n-2}\omega_n}, \tag{2.3.34}$$

in agreement with the result derived in Problem 2.3.2 for Laplace's equation.

2.3.4 *Potential Due to a Dipole*

In a number of applications to be discussed later, we need to solve (2.3.1) with a higher-order singularity on the right-hand side. This is discussed next.

Consider the potential at a point $P = (x, y, 0)$ due to a source of strength C located at $x = \epsilon/2$, $y = 0$, $z = 0$ superposed on the potential of a source of strength $-C$ located at $x = -\epsilon/2$, $y = 0$, $z = 0$. If we denote the sum of the two potentials by w, we have

$$w = \frac{C}{4\pi}\left[\frac{1}{r_2} - \frac{1}{r_1}\right], \tag{2.3.35}$$

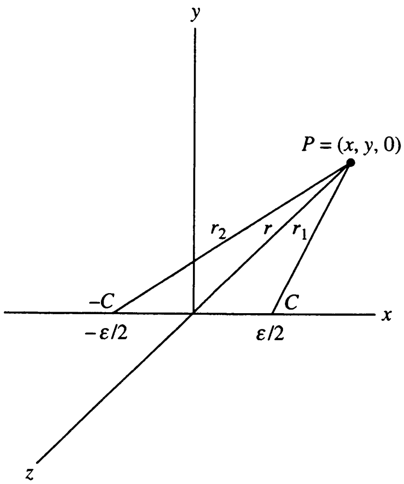

FIGURE 2.7. Dipole along the x-axis

where, as is shown in Figure 2.7,

$$r_2^2 \equiv \left(x + \frac{\epsilon}{2}\right)^2 + y^2 = r^2 + \epsilon r \cos\theta + O(\epsilon^2), \qquad (2.3.36a)$$

$$r_1^2 \equiv \left(x - \frac{\epsilon}{2}\right)^2 + y^2 = r^2 - \epsilon r \cos\theta + O(\epsilon^2). \qquad (2.3.36b)$$

Here we have used the polar coordinates r, θ defined by $x = r\cos\theta$, $y = r\sin\theta$, and have ignored terms proportional to ϵ^2. Using (2.3.36) to calculate $(1/r_2)$ and $(1/r_1)$ gives

$$\frac{1}{r_2} = \frac{1}{r} - \frac{\epsilon}{2r^2}\cos\theta + O(\epsilon^2),$$
$$\frac{1}{r_1} = \frac{1}{r} + \frac{\epsilon}{2r^2}\cos\theta + O(\epsilon^2).$$

Hence,

$$w = -\frac{C\epsilon}{4\pi r^2}\cos\theta + O(\epsilon^2). \qquad (2.3.37)$$

If we consider the limit $\epsilon \to 0$ with $C\epsilon \equiv D$ fixed, the potential reduces to

$$w = -\frac{D}{4\pi r^2}\cos\theta = -\frac{Dx}{4\pi r^3}. \qquad (2.3.38)$$

This limiting configuration is called a *dipole* (sometimes also called a *doublet*) of strength D located at $x = 0$ and *oriented* along the positive x-axis. Note that the orientation of the dipole is determined by the relative location of the positive and negative sources.

For a dipole of strength D located at $x = \xi_0$, $y = z = 0$ and oriented in the positive x-direction, we have the potential

$$w(x, y, z) = -\frac{D}{4\pi}\frac{(x - \xi_0)}{[(x - \xi_0)^2 + y^2 + z^2]^{3/2}} \qquad (2.3.39a)$$

at the point $P = (x, y, z)$. We note that this potential is just

$$w = \frac{\partial}{\partial\xi}\left\{-\frac{D}{4\pi}\frac{1}{[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{1/2}}\right\}\Bigg|_{\xi=\xi_0,\eta=0,\zeta=0}. \qquad (2.3.39b)$$

In general, the potential at $P = (x, y, z)$ due to a unit dipole located at $Q_0 = (\xi_0, \eta_0, \zeta_0)$ and oriented along the unit vector $\mathbf{a} = a_i\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is

$$w = -\frac{1}{4\pi}\operatorname{grad}_Q\left(\frac{1}{r_{PQ}}\right)\cdot\mathbf{a}\Bigg|_{Q=Q_0}, \qquad (2.3.40)$$

where, as usual, $r_{PQ}^2 \equiv (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$, and grad_Q means that in evaluating the gradient, partial derivatives are taken with respect to $Q = (\xi, \eta, \zeta)$.

It is also interesting to observe that the result (2.3.38) is the solution of

$$\Delta w = -D\delta'(x)\delta(y)\delta(z). \qquad (2.3.41)$$

To see this, note that the solution of $\Delta w = C[\delta(x - \epsilon/2) - \delta(x + \epsilon/2)]\delta(y)\delta(z)$ is just (2.3.35), and in the limit as $\epsilon \to 0$ with $\epsilon C \equiv D =$ fixed, this solution tends to (2.3.38). But

$$\lim_{\substack{\epsilon \to 0 \\ \epsilon C \equiv D = \text{fixed}}} C\left\{\delta\left(x - \frac{\epsilon}{2}\right) - \delta\left(x + \frac{\epsilon}{2}\right)\right\} = -D\delta'(x), \tag{2.3.42}$$

as can be verified using any representation of the delta function, for example,

$$\delta(x) \approx \frac{1}{\sqrt{\pi\alpha}} e^{-x^2/\alpha} \tag{2.3.43}$$

with α small [see (A.1.11b)]. Therefore, (2.3.38) solves (2.3.41).

We can construct more complicated limiting singularities from 4, 6, ... sources of zero total strength in various configurations. See Problem 2.3.4. However, these do not play a fundamental role in solving Laplace's equation. As we shall see later on, source and dipole distributions are crucial in describing solutions of the two main boundary-value problems for Laplace's equation.

Problems

2.3.1 In this problem we review some ideas from vector calculus for curvilinear coordinates. Let the functions

$$x = f(\xi, \eta, \zeta), \tag{2.3.44a}$$

$$y = g(\xi, \eta, \zeta), \tag{2.3.44b}$$

$$z = h(\xi, \eta, \zeta), \tag{2.3.44c}$$

be prescribed in some domain in which the Jacobian determinant does not vanish—that is,

$$J \equiv \det \begin{pmatrix} f_\xi & f_\eta & f_\zeta \\ g_\xi & g_\eta & g_\zeta \\ h_\xi & h_\eta & h_\zeta \end{pmatrix} \neq 0. \tag{2.3.45}$$

We may regard ξ, η, ζ as curvilinear coordinates in the Cartesian xyz-space. Holding any of the coordinates ξ, η, or ζ constant in (2.3.44) defines a surface in xyz-space. Holding any pair of curvilinear coordinates constant in (2.3.44) defines a curve in xyz-space.

a. Describe geometrically the following curvilinear coordinate systems:

$$x = \xi \cos\eta, \; y = \xi \sin\eta, \; z = \zeta, \tag{2.3.46a}$$

$$x = \xi \sin\eta \cos\zeta, \; y = \xi \sin\eta \sin\zeta, \; z = \xi \cos\eta, \tag{2.3.46b}$$

$$x = a\cosh\xi \cos\eta, \; y = a\sinh\xi \sin\eta \sin\zeta,$$

$$z = a\sinh\xi \sin\eta \cos\zeta, \; a = \text{constant} > 0, \tag{2.3.46c}$$

$$x = \eta - \xi^3, \; y = \xi + \eta, \; z = \zeta. \tag{2.3.46d}$$

b. Show that the infinitesimal displacement vector from the point $P \equiv (\xi_0, \eta_0, \zeta_0)$ to the neighboring point $Q \equiv (\xi_0 + \Delta\xi, \eta_0, \zeta_0)$ along the direction of increasing ξ is given by

$$\mathbf{PQ} \equiv \left\{ f_\xi(\xi_0, \eta_0, \zeta_0)\mathbf{i} + g_\xi(\xi_0, \eta_0, \zeta_0)\mathbf{j} + h_\xi(\xi_0, \eta_0, \zeta_0)\mathbf{k} \right\} \Delta\xi + O(\Delta\xi^2), \tag{2.3.47}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are Cartesian unit vectors in the x, y, and z directions, respectively.

Therefore,

$$\mathbf{b}_1(\xi_0, \eta_0, \zeta_0) \equiv \lim_{\Delta\xi \to 0} \frac{\mathbf{PQ}}{\Delta\xi} = f_\xi(\xi_0, \eta_0, \zeta_0)\mathbf{i} + g_\xi(\xi_0, \eta_0, \zeta_0)\mathbf{j} + h_\xi(\xi_0, \eta_0, \zeta_0)\mathbf{k} \tag{2.3.48a}$$

is a tangent vector to the curve defined by (2.3.44) with $\eta = \eta_0, \zeta = \zeta_0$ at P; that is, $\mathbf{b}_1$ is a tangent vector in the direction of increasing ξ. Similarly,

$$\mathbf{b}_2(\xi_0, \eta_0, \zeta_0) = f_\eta(\xi_0, \eta_0, \zeta_0)\mathbf{i} + g_\eta(\xi_0, \eta_0, \zeta_0)\mathbf{j} + h_\eta(\xi_0, \eta_0, \zeta_0)\mathbf{k} \tag{2.3.48b}$$

and

$$\mathbf{b}_3(\xi_0, \eta_0, \zeta_0) = f_\zeta(\xi_0, \eta_0, \zeta_0)\mathbf{i} + g_\zeta(\xi_0, \eta_0, \zeta_0)\mathbf{j} + h_\zeta(\xi_0, \eta_0, \zeta_0)\mathbf{k} \tag{2.3.48c}$$

are tangent vectors in the η and ζ directions, respectively.

Denote the scalar product by

$$\mathbf{b}_i \cdot \mathbf{b}_j = g_{ij}, \quad i, j = 1, 2, 3. \tag{2.3.49}$$

Thus, $g_{ij} = g_{ji}$ in general.

Show that the tangent vectors associated with a curvilinear coordinate system (2.3.44) satisfying (2.3.45) are linearly independent and may therefore be regarded as a local basis at each point P. A curvilinear system is said to be orthogonal if $g_{ij} \equiv 0$ for $i \neq j$. One can then define an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, characterized by

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \quad \delta_{ij} = \text{Kronecker delta} \tag{2.3.50a}$$

by setting

$$\mathbf{e}_i = \frac{\mathbf{b}_i}{g_{ii}^{1/2}}, \quad i = 1, 2, 3. \tag{2.3.50b}$$

Calculate the $\{\mathbf{b}_i\}$ basis for each of the curvilinear coordinate systems (2.3.46), evaluate the g_{ij} in each case, and indicate which of these coordinate systems is orthogonal. Calculate the orthonormal basis associated with each orthogonal basis.

c. The infinitesimal displacement vector $d\mathbf{r}$ at a point P in an arbitrary direction has Cartesian components dx, dy, dz; that is,

$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}. \tag{2.3.51a}$$

Show that the components of $d\mathbf{r}$ with respect to the basis $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ defined in (2.3.48) are $d\xi$, $d\eta$, $d\zeta$; that is,

$$d\mathbf{r} = d\xi\mathbf{b}_1 + d\eta\mathbf{b}_2 + d\zeta\mathbf{b}_3. \tag{2.3.51b}$$

In view of the formal similarity between (2.3.51a) and (2.3.51b), the vectors $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ are sometimes called the *natural basis* for the curvilinear coordinate system (2.3.44). Express $d\mathbf{r}$ in terms of the $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ basis for each of the orthogonal systems in (2.3.46).

d. Consider the surface

$$x = f(\xi_1, \xi_2), \tag{2.3.52a}$$

$$y = g(\xi_1, \xi_2), \tag{2.3.52b}$$

$$z = h(\xi_1, \xi_2), \tag{2.3.52c}$$

defined in parameteric form (with the parameters ξ_1, ξ_2) in xyz-space. Such a surface could be defined, for example, by setting one of the curvilinear coordinates ξ, η, or ζ equal to zero in (2.3.44), in which case ξ_1 and ξ_2 describe the remaining two coordinates that vary on the surface. We define the tangent vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ on this surface in the usual way [see (2.3.48)]. Show that the infinitesimal area element dA on this surface is given by

$$dA \equiv |\mathbf{b}_1 \times \mathbf{b}_2| d\xi_1 d\xi_2, \tag{2.3.53}$$

and verify that this result gives

$$dA = \xi^2 \sin \eta d\eta d\zeta$$

for the surface defined by holding ξ = constant in the coordinate system (2.3.46b).

Therefore, given a function $F(x, y, z)$, the surface integral of this function on the portion S of the surface (2.3.52) is

$$I \equiv \iint_S F(f(\xi_1, \xi_2), g(\xi_1, \xi_2), h(\xi_1, \xi_2))dA, \tag{2.3.54}$$

where dA is given in (2.3.53). Use this result and (2.3.46b) to calculate I on the surface of the unit sphere for the case where $F = xy$.

e. Show that the infinitesimal volume element dV for the curvilinear system (2.3.44) is given by

$$dV \equiv |\mathbf{b}_1 \cdot \mathbf{b}_2 \times \mathbf{b}_3| d\xi d\eta d\zeta. \tag{2.3.55}$$

Therefore, the volume integral of some given function $F(x, y, z)$ over some domain G is given by

$$K \equiv \iiint_G F(f(\xi, \eta, \zeta), g(\xi, \eta, \zeta), h(\xi, \eta, \zeta))dV, \tag{2.3.56}$$

where dV is defined in (2.3.55). Specialize (2.3.56) to calculate the integral of $F = xy$ over the interior of the unit sphere.

f. Let F be a scalar function of position. The gradient of F (denoted by grad F) is a vector that is defined independently of the choice of coordinates by

$$dF = \text{grad } F \cdot d\mathbf{r}. \tag{2.3.57}$$

Here dF is the differential of F, $d\mathbf{r}$ is the infinitesimal displacement vector (2.3.51), and a dot denotes the scalar product.

Show that using Cartesian coordinates, setting $F = \phi(x, y, z)$, and expressing grad F in terms of its components with respect to the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ basis in (2.3.57) gives

$$\text{grad } F = \phi_x \mathbf{i} + \phi_y \mathbf{j} + \phi_z \mathbf{k}. \tag{2.3.58}$$

The notation

$$\nabla F \equiv \text{grad } F \tag{2.3.59}$$

is also quite prevalent. This notation is motivated by the expression (2.3.58) for the Cartesian component representation of the gradient, which may be expressed in terms of the "del" operator

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}. \tag{2.3.60}$$

Now express F in terms of the curvilinear coordinates (2.3.44)

$$F(f(\xi, \eta, \zeta), g(\xi, \eta, \zeta), h(\xi, \eta, \zeta)) \equiv \psi(\xi, \eta, \zeta), \tag{2.3.61}$$

and denote

$$\text{grad } F = \Gamma_1 \mathbf{b}_1 + \Gamma_2 \mathbf{b}_2 + \Gamma_3 \mathbf{b}_3. \tag{2.3.62}$$

Substituting (2.3.61), (2.3.62), and (2.3.51b) into (2.3.57) shows that we must have

$$\begin{aligned} \psi_\xi d\xi + \psi_\eta d\eta + \psi_\zeta d\zeta &= (\Gamma_1 \mathbf{b}_1 + \Gamma_2 \mathbf{b}_2 + \Gamma_3 \mathbf{b}_3) \cdot (d\xi \mathbf{b}_1 + d\eta \mathbf{b}_2 \\ &\qquad + d\zeta \mathbf{b}_3) \\ &= (\Gamma_1 g_{11} + \Gamma_2 g_{12} + \Gamma_3 g_{13}) d\xi \\ &\quad + (\Gamma_1 g_{12} + \Gamma_2 g_{22} + \Gamma_3 g_{23}) d\eta \\ &\quad + (\Gamma_1 g_{13} + \Gamma_2 g_{23} + \Gamma_3 g_{33}) d\zeta. \end{aligned} \tag{2.3.63}$$

Identifying the multipliers of $d\xi$, $d\eta$, and $d\zeta$ on both sides of (2.3.63) gives three linear algebraic equations for the three unknowns Γ_1, Γ_2, Γ_3. Solve these to obtain

$$\begin{aligned} \Gamma_1 = \frac{1}{M}[(g_{22}g_{33} - g_{23}^2)\psi_\xi + (g_{13}g_{23} - g_{12}g_{33})\psi_\eta \\ + (g_{12}g_{23} - g_{13}g_{22})\psi_\zeta], \end{aligned} \tag{2.3.64a}$$

where M is the determinant of the matrix $\{g_{ij}\}$; that is,

$$M \equiv g_{11}g_{22}g_{33} - (g_{11}g_{23}^2 + g_{22}g_{13}^2 + g_{33}g_{12}^2) + 2g_{12}g_{13}g_{23}.$$

Permute indices $1 \to 2, 2 \to 3, 3 \to 1$ and variables $\xi \to \eta, \eta \to \zeta$, $\zeta \to \xi$ to obtain

$$\Gamma_2 = \frac{1}{M}[(g_{23}g_{13} - g_{12}g_{33})\psi_\xi + (g_{11}g_{33} - g_{13}^2)\psi_\eta + (g_{13}g_{12} - g_{11}g_{23})\psi_\zeta]. \quad (2.3.64b)$$

A second permutation gives

$$\Gamma_3 = \frac{1}{M}[(g_{12}g_{23} - g_{22}g_{13})\psi_\xi + (g_{13}g_{12} - g_{23}g_{11})\psi_\eta + (g_{22}g_{11} - g_{12}^2)\psi_\zeta]. \quad (2.3.64c)$$

Verify that for an orthogonal curvilinear system, (2.3.64) simplifies to give

$$\text{grad } F = \frac{1}{g_{11}}\psi_\xi \mathbf{b}_1 + \frac{1}{g_{22}}\psi_\eta \mathbf{b}_2 + \frac{1}{g_{33}}\psi_\zeta \mathbf{b}_3 \quad (2.3.65a)$$

or

$$\text{grad } F = \frac{1}{\sqrt{g_{11}}}\psi_\xi \mathbf{e}_1 + \frac{1}{\sqrt{g_{22}}}\psi_\eta \mathbf{e}_2 + \frac{1}{\sqrt{g_{33}}}\psi_\zeta \mathbf{e}_3 \quad (2.3.65b)$$

in terms of the $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ basis (2.3.50b).

Calculate the gradient of $x^2 + 2xy^2 + z^3$ in the curvilinear coordinates (2.3.46b) and (2.3.46d).

g. Suppose the vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is an arbitrarily specified unit vector at the point $P = (x, y, z)$. Let $F = \phi(x, y, z)$ be a given scalar function. Consider a point Q infinitesimally close to P *along* the vector $\mathbf{a}$ from P; that is, $\mathbf{PQ} = \mathbf{a}\Delta s$, where Δs is small. Now, the derivative of ϕ in the $\mathbf{a}$ direction is by definition

$$\lim_{\Delta s \to 0} \frac{1}{\Delta s}[\phi(x + a_1\Delta s, y + a_2\Delta s, z + a_3\Delta s) - \phi(x, y, z)] = \phi_x a_1 + \phi_y a_2 + \phi_z a_3 \equiv \frac{d\phi}{da}. \quad (2.3.66a)$$

We call $d\phi/da$ the *directional derivative* of ϕ in the $\mathbf{a}$ direction. It follows from (2.3.66a) and (2.3.58) that $d\phi/da$ is

$$\frac{d\phi}{da} \equiv \text{grad } \phi \cdot \mathbf{a}, \quad (2.3.66b)$$

independently of the choice of coordinates.

Use (2.3.66) to argue that grad ϕ is a vector normal to the surface $\phi =$ constant at each point on this surface.

h. The divergence of a vector field **W** is a scalar field denoted by div **W** and defined for each point P in the following form, which is independent of the choice of coordinates:

$$\operatorname{div}\mathbf{W} \equiv \lim_{\tau\to 0}\frac{1}{\tau}\iint_S \mathbf{W}\cdot\mathbf{n}\,dA. \tag{2.3.67}$$

Here τ is the volume of an arbitrary domain containing P, S is the surface of τ, **n** is the unit outward normal to S, and dA is the infinitesimal area element on S.

Let **W** have Cartesian components W_1, W_2, W_3, that is,

$$\mathbf{W} = W_1(x,y,z)\mathbf{i} + W_2(x,y,z)\mathbf{j} + W_3(x,y,z)\mathbf{k}. \tag{2.3.68}$$

Choose a prism with sides Δx, Δy, Δz surrounding P to show that

$$\operatorname{div}\mathbf{W} = \frac{\partial W_1}{\partial x} + \frac{\partial W_2}{\partial y} + \frac{\partial W_3}{\partial z}. \tag{2.3.69a}$$

This result for the Cartesian form of div **W** may be written in terms of the ∇ operator as

$$\nabla\cdot\mathbf{W} = \frac{\partial W_1}{\partial x} + \frac{\partial W_2}{\partial y} + \frac{\partial W_3}{\partial z}. \tag{2.3.69b}$$

The definition (2.3.67) may be used to derive the expression for div **W** when **W** is expressed in terms of its components with respect to an arbitrary curvilinear coordinate system. Consider here the special case where the curvilinear system (2.3.44) is orthogonal and denote

$$\mathbf{W} = W_1^*(\xi,\eta,\zeta)\mathbf{e}_1 + W_2^*(\xi,\eta,\zeta)\mathbf{e}_3 + W_3^*(\xi,\eta,\zeta)\mathbf{e}_3, \tag{2.3.70}$$

where the $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the orthonormal unit vectors in the ξ, η, ζ directions defined by (2.3.50b). Denote

$$h_i \equiv (g_{ii})^{1/2}, \quad i = 1,2,3,$$

and show that

$$\operatorname{div}\mathbf{W} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\xi}(h_2h_3W_1^*) + \frac{\partial}{\partial\eta}(h_1h_3W_2^*) + \frac{\partial}{\partial\zeta}(h_1h_2W_3^*)\right]. \tag{2.3.71}$$

Specialize (2.3.71) for the cases (2.3.46b) and (2.3.46c).

i. The Laplacian of a scalar field F is denoted by ΔF and is defined by

$$\Delta F \equiv \operatorname{div}(\operatorname{grad} F). \tag{2.3.72}$$

Again assume that (2.3.44) is orthogonal and show that [see (2.3.65)]

$$\Delta F = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\xi}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{h_1h_3}{h_2}\frac{\partial\psi}{\partial\eta}\right)\right.$$

$$+ \left. \frac{\partial}{\partial \zeta} \left(\frac{h_1 h_2}{h_3} \frac{\partial \psi}{\partial \zeta} \right) \right] . \tag{2.3.73}$$

Specialize this result to the cases (2.3.46b) and (2.3.46c).

In view of the del notation for the gradient and the divergence, the Laplacian is often denoted by ∇^2. The del notation will not be used in this book.

2.3.2 The fundamental solution of the n-dimensional Laplace equation solves

$$\frac{d^2u}{dr^2} + \frac{n-1}{r}\frac{du}{dr} = \delta_n(r), \tag{2.3.74}$$

where δ_n is the n-dimensional delta function defined in (1.6.6)–(1.6.9).

a. Show that if $u(\infty) = 0$, the solution of (2.3.74) for $r > 0$ is

$$u = \frac{C_n}{r^{n-2}}, \quad n \neq 2, \tag{2.3.75}$$

where C_n is a constant. Use the n-dimensional Gauss theorem to evaluate the left-hand side of (2.3.74), and show that

$$C_n = \frac{1}{(2-n)\omega_n}, \quad n \geq 3, \tag{2.3.76}$$

where ω_n is the surface area of the n-dimensional unit sphere defined by (1.6.13). Note the singularity for $n = 2$.

2.3.3a. Show that the constant B_n in (2.3.30a) is given by

$$B_n = -\frac{\pi\Gamma\left(\frac{n}{2}\right)\left(\frac{\sqrt{\lambda}}{2}\right)^{\frac{n-2}{2}}}{\left(\frac{n}{2}-2\right)!(2-n)2\pi^{n/2}} \quad \text{if } n \geq 4,\ n \text{ even}, \tag{2.3.77a}$$

$$B_2 = \frac{1}{4}. \tag{2.3.77b}$$

Verify that as $\lambda \to 0$, the result in (2.3.30a) with B_n as defined above reduces to (2.3.75)–(2.3.76).

b. Show that for $\lambda < 0$ the fundamental solution of the modified Helmholtz equation is given by

$$w(r) = D_n r^{-\frac{n-2}{2}} K_{\frac{n-2}{2}}(\sqrt{-\lambda}r), \tag{2.3.78}$$

where

$$D_n = \frac{2^{\frac{4-n}{2}}\Gamma\left(\frac{n}{2}\right)}{\left(\frac{n}{2}-2\right)!(2-n)2\pi^{n/2}}, \quad n \geq 3, \tag{2.3.79a}$$

$$D_2 = -\frac{1}{2\pi}. \tag{2.3.79b}$$

2.3.4 Consider the three-dimensional Helmholtz equation

$$u_{xx} + u_{yy} + u_{zz} + \lambda u = 0, \tag{2.3.80}$$

where $\lambda = \text{constant} > 0$.

a. Calculate the fundamental solution directly (without using the identity for $J_{-1/2}$ in (2.3.30b)) in the form

$$F(P, Q) = -\frac{1}{4\pi} \frac{\cos(\lambda^{1/2} r_{PQ})}{r_{PQ}}, \tag{2.3.81}$$

where $P = (x, y, z)$, $Q = (\xi, \eta, \zeta)$, and $r_{PQ}^2 = (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2$.

b. Use the method of descent to show that the fundamental solution of the two-dimensional Helmholtz equation for a source located at $x = y = 0$ is

$$u(r) = -\frac{1}{2\pi} \int_0^\infty \frac{\cos\sqrt{\lambda(r^2 + \zeta^2)}}{\sqrt{r^2 + \zeta^2}} d\zeta, \tag{2.3.82}$$

where $r = \sqrt{x^2 + y^2}$. Use the integral representation for Y_0,

$$Y_0(\lambda^{1/2} r) = -\frac{2}{\pi} \int_0^\infty \cos(\lambda^{1/2} r \cosh s) ds, \; r > 0, \tag{2.3.83}$$

to show that (2.3.82) reduces to (2.3.30a) for $n = 2$.

2.3.5 Consider a dipole of strength D, oriented along the x-axis and located at the point $x = \frac{\epsilon}{2}$, $y = 0$, $z = 0$, and a dipole of strength $-D$, oriented along the x-axis and located at the point $x = -\frac{\epsilon}{2}$, $y = 0$, $z = 0$. Write down the expression for the potential of these two dipoles. Take the limit $\epsilon \to 0$, $\epsilon D \equiv E =$ fixed, and show that the limiting potential is given by

$$u = -\frac{E}{4\pi r^3} (3\cos^2\theta - 1). \tag{2.3.84}$$

This configuration is a quadrupole. Show that u satisfies

$$\Delta u = \delta''(x)\delta(y)\delta(z). \tag{2.3.85}$$

2.4 Volume, Surface, and Line Distributions of Sources and Dipoles

In this section we study the effect of distributing sources and dipoles in various configurations. These distributions may directly represent an actual physical state. For example, as discussed in Section 2.4.1, a continuous distribution of mass sources of variable strength defines the gravitational field of a given body, or a distribution of stationary charges in space defines an electrostatic field. In other applications, a distribution of singularities may be used to simulate a given physical situation. For example, the flow past a nonlifting body of revolution may be represented by an appropriate distribution of positive and negative sources of mass along the axis. This simple problem is worked out in Section 2.4.3, and the more general case is discussed in Section 2.7.

2.4.1 Volume Distribution of Sources

Consider the force of gravity acting on a point of mass m located at $P = (x, y, z)$ due to a point of mass μ located at $Q = (\xi, \eta, \zeta)$. According to Newton's law of gravitation, this force $\mathbf{f}$ is given by

$$\mathbf{f} = -\frac{\gamma m \mu}{r_{PQ}^2} \frac{\mathbf{r}_{PQ}}{r_{PQ}}, \tag{2.4.1}$$

where γ (6.67×10^{-8} dyne cm^2 g^{-2}) is the universal gravitational constant and $\mathbf{r}_{PQ}$ is the displacement vector from Q to P. Thus, $\mathbf{f}$ is in the direction opposite $\mathbf{r}_{PQ}$ in Figure 2.8.

Clearly, the specific force $\mathbf{f}/m$ can be derived from the potential V according to

$$\frac{\mathbf{f}}{m} = -\operatorname{grad}_P V(P, Q), \tag{2.4.2}$$

where the subscript P indicates that partial derivatives are taken with respect to the x, y, z coordinates and V is given by

$$V(P, Q) \equiv -\frac{\gamma \mu}{r_{PQ}}. \tag{2.4.3}$$

Thus, V obeys [compare with (2.3.9) and (2.3.10)]

$$\Delta_P V = 4\pi \mu \gamma \delta_3(P, Q). \tag{2.4.4}$$

If we now have an arbitrary distribution of mass (with density $= \rho(x, y, z)$) in some domain G, the gravitational potential at some point P obeys

$$\Delta V = 4\pi \gamma \rho(x, y, z). \tag{2.4.5}$$

Since the fundamental solution (2.3.10) satisfies $\Delta u = \delta_3(P, Q)$, the solution for V is given by superposition in the form of a volume distribution of (mass)

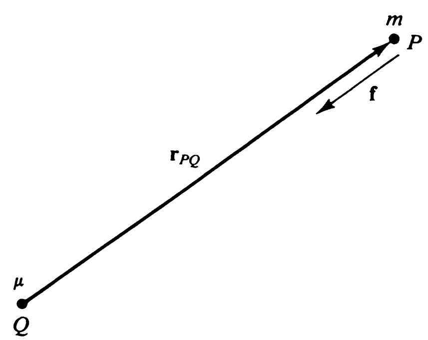

FIGURE 2.8. Gravitational force of Q on P

sources of strength/unit volume (density) ρ:

$$V(x, y, z) = -\gamma \iiint_G \frac{\rho(\xi, \eta, \zeta) d\xi\, d\eta\, d\zeta}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}}. \tag{2.4.6}$$

Some examples are outlined in Problems 2.4.1 and 2.4.2.

2.4.2 Surface and Line Distribution of Sources or Dipoles

Consider now a distribution of sources on a prescribed surface S defined parametrically by

$$x = f(s_1, s_2), \quad y = g(s_1, s_2), \quad z = h(s_1, s_2). \tag{2.4.7}$$

See Problem 2.3.1. If the source strength/unit area is $q(s_1, s_2)$, the potential at a point $P = (x, y, z)$ is just

$$u(x, y, z) = -\frac{1}{4\pi} \iint_S \frac{q(s_1, s_2) dA}{\sqrt{(x-f)^2 + (y-g)^2 + (z-h)^2}}, \tag{2.4.8}$$

where dA is the element of area on S; that is,

$$dA = |\mathbf{b}_1 \times \mathbf{b}_2| ds_1 ds_2, \tag{2.4.9}$$

with $\mathbf{b}_1$ and $\mathbf{b}_2$ the tangent vectors

$$\mathbf{b}_1 = \frac{\partial f}{\partial s_1}\mathbf{i} + \frac{\partial g}{\partial s_1}\mathbf{j} + \frac{\partial h}{\partial s_1}\mathbf{k}, \tag{2.4.10a}$$

$$\mathbf{b}_2 = \frac{\partial f}{\partial s_2}\mathbf{i} + \frac{\partial g}{\partial s_2}\mathbf{j} + \frac{\partial h}{\partial s_2}\mathbf{k}. \tag{2.4.10b}$$

If we have a surface distribution of dipoles of strength $p(s_1, s_2)$/unit area, oriented along the unit vector $\mathbf{a}(s_1, s_2) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, the potential w is

$$\begin{aligned} w &= -\frac{1}{4\pi} \iint_S p(s_1, s_2) \operatorname{grad}_Q \left(\frac{1}{r_{PQ}}\right) \cdot \mathbf{a}\, dA \\ &= -\frac{1}{4\pi} \iint_S \frac{p(s_1, s_2)[(x-f)a_1 + (y-g)a_2 + (z-h)a_3] dA}{[(x-f)^2 + (y-g)^2 + (z-h)^2]^{3/2}}. \end{aligned} \tag{2.4.11}$$

Similarly, if C is the curve defined by

$$x = f(s), \quad y = g(s), \quad z = h(s), \tag{2.4.12}$$

we can compute the potential due to a distribution of sources or dipoles along C in the following forms:

For sources of strength $q(s)$/unit length, we have

$$u = -\frac{1}{4\pi} \int_C \frac{q(s) ds}{[(x-f)^2 + (y-g)^2 + (z-h)^2]^{1/2}}, \tag{2.4.13}$$

and for dipoles of strength $p(s)$/unit length oriented along $\mathbf{a}(s)$, we have

$$w = -\frac{1}{4\pi}\int_C \frac{p(s)[(x-f)a_1 + (y-g)a_2 + (z-h)a_3]ds}{[(x-f)^2 + (y-g)^2 + (z-h)^2]^{3/2}}, \tag{2.4.14}$$

where ds is the element of arc along C.

2.4.3 An Example: Flow over a Nonlifting Body of Revolution

As an illustration of the use of (2.4.13), consider the problem of computing the axisymmetric flow of an incompressible irrotational fluid outside a body of revolution defined by $r \equiv \sqrt{z^2 + y^2} = F(x)$. (See Figure 2.9.) We assume that the flow at $x = -\infty$ is uniform, $U = \mathbf{i}$ in dimensionless units, and we represent the velocity potential for the flow outside the body by its uniform part, $u = x$, plus a disturbance potential in the form

$$u(x, r) = x - \frac{1}{4\pi}\int_0^1 \frac{S(\xi)d\xi}{[(x-\xi)^2 + r^2]^{1/2}}. \tag{2.4.15}$$

Thus, the disturbance potential, which need not be small for finite x and r, is assumed to be due to an *axial* distribution of sources of strength $S(x)$ per unit length along the unit interval. In this example, the sources, which are located inside the body, are outside the domain where we wish to solve for u. Hence, the potential defined by (2.4.15) satisfies $\Delta u = 0$ outside the body, and it is also easily seen that $u \to x$ as $r \to \infty$ or $|x| \to \infty$.

We shall see in Section 2.7 that the problem we wish to solve is a special case of a general boundary-value problem (Neumann's problem) for which the

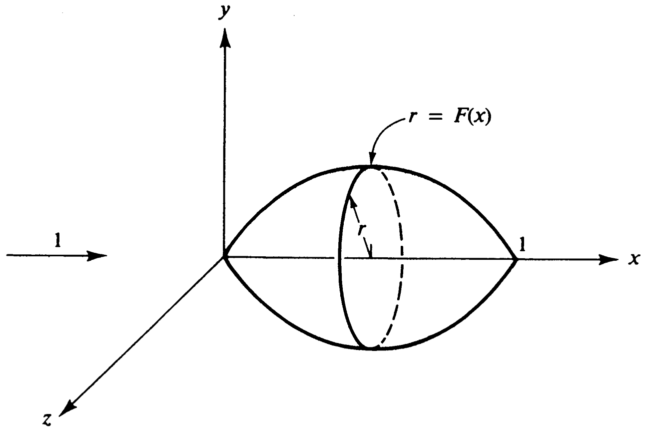

FIGURE 2.9. Nonlifting axisymmetric body

assumed form of solution is appropriate. The question now is how to choose $S(x)$ so that the flow defined by the potential (2.4.15) is tangent to the given surface $r = F(x)$—that is,

$$\frac{u_r(x, F(x))}{u_x(x, F(x))} = F'(x). \tag{2.4.16}$$

This boundary condition is clearly necessary for a frictionless flow at a solid boundary. Using (2.4.15) gives

$$\int_0^1 \frac{[F(x) - F'(x)(x - \xi)]S(\xi)}{[(x - \xi)^2 + F^2(x)]^{3/2}} d\xi = 4\pi F'(x). \tag{2.4.17}$$

Equation (2.4.17) is an integral equation for the unknown $S(\xi)$ in the form

$$\int_0^1 k(x, \xi)S(\xi)d\xi = G(x), \tag{2.4.18}$$

where the kernel k is the following given function of x and ξ:

$$k(x, \xi) \equiv \frac{F(x) - F'(x)(x - \xi)}{[(x - \xi)^2 + F^2(x)]^{3/2}}, \tag{2.4.19}$$

and $G(x) \equiv 4\pi F'(x)$ is also given.

One can solve (2.4.18) numerically by discretizing the integral. Subdivide the unit interval $0 \leq x \leq 1$ into N equal parts and denote

$$\begin{aligned} x_i &\equiv \frac{i}{N}, \ \xi_i \equiv \frac{i}{N}, \ i = 1, \ldots, N - 1, \\ S_i &\equiv S(\xi_i), \ G_i \equiv G(x_i), \\ k_{ij} &\equiv k(x_i, \xi_j). \end{aligned} \tag{2.4.20}$$

Using the trapezoidal rule gives the following system of $N - 1$ linear algebraic equations:

$$\sum_{j=1}^{N-1} k_{ij}S_j = NG_i, \quad i = 1, \ldots, N - 1. \tag{2.4.21}$$

Solving these determines the $N - 1$ unknowns S_j; as a check, we verify that $\sum_{j=1}^{N-1} S_j = 0$, since we have a closed body. *Note*: $S(0) = S(1) = 0$.

This result is studied for the case $N = 4$ in Problem 2.4.3. The perturbation solution for the case of a slender body is given in Section 8.2.4.

2.4.4 Limiting Surface Values for Source and Dipole Distributions

In the applications discussed in Section 2.7, we shall represent the potential at a point $P = (x, y, z)$ by a surface distribution of sources or dipoles as given by the integral expressions (2.4.8) or (2.4.11), respectively. In these problems we shall need to satisfy a specified boundary condition on the surface S itself. For example, we require either the potential or its normal derivative to equal a specified function on S. We note, however, that if P is on S, the integrals (2.4.8) and (2.4.11) (as well as the expressions that result from these for the normal derivative of the potential) become improper because the denominators vanish at P. Corresponding singularities also occur in the integrals (2.4.13) and (2.4.14) for source or dipole distributions on a curve C. This difficulty did not arise in the example worked out in Section 2.4.3 because the sources were distributed on the x-axis, whereas the boundary condition was evaluated on the surface $r = F(x) \geq 0$, which is off the axis in the interval $0 < x < 1$. In general, we cannot avoid evaluating a boundary condition on the surface over which sources or dipoles are distributed.

Even though letting P be on S leads to an improper integral, the actual potential (or its normal derivative) is well behaved in this limit. In fact, if we first evaluate the integral for P, not on S, and then let P approach S, the result is perfectly well defined; it is only when the limit is imposed on the integral representation that we encounter a difficulty.

(i) *Potential due to a planar dipole distribution of constant strength*

To illustrate this situation, consider the special case of a dipole distribution having a strength per unit area equal to a constant p_0 over the entire $z = 0$ plane. The axes of the dipoles are taken in the $+z$ direction, so (2.4.11) (with $p = p_0 =$ constant, $f = s_1 = \xi, g = s_2 = \eta, h = 0, a_1 = a_2 = 0, a_3 = 1$) specializes to

$$w(x, y, z) = -\frac{p_0 z}{4\pi} \int_{\eta=-\infty}^{\infty} \int_{\xi=-\infty}^{\infty} \frac{d\xi\, d\eta}{[(x-\xi)^2 + (y-\eta)^2 + z^2]^{3/2}}. \tag{2.4.22a}$$

Now, if we set $z = 0$ in the integrand of (2.4.22a), the double integral has a nonintegrable singularity at $\xi = x, \eta = y$. This divergent expression is then multiplied by z, which equals zero, and it is not helpful to set $z = 0$ directly in (2.4.22a). However, if we evaluate (2.4.22a) for $z \neq 0$, then let $z \to 0$, the limit exists. For this simple example it is easy to evaluate w for $z \neq 0$. To do so, we introduce $\overline{\xi} = \xi - x$ and $\overline{\eta} = \eta - y$ as integration variables and observe, as expected, that w does not depend on x or y; that is,

$$w(x, y, z) = -\frac{p_0 z}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\overline{\xi}\, d\overline{\eta}}{(\overline{\xi}^2 + \overline{\eta}^2 + z^2)^{3/2}}, \quad z \neq 0,$$

or using polar coordinates (ρ, ϕ) defined by $\overline{\xi} = \rho \cos\phi, \overline{\eta} = \rho \sin\phi$, we obtain

$$\begin{aligned} w(x, y, z) &= -\frac{p_0 z}{4\pi} \int_{\rho=0}^{\infty} \int_{\phi=0}^{2\pi} \frac{\rho\, d\phi\, d\rho}{(\rho^2 + z^2)^{3/2}}, \quad z \neq 0, \\ &= -\frac{p_0 z}{2} \int_{\rho=0}^{\infty} \frac{\rho\, d\rho}{(\rho^2 + z^2)^{3/2}} = \frac{p_0 z}{2} \frac{1}{(\rho^2 + z^2)^{1/2}} \Bigg|_{\rho=0}^{\rho=\infty} \\ &= -\frac{p_0 z}{2} \frac{1}{|z|} = \begin{cases} -p_0/2 & \text{if } z > 0, \\ p_0/2 & \text{if } z < 0. \end{cases} \end{aligned} \tag{2.4.22b}$$

Thus, w is a constant in each half-space, and the limiting value of w as $z \to 0^+$ or $z \to 0^-$ is well-defined; it is just $-p_0/2$ or $p_0/2$, respectively.

In this simple example it was possible to evaluate w explicitly and then take the limit as P approaches S. In general, an explicit result will not be feasible, and we shall need to calculate the limiting expression for the potential or its normal derivative on S by a construction that involves the integral expression itself.

(ii) *Potential due to a planar dipole distribution of variable strength*

To illustrate ideas, let us consider the more general problem of a dipole distribution on the simply connected domain D, that lies in the $z = 0$ plane and has the boundary Γ. Again, we assume that the dipole axes are all normal to the plane D and let their strength per unit area be a specified function $p(x, y)$. The potential w at $P = (x, y, z)$ is then given by

$$w(x, y, z) = -\frac{z}{4\pi} \iint_D \frac{p(\xi, \eta) d\xi \, d\eta}{[(x-\xi)^2 + (y-\eta)^2 + z^2]^{3/2}}, \tag{2.4.23}$$

which generalizes (2.4.22a).

As shown in Figure 2.10, we subdivide the integration domain D into two parts: (1) the interior D_ϵ of a circle of radius ϵ centered at $\xi = x, \eta = y, \zeta = 0$, with ϵ sufficiently small so that D_ϵ is entirely contained in D, and (2) the remainder $D_a = D - D_\epsilon$. Thus, (2.4.23) will now involve two contributions, one from D_ϵ, denoted by w_ϵ, and the other from D_a, denoted by w_a; that is,

$$w(x, y, z) = w_\epsilon(x, y, z; \epsilon) + w_a(x, y, z; \epsilon). \tag{2.4.24}$$

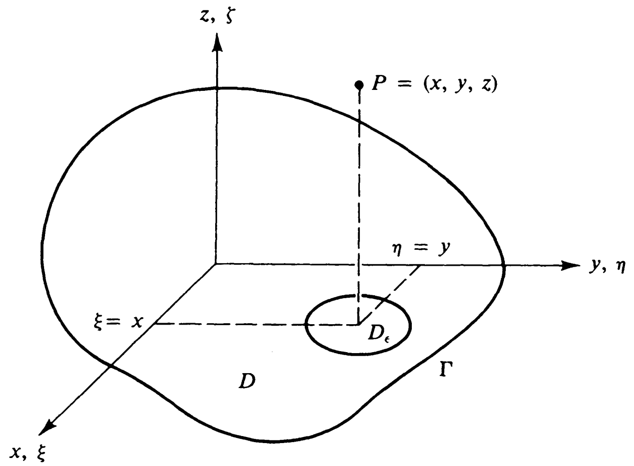

FIGURE 2.10. Decomposition of D for (2.4.23)

We introduce a local polar coordinate system

$$\xi = x + \rho \cos\phi, \;\; \eta = y + \rho \sin\phi,$$

and express w_ϵ and w_a in the form

$$w_\epsilon = -\frac{z}{4\pi}\int_0^{2\pi}\int_0^{\epsilon}\frac{p(x+\rho\cos\phi, y+\rho\sin\phi)}{(\rho^2+z^2)^{3/2}}\rho\, d\rho\, d\phi, \qquad (2.4.25a)$$

$$w_a = -\frac{z}{4\pi}\int_0^{2\pi}\int_\epsilon^{R(\phi,x,y)}\frac{p(x+\rho\cos\phi, y+\rho\sin\phi)}{(\rho^2+z^2)^{3/2}}\rho\, d\rho\, d\phi. (2.4.25b)$$

Here $\rho = R(\phi, x, y)$ is the expression defining the distance between the fixed point (x, y) and a point on Γ in polar coordinates. For example, if Γ is the unit circle centered at the origin, we see from Figure 2.11 that

$$\rho \sin\phi + y = \sin\theta,$$
$$\rho \cos\phi + x = \cos\theta.$$

Therefore, eliminating θ and solving the resulting quadratic for ρ defines R in the form $R(\phi, x, y) = -x\cos\phi - y\sin\phi + [1 - (x\sin\phi - y\cos\phi)^2]^{1/2}$, where $R > 0$ if (x, y) is inside D.

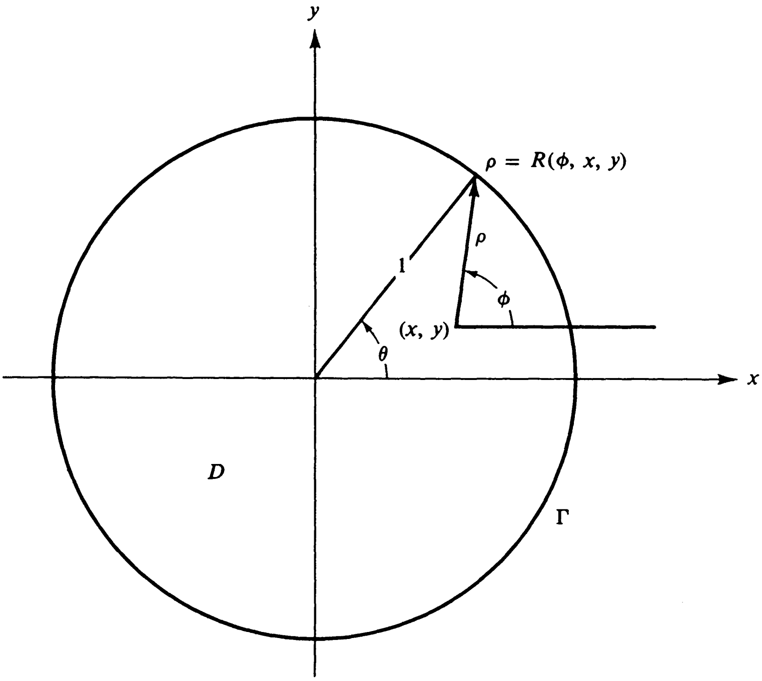

FIGURE 2.11. Local polar coordinates at (x, y)

The individual contributions w_ϵ and w_a depend on ϵ, but their sum does not. The basic idea for our calculation of the limiting value of w as z approaches zero is to regard z as some as yet unspecified function α of ϵ, to be chosen so as to simplify the calculation of the integrals (2.4.25) when z is small. We then obtain the limiting value of w from

$$w(x, y, 0) = \lim_{\epsilon \to 0}\{w_\epsilon(x, y, \alpha(\epsilon); \epsilon) + w_a(x, y, \alpha(\epsilon); \epsilon)\}, \tag{2.4.26}$$

where $\alpha \to 0$ as $\epsilon \to 0$.

Consider first (2.4.25a) for w_ϵ. Since the maximum value of ρ is ϵ, we rescale this variable by setting $\rho = \epsilon\tilde{\rho}$, so that $\tilde{\rho}$ now varies over the interval (0, 1). This gives

$$w_\epsilon(x, y, \alpha; \epsilon) = -\frac{(\alpha/\epsilon)}{4\pi}\int_0^{2\pi}\int_0^1 \frac{p(x + \epsilon\tilde{\rho}\cos\phi, y + \epsilon\tilde{\rho}\sin\phi)}{[\tilde{\rho}^2 + (\alpha/\epsilon)^2]^{3/2}}\tilde{\rho}\, d\tilde{\rho}\, d\phi. \tag{2.4.27}$$

Assuming that p is analytic, we can expand

$$\begin{aligned} p(x + \epsilon\tilde{\rho}\cos\phi, y + \epsilon\tilde{\rho}\sin\phi) &= p(x, y) + \epsilon p_x(x, y)\tilde{\rho}\cos\phi \\ &\quad + \epsilon p_y(x, y)\tilde{\rho}\sin\phi + O(\epsilon^2\tilde{\rho}^2). \end{aligned}$$

Now interchanging the order of integration in (2.4.27) shows that the terms proportional to $\sin\phi$ and $\cos\phi$ do not contribute. (In fact, if the higher-order terms in the series for p are included, only the averages of the various products of trigonometric functions will contribute.) Therefore, the integral (2.4.27) has the approximation

$$w_\epsilon(x, y, \alpha; \epsilon) = -\frac{(\alpha/\epsilon)}{2}\int_0^1 \frac{\tilde{\rho}\, d\tilde{\rho}}{[\tilde{\rho}^2 + (\alpha/\epsilon)^2]^{3/2}}[p(x, y) + O(\epsilon^2\tilde{\rho}^2)]. \tag{2.4.28a}$$

We can evaluate this integral [see (2.4.22b)] and obtain

$$w_\epsilon(x, y, \alpha; \epsilon) = \frac{p(x, y)}{2}\left\{\frac{(\alpha/\epsilon)}{[1 + (\alpha/\epsilon)^2]^{1/2}} - \frac{(\alpha/\epsilon)}{|\alpha/\epsilon|}\right\} + O(\alpha\epsilon). \tag{2.4.28b}$$

So far, we have assumed only that $\alpha(\epsilon) \to 0$; we have made no assumption regarding the behavior of α/ϵ as $\epsilon \to 0$. It is clear from (2.4.28b) that if $(\alpha/\epsilon) \to 0$ as $\epsilon \to 0$, the limiting value of w_ϵ will be independent of (α/ϵ) and is simply given by

$$w_\epsilon(x, y, 0^+; \epsilon) = -\frac{p(x, y)}{2}, \tag{2.4.29a}$$

$$w_\epsilon(x, y, 0^+; \epsilon) = +\frac{p(x, y)}{2}. \tag{2.4.29b}$$

We now show that the choice $(\alpha/\epsilon) \to 0$ also simplifies the calculation for the limiting value of w_a. Changing variables from ρ to $\tilde{\rho}$ in (2.4.25b) gives

$$w_a = -\frac{(\alpha/\epsilon)}{4\pi}\int_0^{2\pi}\int_1^{R(\phi,x,y)/\epsilon} \frac{p(x + \epsilon\tilde{\rho}\cos\phi, y + \epsilon\tilde{\rho}\sin\phi)}{[\tilde{\rho}^2 + (\alpha/\epsilon)^2]^{3/2}}\tilde{\rho}\, d\tilde{\rho}\, d\phi.$$

For $\epsilon \neq 0$, the denominator in the integrand does not vanish over the interval $1 \leq \tilde{\rho} \leq R/\epsilon$, so that the double integral exists, and the factor (α/ϵ) in front implies that $w_a = O(\alpha/\epsilon)$ as $\epsilon \to 0$. Therefore, the choice $(\alpha/\epsilon) \to 0$ gives

$$w_a(x, y, 0; 0) = 0,$$

and we conclude that

$$w(x, y, 0^+) = -\frac{p(x, y)}{2}, \qquad (2.4.30a)$$

$$w(x, y, 0^-) = +\frac{p(x, y)}{2}. \qquad (2.4.30b)$$

We reiterate that the final result (2.4.30) does not depend on the limiting behavior of α; the choice $(\alpha/\epsilon) \to 0$ is made to simplify the calculations. We illustrate this point by reconsidering the simple example $p = p_0 =$ constant, $R = \infty$ discussed earlier. Expressing the integral in (2.4.22b) in terms of the decomposition (2.4.24) gives the exact result

$$w_\epsilon(x, y, \alpha; \epsilon) = \frac{p_0}{2}\left\{\frac{(\alpha/\epsilon)}{[1 + (\alpha/\epsilon)^2]^{1/2}} - \frac{(\alpha/\epsilon)}{|\alpha/\epsilon|}\right\}, \qquad (2.4.31a)$$

$$w_a(x, y, \alpha; \epsilon) = -\frac{p_0}{2}\frac{(\alpha/\epsilon)}{[1 + (\alpha/\epsilon)^2]^{1/2}}. \qquad (2.4.31b)$$

We see that regardless of the choice of the limiting value of (α/ϵ), the first terms in each of the expressions on the right-hand sides of (2.4.31) have opposite signs and cancel in the sum $w_\epsilon + w_a$. For the choice $(\alpha/\epsilon) \to 0$, we have that $w_a \to 0$, and the limiting value of w is the same as the limiting value of w_ϵ.

We have shown in (2.4.30) that for a planar dipole distribution, the potential at any point on the surface depends on the *local* dipole strength $p(x, y)$ only. This result is also easily justified geometrically by recalling that the potential of a dipole vanishes all along the plane normal to the dipole axis (cf. (2.3.7)). Thus, all the dipoles located at points $(x_1, y_1, 0) \neq (x, y, 0)$ do not contribute to the potential at $(x, y, 0)$. This feature does not persist in general for nonplanar dipole distributions; the integrated contribution corresponding to w_a for this case will not vanish in general (see Problem 2.4.6).

(iii) *Potential due to a planar source distribution of variable strength*

Consider now the potential due to a planar distribution of sources of strength per unit area equal to $q(\xi, \eta)$. The expression (2.4.8) specializes to the following integral analogous to (2.4.23):

$$u(x, y, z) = -\frac{1}{4\pi}\iint_D \frac{q(\xi, \eta)d\xi\, d\eta}{[(x-\xi)^2 + (y-\eta)^2 + z^2]^{1/2}}. \qquad (2.4.32)$$

If we decompose D as in Figure 2.10, set $u(x, y, z) = u_\epsilon(x, y, \alpha; \epsilon) + u_a(x, y, \alpha; \epsilon)$, and introduce local polar coordinates, we obtain

$$u_\epsilon = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\epsilon}\frac{q(x + \rho\cos\phi, y + \rho\sin\phi)}{(\rho^2 + \alpha^2)^{1/2}}\rho\, d\rho\, d\phi, \qquad (2.4.33a)$$

$$u_a = -\frac{1}{4\pi}\int_0^{2\pi}\int_\epsilon^{R(\phi,x,y)} \frac{q(x+\rho\cos\phi, y+\rho\sin\phi)}{(\rho^2+\alpha^2)^{1/2}}\rho\, d\rho\, d\phi. \quad (2.4.33b)$$

Again, we change the ρ variable to $\epsilon\tilde{\rho}$ and find that u_ϵ has the approximation

$$u_\epsilon(x, y, \alpha; \epsilon) = -\frac{\epsilon}{4\pi}\int_0^{2\pi}\int_0^1 \frac{1}{[\tilde{\rho}^2+(\alpha/\epsilon)^2]^{1/2}}\{q(x, y) + \epsilon q_x(x, y)\tilde{\rho}\cos\phi + \epsilon q_y(x, y)\tilde{\rho}\sin\phi + O(\epsilon^2\tilde{\rho}^2)\}\tilde{\rho}\, d\tilde{\rho}\, d\phi \quad (2.4.34)$$

if q is analytic. Integrating with respect to ϕ first shows that the $O(\epsilon)$ terms in the integrand give no contributions, and integrating with respect to $\tilde{\rho}$ gives

$$u_\epsilon(x, y, \alpha; \epsilon) = O(\epsilon) + O(\alpha). \quad (2.4.35)$$

Thus, u_ϵ gives no contribution as long as $\alpha \to 0$ as $\epsilon \to 0$. The limit of the ratio (α/ϵ) does not affect this result and can therefore not affect the limiting value of u_a. In fact, this value is uniquely given by

$$\begin{aligned} u_a(x, y, 0; 0) &= -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{R(\phi,x,y)} q(x+\rho\cos\phi, y+\rho\sin\phi)d\rho\, d\phi \\ &\equiv u(x, y, 0). \end{aligned} \quad (2.4.36)$$

In contrast with the result (2.4.30) for dipoles, we see that the potential at a point on the surface *depends on the entire distribution.*

For the special case $q = q_0 =$ constant over the unit disk centered at the origin, (2.4.36) becomes

$$u(x, y, 0) = -\frac{q_0}{4\pi}\int_0^{2\pi}\{-x\cos\phi - y\sin\phi + [1-(x\sin\phi - y\cos\phi)^2]^{1/2}\}d\phi,$$

which simplifies to a function of $r \equiv (x^2+y^2)^{1/2}$ only:

$$\begin{aligned} u(r, 0) &= -\frac{q_0}{\pi}\int_0^{\pi/2}(1-r^2\sin^2\phi)^{1/2}d\phi \\ &= -\frac{q_0}{\pi}E(r^2). \end{aligned} \quad (2.4.37a)$$

Here E is the complete elliptic integral of the second kind (see pp. 590–591 of [3]). Using the expansion for $E(r^2)$, valid if $r^2 < 1$, gives

$$u(r, 0) = -\frac{q_0}{2}\left[1 - \left(\frac{1}{2}\right)^2\frac{r^2}{1} - \left(\frac{1}{2}\cdot\frac{3}{4}\right)^2\frac{r^4}{3} - \left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right)^2\frac{r^6}{5} - \cdots\right], \quad (2.4.37b)$$

and, in particular, $u(0, 0) = -q_0/2$.

The normal derivative of the potential (2.4.32) on the $z = 0$ plane is just the z-derivative, which for $z \neq 0$ is

$$u_z(x, y, z) = \frac{z}{4\pi}\iint_D \frac{q(\xi, \eta)d\xi\, d\eta}{[(x-\xi)^2+(y-\eta)^2+z^2]^{3/2}}. \quad (2.4.38)$$

This is formally the same expression as the potential due to a dipole distribution of strength $-q$ [see (2.4.23)]. Therefore, it follows from (2.4.30) that

$$u_z(x, y, 0^+) = \frac{q(x, y)}{2}, \tag{2.4.39a}$$

$$u_z(x, y, 0^-) = -\frac{q(x, y)}{2}. \tag{2.4.39b}$$

Again, the local nature of this result is geometrically obvious for a planar distribution if we interpret the sources as mass sources, and u as the velocity potential. The velocity normal to the plane $z = 0$ at the point $P = (x, y, 0)$ is $u_z(x, y, 0)$. A source at a point $Q = (x_1, y_1, 0) \neq P$ produces a radial velocity field centered at Q. This field has no component in the z direction at P.

(iv) *Normal derivative of the potential due to a planar distribution of dipoles of variable strength*

Finally, let us study the limiting value (as $z \to 0^+$) of the normal derivative of the potential for the dipole distribution (2.4.23). It follows from (2.4.28b) that

$$\frac{\partial w_\epsilon}{\partial z} = \frac{p(x, y)}{2} \frac{\epsilon^2}{(\epsilon^2 + z^2)^{3/2}} + O(\epsilon), \ z > 0,$$

which implies that for $\epsilon \to 0$ and $(z/\epsilon) \to 0^+$, we have the singular behavior

$$\frac{\partial w_\epsilon}{\partial z} = \frac{p(x, y)}{2\epsilon} + O(\epsilon), \quad \text{as } \epsilon \to 0. \tag{2.4.40}$$

We therefore anticipate finding a corresponding singularity of opposite sign in $\partial w_a/\partial z$. Now, as $z = \alpha \to 0$, with $(\alpha/\epsilon) \to 0$, $\partial w_a/\partial z$ is of the form

$$\frac{\partial w_a(x, y, \alpha; \epsilon)}{\partial z} = -\frac{1}{4\pi} \int_0^{2\pi} \int_\epsilon^{R(\phi,x,y)} \frac{p(x + \rho\cos\phi, y + \rho\sin\phi)}{\rho^2} d\rho\, d\phi + O(\alpha). \tag{2.4.41a}$$

To exhibit the singular behavior near the lower limit $\rho = \epsilon$, we subtract and add the first term in the expansion of p to obtain the following expression, which is identical to (2.4.41a):

$$\frac{\partial w_a(x, y, \alpha; \epsilon)}{\partial z} = -\frac{1}{4\pi} \int_0^{2\pi} \int_\epsilon^{R(\phi,x,y)} \frac{1}{\rho^2} [p(x + \rho\cos\phi, y + \rho\sin\phi) - p(x, y)] d\rho d\phi - \frac{p(x, y)}{4\pi} \int_0^{2\pi} \int_\epsilon^{R(\phi,x,y)} \frac{1}{\rho^2} d\rho\, d\phi + O(\alpha). \tag{2.4.41b}$$

Now, as $\epsilon \to 0$, the first integral is well behaved. In fact, it is $O(1)$ as $\epsilon \to 0$ because the second term in the development of p gives a zero contribution when integrated with respect to ϕ. Evaluating the second integral gives

$$\frac{\partial w_a(x, y, \alpha; \epsilon)}{\partial z}$$
$$= -\frac{1}{4\pi}\int_0^{2\pi}\int_\epsilon^{R(\phi,x,y)} \frac{1}{\rho^2}[p(x+\rho\cos\phi, y+\rho\sin\phi)$$
$$- p(x, y)]d\rho\, d\phi + \frac{p(x, y)}{4\pi}\int_0^{2\pi}\frac{d\phi}{R(\phi, x, y)} - \frac{p(x, y)}{2\epsilon}, \quad (2.4.41c)$$

and we exhibit the needed $O(\epsilon^{-1})$ singularity.

Thus, adding (2.4.40) to (2.4.41c) and taking the limit as $\epsilon \to 0$ gives

$$w_z(x, y, 0^+) = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{R(\phi,x,y)} l(\rho, \phi, x, y)d\rho\, d\phi$$
$$+ \frac{p(x, y)}{4\pi}\int_0^{2\pi}\frac{d\phi}{R(\phi, x, y)}, \quad (2.4.42)$$

where

$$l(\rho, \phi, x, y) \equiv \frac{1}{\rho^2}[p(x+\rho\cos\phi, y+\rho\sin\phi) - p(x, y)]. \quad (2.4.43)$$

For the special case $p = p_0$ = constant, we have $l = 0$, and (2.4.42) simplifies to

$$w_z(x, y, 0^+) = \frac{p_0}{4\pi}\int_0^{2\pi}\frac{d\phi}{R(\phi, x, y)}. \quad (2.4.44)$$

Problems

2.4.1a. Consider a spherical mass of radius R having a radially varying density ρ

$$\rho = \begin{cases} f(r), & 0 \le r \le R, \\ 0, & r > R. \end{cases} \quad (2.4.45)$$

Calculate the force of gravity on a point P of mass m when P is either outside or inside the sphere $0 \le r \le R$. Show that this force is exactly the same as if the entire mass of the portion of the sphere inside P were concentrated at the origin. Specialize your result to the case of a hollow sphere where $f(r) = 0$ for all r such that $0 \le r \le a < R$.

b. Now let the density distribution be axisymmetric, i.e.,

$$\rho = \begin{cases} f(r, \phi) = \text{prescribed} & \text{if } 0 \le r \le R, 0 \le \phi \le \pi, \\ 0 & \text{if } r > R, \end{cases} \quad (2.4.46)$$

where ϕ is the polar angle measured from the axis of symmetry. Consider the case $(r/R) > 1$, and normalize distances so $R = 1$. Show that

$$V(r, \phi) = -\gamma \int_{\theta'=0}^{2\pi} d\theta' \int_{\phi'=0}^{\pi} \sin\phi' d\phi' \int_{r'=0}^{1} \frac{f(r', \phi')dr'}{\sqrt{r^2 + r'^2 - 2rr' \cos\gamma}}, \tag{2.4.47}$$

where $\cos\gamma = \sin\phi \sin\phi' \cos\theta' + \cos\phi \cos\phi'$. Assume that for $r > 1$, f has the series expansion

$$f(r', \phi') = \sum_{n=0}^{\infty} A_n(\phi') r'^n \tag{2.4.48}$$

for given A_n. Use the generating function for Legendre polynomials (for example, see pp. 102–103 of [8]) to express

$$\frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos\gamma}} = \frac{1}{r} \frac{1}{\sqrt{1 + \left(\frac{r'}{r}\right)^2 - \frac{2r'}{r} \cos\gamma}}$$

$$= \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\gamma), \tag{2.4.49}$$

where P_n are the Legendre polynomials. Develop the product of (2.4.48) and (2.4.49) in a power series in (r'/r) and show that

$$\int_{r'=0}^{1} \frac{f(r', \phi')dr'}{\sqrt{r^2 + r'^2 - 2rr' \cos\gamma}} = \sum_{n=0}^{\infty} C_n r^{-(n+1)}, \tag{2.4.50}$$

where

$$C_n(\phi, \theta', \phi') = P_n(\cos\gamma) \sum_{m=0}^{\infty} \frac{A_m}{m + n + 1}. \tag{2.4.51}$$

Therefore, (2.4.47) has the power series representation

$$V(r, \phi) = \sum_{n=0}^{\infty} D_n r^{-(n+1)},$$

where

$$D_n = -\gamma \int_0^{2\pi} d\theta' \int_0^{\pi} C_n(\phi, \theta', \phi') \sin\phi' d\phi', \quad r > 1. \tag{2.4.52}$$

2.4.2 Consider a body occupying the finite domain G and having a prescribed density distribution. If $P(x, y, z)$ is a point at a distance R from O, the center of mass of G, and R is large compared to the dimensions of G, show that the gravitational potential at P is given approximately by

$$V = -\frac{\gamma M}{R} - \frac{\gamma}{2R^3}(A + B + C - 3I), \tag{2.4.53}$$

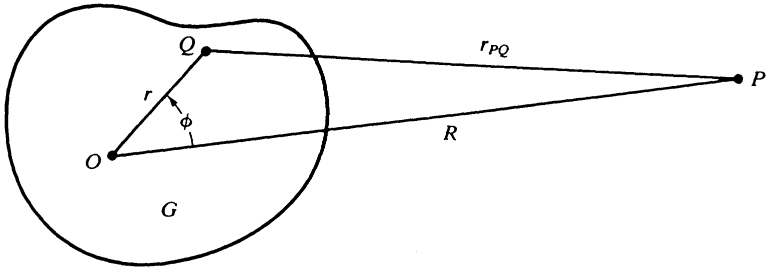

FIGURE 2.12. Geometry for Problem 2.4.2

where M is the mass of G. The constants A, B, and C are the principal moments of inertia, and I is the moment of inertia about the axis OP.
Hint: Introduce spherical polar coordinates about the OP axis: $\xi = r\sin\phi\cos\theta$; $\eta = r\sin\phi\sin\theta$; $\zeta = r\cos\phi$; then expand (2.4.6) for small values of r/R using $r_{PQ} \equiv \sqrt{R^2 + r^2 - 2rR\cos\phi}$ for the distance between P and the variable point of integration Q, as shown in Figure 2.12.

2.4.3 Derive the differential equation and boundary condition defining the axisymmetric body generated by a unit point source of mass at the origin in the presence of a uniform freestream velocity $\mathbf{i}$. Derive the approximate shape of this body near $r = 0$.

2.4.4 As an application of the result (2.4.21), consider the case of an ellipsoid of revolution with $F(x) = 2b[x(1-x)]^{1/2}$; $0 < b \leq 1$. Introduce three point sources at $x = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ with strengths S_1, S_2, S_3, respectively. Solve (2.4.21) for S_1, S_2, S_3 and calculate $u(x, r)$ by discretizing the integral in (2.4.15). How well does the result satisfy the boundary condition (2.4.16) over the entire surface $r = F(x)$?

2.4.5 Let x, r, θ be the cylindrical polar coordinates of the point $P = (x, y, z)$ and let ξ, ρ, ϕ be the corresponding cylindrical polar coordinates of the point $Q = (\xi, \eta, \zeta)$. (see Figure 2.13):

$$y = r\cos\theta, \quad z = r\sin\theta, \tag{2.4.54a}$$

$$\eta = \rho\cos\phi, \quad \zeta = \rho\sin\phi. \tag{2.4.54b}$$

Let D be the domain *outside* the cylindrically symmetric surface Γ of unit length defined by $\rho = F(\xi)$, with $F(0) = F(1) = 0$, for a given function F. Let $\mathbf{n}$ be the unit normal in the direction indicated (Figure 2.14). Show that the potential at P due to a dipole distribution of strength per unit area equal to $\mu(\xi, \phi)$ on Γ (The dipole axes are oriented along $\mathbf{n}$) is given by

$$u(x, r, \theta) = \int_{\xi=0}^{1}\int_{\phi=0}^{2\pi} \mu(\xi, \phi) L(x, r, \theta, \xi, \phi)\, d\phi\, d\xi, \tag{2.4.55}$$

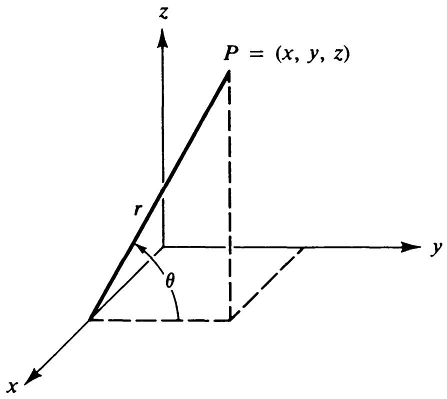

FIGURE 2.13. Cylindrical polar coordinates

where

$$L = \frac{F(\xi)[r\cos(\theta-\phi)+(\xi-x)F'(\xi)-F(\xi)]}{[r^2+\rho^2-2r\rho\cos(\theta-\phi)+(x-\xi)^2]^{3/2}\sqrt{1+F'^2(\xi)}}. \tag{2.4.56}$$

2.4.6 Consider a uniform distribution of dipoles on the surface of the unit hemisphere $z = (1 - x^2 - y^2)^{1/2}$. Assume that the dipole strength per unit area equals unity and that the axes are oriented along the outward normal to the surface. Show that the potential $w(0, 0, z)$ along the z-axis, as given by

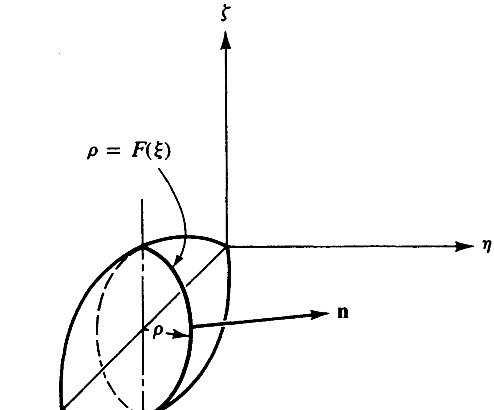

FIGURE 2.14. Geometry for Problem 2.4.5

(2.4.11), reduces to

$$w(0,0,z) = -\frac{1}{2}\int_0^{\pi/2} \frac{(z\cos\phi - 1)\sin\phi}{[1+z^2-2z\cos\phi]^{3/2}}\,d\phi$$
$$= \frac{1}{2}\left[\frac{1-z}{|1-z|} + \frac{z}{(1+z^2)^{1/2}}\right]. \qquad (2.4.57)$$

The preceding implies that as $z \to 1^{\pm}$, we have $w(0,0,1^+) = -\frac{1}{2}(1 - 2^{-1/2})$ and $w(0,0,1^-) = \frac{1}{2}(1+2^{-1/2})$. Denote

$$w_\epsilon = -\frac{1}{2}\int_0^{\epsilon} \frac{(z\cos\phi - 1)\sin\phi}{[1+z^2-2z\cos\phi]^{3/2}}\,d\phi,$$
$$w_a = -\frac{1}{2}\int_\epsilon^{\pi/2} \frac{(z\cos\phi - 1)\sin\phi}{[1+z^2-2z\cos\phi]^{3/2}}\,d\phi,$$

and show that as $\epsilon \to 0^+$ and $z - 1 \equiv \alpha \to 0^{\pm}$, we have the following behavior

$$w_\epsilon = \mp\frac{1}{2} + \frac{1}{2}\frac{\epsilon/2 + (\alpha/\epsilon)}{\sqrt{1+(\alpha/\epsilon)^2}} + \dots,$$
$$w_a = \frac{1}{2^{3/2}} - \frac{1}{2}\frac{\epsilon/2 + (\alpha/\epsilon)}{\sqrt{1+(\alpha/\epsilon)^2}} + \dots.$$

Thus, if $(\alpha/\epsilon) \to 0$, then $w_\epsilon \to \mp\frac{1}{2}$, and this is the same limiting value that we calculated for a planar distribution. But now we have $w_a = 2^{-3/2} \neq 0$ unlike the planar case. In fact, the only choice for which $w_a = 0$ is to require $(\alpha/\epsilon) \to 1$ as $\epsilon \to 0$.

2.5 Green's Formula and Applications

In this section we derive a number of general results concerning properties of solutions of Laplace's equation. The starting point is the familiar Gauss theorem of vector calculus.

Consider a one-valued vector field $\mathbf{F}$ defined in a domain G with boundary Γ on which $\mathbf{n}$ is an outward unit normal. Gauss' theorem states that if $\mathbf{F}$ has continuous first partial derivatives in G, then

$$\iiint_G \operatorname{div} \mathbf{F}\,dV = \iint_\Gamma \mathbf{F}\cdot\mathbf{n}\,dA. \qquad (2.5.1)$$

This result is valid for multiply connected domains as long as Γ includes all the boundaries of G. If G is infinite, we assume that $|\mathbf{F}| \ll r^{-2}$ as $r \to \infty$, where r is the scalar distance, in order to ensure the existence of the integrals.

2.5.1 Green's Formula

To derive Green's formula, we choose $\mathbf{F} = v \operatorname{grad} u$ for prescribed scalar functions $u(x, y, z)$ and $v(x, y, z)$ having continuous second partial derivatives. Then $\operatorname{div} \mathbf{F} = \operatorname{div}(v \operatorname{grad} u) = (\operatorname{grad} u \cdot \operatorname{grad} v) + v\Delta u$, where Δ denotes the Laplacian, and (2.5.1) reduces to Green's formula

$$\iiint_G [\operatorname{grad} u \cdot \operatorname{grad} v + v\Delta u] dV = \iint_\Gamma v \frac{\partial u}{\partial n} dA. \tag{2.5.2}$$

Here $\partial u/\partial n$ is the directional derivative of u in the outward normal direction,

$$\frac{\partial u}{\partial n} \equiv \operatorname{grad} u \cdot \mathbf{n}. \tag{2.5.3}$$

Interchanging u and v in (2.5.2) and subtracting the result gives the symmetric form of Green's formula:

$$\iiint_G [u\Delta v - v\Delta u] dV = \iint_\Gamma \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dA. \tag{2.5.4}$$

The two-dimensional version of (2.5.1), called Green's theorem, states that for a vector field $\mathbf{F}$ defined in the planar domain D bounded by the curve C, we have

$$\iint_D \operatorname{div} \mathbf{F} dA = \oint_C \mathbf{F} \cdot \mathbf{n} ds, \tag{2.5.5}$$

where dA is the element of area, $\mathbf{n}$ is the outward unit normal, ds is the element of arc along C, and the contour C is traversed in the counterclockwise sense in the integral on the right-hand side. The formula corresponding to (2.5.4) in two dimensions is

$$\iint_D (u\Delta v - v\Delta u) dA = \oint_C \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds. \tag{2.5.6}$$

This result is equivalent to Green's lemma,

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \int_C P dx + \int_C Q dy, \tag{2.5.7}$$

if we introduce the Cartesian coordinates x, y and identify

$$P(x, y) \equiv vu_y - uv_y, \quad Q(x, y) \equiv uv_x - vu_x. \tag{2.5.8}$$

Note that the two integrals on the right-hand side of (2.5.7) are line integrals along the curve C.

2.5.2 Gauss' Integral Theorem

If $\Delta u = 0$ and $v = 1$, (2.5.2) gives

$$\iint_{\Gamma} \frac{\partial u}{\partial n} dA = 0. \tag{2.5.9}$$

Thus, the surface integral over Γ of the normal derivative of u vanishes if Γ encloses a region where u is harmonic. This is intuitively obvious if we interpret u as a velocity potential. Equation (2.5.9) states that for an incompressible, irrotational flow with no sources, the *net* mass flow through a prescribed boundary Γ is zero.

2.5.3 Energy Theorem and Corollaries

Setting $u = v$ with $\Delta u = 0$ in (2.5.2) gives the energy theorem

$$\frac{1}{2} \iiint_{G} (\text{grad } u)^2 dV = \frac{1}{2} \iint_{\Gamma} u \frac{\partial u}{\partial n} dA, \tag{2.5.10}$$

relating the total kinetic energy in the interior to the integral of $u(\partial u/\partial n)$ on the boundary.

A corollary to (2.5.10) is that if u is harmonic in G and vanishes on the boundary Γ, then u must be identically equal to zero in G. This follows from the fact that $u = 0$ on Γ implies that the right-hand side of (2.5.10) vanishes identically. But since the integrand on the left-hand side is non-negative, (2.5.10) can be true only if $\text{grad } u \equiv 0$ in G, i.e., $u =$ constant in G. This, combined with the fact that $u = 0$ on Γ, implies that $u \equiv 0$ in G.

A second corollary to (2.5.10) is that if u is harmonic in G and $\partial u/\partial n$ vanishes on Γ, then u must be a constant throughout the interior.

It is important to keep in mind that this result applies only to *one-valued* functions u. For example, consider the two-dimensional Laplacian in the annular region D: $0 < r_1 \leq r \leq r_2$ contained between the two concentric circles with radii $r = r_1$ and $r = r_2$ centered at the origin. With r and θ denoting polar coordinates, we note that the multivalued function $w = c\theta \equiv c \tan^{-1}(y/x)$ (where c is an arbitrary constant) is harmonic in D. The normal derivative of w on the two boundary circles is the radial derivative, and this vanishes. Thus, our assertion that $\Delta w = 0$ in D and $\partial w/\partial n = 0$ on Γ implies that w is a constant does not apply because w is multivalued. Of course, we may render w one-valued by introducing the radial cut along $\theta = \theta_1$ and restricting allowable values of θ according to $\theta_1 \leq \theta < \theta_1 + 2\pi$. In this case, the barrier at $\theta = \theta_1$ *becomes part of the boundary* on which w satisfies

$$\frac{\partial w}{\partial n} = \frac{1}{r}\frac{\partial w}{\partial \theta} = \frac{c}{r}.$$

Hence, in order to have the normal derivative of w vanish on *all* the boundaries, we must set $c = 0$, and this result is indeed consistent with our claim.

2.5.4 Uniqueness Theorems

As a direct consequence of the foregoing results concerning harmonic functions with zero boundary conditions, we now prove the following uniqueness theorems for certain boundary-value problems satisfying $\Delta u = 0$ in G.

(i) *Dirichlet's problem*

In Dirichlet's problem, we have $\Delta u = 0$ in some domain G with u prescribed on the boundary Γ. The solution is unique, i.e., if two functions u_1 and u_2 are harmonic in G and coincide on Γ, they are identical in G. To see this, we note that $u = u_1 - u_2$ is also harmonic in G and vanishes on Γ; therefore, according to the first corollary to (2.5.10), $u \equiv 0$ in G, i.e., $u_1 = u_2$ in G.

(ii) *Neumann's problem*

In Neumann's problem, we have $\Delta u = 0$ in some domain G with the normal derivative of u prescribed on the boundary Γ. The solution is unique to within a constant, i.e., if two one-valued functions u_1 and u_2 are harmonic in G and their normal derivatives coincide on Γ, then u_1 and u_2 differ by at most a constant in G. This is also an immediate consequence of the second corollary to (2.5.10).

(iii) *Mixed boundary-value problem*

If two functions u_1 and u_2 are harmonic in G and satisfy the same mixed boundary condition (where u is prescribed on part of Γ and $\partial u/\partial n$ is prescribed on the remainder), then u_1 and u_2 coincide in G. The specification of u_1 on part of the boundary eliminates the arbitrary constant that arises if only $\partial u/\partial n$ is prescribed.

(iv) *General linear boundary-value problem*

If two functions u_1 and u_2 are harmonic in G and satisfy the same general linear boundary condition on Γ,

$$u_i + f\frac{\partial u_i}{\partial n} = g, \quad i = 1, 2, \tag{2.5.11a}$$

where f and g are specified functions on Γ, then $u_1 \equiv u_2$ in G if $f \geq 0$ everywhere on Γ.

To show this let $u \equiv u_1 - u_2$, which is harmonic in G and satisfies

$$u + f\frac{\partial u}{\partial n} = 0 \tag{2.5.11b}$$

on Γ. Therefore, using (2.5.11b) to express u in terms of $(\partial u/\partial n)$ on Γ in the expression (2.5.10) gives

$$\iint_G (\operatorname{grad} u)^2 dV = -\iint_\Gamma f\left(\frac{\partial u}{\partial n}\right)^2 dA. \tag{2.5.12}$$

Since $f \geq 0$ on Γ, the left-hand side of (2.5.12) is less than or equal to 0. But, the right-hand side of (2.5.12) is greater than or equal to 0. Therefore, (2.5.12) can be true only if $u =$ constant in G and on Γ. This constant must be zero in order that (2.5.11b) hold. Thus, $u_1 \equiv u_2$.

If u is interpreted as the steady-state temperature in a solid, the boundary condition (2.5.11a) with $f \geq 0$ corresponds to Newton's law for cooling (compare with (1.4.34)), and the choice of $f < 0$ is unphysical. In Problem 2.5.3 we use a simple two-dimensional example to illustrate the fact that the solution may not exist if $f < 0$, and that even if it exists it may not be unique.

2.5.5 Mean Value Theorem, Maximum–Minimum Theorem

Let $P(x, y, z)$ be a fixed point inside G and let $\Delta u = 0$ in G. Consider the symmetric form of Green's formula, (2.5.4), with respect to the integration variables $Q = (\xi, \eta, \zeta)$ and regard u as a function of Q. Thus, $\Delta u = \Delta_Q u \equiv u_{\xi\xi} + u_{\eta\eta} + u_{\zeta\zeta} = 0$. Also, in (2.5.4), let $v(P, Q) = -1/4\pi r_{PQ}$, where $r_{PQ}^2 \equiv (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$. Let G_1 be a sphere of radius R centered at P with boundary Γ_1 lying entirely inside G. Clearly, the structure of the fundamental solution implies that $\Delta_Q v = \Delta_P v = \delta_3(P, Q)$ [see (2.3.9)–(2.3.10)]. Therefore, the left-hand side of (2.5.4) evaluated inside G_1 reduces to

$$\iiint_{G_1} [u(Q)\Delta_Q v - v(P, Q)\Delta_Q u] dV_Q = \iiint_{G_1} u(Q)\delta_3(P, Q) dV_Q = u(P). \tag{2.5.13}$$

The right-hand side is

$$\begin{aligned} \iint_{\Gamma_1} \left(u \frac{\partial v}{\partial n_Q} - v \frac{\partial u}{\partial n_Q} \right) dA_Q = & \iint_{\Gamma_1} u \frac{\partial}{\partial \rho} \left(-\frac{1}{4\pi\rho} \right) \Bigg|_{\rho=R} dA_Q \\ & - \iint_{\Gamma_1} v \frac{\partial u}{\partial n_Q} dA_Q. \end{aligned} \tag{2.5.14}$$

The second term on the right-hand side of (2.5.14) reduces to

$$-\frac{1}{4\pi R} \iint_{\Gamma_1} \frac{\partial u}{\partial n_Q} dA_Q = 0 \tag{2.5.15}$$

according to Gauss' integral theorem (2.5.9). Setting

$$\frac{\partial}{\partial \rho} \left(-\frac{1}{4\pi\rho} \right) \Bigg|_{\rho=R} = \frac{1}{4\pi R^2}$$

in (2.5.14), using (2.5.15), and equating (2.5.13) and (2.5.14) gives

$$u(P) = \frac{1}{4\pi R^2} \iint_{\Gamma_1} u \, dA. \tag{2.5.16}$$

Thus, the value of a harmonic function at any point P is the average of the values it takes on any sphere surrounding that point.

As a corollary of (2.5.16), called the *maximum–minimum theorem*, it follows that the maximum and minimum values of a harmonic function must occur on the boundary; in particular, if u is constant on the boundary, then it is constant everywhere in G.

2.5.6 Surface Distribution of Sources and Dipoles

Any harmonic function in G can be represented by a distribution of sources and dipoles on the boundary Γ.

In the derivation of the mean value theorem, we restricted our attention to a spherical domain G_1 inside G, and this implied (2.5.15). If we repeat the calculations for the entire domain G and boundary Γ, (2.5.13) still holds, but the right-hand side is more complicated. In fact, we find that

$$u(P) = \iint_\Gamma \left(-\frac{\partial u}{\partial n_Q}\right)\left(-\frac{1}{4\pi r_{PQ}}\right) dA_Q + \iint_\Gamma u \frac{\partial}{\partial n_Q}\left(-\frac{1}{4\pi r_{PQ}}\right) dA_Q, \tag{2.5.17}$$

which means that $u(P)$ can be regarded as the potential due to a surface distribution of sources of strength $-\partial u/\partial n$, as given by the first term on the right-hand side of (2.5.17), plus a distribution of dipoles oriented along the outward normal to Γ having strength u. Since arbitrarily prescribing both u and $\partial u/\partial n$ on the boundary leads to an ill-posed problem for Laplace's equation (see the two-dimensional example discussed in Section 4.4.5), (2.5.17) does not provide the solution of a realistic boundary-value problem. Rather, it should be interpreted as an integral equation for $\partial u/\partial n$ on the boundary if u is prescribed there (or vice versa). This point of view is discussed in more detail in Section 2.7. We shall also use the general result (2.5.17) in interpreting solutions of Dirichlet's and Neumann's problems for simple geometries in Section 2.6.

Strictly speaking, the derivation leading to (2.5.16) or (2.5.17) is suspect because we have used Gauss' theorem for functions that are singular at $P = Q$. Let us verify that (2.5.17) is indeed correct using a more careful derivation that avoids delta functions.

Let G_ϵ be a sphere of radius ϵ centered at P and lying entirely in G. We apply Green's formula now in the domain *outside* G_ϵ and inside G. Taking u and v in (2.5.4) again to be $u(\xi, \eta, \zeta)$: $\Delta_Q u = 0$ and $v = -1/4\pi r_{PQ}$, the left-hand side of (2.5.4) vanishes because the Laplacians of both u and v are equal to zero in $G - G_\epsilon$. To compute the right-hand side, we include the boundary contributions for both Γ and Γ_ϵ and obtain

$$0 = \iint_\Gamma \left[u \frac{\partial}{\partial n_Q}\left(-\frac{1}{4\pi r_{PQ}}\right) + \frac{1}{4\pi r_{PQ}} \frac{\partial u}{\partial n_Q}\right] dA_Q$$

$$-\iint_{\Gamma_\epsilon}\left[u\frac{\partial}{\partial n_Q}\left(-\frac{1}{4\pi r_{PQ}}\right)+\frac{1}{4\pi r_{PQ}}\frac{\partial u}{\partial n_Q}\right]dA_Q. \quad (2.5.18)$$

In the second integral of (2.5.18) we must use the direction of *increasing* radius as the normal $\mathbf{n}_Q$, since we have introduced the minus sign in front.

To evaluate the integral over Γ_ϵ, we change variables from $Q = (\xi, \eta, \zeta)$ to spherical polar coordinates ρ, ϕ, θ centered at $P = (x, y, z)$, that is, $\xi = x + \rho\sin\phi\cos\theta$, $\eta = y + \rho\sin\phi\sin\theta$, $\zeta = z + \rho\cos\phi$. Therefore,

$$\iint_{\Gamma_\epsilon}\left[u\frac{\partial}{\partial n_Q}\left(-\frac{1}{4\pi r_{PQ}}\right)+\frac{1}{4\pi r_{PQ}}\frac{\partial u}{\partial n_Q}\right]dA_Q$$

$$=\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\left[\frac{u(x+\epsilon\sin\phi\cos\theta,\, y+\epsilon\sin\phi\sin\theta,\, z+\epsilon\cos\phi)}{4\pi\epsilon^2}\right.$$

$$\left.+\frac{1}{4\pi\epsilon}\left(\frac{\partial u}{\partial\rho}\right)_{\rho=\epsilon}\right]\epsilon^2\sin\phi\, d\phi\, d\theta. \quad (2.5.19)$$

As $\epsilon \to 0$, the second term on the right-hand side does not contribute, and the first term tends to $u(x, y, z)$. Using this result in (2.5.18) gives (2.5.17).

In subsequent calculations, we shall mainly rely on formal derivations with the aid of delta functions without this type of justification, as these calculations are significantly simpler.

2.5.7 Potential Due to a Dipole Distribution of Unit Strength

As an application of result (2.5.15), consider a given surface Γ, not necessarily closed, and let P be a point not on Γ. If Γ is closed, let C be an arbitrary simple closed curve on Γ, and if Γ is open, let C be its boundary. We generate a cone by running a straight line from P along C (Figure 2.15). Exclude from this cone the spherical cap K_ϵ generated by the sphere of radius ϵ centered at P, and denote what remains by Ω. In the region G bounded by Γ, Ω, K_ϵ, the function $u = -1/4\pi r_{PQ}$ is harmonic because $\Delta_Q u = \delta_3(Q, P)$ and P is outside G. Thus, using Gauss' integral theorem (2.5.9), we have

$$\iint_{\Gamma+\Omega+K_\epsilon}\frac{\partial u}{\partial n}dA = 0. \quad (2.5.20)$$

Now, on Γ,

$$\iint_{\Gamma}\frac{\partial u}{\partial n}dA = \iint_{\Gamma}\frac{\partial}{\partial n}\left(-\frac{1}{4\pi r_{PQ}}\right)dA \equiv v(P), \quad (2.5.21)$$

where it is clear from the definition of v that it is the potential at P due to a distribution of *dipoles of unit strength* oriented along the outward normal on Γ.

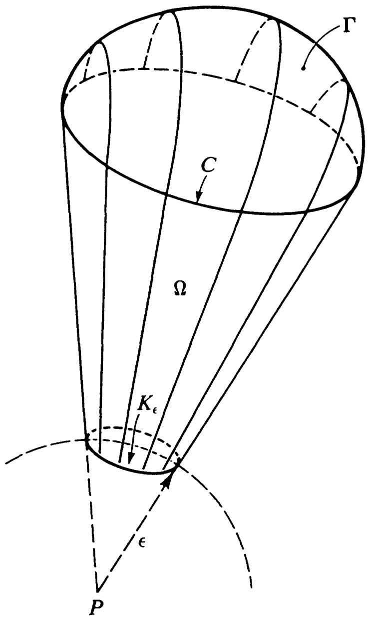

FIGURE 2.15. Cone generated from P and curve C

The contribution to (2.5.20) from Ω vanishes,

$$\iint\limits_{\Omega} \frac{\partial u}{\partial n} dA = 0,$$

because on Ω, $\operatorname{grad}(1/r_{PQ})$ is perpendicular to the normal to Ω. To evaluate the contribution form K_ϵ, we note that the outward normal on K_ϵ is in the direction of decreasing radius. Therefore, on K_ϵ,

$$\frac{\partial u}{\partial n} = \frac{\partial}{\partial(-\rho)} \left(-\frac{1}{4\pi\rho} \right) \Bigg|_{\rho=\epsilon} = -\frac{1}{4\pi\epsilon^2}.$$

Using this together with the definition of v in (2.5.21) gives

$$v(P) = \frac{1}{4\pi\epsilon^2} \iint\limits_{K_\epsilon} \epsilon^2 d\omega, \tag{2.5.22}$$

where $d\omega$ is the area element on the unit sphere centered at P. In other words, $v(P)$ equals the ratio of the area of K_ϵ to the area $4\pi\epsilon^2$ of the entire ϵ sphere, that is, the solid angle subtended by C.

This result also implies that the actual shape of Γ is irrelevant; the potential at P is the same for all possible surface distributions having a given constant strength and boundary C. If C lies in a plane, the solid angle tends to $\frac{1}{2}$ as P approaches this plane and is independent of C. This is just a special case, with $p = \pm 1$, of our earlier result (2.4.30). If Γ is a closed surface, we can generate C by intersecting Γ with an arbitrary plane. In this case, if P is an interior point, the contributions to v from the two portions of Γ add, and we have $v = 1$. Conversely, if P is outside Γ, these contributions cancel, and we have $v = 0$. Thus, for a unit dipole distribution on a closed surface, v approaches 1 or 0 as P approaches the surface from inside or outside, respectively.

Problems

2.5.1 Verify that (2.4.57) follows from (2.5.22).

2.5.2 We wish to solve the two-dimensional Laplace equation in the rectangular domain $0 \leq x \leq \ell$; $0 \leq y \leq h$, subject to prescribed values of u on the perimeter of the rectangle. Use a finite-difference scheme based on the mean value theorem and let $h = M\xi$ and $\ell = N\xi$, where ξ is a small constant and M and N are positive integers. Subdivide the rectangle into $M \times N$ squares and denote

$$u(i\xi, j\xi) \equiv u_{ij}. \tag{2.5.23}$$

Derive a linear system of $(M - 1) \times (N - 1)$ algebraic equations for the interior values of u_{ij} in terms of the known boundary values.

2.5.3 Consider the two-dimensional Laplace equation in polar coordinates in the interior of the unit circle

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 \leq r \leq 1. \tag{2.5.24}$$

The boundary condition on $r = 1$ is

$$u(1, \theta) + f u_r(1, \theta) = g(\theta), \tag{2.5.25}$$

where f is a constant and g is prescribed on $0 \leq \theta < 2\pi$.

a. Use separation of variables to calculate the solution in the form

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta), \tag{2.5.26}$$

where

$$(1 + nf)a_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos n\theta d\theta, \quad n = 0, 1, \ldots, \tag{2.5.27a}$$

$$(1 + nf)b_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin n\theta d\theta, \quad n = 1, 2, \ldots. \tag{2.5.27b}$$

Thus if $f \geq 0$, a_n and b_n are uniquely defined, as proven in general in Section 2.5.4.

b. Now let $f = -1/m$, where m is a positive integer. Show that if $g(\theta)$ is orthogonal to $\cos m\theta$ and $\sin m\theta$, i.e., if the Fourier series for $g(\theta)$

does not contain the cos $m\theta$ and sin $m\theta$ terms, then the solution (2.5.26) exists but is not unique. Next, show that if $g(\theta)$ contains either the cos $m\theta$ or sin $m\theta$ term, then the solution does not exist.

2.6 Green's and Neumann's Functions

2.6.1 Green's Function

Given a domain G with boundary Γ and two points P, Q in G, Green's function $K(P, Q)$ satisfies

$$\Delta_P K(P, Q) = \delta_3(P, Q) \tag{2.6.1}$$

with boundary condition

$$K(P_\Gamma, Q) = 0, \tag{2.6.2}$$

where P_Γ denotes a point P on the boundary Γ and $\delta_3(P, Q)$ is defined in (2.3.8b). Thus, K consists of the fundamental solution plus a harmonic function chosen so that it cancels out the value of the fundamental solution on Γ. In particular, finding K is very much dependent on how complicated the domain G is.

It is easy to show that K is *symmetric*; that is,

$$K(P, Q) = K(Q, P). \tag{2.6.3}$$

To prove (2.6.3), we use the symmetric form of Green's formula (2.5.4) with respect to the integration variables $R = (\alpha, \beta, \gamma)$ and regard P and Q as *fixed* points in G. Let $u = K(R, P)$ and $v = K(R, Q)$ in (2.5.4). Since δ_3 is the product of three one-dimensional delta functions, each of which is an even function of its argument, we have $\delta_3(R, Q) = \delta_3(Q, R)$ and $\delta_3(R, P) = \delta_3(P, R)$. Therefore, $\Delta_R K(R, Q) = \delta_3(R, Q) = \delta_3(Q, R)$, $\Delta_R K(R, P) = \delta_3(R, P) = \delta_3(P, R)$, and the left-hand side of (2.5.4) becomes

$$\iiint_G [K(R, P)\delta_3(Q, R) - K(R, Q)\delta_3(P, R)]dV_R = K(Q, R) - K(P, Q). \tag{2.6.4}$$

The right-hand side of (2.5.4) vanishes because $K = 0$ on the boundary, and (2.6.3) follows.

Green's function for a given domain is *unique*, as can be seen by assuming the contrary. If there exist two Green's functions K_1 and K_2 for a given domain G, the difference between K_1 and K_2 is harmonic *everywhere* inside G even though K_1 and K_2 individually fail to be harmonic at $P = Q$. Therefore, according to the first corollary to the energy theorem, $K_1 - K_2 = 0$ in G.

2.6.2 Neumann's Function

Neumann's function is denoted by $N(P, Q)$ and satisfies

$$\Delta_P N(P, Q) = \delta_3(P, Q) \tag{2.6.5}$$

in G with boundary condition

$$\frac{\partial N}{\partial n_P}(P_\Gamma, Q) = C = \text{constant} \tag{2.6.6}$$

on Γ. Here $(\partial N/\partial n_P)$ denotes $\text{grad}_P\, N \cdot \mathbf{n}$, and P_Γ denotes a point P on the boundary Γ.

Actually, the constant C is *not arbitrary*, because integrating the left-hand side of (2.6.5) over the interior of G gives, according to Gauss' theorem,

$$\iiint\limits_G \text{div grad}_P\, N(P, Q) dV_P = \iint\limits_\Gamma \frac{\partial N}{\partial n_P} dA_P = C \iint\limits_\Gamma dA_P.$$

But the volume integral on the left-hand side is just $\iiint\limits_G \delta_3(P, Q) dV_P = 1$. Therefore,

$$C = \frac{1}{\text{area of } \Gamma}. \tag{2.6.7}$$

We also note that the solution of the boundary-value problem (2.6.5)–(2.6.6) is not unique in the sense that given one solution N_1 we can define a second solution N_2 that differs from N_1 by an arbitrary constant. This result, analogous to the uniqueness theorem of Section 2.5.4, also follows immediately from the second corollary to the energy theorem applied to the difference $N_1 - N_2$. It will be convenient to make Neumann's function unique by appending the normalizing condition

$$\iint\limits_\Gamma N(P_\Gamma, Q) dA_P = 0, \tag{2.6.8}$$

whenever the integral (2.6.8) exists.

As in (2.6.3), it is also easy to show that Neumann's function is symmetric; that is,

$$N(P, Q) = N(Q, P). \tag{2.6.9}$$

2.6.3 Dirichlet's Problem

Dirichlet's problem, also called the boundary-value problem of the first kind, consists of solving $\Delta u = 0$ subject to prescribed values of u on the boundary; that is,

$$\Delta u = 0 \quad \text{in } G, \tag{2.6.10a}$$

$$u = f = \quad \text{prescribed on} \quad \Gamma. \tag{2.6.10b}$$

If G is infinite, we assume that $u \to 0$ at infinity.

Once Green's function for G is known, we can write down the solution of (2.6.10) immediately. To see this, we use (2.5.4), the symmetric form of Green's formula, where we let the coordinates Q be the integration variables, whereas P is regarded as a fixed point in G. Let $u(Q)$ be the solution of (2.6.10) and let $v = K(P, Q) = K(Q, P)$ be Green's function for G. We then have

$$\iiint_G [u(Q)\delta_3(P, Q) - K(Q, P)\Delta_Q u]dV_Q$$

$$= \iint_\Gamma \left[u(Q_\Gamma)\frac{\partial K}{\partial n_Q}(P, Q) - K(P_\Gamma, Q)\frac{\partial u}{\partial n_Q}\right]dA_Q. \quad (2.6.11)$$

Since $\Delta_Q u = 0$ and $K(P_\Gamma, Q) = 0$, (2.6.11) reduces to

$$u(P) = \iint_\Gamma f(Q)\frac{\partial K}{\partial n_Q}(P, Q)dA_Q, \quad (2.6.12)$$

which is called the generalized Poisson formula. It gives u at any interior point P by quadrature once the boundary values f are prescribed *as long as K is known for G*.

2.6.4 Neumann's Problem

Neumann's problem consists of

$$\Delta u = 0 \quad \text{in } G, \quad (2.6.13a)$$

$$\frac{\partial u}{\partial n} = g = \text{prescribed on } \Gamma. \quad (2.6.13b)$$

Here, the function g cannot be prescribed arbitrarily, because (2.5.9) requires that $\iint_\Gamma g dA = 0$. Also, as in (2.6.8), we introduce the normalizing condition

$$\iint_\Gamma u dA = 0, \quad (2.6.14)$$

to make the solution unique if the integral (2.6.14) exists.

Assuming that we have found Neumann's function $N(P, Q)$ for the given domain, we again use (2.5.4) with $u = u(Q)$, $v = N(P, Q)$ to obtain

$$\iiint_G [u(Q)\delta_3(P, Q) - N(Q, P)\Delta_Q u]dV_P$$

$$= \iint_\Gamma \left[u(Q)\frac{\partial N}{\partial n_Q} - N(P, Q)\frac{\partial u}{\partial n_Q}\right]dA_Q. \quad (2.6.15)$$

The left-hand side of (2.6.15) reduces to $u(P)$, and the first integral on the right-hand side is the constant $C \iint_\Gamma u(Q)dA$, which vanishes for the choice of normalization (2.6.14). If the domain is infinite and $\iint_\Gamma u(Q)dA$ does not exist, then $C \to 0$, as it is the reciprocal area of Γ, and the product vanishes.

Thus, we obtain

$$u(P) = -\iint_\Gamma g(Q)N(P, Q)dA_Q. \tag{2.6.16}$$

Again, we emphasize that the solutions (2.6.12) and (2.6.16) are quadratures, once K or N has been derived for the given domain. Moreover, faced with the problem of computing u for a given domain and different boundary data, these formulas are most convenient, as one need only compute K or N once.

2.6.5 Examples of Green's and Neumann's Functions

(i) *Upper half-plane, $y \geq 0$ (two dimensions)*

Green's function for the upper half-plane may be interpreted as the deflection of a membrane (which is clamped all along the x-axis), as measured at a point $P = (x, y)$, due to a unit concentrated force at $Q = (\xi, \eta)$. It is clear from symmetry that the zero boundary condition on $y = 0$ can be achieved by adding an image force of unit negative strength at the point $\overline{Q} = (\xi, -\eta)$ (see Figure 2.16). Thus, we have

$$K(P, Q) = \frac{1}{2\pi} \log r_{PQ} - \frac{1}{2\pi} \log r_{P\overline{Q}}, \tag{2.6.17}$$

where

$$r_{PQ}^2 \equiv (x - \xi)^2 + (y - \eta)^2, \; r_{P\overline{Q}}^2 \equiv (x - \xi)^2 + (y + \eta)^2. \tag{2.6.18}$$

Note that in (2.6.17), K consists of the fundamental solution $(1/2\pi) \log r_{PQ}$, plus a *harmonic* function in the upper half-plane (because the singularity of $\log r_{P\overline{Q}}$ is in the lower half-plane), which cancels out the values of the fundamental solution on the boundary and thus satisfies the requirement $K(P_\Gamma, Q) = 0$.

Neumann's function in the upper half-plane is simply

$$N(P, Q) = \frac{1}{2\pi} \log r_{PQ} + \frac{1}{2\pi} \log r_{P\overline{Q}}, \tag{2.6.19}$$

and it may be interpreted as the velocity potential due to a two-dimensional unit source at Q and an image source of equal strength at $\overline{Q}$. Now, $\partial N/\partial n = -\partial N/\partial y$, and it is clear that on $y = 0$ the normal components of velocity due to the two sources are exactly opposite and cancel out (see Figure 2.17). Note that since the domain is infinite, the integral (2.6.8) does not exist and we have normalized N arbitrarily by requiring its minimum value on Γ to be $(1/\pi) \log \eta$.

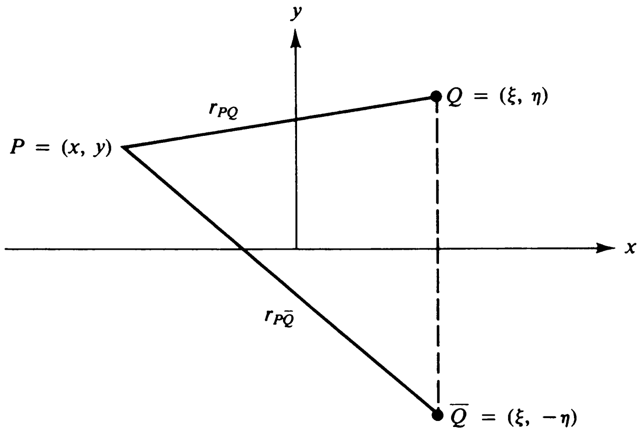

FIGURE 2.16. Positive source at Q and negative image source at $\overline{Q}$

(ii) *Upper half-space, $z \geq 0$ (three dimensions)*

Here we replace (2.6.17)–(2.6.19) by the appropriate expressions using three-dimensional sources, that is,

$$K(P, Q) = -\frac{1}{4\pi}\left(\frac{1}{r_{PQ}} - \frac{1}{r_{P\overline{Q}}}\right), \tag{2.6.20a}$$

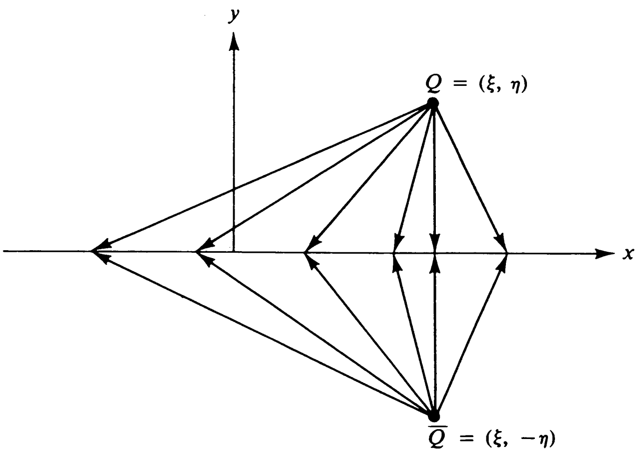

FIGURE 2.17. Positive sources at Q and $\overline{Q}$

$$N(P, Q) = -\frac{1}{4\pi}\left(\frac{1}{r_{PQ}} + \frac{1}{r_{P\overline{Q}}}\right), \tag{2.6.10b}$$

where

$$r_{PQ}^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2,\; r_{P\overline{Q}}^2 = (x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2.$$

Suppose we wish to solve Dirichlet's problem in the upper half-space $z \geq 0$, that is,

$$\Delta u = 0 \quad \text{in} \quad z \geq 0 \tag{2.6.21a}$$

with boundary condition

$$u(x, y, 0^+) = f(x, y) = \text{prescribed}, \tag{2.6.21b}$$

using the generalized Poisson formula (2.6.12). We need to compute $\partial K/\partial n_Q$, that is, $-\partial K/\partial \zeta$, on $\zeta = 0$. Using (2.6.20a), we obtain

$$\frac{\partial K}{\partial \zeta} = -\frac{1}{4\pi}\left(\frac{z-\zeta}{r_{PQ}^3} + \frac{z+\zeta}{r_{P\overline{Q}}^3}\right).$$

Therefore, on the boundary ($\zeta = 0$),

$$-\frac{\partial K}{\partial \zeta} = \frac{z}{2\pi}\frac{1}{[(x-\xi)^2 + (y-\eta)^2 + z^2]^{3/2}},$$

and (2.6.12) becomes

$$u(x, y, z) = \frac{z}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(\xi, \eta)d\xi\, d\eta}{[(x-\xi)^2 + (y-\eta)^2 + z^2]^{3/2}}. \tag{2.6.22}$$

This result may be interpreted as the potential at $P = (x, y, z)$ due to a distribution of dipoles on the $z = 0$ plane. The strength of this distribution is $-2f(x, y)$, and the axes of the dipoles are all along the $+z$ direction; see (2.4.23). Moreover, (2.4.30a) confirms that the boundary condition (2.6.21b) on $z = 0$ is indeed satisfied. In Section 2.7 we shall extend this idea to general Dirichlet problems and seek a representation of the solution in terms of a dipole distribution of unknown strength on the boundary surface.

It is instructive to rederive (2.6.22) and the corresponding formula for the Neumann problem in the upper half-space directly using the general result (2.5.17) for a harmonic function. Assume $\Delta u = 0$ in $z \geq 0$, and let $P = (x, y, z)$ and $Q = (\xi, \eta, \zeta)$ be two interior points in $z > 0$, whereas $Q_0 = (\xi, \eta, 0)$ is a point on the boundary. Specializing the integral formula (2.5.17) for this case, we have

$$r_{PQ} \equiv [(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{1/2},$$
$$r_{PQ_0} \equiv [(x-\xi)^2 + (y-\eta)^2 + z^2]^{1/2}.$$

We compute

$$\frac{\partial}{\partial n_Q}\left(\frac{1}{r_{PQ}}\right) = \frac{-1}{r_{PQ}^2}\frac{\partial r_{PQ}}{\partial(-\zeta)} = -\frac{(z-\zeta)}{r_{PQ}^3},$$

$$\left[\frac{\partial}{\partial n_Q}\left(\frac{1}{r_{PQ}}\right)\right]_{Q=Q_0} = -\frac{z}{r^3_{PQ_0}}.$$

Therefore, noting that $\partial u/\partial n_Q = -\partial u/\partial \zeta$, (2.5.17) becomes

$$u(x, y, z) = \frac{1}{4\pi}\iint_{-\infty}^{\infty}\left[\frac{u(\xi, \eta, 0^+)z}{r^3_{PQ_0}} - \frac{\frac{\partial u}{\partial \zeta}(\xi, \eta, 0^+)}{r_{PQ_0}}\right] d\xi\, d\eta. \qquad (2.6.23)$$

Now consider (2.5.4) with $u(Q)$ denoting the solution of (2.6.21a) and $v = 1/4\pi r_{P\overline{Q}}$. Here $\overline{Q} = (\xi, \eta, -\zeta)$ is the mirror image of Q with respect to the $z = 0$ plane, and

$$r_{P\overline{Q}} \equiv [(x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2]^{1/2}.$$

Since $\overline{Q}$ is in the lower half-space, $r_{P\overline{Q}}$ does not vanish for $z \geq 0$ and $\Delta_Q v = 0$. Thus, the left-hand side of (2.5.4) equals zero. To compute the right-hand side, we note that

$$\left(\frac{\partial u}{\partial n_Q}\right)_{Q=Q_0} = \frac{\partial u(\xi, \eta, 0^+)}{\partial(-\zeta)} = -\frac{\partial u(\xi, \eta, 0^+)}{\partial \zeta},$$

$$\left(\frac{\partial v}{\partial n_Q}\right)_{Q=Q_0} = \frac{1}{4\pi}\left[\frac{\partial}{\partial(-\zeta)}\left(\frac{1}{r_{P\overline{Q}}}\right)\right]_{Q=Q_0} = \frac{1}{4\pi}\frac{z}{r^3_{PQ_0}}.$$

Therefore, (2.5.4) reduces to

$$0 = \frac{1}{4\pi}\iint_{-\infty}^{\infty}\left[\frac{u(\xi, \eta, 0^+)z}{r^3_{PQ_0}} + \frac{\frac{\partial u}{\partial \zeta}(\xi, \eta, 0^+)}{r_{PQ_0}}\right] d\xi\, d\eta. \qquad (2.6.24)$$

Adding (2.6.13) and (2.6.24) gives Poisson's formula (2.6.22), and subtracting gives

$$u(x, y, z) = -\frac{1}{2\pi}\iint_{-\infty}^{\infty}\frac{\frac{\partial u}{\partial \zeta}(\xi, \eta, 0^+)}{r_{PQ_0}}\, d\xi\, d\eta, \qquad (2.6.25)$$

which is the solution of the following Neumann problem in the upper half-space (see Problem 2.6.3):

$$\Delta u = 0 \quad \text{in} \quad z \geq 0, \qquad (2.6.26a)$$

$$\frac{\partial u}{\partial z}(x, y, 0^+) = g(x, y) = \text{prescribed}. \qquad (2.6.26b)$$

The result (2.6.25) may be interpreted as the potential at $P = (x, y, z)$ due to a distribution of sources of strength $q(x, y) \equiv 2g(x, y)$ on the $z = 0$ plane [compare (2.6.15) with (2.4.32) and (2.6.26b) with (2.4.39a) to confirm that the boundary condition is satisfied]. In Section 2.7 we shall also extend this idea to solve Neumann's problem in a more general domain with a prescribed value of the normal derivative on the surface by using an unknown distribution of sources on the boundary.

(iii) *Interior (exterior) of unit sphere or circle*

Consider a unit sphere, two points $P(r, \theta, \phi)$ and $Q = (\rho, \theta', \phi')$ in the interior, and the point $\overline{Q} = (1/\rho, \theta', \phi')$ outside the sphere, as shown in Figure 2.18. If we denote the angle POQ by γ, the distance PQ by r_{PQ}, and the distance $P\overline{Q}$ by $r_{P\overline{Q}}$, we have

$$\cos\gamma \equiv \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi'), \tag{2.6.27a}$$

$$r_{PQ}^2 \equiv r^2 + \rho^2 - 2r\rho\cos\gamma, \tag{2.6.27b}$$

$$r_{P\overline{Q}}^2 \equiv r^2 + \frac{1}{\rho^2} - \frac{2r}{\rho}\cos\gamma. \tag{2.6.27c}$$

It is easily seen from (2.6.27) that if P is on the surface of the sphere, that is, if $r = 1$, then

$$\left.\frac{r_{PQ}}{r_{P\overline{Q}}}\right|_{r=1} = \rho. \tag{2.6.28}$$

Using this result, we can immediately derive the following Green's functions:

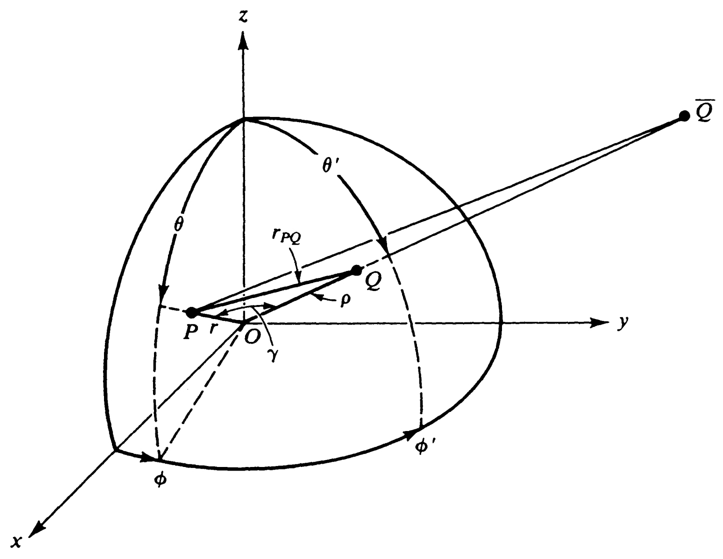

FIGURE 2.18. Geometry for spherical domain

Interior of unit circle

$$K = \frac{1}{2\pi} \log \frac{r_{PQ}}{\rho r_{P\bar{Q}}}, \quad \theta = \theta' = \pi/2. \tag{2.6.29}$$

Exterior of unit circle

$$K = \frac{1}{2\pi} \log \frac{\rho r_{P\bar{Q}}}{r_{PQ}}, \quad \theta = \theta' = \pi/2. \tag{2.6.30}$$

Interior of unit sphere

$$K = -\frac{1}{4\pi} \left[\frac{1}{r_{PQ}} - \frac{1}{\rho r_{P\bar{Q}}} \right]. \tag{2.6.31}$$

Exterior of unit sphere

$$K = -\frac{1}{4\pi} \left[\frac{1}{r_{P\bar{Q}}} - \frac{\rho}{r_{PQ}} \right]. \tag{2.6.32}$$

(iv) *Poisson's formula for the sphere*

To solve the Dirichlet problem for Laplace's equation in spherical polar coordinates,

$$\Delta u \equiv u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r^2 \sin^2 \theta} u_{\phi\phi} + \frac{\cot \theta}{r^2} u_\theta = 0 \tag{2.6.33a}$$

with

$$u(1, \theta, \phi) = f(\theta, \phi) = \text{prescribed}, \tag{2.6.33b}$$

in the interior of the unit sphere, we use (2.6.31) for K in (2.6.12). First we calculate

$$\frac{\partial K}{\partial n_Q} = \frac{\partial K}{\partial \rho} = \frac{1}{4\pi} \left(\frac{\rho - r \cos \gamma}{r_{PQ}^3} - \frac{\rho r^2 - r \cos \gamma}{\rho^3 r_{P\bar{Q}}^3} \right).$$

Therefore, on the boundary $\rho = 1$, we have

$$\left. \frac{\partial K}{\partial \rho} \right|_{\rho=1} = \frac{1 - r^2}{4\pi r_B^3}, \quad r_B^2 \equiv 1 + r^2 - 2r \cos \gamma,$$

and (2.6.12) becomes

$$u(r, \theta, \phi) = \frac{1 - r^2}{4\pi} \int_{\theta'=0}^{\pi} \int_{\phi'=0}^{2\pi} \frac{f(\theta', \phi') \sin \theta' d\phi' d\theta'}{[1 + r^2 - 2r \cos \gamma]^{3/2}}. \tag{2.6.34}$$

This is called *Poisson's* formula. Now, unlike the result (2.6.22) for the planar problem, (2.6.34) is not the potential only of a surface distribution of dipoles; it also includes a surface distribution of sources and is a special case of the general result (2.5.17). In fact, it is easy to show that (2.6.34) consists of a surface dipole distribution of strength $2f(\theta', \phi')$ plus a surface source distribution of strength $f(\theta', \phi')$ (see Problem 2.6.4).

It is easy to show by direct substitution that if $u = F(r, \theta, \phi)$ is harmonic inside the unit sphere, then $v = (1/r)F(1/r, \theta, \phi)$ is harmonic outside, and $v \to 0$ as $r \to \infty$ (see Problem 2.6.9).

Using this result, or explicit calculation with K given by (2.6.32), we obtain the following Poisson formula for the exterior problem:

$$u(r, \theta, \phi) = \frac{r^2 - 1}{4\pi} \int_{\phi'=0}^{2\pi} \int_{\theta'=0}^{\pi} \frac{f(\theta', \phi') \sin\theta' d\theta' d\phi'}{[1 + r^2 - 2r\cos\gamma]^{3/2}}, \tag{2.6.35}$$

which differs from (2.6.34) only by a minus sign.

The corresponding result for the interior of the circle is [see (2.2.15)]

$$u(r, \phi) = \frac{1 - r^2}{2\pi} \int_0^{2\pi} \frac{u(1, \phi')d\phi'}{1 + r^2 - 2r\cos(\phi - \phi')}, \tag{2.6.36}$$

and a change in sign gives the formula for the exterior problem.

The formulas for the Neumann problems inside or outside the sphere and circle are left as an exercise (Problem 2.6.5).

2.6.6 Connection Between Green's Function and Conformal Mapping (Two Dimensions)

Let D be a given domain in the xy-plane with boundary Γ consisting of at least two points. Let (x_0, y_0) be a fixed point inside D. According to Riemann's mapping theorem (for example, see p. 175 of [40]), there exists a conformal map of D onto the interior of the unit circle in the ζ-plane that sends the point $z_0 \equiv x_0 + iy_0$ to the origin $\zeta = 0$. Denote this conformal mapping by

$$\zeta = f(z, z_0); \quad \zeta \equiv \xi + i\eta, \quad z \equiv x + iy. \tag{2.6.37}$$

Clearly, Green's function for the interior of the unit circle with the source at the origin is $(1/2\pi)\log|\zeta|$. Therefore, Green's function for D is

$$K(x, y, x_0, y_0) = \frac{1}{2\pi} \log|f(z, z_0)|. \tag{2.6.38}$$

Thus, knowing Green's function for a two-dimensional domain is equivalent to knowing the mapping of the domain to the interior of a unit circle and vice versa (see Problem 2.6.10).

2.6.7 Series Representations; Connection with Separation of Variables

Consider Poisson's formula (2.6.34) for the interior of a unit sphere. We wish to develop this result in series form valid for $r < 1$. A useful identity for the kernel in Poisson's formula is

$$J \equiv \frac{1 - r^2}{[1 + r^2 - 2r\cos\gamma]^{3/2}} = \left[-\frac{1}{r_{PQ}} - 2\frac{\partial}{\partial\rho}\left(\frac{1}{r_{PQ}}\right) \right]_{\rho=1}, \qquad (2.6.39)$$

which can be verified immediately using the definition for r_{PQ} in (2.6.27b).

Recall the generating function for Legendre polynomials (for example, see pp. 102–103 of [8]):

$$\begin{aligned} \frac{1}{r_{PQ}} &= \rho^{-1}\left[1 - \frac{2r}{\rho}\cos\gamma + \left(\frac{r}{\rho}\right)^2\right]^{-1/2} \\ &= \frac{1}{\rho}\sum_{n=0}^{\infty} P_n(\cos\gamma)\left(\frac{r}{\rho}\right)^n, \quad \left(\frac{r}{\rho}\right) < 1, \end{aligned} \qquad (2.6.40)$$

where P_n denotes the Legendre polynomials

$$P_0(\cos\gamma) \equiv 1, \;\; P_1(\cos\gamma) \equiv \cos\gamma, \;\; P_2(\cos\gamma) \equiv \frac{3}{2}(\cos^2\gamma - \frac{1}{3}), \ldots .$$

We compute

$$\frac{\partial}{\partial\rho}\left(\frac{1}{r_{PQ}}\right) = -\frac{1}{\rho^2}\sum_{n=0}^{\infty} P_n(\cos\gamma)\left(\frac{r}{\rho}\right)^n (n+1).$$

Therefore, using (2.6.39) gives

$$J = \sum_{n=0}^{\infty}(2n+1)P_n(\cos\gamma)r^n. \qquad (2.6.41)$$

This series converges uniformly if $r < 1$, and we can interchange the order of summation and integration in (2.6.34) to obtain

$$u(r, \theta, \phi) = \sum_{n=0}^{\infty} \frac{r^n(2n+1)}{4\pi}\left\{\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi} f(\theta', \phi')P_n(\cos\gamma)\sin\theta' d\phi' d\theta'\right\}. \qquad (2.6.42)$$

This is a power series in r^n with coefficients C_n given by the double integral times $(2n+1)/4\pi$.

If the prescribed data are axisymmetric, that is, $f(\theta', \phi') = g(\theta')$, the expression in (2.6.42) involves ϕ' only through the P_n term. We can use the identity (for example, see pp. 326–328 of [43])

$$\frac{1}{2\pi}\int_0^{2\pi} P_n(\cos\gamma)d\phi' = P_n(\cos\theta)P_n(\cos\theta') \qquad (2.6.43)$$

to carry out the ϕ' integration, and (2.6.42) reduces to

$$u(r, \theta) = \sum_{n=0}^{\infty} \frac{r^n(2n+1)}{2}\left\{\int_0^{\pi} g(\theta')P_n(\cos\theta')\sin\theta' d\theta'\right\} P_n(\cos\theta). \qquad (2.6.44)$$

With the change of variable $s = \cos\theta'$, this becomes

$$u(r,\theta) = \sum_{n=0}^{\infty} C_n P_n(\cos\theta) r^n, \tag{2.6.45a}$$

where

$$C_n = \frac{2n+1}{2}\int_{-1}^{1} g(\cos^{-1} s) P_n(s)\,ds. \tag{2.6.45b}$$

This result also follows directly by solving the Dirichlet problem for Laplace's equation (2.6.33a)with axial symmetry ($u_\phi = 0$); that is,

$$u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{\cot\theta}{r^2}u_\theta = 0, \tag{2.6.46a}$$

$$u(r,\theta) = g(\theta) \tag{2.6.46b}$$

by separation of variables. Assuming $u(r,\theta) = R(r)\Theta(\theta)$ gives

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr} = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} - \frac{\cot\theta}{\Theta}\frac{d\Theta}{d\theta} = \lambda = \text{constant}. \tag{2.6.47a}$$

Therefore,

$$\frac{d^2\Theta}{d\theta^2} + (\cot\theta)\frac{d\Theta}{d\theta} + \lambda\Theta = 0, \tag{2.6.48}$$

which is Legendre's equation written in terms of θ. The conventional form of Legendre's equation is in terms of the independent variable $x = \cos\theta$ and has the form (with $\Theta(\theta) \equiv y(x)$, $' \equiv d/dx$)

$$(1 - x^2)y'' - 2xy' + \lambda y = 0. \tag{2.6.49}$$

Bounded solutions exist only for $\lambda = n(n+1)$ with n an integer. Then

$$\Theta(\theta) = P_n(\cos\theta).$$

The equation for R is equidimensional, with solutions r^n and r^{-n-1}. We discard the r^{-n-1} solutions because they become singular for the interior problem, and we have the series:

$$u(r,\theta) = \sum_{n=0}^{\infty} C_n r^n P_n(\cos\theta),$$

which corresponds to (2.6.45a). Using the boundary condition gives

$$g(\theta) = \sum_{n=0}^{\infty} C_n P_n(\cos\theta).$$

The orthogonality of the P_n gives

$$C_n = \frac{2n+1}{2}\int_0^{\pi} g(\theta')\sin\theta' P_n(\cos\theta')\,d\theta',$$

which reduces to (2.6.45b) if we set $s = \cos\theta'$.

Problems

2.6.1 Use the fundamental solution of the Helmholtz equation given in (2.3.81) to show that every solution of (2.3.80) with $\lambda \geq 0$ satisfies the following mean value theorem that generalizes (2.5.16):

$$\frac{\sin R\sqrt{\lambda}}{R\sqrt{\lambda}} u(P) = \frac{1}{4\pi R^2} \iint_{\Gamma_1} u\, dA. \tag{2.6.50}$$

Here Γ_1 is the surface of the sphere of radius R centered at the point P.

2.6.2a Consider the interior Dirichlet problem for the Helmholtz equation

$$\Delta u + \lambda u = 0 \quad \text{in} \quad G,\ \lambda > 0, \tag{2.6.51a}$$

$$u = f \quad \text{on} \quad \Gamma. \tag{2.6.51b}$$

Assume that you have computed Green's function for (2.6.51a) for G, i.e., you have the function $K(P, Q)$ that satisfies

$$\Delta_P K(P, Q) + \lambda K(P, Q) = \delta_3(P, Q) \tag{2.6.52a}$$

for any Q in G and any P in $G + \Gamma$. The boundary condition that K satisfies is

$$K(P, Q) = 0 \quad \text{if } P \quad \text{is on} \quad \Gamma. \tag{2.6.52b}$$

Multiply (2.6.51a) by $-K$ and (2.6.52a) by u, add and integrate the result over G; then use (2.5.4) to show that the solution of (2.6.51) is given by the *same* generalized Poisson formula (2.6.12) that we derived for Laplace's equation.

b. Show that if G is the upper half-space $z \geq 0$, Green's function for the Helmholtz equation is given by

$$K(P, Q) = -\frac{1}{4\pi}\left[\frac{\cos(\lambda^{1/2} r_{PQ})}{r_{PQ}} - \frac{\cos(\lambda^{1/2} r_{P\overline{Q}})}{r_{P\overline{Q}}}\right], \tag{2.6.53}$$

where r_{PQ} and $r_{P\overline{Q}}$ are defined in (2.6.18).

c. Use (2.6.53) to show that the solution of (2.6.51) for the upper half-space $z \geq 0$ is given by

$$u(x, y, z) = \frac{z}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \eta)\left[\frac{\cos(\lambda^{1/2} r_0)}{r_0^3} + \frac{\lambda^{1/2}\sin(\lambda^{1/2} r_0)}{r_0^2}\right] d\xi\, d\eta, \tag{2.6.54}$$

where $r_0^2 = (x - \xi)^2 + (y - \eta)^2 + z^2$.

2.6.3 Show that if G is the upper half-space $z \geq 0$, the expression (2.6.16) for the solution of Neumann's problem reduces to

$$u(x, y, z) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{g(\xi, \eta)}{[(x-\xi)^2 + (y-\eta)^2 + z^2]^{3/2}} \, d\xi \, d\eta, \tag{2.6.55}$$

the result we derived in (2.6.25).

2.6.4 Write (2.4.11) in spherical polar coordinates r, θ, ϕ as defined in Figure 2.15. Let $p(\theta', \phi')$ be the strength/unit area of dipoles distributed on the surface of the unit sphere, and assume that these dipoles are oriented along the outward normal to the sphere. Show that the potential reduces to

$$w(r, \theta, \phi) = \frac{1}{4\pi} \int_{\theta'=0}^{\pi} \int_{\phi'=0}^{2\pi} \frac{(1 - r\cos\gamma) p(\theta', \phi') \sin\theta' d\phi' d\theta'}{[1 + r^2 - 2r\cos\gamma]^{3/2}}. \tag{2.6.56}$$

Specialize (2.4.8) to the same geometry and show that it reduces to

$$u(r, \theta, \phi) = -\frac{1}{4\pi} \int_{\theta'=0}^{\pi} \int_{\phi'=0}^{2\pi} \frac{q(\theta', \phi') \sin\theta' d\phi' d\theta'}{[1 + r^2 - 2r\cos\gamma]^{1/2}}. \tag{2.6.57}$$

Use this to show that Poisson's formula (2.6.34) is the potential due to a surface distribution of dipoles of strength $p(\theta', \phi') \equiv 2f(\theta', \phi')$ plus the potential due to surface distribution of sources of strength $q(\theta', \phi') \equiv f(\theta', \phi')$.

2.6.5a Show that Neumann's function for the interior of the unit circle is

$$N(P, Q) = \frac{1}{2\pi} \log r_1 r_2 \rho, \tag{2.6.58}$$

where, as usual, P has the polar coordinates r, θ and Q has the polar coordinates ρ, θ'. Also,

$$r_1^2 \equiv r^2 + \rho^2 - 2r\rho\cos(\theta - \theta'), \tag{2.6.59a}$$

$$r_2^2 \equiv r^2 + \frac{1}{\rho^2} - \frac{2r}{\rho}\cos(\theta - \theta'). \tag{2.6.59b}$$

Thus,

$$\Delta_P N(P, Q) = \delta_2(P, Q), \tag{2.6.60}$$

and

$$\frac{\partial N}{\partial r} = \frac{1}{2\pi} \quad \text{for } P \quad \text{on} \quad r = 1. \tag{2.6.61}$$

What is N for the exterior problem?

b. Show that Neumann's function for the interior of the unit sphere is

$$N(P, Q) = -\frac{1}{4\pi} \left[\frac{1}{r_{PQ}} + \frac{1}{\rho r_{P\overline{Q}}} + \log \frac{C}{1 - \rho r\cos\gamma + (r^2\rho^2 + 1 - 2r\rho\cos\gamma)^{1/2}} \right], \tag{2.6.62}$$

where the notation is defined in (2.6.27) and where C is an arbitrary constant that may be chosen to satisfy (2.6.8).

2.6.6 Consider the Dirichlet problem for the two-dimensional Laplacian in the upper half-plane, that is,

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty; \quad 0 \leq y < \infty, \tag{2.6.63a}$$

with boundary conditions

$$u(x, 0) = f(x), \quad f \to 0 \quad \text{as } |x| \to \infty, \tag{2.6.63b}$$

$$u(x, \infty) = 0. \tag{2.6.63c}$$

Derive the solution in the form (see Problem 2.3.3)

$$u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)d\xi}{(x-\xi)^2 + y^2} \tag{2.6.64}$$

using

a. Green's function (2.6.17) in the two-dimensional version of (2.6.12).
b. Green's function (2.6.17), and superposition for Poisson's equation with homogeneous boundary condition $w(x, 0) = 0$, which results for $w(x, y) \equiv u(x, y) - f(x)$.
c. Fourier transforms with respect to x.
d. Specialize (2.6.64) to the case where

$$f(x) = \begin{cases} 1 & \text{if } -a \leq x \leq b, \\ 0 & \text{otherwise,} \end{cases}$$

and relate your result to Cauchy's integral formula in complex variables.

2.6.7a The two-dimensional version of the generalized Poisson formula (2.6.12) is

$$u(P) = \int_{\Gamma} f(Q) \frac{\partial K}{\partial n_Q} ds, \tag{2.6.65}$$

which reduces to

$$u(r, \theta) = \int_0^{2\pi} f(\theta') \left.\frac{\partial K}{\partial \rho}\right|_{\rho=1} d\theta' \tag{2.6.66}$$

for Dirichlet's problem in the interior of the unit circle. Show that when (2.6.29) is used to compute $(\partial K/\partial \rho)$, (2.6.66) reduces to (2.6.36).

b. Verify by direct substitution that (2.6.36) solves

$$\Delta u \equiv u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \tag{2.6.67a}$$

in $r \leq 1$, with the boundary condition

$$u(1, \theta) = f(\theta). \tag{2.6.67b}$$

c. Solve (2.6.67) using separation of variables to obtain the Fourier series

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta), \tag{2.6.68}$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta, \; b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta. \quad (2.6.69)$$

Show that this result follows from (2.6.36) if the integrand is expanded in a power series in r and then integrated term by term.

d. Calculate $u_r(r, \theta)$ using (2.6.36), and simplify the improper integral that results as $r \to 1^-$ to show that

$$u_r(1^-, \theta) = \frac{1}{2\pi}\lim_{\alpha\to 0}\int_0^{2\pi} \frac{f(\theta) - f(\theta')}{1 - (1-\alpha)\cos(\theta - \theta')}d\theta'. \quad (2.6.70)$$

Verify that (2.6.70) gives the correct result for the two special cases where $f(\theta) = \cos n\theta$ and $f(\theta) = \sin n\theta$, for which the solutions are $u(r, \theta) = r^n \cos n\theta$ and $u(r, \theta) = r^n \sin n\theta$, respectively.

2.6.8a Show that the two-dimensional version of (2.6.16) for the interior of the unit circle is

$$u(r, \theta) = -\int_0^{2\pi} g(\theta')N(r, \theta, 1, \theta')d\theta', \quad (2.6.71)$$

where according to (2.6.58),

$$N(r, \theta, 1, \theta') = \frac{1}{2\pi}\log[r^2 + 1 - 2r\cos(\theta - \theta')]. \quad (2.6.72)$$

b. Verify by direct substitution that (2.6.71) satisfies (2.6.67a) with the boundary condition $u_r(1, \theta) = g(\theta)$. In verifying the boundary condition you will encounter an improper integral if $r \to 1$. Handle this integral using the approach discussed in Section 2.4.4. In particular, show that the only contribution to u_r as $r \to 1$ occurs from the integral

$$I(r, \theta; \epsilon) \equiv \int_{\theta-\epsilon}^{\theta+\epsilon} g(\theta')\frac{r - \cos(\theta - \theta')}{1 + r^2 - 2r\cos(\theta - \theta')}d\theta' \quad (2.6.73)$$

if $(1 - r)/\epsilon \to 0$ as $\epsilon \to 0$.

2.6.9 Show that if $u = F(r, \theta, \phi)$ satisfies (2.6.33a), then $v(r, \theta, \phi) \equiv (1/r)F(1/r, \theta, \phi)$ satisfies $\Delta v = 0$.

2.6.10 This problem concerns the connection between conformal mapping and Green's function of the first kind for various two-dimensional domains D, that is, solutions of

$$u_{xx} + u_{yy} = \delta(x - \xi)\delta(y - \eta) \quad \text{in } D, \quad (2.6.74a)$$

$$u = 0 \quad \text{on the boundary of } D, \quad (2.6.74b)$$

where ξ and η are the coordinates of a point in D.

a. Consider the corner domain $D_1 : x \geq 0, y \geq 0$, and use image sources to show that Green's function is given by

$K(x, y, \xi, \eta)$

$$= \frac{1}{2\pi} \log \frac{[(x-\xi)^2 + (y-\eta)^2]^{1/2}[(x+\xi)^2 + (y+\eta)^2]^{1/2}}{[(x-\xi)^2 + (y+\eta)^2]^{1/2}[(x+\xi)^2 + (y-\eta)^2]^{1/2}}, \tag{2.6.75}$$

where $\xi > 0, \eta > 0$.

Use the mapping

$$w = z^2 \tag{2.6.76}$$

of D_1 onto the upper half-plane $y \geq 0$ and Green's function (2.6.17) for $y \geq 0$ to derive (2.6.75).

b. Consider the strip domain $D_2 : -\infty < x < \infty, 0 \leq y \leq \pi$. Use (2.6.17) and the mapping

$$w = e^z \tag{2.6.77}$$

of D_2 onto the upper half-plane to derive Green's function for D_2 in the form

$$K = \frac{1}{4\pi} \log \frac{(e^x \cos y - e^\xi \cos \eta)^2 + (e^x \sin y - e^\xi \sin \eta)^2}{(e^x \cos y - e^\xi \cos \eta)^2 + (e^x \sin y + e^\xi \sin \eta)^2}. \tag{2.6.78}$$

Now use image sources to calculate Green's function in the form

$$K = \frac{1}{4\pi} \sum_{n=1}^{\infty} \log \frac{(x-\xi)^2 + (y - 2n\pi - \eta)^2}{(x-\xi)^2 + (y - 2n\pi + \eta)^2}. \tag{2.6.79}$$

Compare (2.6.78) to (2.6.79) to deduce the identity

$$\frac{\cosh x - \cos y}{\cosh x - \cos z} = \prod_{n=-\infty}^{\infty} \frac{x^2 + (y - 2n\pi)^2}{x^2 + (z - 2n\pi)^2} \tag{2.6.80}$$

for real variables x, y, z.

c. Use the results in part (b) and the mapping defined by

$$w = \log \left(\frac{1+z}{1-z} \right)^2 \tag{2.6.81}$$

to solve (2.6.74) in the interior of the half disk: $x^2 + y^2 \leq 1, y \geq 0$.

2.6.11 Use the Poisson formula (2.6.34) for the interior of the unit sphere to calculate the potential on the z-axis in the form

$$u(r, 0, \phi) = \frac{1 - r^2}{r} \left[\frac{1}{1 - r^2} - \frac{1}{\sqrt{1 + r^2}} \right] \tag{2.6.82}$$

for the boundary condition $u = \operatorname{sign} z$ on the surface of the sphere.

2.6.12 Consider the following Neumann problem for the three-dimensional Laplacian in $z \geq 0$:

$$\Delta u = 0 \quad \text{in } z \geq 0, \tag{2.6.83a}$$

$$u_z(x, y, 0^-) = \begin{cases} 1 & \text{on } \sqrt{x^2 + y^2} < 1, \\ 0 & \text{on } \sqrt{x^2 + y^2} > 1. \end{cases} \tag{2.6.83b}$$

Use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ in (2.6.25) to obtain the solution in the form

$$u(r,z) = -\frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \frac{\rho d\rho d\phi}{(\rho^2 + r^2 - 2r\rho\cos\phi + z^2)^{1/2}}. \qquad (2.6.84)$$

Show that for $r > 1$ and $z = 0^+$, (2.6.84) can be expressed in the form

$$u(r,0^+) = -\frac{1}{2\pi}\sum_{n=0}^{\infty}\int_0^{2\pi}\int_0^1 \left(\frac{\rho}{r}\right)^{n+1} P_n(\cos\phi)d\rho d\phi. \qquad (2.6.85)$$

Use the identity (see (22.13.6) of [3])

$$\frac{1}{2\pi}\int_0^{2\pi} P_{2n}(\cos\phi)d\phi = \frac{1}{16^n}\frac{[(2n)!]^2}{(n!)^4},\ n = 0, 1, 2, \ldots, \qquad (2.6.86a)$$

$$\int_0^{2\pi} P_{2n+1}(\cos\phi)d\phi = 0,\ n = 0, 1, 2, \ldots, \qquad (2.6.86b)$$

to show that

$$u(r,0^+) = -\sum_{n=0}^{\infty}\frac{[(2n)!]^2}{2(n+1)16^n(n!)^4}\left(\frac{1}{r^{2n+1}}\right),\ r > 1. \qquad (2.6.87)$$

2.6.13a Use symmetry to show that Green's function for the corner domain G : $x \geq 0, y \geq 0, z \geq 0$ is

$$K(P,Q) = -\frac{1}{4\pi}\left(\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4} - \frac{1}{r_5} + \frac{1}{r_6} - \frac{1}{r_7} + \frac{1}{r_8}\right), \qquad (2.6.88)$$

where the source location Q_i for each of the r_i is

$$Q_1 = (\xi, \eta, \zeta),\ Q_2 = (\xi - \eta, \zeta), \qquad (2.6.89a)$$

$$Q_3 = (-\xi, -\eta, \zeta),\ Q_4 = (-\xi, \eta, \zeta), \qquad (2.6.89b)$$

$$Q_5 = (\xi, \eta, -\zeta),\ Q_6 = (\xi, -\eta, -\zeta), \qquad (2.6.89c)$$

$$Q_7 = (-\xi, -\eta, -\zeta),\ Q_8 = (-\xi, \eta, -\zeta). \qquad (2.6.89d)$$

b. Use (2.6.12) and (2.6.88) to express the solution of

$$\Delta u = 0 \text{ in } G, \qquad (2.6.90a)$$

$$u(x,y,0) = f(x,y),\ u(x,0,z) = u(0,y,z) = 0, \qquad (2.6.90b)$$

in the form

$$u(x,y,z) = \frac{z}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\left(\frac{1}{r_{10}^3} - \frac{1}{r_{30}^3} + \frac{1}{r_{30}^3} - \frac{1}{r_{40}^3}\right) d\xi\, d\eta, \qquad (2.6.91)$$

where r_{i0} is the value of r_i for $\zeta = 0$.

2.7 Solutions in Terms of Integral Equations

In Section 2.6 we saw that the solution of Dirichlet's and Neumann's problems could be expressed as a quadrature in terms of Green's function K, and Neumann's function N, respectively. The principal task in solving either of these boundary-value problems is the actual computation of K or N for the given domain. For the simple geometries considered in Section 2.6.5 (see also Problem 2.6.10), symmetry arguments easily defined K or N. We also observed that for planar boundaries, our final result for u in Dirichlet's problem turned out to be the potential due to a distribution of dipoles *only*, and in Neumann's problem a distribution of sources *only*. For the case of a spherical boundary, our result for Dirichlet's problem involved a surface distribution of dipoles as well as sources. In summary, for the simple geometries that we considered, it was easy to obtain, a posteriori, expressions linking the strengths of these distributions to the boundary data.

In this section we propose to bypass the calculation of K and N (as these may be impractical for a nontrivial geometry) and to proceed *directly* to representations in terms of dipole or source distributions of *unknown* strength. We shall show that one can then derive *integral equations* linking these unknown distribution strengths to the given boundary data.

2.7.1 Dirichlet's Problem

Consider Dirichlet's problem in an interior domain G bounded by the smooth surface Γ. We wish to solve

$$\Delta u = 0 \quad \text{in } G \tag{2.7.1a}$$

subject to

$$u = f, \tag{2.7.1b}$$

a prescribed function on Γ. Assume that the solution can be represented by a surface distribution of dipoles of strength per unit area μ oriented in the outward normal direction on Γ, that is,

$$u(P) = \iint_{\Gamma} \mu(Q_\Gamma) \frac{\partial}{\partial n_Q} \left(\frac{-1}{4\pi r_{PQ}} \right) dA_Q. \tag{2.7.2}$$

In order to apply the boundary condition $u = f$ on Γ, we must evaluate (2.7.2) for $P \to p$, a point on the boundary. Because $r_{PQ} \to r_{pQ}$ and $r_{pQ} \to 0$ for $Q = p$, (2.7.2) leads to an improper integral, as discussed in Section 2.4.4 for the case of a planar boundary. We show next how one may evaluate $u(p)$ by a process similar to the one used for the planar boundary.

Consider a small sphere of radius ϵ centered at p, as shown in Figure 2.19. This sphere intersects Γ along a curve C and excludes the small "disk" Γ_ϵ from Γ. Now with $\epsilon \neq 0$, subdivide the integration over Γ in (2.7.2) into a contribution due to Γ_ϵ and one due to $\Gamma - \Gamma_\epsilon$; then let $\epsilon \to 0$ and $P \to p$. If $\alpha(\epsilon)$, the distance

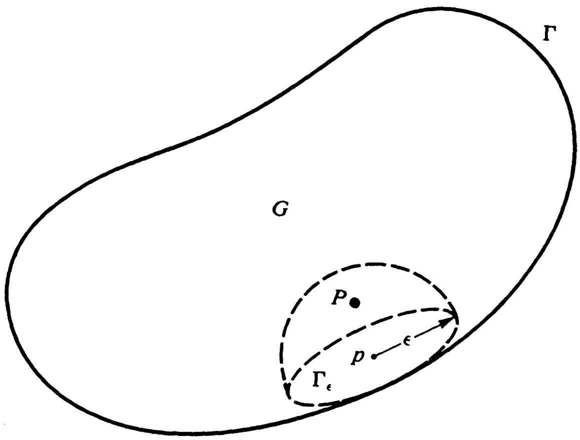

FIGURE 2.19. Sphere of radius ϵ centered at a point p on Γ

between P and p, tends to zero faster than ϵ, the potential at p may be expressed in the form

$$u(p) = \lim_{\epsilon\to 0}\left\{\iint_{\Gamma_\epsilon}\mu(Q)\frac{\partial}{\partial n_Q}\left(-\frac{1}{4\pi r_{PQ}}\right)dA_Q + \iint_{\Gamma-\Gamma_\epsilon}\mu(Q)\frac{\partial}{\partial n_Q}\left(-\frac{1}{4\pi r_{pQ}}\right)dA_Q\right\}. \qquad (2.7.3)$$

Note carefully the distinction between r_{PQ} and r_{pQ} in (2.7.3). In particular, it is important to note that we evaluate both integrals with ϵ fixed and not equal to 0. In the integral over Γ_ϵ, the observer is at $P(\epsilon) \neq 0$ with $\epsilon \neq 0$, whereas in the integral over $\Gamma - \Gamma_\epsilon$, we have set $P = p$ in the integrand. We then take the limit $\epsilon \to 0$ for the two expressions. It is easy to verify that this approach gives exactly the same result for $u(p)$ as would be derived from the more elaborate limit process discussed in Section 2.4.4 for the case of a planar boundary. In effect, setting $P = p$ in the second integrand in (2.7.3) corresponds to choosing the function $P(\epsilon)$ such that $(r_{Pp}/\epsilon) \to 0$.

We recall the result (2.5.22) for the potential of a dipole distribution of unit strength and note that as $\epsilon \to 0$, with $(r_{Pp}/\epsilon) \to 0$ also, the solid angle subtended from P tends to $\frac{1}{2}$ if Γ is smooth at p. Therefore, the contribution to (2.7.3) from

Γ_ϵ is

$$\lim_{\epsilon \to 0} \iint\limits_{\Gamma_\epsilon} \mu(Q) \frac{\partial}{\partial n_Q} \left(-\frac{1}{4\pi r_{PQ}} \right) dA_Q = \frac{\mu(p)}{2}. \tag{2.7.4}$$

This result also follows directly from (2.4.30b), since for ϵ sufficiently small and Γ smooth, Γ_ϵ is nearly planar, and the axis of the dipole at p points outside G for $\mu(p) > 0$.

Consider now the contribution to $u(p)$ from $\Gamma - \Gamma_\epsilon$. Since the disk Γ_ϵ is excluded from the second integral, $r_{pQ} > 0$, and the integrand is not singular as long as $\epsilon > 0$. The question then arises as to the behavior of $(\partial/\partial n_Q)(-1/4\pi r_{pQ})$ along the critical curve C as $\epsilon \to 0$ (see Figure 2.20). Now,

$$\frac{\partial}{\partial n_Q} \left(-\frac{1}{4\pi r_{pQ}} \right) = -\frac{1}{4\pi} \operatorname{grad} \left(\frac{1}{r_{pQ}} \right) \cdot \mathbf{n}(Q), \tag{2.7.5}$$

where $\mathbf{n}(Q)$ is the unit outward normal to Γ along C. Clearly, as $\epsilon \to 0$, $\operatorname{grad}(1/r_{pQ})$ and $\mathbf{n}(Q)$ become orthogonal and (2.7.5) vanishes on C, so there is no difficulty associated with the contribution to (2.7.5) near C, as C shrinks to the point p.

We have shown that the integral equation to be solved is

$$f = \frac{\mu(p)}{2} + \iint\limits_{\Gamma} \mu(Q) \frac{\partial}{\partial n_Q} \left(-\frac{1}{4\pi r_{pQ}} \right) dA_Q, \tag{2.7.6}$$

where the integral is to be interpreted in the limiting sense just discussed of having the point p "removed" from the surface of integration Γ. For the exterior problem

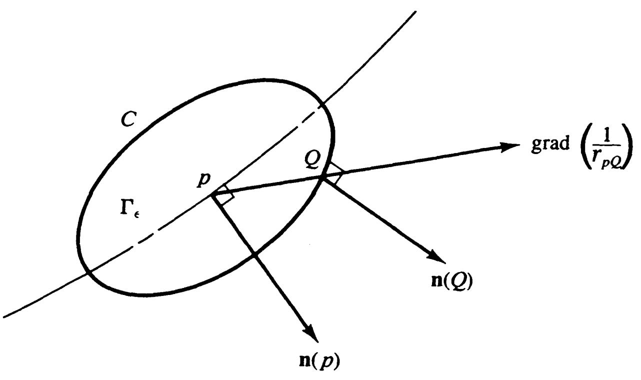

FIGURE 2.20. Orthogonality of $\mathbf{n}(Q)$ and $\operatorname{grad}(1/r_{pQ})$ as $\epsilon \to 0$

and the same representation (2.7.2), we obtain the analogous result

$$f = -\frac{\mu(p)}{2} + \iint_{\Gamma} \mu(Q) \frac{\partial}{\partial n_Q} \left(-\frac{1}{4\pi r_{pQ}} \right) dA_Q, \tag{2.7.7}$$

with the opposite sign for the contribution due to Γ_ϵ. Once μ is known, the solution can be obtained from (2.7.2) by quadrature. Note that if Γ is not smooth at p, the contribution of the first term on the right-hand side of (2.7.6) or (2.7.7) is not $\pm\mu(p)/2$ and must be worked out in detail. Also, if the entire surface Γ is nearly planar, we expect the contribution due to the surface integral in (2.7.6) or (2.7.7) to be small in absolute value relative to $|\mu(p)/2|$.

Let us next verify that the result (2.7.6) gives the correct solution (2.6.22) for Dirichlet's problem in the upper half-space. In this case, Γ is the plane $z = 0$, and $\partial/\partial n_Q = -\partial/\partial\zeta$. The assumed solution (2.7.2) using a surface ($z = 0$) distribution of dipoles with axes oriented along the $-z$ direction and with strength per unit area μ becomes

$$u = \frac{1}{4\pi} \iint_{-\infty}^{\infty} \mu(\xi, \eta) \frac{\partial}{\partial\zeta} \left(\frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}} \right) \Bigg|_{\zeta=0} d\xi\, d\eta$$

$$= \frac{z}{4\pi} \iint_{-\infty}^{\infty} \frac{\mu(\xi, \eta) d\xi\, d\eta}{[(x-\xi)^2 + (y-\eta)^2 + z^2]^{3/2}}. \tag{2.7.8}$$

This is the same expression as (2.4.23) with $p = -\mu$. The sign change is needed because we have the dipole axes in (2.7.2) pointing along the negative z direction, whereas those in (2.4.23) are along $+z$.

To evaluate (2.7.8) on the boundary, we need to let $z \to 0^+$; we obtain

$$u(x, y, 0^+) = \frac{\mu(x, y)}{2} \tag{2.7.9}$$

according to (2.4.30a). Thus, for the planar problem, the double integral in (2.7.6) or (2.7.7) equals zero, and we need not solve an integral equation; the dipole strength μ is given explicitly by

$$\mu(x, y) = 2f(x, y). \tag{2.7.10}$$

For nearly planar surfaces we expect the contributions of the integrals in (2.7.6) or (2.7.7) to be small in magnitude compared with $|\mu/2|$. This is, in fact, why we chose to solve Dirichlet's problem using a surface dipole distribution. We could equally well have used a surface source distribution, but the expression corresponding to (2.7.10) for the source strength is more complicated.

To see this, consider the general result (2.5.17) as it applies to the solution of the planar boundary-value problem (2.6.21). The value of $u(x, y, 0^+)$ is specified, but $(\partial u/\partial z)(x, y, 0^+)$ is not known. Therefore, the second term on the right-hand side of (2.5.17) is known, but the first is not. In fact, (2.5.17) reduces to

$$u(x, y, z) = -\frac{1}{4\pi}\iint\limits_{-\infty}^{\infty} \frac{\lambda(\xi, \eta)d\xi\, d\eta}{[(x-\xi)^2 + (y-\eta0^2 + z^2]^{1/2}}$$
$$+ \frac{z}{4\pi}\iint\limits_{-\infty}^{\infty} \frac{f(\xi, \eta)d\xi\, d\eta}{[(x-\xi)^2 + (y-\eta)^2 + z^2]^{3/2}}, \qquad (2.7.11)$$

where f is the prescribed boundary condition (2.6.21b) and $\lambda(x, y) = (\partial u/\partial z)(x, y, 0^+)$ is unknown.

To determine $\lambda(x, y)$, we evaluate (2.7.11) on the $z = 0^+$ plane. The left-hand side must equal $f(x, y)$ if u is to solve (2.6.21), and the second term on the right-hand side tends to $f(x, y)/2$ according to (2.4.30a) [see (2.4.23) with $p = -f$]. Therefore, $\lambda(x, y)$ satisfies the integral equation (in the limiting sense discussed earlier)

$$f(x, y) = -\frac{1}{2\pi}\iint\limits_{-\infty}^{\infty} \frac{\lambda(\xi, \eta)d\xi\, d\eta}{[(x-\xi)^2 + (y-\eta^2)]^{1/2}}. \qquad (2.7.12)$$

Recognizing that the second term on the right-hand side of (2.7.11) is just $u(x, y, z)/2$, we note that (2.7.11) gives the following representation of the solution of (2.6.21) in terms of a surface distribution of sources

$$u(x, y, z) = -\frac{1}{2\pi}\iint\limits_{-\infty}^{\infty} \frac{\lambda(\xi, \eta)d\xi\, d\eta}{[(x-\xi)^2 + (y-\eta)^2 + z^2]^{1/2}}. \qquad (2.7.13)$$

Clearly, for this example, the dipole representation (2.7.8) is significantly simpler to implement than the source representation (2.7.13). This motivates the use of dipoles for the general Dirichlet problem (2.7.1). We could, in principle, choose an *arbitrary mix* of surface distributions of dipoles and sources to represent the solution. In fact, we saw in interpreting (2.6.24) that the solution of Dirichlet's problem for the sphere is most easily represented by a particular combination of dipoles and sources on the surface.

2.7.2 Neumann's Problem

Neumann's problem has

$$\Delta u = 0 \quad \text{in} \quad G \qquad (2.7.14a)$$

with the boundary condition

$$\frac{\partial u}{\partial n} = g \quad \text{on} \quad \Gamma, \qquad (2.7.14b)$$

where the prescribed function g must satisfy the condition

$$\iint_{\Gamma} g dA = 0. \tag{2.7.14c}$$

We assume an unknown source distribution of strength $\nu(Q)$ on Γ. Hence, let

$$u(P) = \iint_{\Gamma} \nu(Q) \left(-\frac{1}{4\pi r_{PQ}} \right) dA_Q. \tag{2.7.15}$$

Now, applying the boundary condition (2.7.14b) gives [see (2.4.39b)]

$$g(p) = -\frac{\nu(p)}{2} + \iint_{\Gamma} \nu(Q) \frac{\partial}{\partial n_P} \left(-\frac{1}{4\pi r_{PQ}} \right) dA_Q \tag{2.7.16}$$

for the interior problem, where $\partial/\partial n_P$ is now with respect to the coordinates P. For the exterior problem, we have the same formulas, except that the first term on the right-hand side of (2.7.16) now has a positive sign.

For the simple example with axial symmetry, discussed earlier in Section 2.4.3, the axial distribution of sources suffices. In fact, it is intuitively obvious that for any *axisymmetric surface* distribution of sources, there corresponds an equivalent unique *axial* distribution such that the potential at every point in the domain G is identical in *both cases and satisfies* the same boundary condition.

Problems

2.7.1 Consider the axisymmetric three-dimensional Laplacian

$$u_{rr} + \frac{1}{r} u_r + u_{zz} = 0 \quad \text{on} \quad z \geq 0, \tag{2.7.17a}$$

with the mixed boundary conditions

$$u(r, 0^+) = 0 \quad \text{on} \quad r \equiv \sqrt{x^2 + y^2} \geq 1, \tag{2.7.17b}$$

$$u_z(r, 0^+) = f(r) \quad \text{on} \quad 0 \leq r < 1. \tag{2.7.17c}$$

a. Show that the solution $u(r, z)$ may be represented using a distribution of dipoles of strength per unit area equal to $h(r)$ on the unit disk $0 \leq r \leq 1$, $z = 0$ in the form

$$u(r, z) = -\frac{z}{4\pi} \int_0^{2\pi} \int_0^1 \frac{h(\rho)\rho \, d\rho \, d\phi}{(\rho^2 + r^2 + z^2 - 2r\rho \cos \phi)^{3/2}}. \tag{2.7.18}$$

In particular, show that dipoles on the unit disk automatically satisfy (2.7.17b).

b. Derive the following integral equation for $h(r)$:

$$f(r) = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{R(\phi,r)} \ell(\rho,\phi,r)d\rho\, d\phi + \frac{h(r)}{4\pi}\int_0^{2\pi}\frac{d\phi}{R(\phi,r)}, \tag{2.7.19}$$

where

$$R(\phi,r) = -r\cos\phi + 1 - r^2\sin^2\phi, \tag{2.7.20a}$$

$$\ell(\rho,\phi,r) = \frac{1}{\rho^2}\left[h(\sqrt{r^2+\rho^2+2r\rho\cos\phi}) - h(r)\right]. \tag{2.7.20b}$$

c. Now consider (2.7.17a) with the mixed boundary conditions

$$u(r,0^+) = f(r) \quad \text{on} \quad r \geq 1, \tag{2.7.21a}$$

$$u_z(r,0^+) = 0 \quad \text{on} \quad 0 \leq r < 1. \tag{2.7.21b}$$

Show that the solution may be represented by a source distribution of strength per unit area $h(r)$ on the unit disk in the form

$$u(r,z) = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^1 \frac{h(\rho)\rho\, d\rho\, d\phi}{(\rho^2+r^2+z^2-2r\rho\cos\phi)^{1/2}}, \tag{2.7.22}$$

where $h(r)$ satisfies the integral equation

$$f(r) = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{R(\phi,r)} h(\sqrt{r^2+\rho^2+2r\rho\cos\phi})\rho\, d\rho\, d\phi, \tag{2.7.23}$$

and $R(\phi, r)$ is given by (2.7.20a).

2.7.2 The two-dimensional problem corresponding to (2.7.17) is

$$u_{xx} + u_{yy} = 0 \quad \text{on} \quad y \geq 0, \tag{2.7.24a}$$

$$u(x,0^+) = 0 \quad \text{on} \quad |x| \geq 1, \tag{2.7.24b}$$

$$u_y(x,0^+) = f(x) \quad \text{on} \quad |x| < 1. \tag{2.7.24c}$$

Use a two-dimensional dipole distribution of strength per unit length equal to $h(x)$ on the interval $-1 \leq x \leq 1$ to express the solution in the form

$$u(x,y) = -\frac{y}{2\pi}\int_{-1}^1 \frac{h(\xi)}{(x-\xi)^2+y^2}d\xi. \tag{2.7.25a}$$

This automatically satisfies (2.7.24b).

Integrate (2.7.25) by parts to express u in the form

$$u(x,y) = \frac{1}{2\pi}\left[h(-1)\tan^{-1}\left(\frac{x+1}{y}\right) - h(1)\tan^{-1}\left(\frac{x-1}{y}\right) + \int_{-1}^1 h'(\xi)\tan^{-1}\left(\frac{\xi - x}{y}\right)d\xi\right]. \tag{2.7.25b}$$

Impose the boundary condition (2.7.24c) to obtain

$$f(x) = \frac{1}{2\pi}\left[\frac{h(1)}{x-1} - \frac{h(-1)}{x+1}\right] - \frac{1}{2\pi}\lim_{y\to 0^+}\int_{-1}^{1}\frac{(\xi - x)h'(\xi)}{y^2 + (\xi - x)^2}\,d\xi. \tag{2.7.26}$$

Show that when the limit $y \to 0^+$ is taken, (2.7.26) reduces to the following integral equation for $h(x)$

$$f(x) = \frac{1}{2\pi}\left[\frac{h(1)}{x-1} - \frac{h(-1)}{x+1}\right] - \frac{1}{2\pi}h'(x)\log\left(\frac{1-x}{1+x}\right)$$
$$- \frac{1}{2\pi}\int_{-1-x}^{1-x}\frac{h'(x+t) - h'(x)}{t}\,dt. \tag{2.7.27}$$

Thus, if $f(1)$ and $f(-1)$ are finite, we must set $h(0) = h(1) = h'(0) = h'(1) = 0$ to eliminate the singularities in the first two terms. The integral equation that results for $h'(x)$ is then

$$2\pi f(x) = -h'(x)\log\left(\frac{1-x}{1+x}\right) - \int_{-1-x}^{1-x}\frac{h'(x+t) - h'(x)}{t}\,dt, \tag{2.7.28}$$

which has to be solved subject to $h'(0) = h'(1) = 0$.

3

The Wave Equation

In Chapters 1 and 2, the applications that we studied were directly modeled by a linear equation (the diffusion equation or Laplace's equation). Here, however, most of the meaningful physical applications are governed by essentially quasilinear partial differential equations. We begin our discussion by considering in some detail the mathematical modeling for three such problems. We shall study the solution of these quasilinear equations in Chapters 7 and 8, and we devote the remainder of this chapter to solving the linear wave equations that result when we assume a small disturbance approximation.

3.1 The Vibrating String

Consider a string of infinitesimal thickness stretched initially under constant tension τ_0 (g cm/s^2) in the interval $0 \leq X \leq L_0$. Let the density of the string be $\rho \equiv \rho_0 r(X/L_0)$ (g/cm). Assume that the string is deflected under the influence of a prescribed loading $P \equiv P_0 p(X/L_0, T/T_0)$ (g/s^2), which always remains vertical. Here T_0 is a characteristic time scale, to be defined later, and capital letters indicate dimensional quantities. Thus, r and p are prescribed dimensionless functions that define the variation of the density and loading, respectively. Focus on a particular increment of length ΔX in the equilibrium state, as shown at the top of Figure 3.1, and examine the state of this increment when the string is in motion. In general, the increment will have been translated so that its left end has moved a horizontal distance U and a vertical distance V, as shown at the bottom of Figure 3.1. Also, the element will have rotated by an angle θ with respect to the horizontal and stretched (or compressed). The length of the deflected element is denoted by ΔL, and the tension τ in the deflected state acts at the angle θ (that is, along the direction of the local tangent).

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3.1.1 Equations of Motion

The equations of motion (force = mass × acceleration) for the horizontal and vertical deflections are therefore given by [neglecting terms of order $(\Delta X)^2$]

$$\rho_0 r \frac{\partial^2 U}{\partial T^2} = \frac{\partial}{\partial X}(\tau \cos\theta), \tag{3.1.1a}$$

$$\rho_0 r \frac{\partial^2 V}{\partial T^2} = P_0 p\left(\frac{X}{L_0} + \frac{U}{L_0}, \frac{T}{T_0}\right)\left(1 + \frac{\partial U}{\partial X}\right) + \frac{\partial}{\partial X}(\tau \sin\theta). \tag{3.1.1b}$$

The tension is a function of the strain σ, the change in length divided by the original length of a given element. Thus, we denote $\tau \equiv \tau_0[1 + f(\sigma)]$, where σ_0 is the value of the strain when the string is in equilibrium at the initial tension τ_0. Thus, $f(\sigma_0) = 0$ (see Figure 3.2).

We have $\sigma - \sigma_0 = (\Delta L - \Delta X)/\Delta X$, and referring to Figure 3.1, we see that

$$\sigma - \sigma_0 = \left[\left(1 + \frac{\partial U}{\partial X}\right)^2 + \left(\frac{\partial V}{\partial X}\right)^2\right]^{1/2} - 1, \tag{3.1.2}$$

$$\sin\theta = \frac{(\partial V/\partial X)}{[(1 + \partial U/\partial X)^2 + (\partial V/\partial X)^2]^{1/2}}, \tag{3.1.3}$$

$$\cos\theta = \frac{1 + (\partial U/\partial X)}{[(1 + \partial U/\partial X)^2 + (\partial V/\partial X)^2]^{1/2}}. \tag{3.1.4}$$

Let us introduce the dimensionless variables

$$u(x,t) \equiv \frac{U}{L_0}, \quad v(x,t) \equiv \frac{V}{L_0}, \tag{3.1.5}$$

$$x \equiv \frac{X}{L_0}, \quad t \equiv \frac{T}{T_0}, \tag{3.1.6}$$

where $T_0 \equiv L_0\sqrt{\rho_0/\tau_0}$; that is, we have used the characteristic speed $\sqrt{\tau_0/\rho_0}$ associated with the equilibrium state. The dimensionless equations of motion become

$$r(x)u_{tt} = \{[1 + f(\sigma)]\cos\theta\}_x, \tag{3.1.7a}$$

$$r(x)v_{tt} = \alpha p(x + u, t)(1 + u_x) + \{[1 + f(\sigma)]\sin\theta\}_x, \tag{3.1.7b}$$

where

$$\sigma = \sigma_0 + [(1 + u_x)^2 + v_x^2]^{1/2} - 1, \tag{3.1.8}$$

$$\sin\theta = \frac{v_x}{[(1 + u_x)^2 + v_x^2]^{1/2}}, \tag{3.1.9a}$$

$$\cos\theta = \frac{1 + u_x}{[(1 + u_x)^2 + v_x^2]^{1/2}}, \tag{3.1.9b}$$

and the dimensionless parameter $\alpha \equiv P_0 L_0/\tau_0$ gives a measure of the magnitude of the load in comparison with the equilibrium tension. Thus, for weak loading, $\alpha \ll 1$.

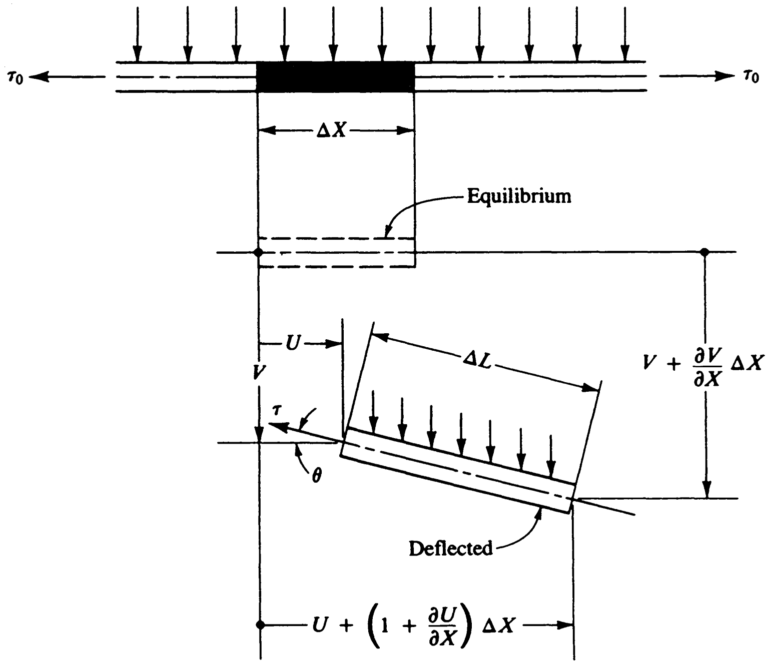

FIGURE 3.1. Deflection of an element ΔX

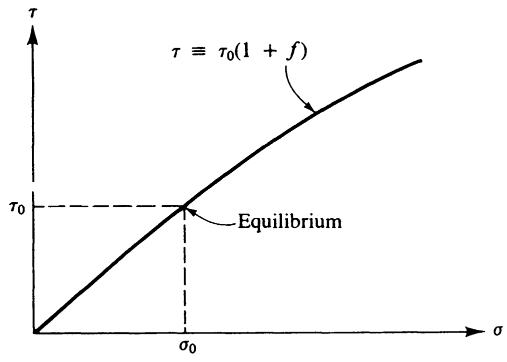

FIGURE 3.2. Tension as a function of the strain

3.1.2 Small-Amplitude Solutions

Let us restrict attention to the special case where the initial displacement and velocity of the string are small. Thus,

$$u(x, 0; \epsilon) = \epsilon a(x; \epsilon), \; u_t(x, 0; \epsilon) = \epsilon b(x; \epsilon),$$
$$v(x, 0; \epsilon) = \epsilon c(x; \epsilon), \; v_t(x, 0; \epsilon) = \epsilon d(x; \epsilon), \tag{3.1.10}$$

where ϵ is a small dimensionless parameter, and a, b, c, d are prescribed functions that are $O(1)$ as $\epsilon \to 0$. Assume also that α is small and comparable to ϵ, say $\alpha = \overline{\alpha}\epsilon$, where $\overline{\alpha}$ is an $O(1)$ constant. We exhibit the fact that the solution depends on the small parameter ϵ by our notation $u(x, t; \epsilon)$ and $v(x, t; \epsilon)$. It is convenient to rescale the dependent variables as follows:

$$u(x, t; \epsilon) = \epsilon \overline{u}(x, t; \epsilon), \quad v(x, t; \epsilon) = \epsilon \overline{v}(x, t; \epsilon). \tag{3.1.11}$$

Thus, the initial condition for $\overline{u}$, $\overline{v}$, and their derivatives are $O(1)$:

$$\overline{u}(x, 0; \epsilon) = a(x; \epsilon), \; \overline{u}_t(x, 0; \epsilon) = b(x; \epsilon),$$
$$\overline{v}(x, 0; \epsilon) = c(x; \epsilon), \; \overline{v}_t(x, 0; \epsilon) = d(x; \epsilon). \tag{3.1.12}$$

In terms of these rescaled variables, the various expressions appearing in (3.1.7) have the following expansions:

$$\begin{aligned} \sigma &= \sigma_0 + \sqrt{1 + 2\epsilon\overline{u}_x + \epsilon^2(\overline{u}_x^2 + \overline{v}_x^2)} - 1 \\ &= \sigma_0 + \epsilon\overline{u}_x + \frac{\epsilon^2}{2}\overline{v}_x^2 + O(\epsilon^3), \end{aligned} \tag{3.1.13a}$$

$$f(\sigma) = \epsilon f'(\sigma_0)\overline{u}_x + \frac{\epsilon^2}{2}[f'(\sigma_0)\overline{v}_x^2 + f''(\sigma_0)\overline{u}_x^2] + O(\epsilon^3), \tag{3.1.13b}$$

$$\sin\theta = \epsilon\overline{v}_x - \epsilon^2\overline{u}_x\overline{v}_x + O(\epsilon^3), \tag{3.1.13c}$$

$$\cos\theta = 1 - \frac{\epsilon^2}{2}\overline{v}_x^2 + O(\epsilon^3), \tag{3.1.13d}$$

$$\alpha p(x + \epsilon\overline{u}, t) = \overline{\alpha}\epsilon p(x, t) + \overline{\alpha}\epsilon^2 p_x(x, t)\overline{u} + O(\epsilon^3), \tag{3.1.13e}$$

where we have set $f(\sigma_0) = 0$ in the expansion (3.1.13b). Substituting the above expansions into (3.1.7), canceling out an ϵ, and retaining the leading nonlinearity gives

$$r(x)\overline{u}_{tt} - f'(\sigma_0)\overline{u}_{xx} = \frac{\epsilon}{2}\{[f'(\sigma_0) - 1]\overline{v}_x^2 + f''(\sigma_0)\overline{u}_x^2\}_x + O(\epsilon^2), \tag{3.1.14a}$$

$$\begin{aligned} r(x)\overline{v}_{tt} - \overline{v}_{xx} &= \overline{\alpha}p(x, t) + \epsilon\{\overline{\alpha}p(x, t)\overline{u} + [f'(\sigma_0) - 1]\overline{u}_x\overline{v}_x\}_x \\ &\quad + O(\epsilon^2). \end{aligned} \tag{3.1.14b}$$

We see that if we ignore the terms of order ϵ in (3.1.14), $\overline{u}$ and $\overline{v}$ obey decoupled linear wave equations. We can also derive the equation and initial conditions governing the correction terms of order ϵ. The details are very similar to those discussed in the next section, and are not given here.

For future reference, let us also consider the effect of introducing an elastic restoring force for the vertical component of the motion. For example, we can imagine the stretched string in equilibrium to be resting on a continuous elastic support, as shown in Figure 3.3. The support exerts a force per unit length given by $F \equiv F_0\phi(v)$ (g/s^2), in the direction opposite v. This force depends only on v with $\phi(0) = 0$. In this case, we must add $(-F)$ to the right-hand side of (3.1.1b), and this results in the added term $-\lambda\phi(v)$ to the right-hand side of (3.1.7b), where λ is the dimensionless parameter $\lambda \equiv F_0L_0/\tau_0$. The linear part of (3.1.14b) then becomes

$$r(x)\overline{v}_{tt} - \overline{v}_{xx} + \lambda\phi'(0)\overline{v} = \overline{\alpha}p(x,t), \qquad (3.1.15)$$

which we shall study in Section 3.7 for the case $r = 1$.

3.1.3 Boundary Conditions

If the string is fixed at the two ends, the boundary conditions are

$$\overline{u}(0,t;\epsilon) = \overline{u}(1,t;\epsilon) = \overline{v}(0,t;\epsilon) = \overline{v}(1,t;\epsilon) = 0. \qquad (3.1.16)$$

Either end may be subjected to an arbitrarily prescribed deflection. For example, if the left end is constrained to deflect in a prescribed manner vertically, the boundary condition at $x = 0$ would be

$$\overline{u}(0,t;\epsilon) = 0; \ \overline{v}(0,t;\epsilon) = g(t) = \text{prescribed}. \qquad (3.1.17)$$

A more elaborate arrangement is to attach one or both ends of the string to a point mass M supported by a spring with spring constant K. If the spring–mass system is attached to the left end and constrained to deflect vertically, the boundary condition at $x = 0$ becomes

$$U(0,T) = 0, \qquad (3.1.18a)$$

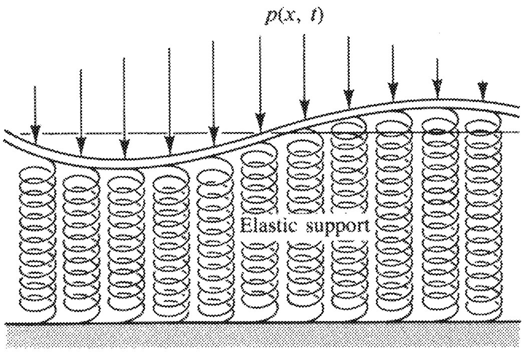

FIGURE 3.3. Vibrating string on an elastic support

$$MV_{TT}(0, T) + KV(0, T) = \tau_0(\tau \sin\theta), \tag{3.1.18b}$$

where we have assumed that the spring is in equilibrium when $V(0, T) = 0$. Introducing the dimensionless variables (3.1.5)–(3.1.6) gives

$$u(0, t; \epsilon) = 0, \tag{3.1.19a}$$

$$\mu v_{tt}(0, t; \epsilon) + kv(0, t; \epsilon) = (1 + f(\sigma))\sin\theta, \tag{3.1.19b}$$

where μ and k are the dimensionless parameters

$$\mu \equiv \frac{M}{L_0 \rho_0}, \quad k \equiv \frac{KL_0}{\tau_0}. \tag{3.1.20}$$

In terms of the rescaled $\overline{u}$, $\overline{v}$ variables, we have

$$\mu \overline{v}_{tt}(0, t; \epsilon) + k\overline{v}(0, t; \epsilon) - \overline{v}_x(0, t; \epsilon) = \epsilon \overline{u}_x \overline{v}_x + O(\epsilon^2). \tag{3.1.21}$$

We note the following limiting cases: If $k \to \infty$, i.e., for a very stiff spring, the effective boundary condition is that of zero deflection, $\overline{v}(0, t; \epsilon) = 0$, as expected. The linearized problem satisfies (3.1.21) with zero right-hand side, and in the limit $\mu \to 0$, we have the general linear boundary condition

$$k\overline{v}(0, t; 0) - \overline{v}_x(0, t; 0) = 0, \tag{3.1.22}$$

that we also studied for the diffusion equation in Section 1.4.7. The limit $k \to 0$ in (3.1.22) corresponds to a "free" end where we must set $\overline{v}_x = 0$.

3.2 Shallow-Water Waves

The problem of wave propagation in shallow water is perhaps the simplest problem in fluid mechanics that exhibits all the features that we intend to discuss in this chapter for the linearized problem and in Chapters 7 and 8 for the quasilinear case.

3.2.1 Assumptions

The basic equations follow from elementary physical concepts under the following assumptions.

The *shallow-water*, or *long-wave*, approximation is characterized by disturbances that have a long wavelength L_0 compared to the undisturbed depth H. In particular, this implies that vertical motions are ignored, and we have hydrostatic balance in the vertical, Y, direction. In addition, we neglect surface tension and viscosity, and we take the density ρ to be a constant. We shall also assume that flow quantities do not vary in the lateral direction (one-dimensional flow), and we simplify the geometry by taking a flat horizontal bottom. The last two assumptions are not necessary for deriving a shallow-water theory. Thus, in terms of an average (over depth) velocity vector, we may study a two-dimensional problem and also

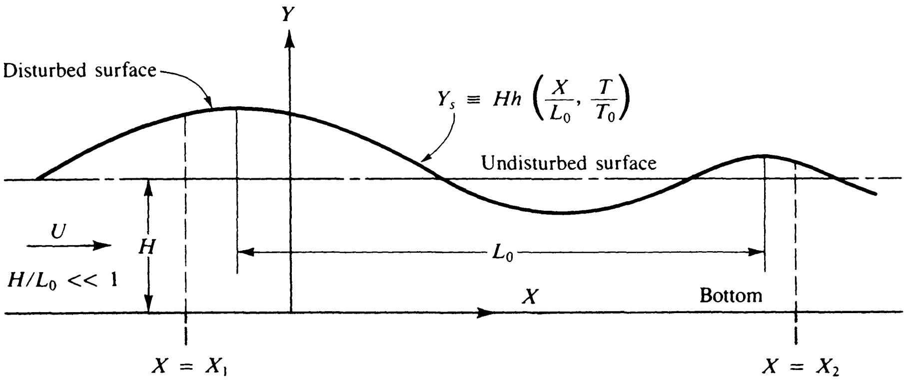

FIGURE 3.4. Shallow-water flow

account for a variable bottom (see Problem 3.2.1). The geometry for our simple one-dimensional problem is sketched in Figure 3.4.

The characteristic length and time scales L_0 and T_0 will be defined for some specific examples later on.

3.2.2 Hydrostatic Balance

Hydrostatic balance means that any element of water with volume $\Delta X \Delta Y \cdot 1$ is in static equilibrium in the vertical direction under the influence of gravity and pressure forces (see Figure 3.5). Thus, the net upward force due to the pressure difference on the upper and lower surfaces must be balanced by the weight of the element; that is,

$$P(X, Y, T)\Delta X - P(X, Y + \Delta Y, T)\Delta X - \rho g \Delta X \Delta Y = 0, \qquad (3.2.1)$$

where P is the pressure and g is the acceleration of gravity. In the limit as $\Delta Y \to 0$, we obtain

$$\frac{\partial P}{\partial Y} = -\rho g = \text{constant}, \qquad (3.2.2)$$

and integrating this expression from the free surface $Y_S = Hh$, where $P = 0$, to Y gives

$$\int_0^{P(X,Y,T)} dP' = -\rho g \int_{Hh}^{Y} dY',$$

or

$$P(X, Y, T) = \rho g(Hh - Y). \qquad (3.2.3)$$

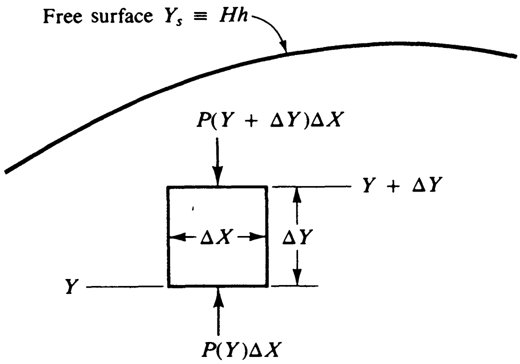

FIGURE 3.5. Hydrostatic balance

3.2.3 Conservation of Mass

Let X_1 and X_2 be two fixed positions ($X_1 < X_2$) through which the water flows. Let G denote the domain of water bounded by the bottom, the free surface, and the fixed vertical planes $X = X_1$ and $X = X_2$. Since the upper boundary of G (the free surface) varies with time, so does the mass of G. In order to satisfy mass conservation, we must require that the time rate of change of mass be equal to the net inflow of mass through the boundaries of G. Since mass flows only in (or out)

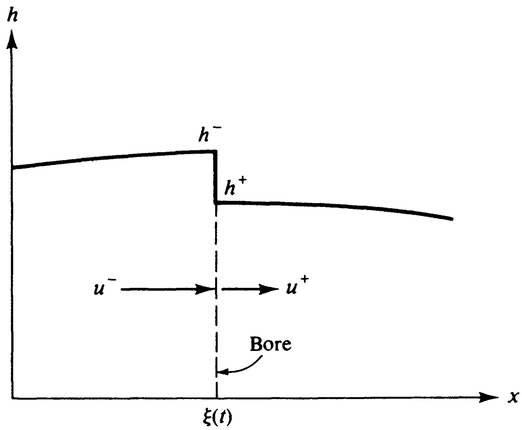

FIGURE 3.6. Solution with a discontinuity

from X_1 and X_2, we have

$$\frac{d}{dT}\int_{X_1}^{X_2} \rho H h\, dX = -\rho U H h\Big|_{X=X_1}^{X=X_2}, \tag{3.2.4}$$

where U is the vertically averaged component of velocity in the X direction.

It is convenient to introduce the dimensionless variables

$$x \equiv \frac{X}{L_0},\quad y \equiv \frac{Y}{H},\quad t \equiv \frac{T}{T_0},\quad u \equiv \frac{U}{(gH)^{1/2}},\quad p \equiv \frac{P}{\rho g H}, \tag{3.2.5}$$

and to choose $T_0 \equiv L_0/(gH)^{1/2}$. Then (3.2.4) becomes

$$\frac{d}{dt}\int_{x_1}^{x_2} h(x,t)dx + u(x_2,t)h(x_2,t) - u(x_1,t)h(x_1,t) = 0. \tag{3.2.6}$$

Note that (3.2.6) is valid even if u and h are not continuous inside (x_1, x_2). For example, consider a typical discontinuity in u and h at some value of ξ: $x_1 < \xi < x_2$, as sketched in Figure 3.6. Such a discontinuity is called a bore and is physically quite relevant, as is discussed in Chapters 5, 7, and 8. The values of u and h on either side of the bore, indicated by $\pm$ superscripts, are not equal. Moreover, the bore will be moving, that is, the bore location is a function of time, $\xi(t)$. In Chapter 5, we shall use (3.2.6) to derive relations linking u^+, u^-, h^+, h^- with $d\xi/dt$.

3.2.4 Conservation of Momentum in the X Direction

According to Newton's law of conservation of momentum, the time rate of change of momentum of G is balanced by the net inflow of momentum through the boundaries plus the forces exerted by the boundaries on G.

Since the pressure is zero on the free surface, we have no force there. Also, the stress is normal to the bottom (because we have an inviscid fluid and there is no shear stress on the bottom). Hence the pressure on a flat horizontal bottom does not contribute a horizontal force; this is no longer true if the bottom varies (see Problem 3.2.1). In the present case, the only horizontal forces acting on the boundaries of G are the pressure forces on the vertical planes $X = X_1$ and $X = X_2$. The integral law of horizontal momentum conservation is therefore given by

$$\frac{d}{dT}\int_{X_1}^{X_2} \rho U H h\, dX + \rho U^2 H h\Big|_{X=X_1}^{X=X_2} + \int_0^{Hh_2} P_2\, dY - \int_0^{Hh_1} P_1\, dY = 0, \tag{3.2.7a}$$

where $h_i \equiv h(x_i, t)$, $P_i \equiv P(X_i, Y, T)$, $i = 1, 2$. In dimensionless form, this becomes

$$\frac{d}{dt}\int_{x_1}^{x_2} uh\, dx + u^2 h\Big|_{x=x_1}^{x=x_2} + \int_0^{h_2} p_2\, dy - \int_0^{h_1} p_1\, dy = 0. \tag{3.2.7b}$$

Using the dimensionless form of (3.2.2),

$$p = h - y, \tag{3.2.8}$$

we calculate

$$\int_0^{h_i} p_i\,dy = \int_0^{h_i} (h_i - y)\,dy = \left(h_i y - \frac{y^2}{2}\right)\Bigg|_{y=0}^{y=h_i} = \frac{h_i^2}{2}, \quad i = 1, 2. \quad (3.2.9)$$

Therefore, the integral law of momentum conservation reduces to

$$\frac{d}{dt}\int_{x_1}^{x_2} uh\,dx + \left(u^2 h + \frac{h^2}{2}\right)\Bigg|_{x=x_1}^{x=x_2} = 0. \qquad (3.2.10)$$

Here again (3.2.10) remains valid for the discontinuous solutions in G, as sketched in Figure 3.6.

3.2.5 Smooth Solutions

If u and h are smooth (that is, continuous and have continuous first partial derivatives with respect to x and t), (3.2.6) and (3.2.10) reduce to the differential conservation relations

$$h_t + (uh)_x = 0, \qquad (3.2.11a)$$

$$(uh)_t + \left(u^2 h + \frac{h^2}{2}\right)_x = 0. \qquad (3.2.11b)$$

Expressions in this form are often called divergence relations. Equation (3.2.11b) simplifies if h_t is eliminated using (3.2.11a), and we obtain

$$h_t + (uh)_x = 0, \qquad (3.2.12a)$$

$$u_t + h_x + uu_x = 0. \qquad (3.2.12b)$$

This pair of quasilinear (derivatives occur in a linear way) equations will be discussed in detail in Chapter 7 when we study hyperbolic systems of equations. It is important to bear in mind that since we have ignored vertical motions and have averaged the horizontal velocity, (3.2.10) provides only an approximate description of the flow. A more systematic derivation of these equations as the leading approximation for a long-wave (that is, $HL_0^{-1} \ll 1$) theory can be found in Section 5.2 of [27], where it is shown that terms of order $(H/L_0)^2$ have been ignored in (3.2.10). See also Section 8.3.2.

In the present chapter we are primarily concerned with the linearized equations that result to leading order when we assume small disturbances (see Section 3.2.9). We shall also see in Section 3.3.4 that analogous equations describe the one-dimensional flow of a compressible perfect gas.

3.2.6 Energy Conservation

It is natural to ask at this point whether the flow described by the solution of the system (3.2.12) is compatible with the law of energy conservation. Note that in the ensuing derivation we are restricted to a narrow definition of energy, since our model does not admit dissipation.

Consider a column of water of width ΔX and unit breadth extending from the bottom $Y = 0$ to the free surface $Y_S = Hh$. The kinetic energy of this column is

$$\text{K.E.} \equiv \frac{\rho \Delta X H h U^2}{2}. \tag{3.2.13}$$

The potential energy is

$$\text{P.E.} \equiv \frac{\rho g \Delta X H^2 h^2}{2}. \tag{3.2.14}$$

In dimensionless form, the total mechanical energy is

$$\text{K.E.} + \text{P.E.} = \int_{x_1}^{x_2} \left(\frac{u^2 h}{2} + \frac{h^2}{2} \right) dx. \tag{3.2.15}$$

Thus, equating the rate of change of energy to the net inflow of energy plus the work done on G by the pressure forces acting at x_1 and x_2, we obtain the integral law of energy conservation:

$$\frac{d}{dt} \int_{x_1}^{x_2} \left(\frac{u^2 h}{2} + \frac{h^2}{2} \right) dx + \left. \frac{u^3 h}{2} \right|_{x=x_1}^{x=x_2} + \left. \frac{u h^2}{2} \right|_{x=x_1}^{x=x_2}$$

$$+ u_2 \int_0^{h_2} p_2 \, dy - u_1 \int_0^{h_1} p_1 \, dy = 0. \tag{3.2.16}$$

Using (3.2.9), this reduces to

$$\frac{d}{dt} \int_{x_1}^{x_2} \left(\frac{u^2 h}{2} + \frac{h^2}{2} \right) dx + \left. \frac{u^3 h}{2} \right|_{x=x_1}^{x=x_2} + \left. u h^2 \right|_{x=x_1}^{x=x_2} = 0. \tag{3.2.17}$$

The corresponding differential conservation relation for smooth solutions is

$$\left(\frac{u^2 h}{2} + \frac{h^2}{2} \right)_t + \left(\frac{u^3 h}{2} + u h^2 \right)_x = 0. \tag{3.2.18}$$

It is easy to verify that if (3.2.12) is used in (3.2.18), the latter reduces to an identity. Thus, under the assumption of continuous partial derivatives, conservation of mass and momentum imply *conservation of mechanical energy*. We shall explore this result further when we discuss discontinuous solutions in Section 5.3.4.

3.2.7 Initial-Value Problem

We now consider an initial state of water of infinite extent, $-\infty < x < \infty$, which will define the solution for all later times. It is intuitively obvious that if we specify the entire field u and h at $t = 0$, we then ought to be able to use (3.2.12) to determine u and h for all subsequent times and all x. A very simple initial state is to have the water at rest ($u = 0$) in some nonequilibrium state. For instance, imagine pressing down on the free surface with a solid sheet of width L_0 and shape defined by $Y = H + Af(X/L_0)$ with $Af < 0$. Here A is a constant characterizing the amplitude of the surface distortion. Thus the water, while still at rest, is forced below the equilibrium level H by the amount Af over an interval L_0 wide. Now at $t = 0$, we suddenly remove the surface constraint.

We use the width L_0 of the initial surface disturbance in the normalizations (3.2.5) and conclude that the initial conditions are

$$u(x, 0; \epsilon) = 0, \quad h(x, 0; \epsilon) = 1 + \epsilon f(x), \tag{3.2.19}$$

where $\epsilon \equiv A/H$, and we again exhibit the dependence of the solution on ϵ explicitly by the notation $u(x, t; \epsilon), h(x, t; \epsilon)$. A number of other initial disturbance mechanisms are possible.

3.2.8 Signaling Problem

Another means of generating a disturbance is to install a vertical wavemaker located initially at some point, say $X = 0$, and to consider the motion generated in the water to one side of the wavemaker as this moves in some prescribed fashion (see Figure 3.7).

The wavemaker displacement, X_w, as a function of time may be defined in the form $X_w \equiv L_1 s(T/T_0)$, where L_1 is a characteristic amplitude for the wavemaker displacement and T_0 is a characteristic time. For example, we may have $X_w = L_1 \sin(T/T_0)$, as sketched in Figure 3.8. Using $L_0 \equiv \sqrt{gH}T_0$ and T_0 as the characteristic length and time scales in (3.2.5), the dimensionless form of the wavemaker displacement becomes

$$x_w \equiv \nu s(t), \quad \nu \equiv \frac{L_1}{\sqrt{gH}T_0}, \tag{3.2.20}$$

where the dimensionless parameter ν is the ratio of the wavemaker characteristic speed (L_1/T_0) to the characteristic flow speed $\sqrt{gH}$.

Now, the appropriate boundary condition at the wavemaker is that the velocity of the water u must equal the velocity of the wavemaker $dx_w/dt = \nu\dot{s}(t)$.

Thus, we have initial conditions

$$u(x, 0; \nu) = 0, \quad h(x, 0; \nu) = 1, \tag{3.2.21}$$

and the boundary condition

$$u(\nu s(t), t; \nu) = \nu\dot{s}(t), \quad t > 0. \tag{3.2.22}$$

3.2.9 Small-Amplitude Solutions

In each of the preceding examples, the solution involves one dimensionless parameter (ϵ or ν). Henceforth, we denote ν by ϵ. The case $\epsilon \ll 1$ corresponds

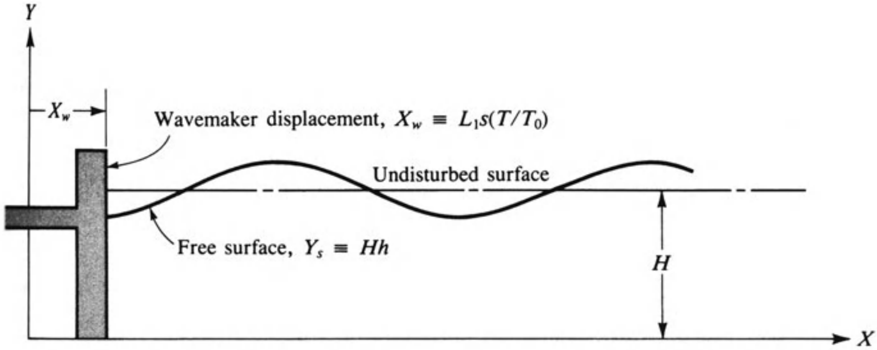

FIGURE 3.7. Signaling problem for a wavemaker

to small-amplitude initial (boundary) disturbances. Since for $\epsilon = 0$ the solution is $u(x, t; \epsilon) = 0$, $h(x, t; \epsilon) = 1$, we again rescale the dependent variables accordingly:

$$u(x, t; \epsilon) = \epsilon \bar{u}(x, t; \epsilon), \tag{3.2.23a}$$

$$h(x, t; \epsilon) = 1 + \epsilon \bar{h}(x, t; \epsilon), \tag{3.2.23b}$$

and write the system (3.2.12) in the form

$$\bar{h}_t + \bar{u}_x = -\epsilon (\bar{h}\bar{u})_x, \tag{3.2.24a}$$

$$\bar{u}_t + \bar{h}_x = -\epsilon \bar{u}\, \bar{u}_x. \tag{3.2.24b}$$

The initial conditions (3.2.19) then become

$$\bar{u}(x, 0; \epsilon) = 0, \quad \bar{h}(x, 0; \epsilon) = f(x). \tag{3.2.25}$$

The signaling problem has zero initial conditions,

$$\bar{u}(x, 0; \epsilon) = 0, \quad \bar{h}(x, 0; \epsilon) = 0, \tag{3.2.26}$$

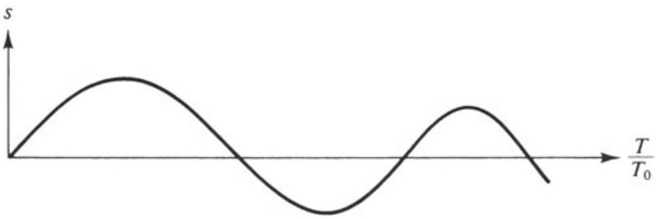

FIGURE 3.8. Wavemaker displacement

and the boundary condition

$$\bar{u}(\epsilon s(t), t; \epsilon) = \dot{s}(t), \ t > 0. \tag{3.2.27}$$

We now expand $\bar{u}$ and $\bar{h}$ in a series in powers of ϵ and retain only the first two terms in the series:

$$\bar{u}(x, t; \epsilon) = u_1(x, t) + \epsilon u_2(x, t) + O(\epsilon^2), \tag{3.2.28a}$$

$$\bar{h}(x, t; \epsilon) = h_1(x, t) + \epsilon h_2(x, t) + O(\epsilon^2). \tag{3.2.28b}$$

Actually, it only makes sense to look at u_1 and h_1 as predicted by this model because if we want accuracies of order ϵ^2 in u and h (i.e., of order ϵ in $\bar{u}$ and $\bar{h}$), we must also retain a term in (3.2.12b) that was ignored in our simple model. However, for the sake of simplicity, and in order to illustrate the procedure for deriving the equations for the higher-order terms, let us regard (3.2.12) as an "exact" model of the physical situation.

To derive governing equations for $u_1, u_2, h_1, h_2, \ldots$, we substitute the expansions (3.2.28) into (3.2.24) and collect terms of order ϵ and ϵ^2. The result is

$$h_{1_t} + u_{1_x} + \epsilon[h_{2_t} + u_{2_x} + (u_1 h_1)_x] = O(\epsilon^2), \tag{3.2.29a}$$

$$u_{1_t} + h_{1_x} + \epsilon[u_{2_t} + h_{2_x} + \frac{1}{2}(u_1^2)_x] = O(\epsilon^2). \tag{3.2.29b}$$

This perturbation series must be valid for ϵ arbitrary (but small). Therefore, the coefficient of each power of ϵ must vanish, and we obtain the following linear first-order system governing u_1, h_1:

$$h_{1_t} + u_{1_x} = 0, \tag{3.2.30a}$$

$$u_{1_t} + h_{1_x} = 0. \tag{3.2.30b}$$

The terms u_2, h_2 obey

$$h_{2_t} + u_{2_x} = -(u_1 h_1)_x, \tag{3.2.31a}$$

$$u_{2_t} + h_{2_x} = -\frac{1}{2}(u_1^2)_x, \tag{3.2.31b}$$

and so on. The homogeneous operator is the same linear one to all orders, and the right-hand sides of the equations for u_n, h_n depend only on $u_{n-1}, u_{n-2}, \ldots, u_1, h_{n-1}, h_{n-2}, \ldots, h_1$.

(i) *Initial-value problem*

Consider first the initial-value problem on $-\infty < x < \infty$ resulting from (3.2.25). Applying the expansion (3.2.28) evaluated at $t = 0$ to (3.2.25) implies that

$$u_1(x, 0) = 0, \ h_1(x, 0) = f(x), \tag{3.2.32}$$

$$u_2(x, 0) = 0, \ h_2(x, 0) = 0. \tag{3.2.33}$$

Thus, we first need to solve (3.2.30) subject to the initial conditions (3.2.32). One approach is to eliminate either u_1 or h_1 to derive a second-order equation for the other variable. For example, if we differentiate (3.2.30b) with respect to x and subtract the result from the partial derivative of (3.2.30a) with respect to t, we obtain the *wave equation* for h_1,

$$h_{1_{tt}} - h_{1_{xx}} = 0, \quad -\infty < x < \infty, \quad t \geq 0. \tag{3.2.34}$$

Based on our experience so far in choosing appropriate initial conditions, we expect (3.2.34) to require *two* conditions at $t = 0$, as the operator is second-order in t. In Chapter 4 this will be fully justified. Now, one initial condition given in (3.2.32) is $h_1(x, 0) = f(x)$. We obtain the other from evaluating (3.2.30a) at $t = 0$ and noting that $u_1(x, 0) = 0$ implies $u_{1_x}(x, 0) = 0$. Thus,

$$h_{1_t}(x, 0) = -u_{1_x}(x, 0) = 0. \tag{3.2.35}$$

The solution of (3.2.34), subject to the two initial conditions $h_1(x, 0) = f(x)$, $h_{1_t}(x, 0) = 0$, is discussed in Section 3.4.2. Once $h_1(x, t)$ is known, we can compute $u_1(x, t)$ from (3.2.30b) by quadrature in the form

$$u_1(x, t) = -\int_0^t h_{1_x}(x, \tau) d\tau, \tag{3.2.36}$$

since $u_1(x, 0) = 0$.

A second alternative for solving (3.2.30) is to eliminate h_1 from the system to derive the following initial-value problem for u_1:

$$u_{1_{tt}} - u_{1_{xx}} = 0, \quad -\infty < x < \infty, \ t \geq 0, \tag{3.2.37a}$$

$$u_1(x, 0) = 0, \tag{3.2.37b}$$

$$u_{1_t}(x, 0) = -f'(x). \tag{3.2.37c}$$

Again, the solution of this problem is discussed in Section 3.4.2.

It turns out that it is more efficient to compute the solution of (3.2.30) proceeding directly from the system of first-order equations. This so-called method of characteristics is illustrated in Section 3.4.3 for (3.2.30) and a particular choice of initial data. It is discussed in more generality in Section 4.5.

We can now also derive equations for the perturbation terms u_2 and h_2 in a similar manner. Thus, eliminating u_2 from (3.2.31) gives

$$h_{2_{tt}} - h_{2_{xx}} = \left\{ \frac{1}{2}(u_1^2)_x - (u_1 h_1)_t \right\}_x, \quad -\infty < x < \infty, \ t \geq 0, \tag{3.2.38a}$$

and using (3.2.30), the right-hand side may also be written in the alternative form

$$h_{2_{tt}} - h_{2_{xx}} = \left\{ \frac{1}{2}(h_1^2)_x - 2u_1 h_{1_t} \right\}_x. \tag{3.2.38b}$$

The initial conditions are now $h_2(x, 0) = 0$ and

$$h_{2_t}(x, 0) = -u_{2_x}(x, 0) - \{u_1(x, 0) h_1(x, 0)\}_x = 0. \tag{3.2.39}$$

The right-hand side of (3.2.39) vanishes because $u_2(x, 0) = 0$ and $u_1(x, 0) = 0$. Once u_1 and h_1 have been calculated, the right-hand side of (3.2.38) will be a known function of x, t, and we have to solve an inhomogeneous wave equation for h_2 (or u_2) with zero initial conditions. This is also discussed in Section 3.4.2. Once h_2 is known, u_2 can be calculated by quadrature with respect to t using (3.2.31b). Of course, the alternative approaches of deriving a wave equation for u_2 or proceeding from the system (3.2.31) directly also apply.

(ii) *Signaling problem*

For the signaling problem formulated in Section 3.2.8, the boundary condition at the left end involves u; hence it is more convenient to eliminate h_1 from (3.2.30). Thus, we need to solve

$$u_{1_{tt}} - u_{1_{xx}} = 0, \quad 0 \leq x < \infty, \; 0 \leq t, \tag{3.2.40}$$

subject to the zero initial conditions

$$u_1(x, 0) = u_{1_t}(x, 0) = 0, \tag{3.2.41}$$

which result from (3.2.21) and (3.2.30b). The boundary condition for u_1 [with $\nu = \epsilon$ in (3.2.22)] is just

$$u_1(0, t) = \dot{s}(t), \; t \geq 0, \tag{3.2.42}$$

when the leading term of the expansion (3.2.28a) is used. The solution of (3.2.40)–(3.2.42) is worked out in Section 3.5.4 first using the method of characteristics and then using Green's function. See also Problem 3.5.1, where the solution by Laplace transforms is outlined. Knowing u_1, we obtain h_1 from (3.2.30a) by quadrature in the form

$$h_1(x, t) = -\int_0^t u_{1_x}(x, \tau)d\tau, \tag{3.2.43}$$

since $h_1(x, 0) = 0$.

To derive the equation governing u_2, we eliminate h_2 from the system (3.2.31) and obtain

$$u_{2_{tt}} - u_{2_{xx}} = \{(u_1 h_1)_x - \frac{1}{2}(u_1^2)_t\}_x, \quad 0 \leq x, \; 0 \leq t. \tag{3.2.44}$$

The right-hand side of (3.2.44) is a known function of x, t once u_1 and h_1 have been calculated. The initial conditions for u_2 are also zero,

$$u_2(x, 0) = u_{2_t}(x, 0) = 0, \tag{3.2.45}$$

but the boundary condition at the left end must be derived with some care. The exact boundary condition (3.2.27) involves ϵ, first as a multiplier of s and $\dot{s}$, and also because $\overline{u}$ itself depends on ϵ. Thus, expanding $\overline{u}$ according to (3.2.28a) implies that

$$\epsilon u_1(\epsilon s, t) + \epsilon^2 u_2(\epsilon s, t) + O(\epsilon^3) = \epsilon \dot{s}. \tag{3.2.46a}$$

Now, expanding u_1 and u_2 around $x = 0$ gives

$$\epsilon[u_1(0, t) + u_{1_x}(0, t)\epsilon s + O(\epsilon^2)] + \epsilon^2 u_2(0, t) + O(\epsilon^3) = \epsilon \dot{s}. \tag{3.2.46b}$$

Therefore, the $O(\epsilon^2)$ boundary condition is

$$u_2(0, t) = -u_{1_x}(0, t)s(t), \ t > 0, \tag{3.2.47}$$

and the right-hand side is known once $u_1(x, t)$ has been determined. The solution of (3.2.44) subject to (3.2.45) and (3.2.47) is also discussed in Section 3.5.4. We observe that at each stage u_n obeys an inhomogeneous wave equation with zero initial conditions and a prescribed boundary condition similar to (3.2.47) involving known lower-order terms $u_{n-1}, u_{n-2}, \ldots, u_1$.

Although a perturbation solution of the form (3.2.28) may be derived to any order, in principle, we must be aware of various stipulations regarding the validity of such a solution. As pointed out earlier, the mathematical model (3.2.12) is not accurate to $O(\epsilon^2)$. This shortcoming is easily remedied by inclusion of the appropriate correction term on the right-hand side (3.2.12b). [See Section 5.2.4 of [27] and Section 8.3.2 of this book]. However, it is also pointed out in Section 8.3.2 that a perturbation expansion in the form (3.2.28) breaks down when t (or x) become as large as $O(\epsilon^{-1})$ for the initial-value (or signaling) problem. The evidence for this breakdown is the occurrence of terms proportional to t (or x) in u_2 and h_2. (See the details of the solution for the signaling problem in Section 3.5.4i.) A perturbation series that remains valid for t large will be derived for the initial-value problem (3.2.19) in Section 8.3.2.

Problem

3.2.1. Consider shallow-water flow in the X and Z directions over a variable bottom defined by the surface

$$Y_b \equiv Ab\left(\frac{X}{L_0}, \frac{Z}{L_0}, \frac{T}{T_0}\right). \tag{3.2.48a}$$

Here L_0 and T_0 are characteristic length and time scales for the bottom surface motion, and A is a characteristic amplitude. We assume, as in Section 3.2.1, that vertical motions are negligible, the density is constant, and that viscosity and surface tension may be ignored. The geometry is sketched in Figure 3.9, where $X = X_1$, $X = X_2$, $Z = Z_1$, and $Z = Z_2$ are fixed vertical planes that bound the sides of the domain of interest G. The free surface is defined by

$$Y_s \equiv Y_b + Hh\left(\frac{X}{L_0}, \frac{Z}{L_0}, \frac{T}{T_0}\right). \tag{3.2.48b}$$

Thus, Ab (where b is dimensionless) is the height of the bottom surface measured from the $Y = 0$ plane, and Hh (where h is dimensionless) is the height of the free surface measured from the bottom surface.

Denote the components of the flow velocity in the X and Z directions by $U(X/L_0, Z/L_0, T/T_0)$ and $W(X/L_0, Z/L_0, T/T_0)$, respectively.

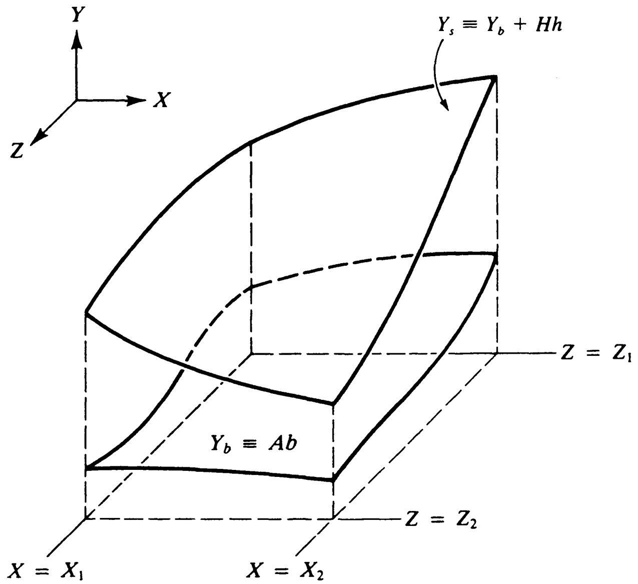

FIGURE 3.9. Shallow-water flow in two dimensions over a variable bottom

a. Use the dimensionless variables

$$x \equiv \frac{X}{L_0}, \quad y \equiv \frac{Y}{H}, \quad z \equiv \frac{Z}{L_0}, \quad t \equiv \frac{T(gH)^{1/2}}{L_0},$$
$$u \equiv \frac{U}{(gH)^{1/2}}, \quad w \equiv \frac{W}{(gH)^{1/2}}, \quad p \equiv \frac{P}{\rho g H}, \tag{3.2.49}$$

and show that the integral law of mass conservation is

$$\frac{d}{dt}\int_{x_1}^{x_2}\int_{z_1}^{z_2} h(x,z,t)dz\,dx$$
$$+\int_{z_1}^{z_2}[u(x_2,z,t)h(x_2,z,t) - u(x_1,z,t)h(x_1,z,t)]dz$$
$$+\int_{x_1}^{x_2}[w(x,z_2,t)h(x,z_2,t) - w(x,z_1,t)h(x,z_1,t)]dx = 0. \tag{3.2.50}$$

For smooth solutions, show that (3.2.50) reduces to

$$h_t + (uh)_x + (wh)_z = 0. \tag{3.2.51}$$

b. Show that the pressure (hydrostatic) is given by

$$p(x,y,z,t) = \epsilon b(x,z,t) + h(x,z,t) - y, \tag{3.2.52}$$

where $\epsilon \equiv A/H$, and that the x and z components of the pressure force exerted by the water in G on that portion of the bottom surface in contact with G are given by $\epsilon \int_{x_1}^{x_2} \int_{z_1}^{z_2} h(x, z, t) b_x(x, z, t) dz\, dx$ and $\epsilon \int_{x_1}^{x_2} \int_{z_1}^{z_2} h(x, z, t) b_z(x, z, t) dz\, dx$, respectively. Notice that in the absence of viscosity, the only stress on the bottom is the hydrostatic pressure that acts normal to the bottom.

c. Use the results in (b) to derive the following integral law of momentum conservation in the x direction:

$$\frac{d}{dt} \int_{x_1}^{x_2} \int_{z_1}^{z_2} u(x, z, t) h(x, z, t) dz\, dx + \epsilon \int_{x_1}^{x_2} \int_{z_1}^{z_2} h(x, z, t) b_x(x, z, t) dz\, dx$$
$$+ \int_{z_1}^{z_2} \left[u^2(x, z, t) h(x, z, t) + \frac{h^2}{2}(x, z, t) \right]_{x=x_1}^{x=x_2} dz$$
$$+ \int_{x_1}^{x_2} \left[u(x, z, t) w(x, z, t) h(x, z, t) \right]_{z=z_1}^{z=z_2} dx = 0. \tag{3.2.53}$$

The formula for momentum conservation in the z direction follows from (3.2.53) by replacing $u \to w$, $w \to u$, $x \to z$, and $z \to x$.

d. For smooth solutions, show that (3.2.53) and the corresponding formula for momentum conservation in the z direction reduce to

$$(uh)_t + \left(u^2 h + \frac{h^2}{2} \right)_x + \epsilon h b_x + (uwh)_z = 0, \tag{3.2.54a}$$

$$(wh)_t + \left(w^2 h + \frac{h^2}{2} \right)_z + \epsilon h b_z + (uwh)_x = 0. \tag{3.2.54b}$$

Use (3.2.51) to simplify these to the form

$$u_t + u u_x + w u_z + (h + \epsilon b)_x = 0, \tag{3.2.55a}$$

$$w_t + u w_x + w w_z + (h + \epsilon b)_z = 0. \tag{3.2.55b}$$

d. Specialize your results in (3.2.51) and (3.2.55) to the case where the bottom surface is axisymmetric, that is,

$$y_b = \epsilon b(r, t), \quad r \equiv \sqrt{x^2 + z^2}. \tag{3.2.56}$$

Assume that the flow is also axisymmetric with radial velocity $q = \sqrt{u^2 + w^2}$. Show that (3.2.51) and (3.2.55) reduce in this case to the pair of equations

$$h_t + (hq)_r + \frac{1}{r} hq = 0, \tag{3.2.57a}$$

$$q_t + q q_r + (h + \epsilon b)_r = 0. \tag{3.2.57b}$$

3.3 Compressible Flow

In this section we study a third application area, which leads to the wave equation in the limit of small disturbances. Again, we proceed from the general laws of mass, momentum, and energy conservation.

3.3.1 Conservation Laws

Consider a domain G that is fixed in space and time, has a bounding surface S, and is occupied by a gas with density $\rho(\mathbf{x}, t)$ and velocity $\mathbf{u}(\mathbf{x}, t)$. Let $\mathbf{n}$ denote the outward unit normal vector on S and let $\mathbf{F}(\mathbf{x}, t)$ denote the body force (usually gravity) per unit mass. Denote the stress by $\boldsymbol{\tau}(\mathbf{x}, t)$, and let the internal energy per unit mass for the molecular motion be $e(\mathbf{x}, t)$. Let the heat flow per unit area across the boundary be denoted by $\mathbf{q}(\mathbf{x}, t)$.

The laws of conservation of mass, momentum, and energy are (see Section 6.1 of [42])

$$\frac{d}{dt}\iiint_G \rho\, dV + \iint_S \rho u_j n_j\, dS = 0, \quad \text{(mass)} \tag{3.3.1a}$$

$$\frac{d}{dt}\iiint_V \rho u_i\, dV + \iint_S (\rho u_i n_j u_j - \tau_i)\, dS = \iiint_V \rho F_i\, dV,$$

$$i = 1, 2, 3, \quad \text{(momentum)} \tag{3.3.1b}$$

$$\frac{d}{dt}\iiint_V \left(\frac{1}{2}\rho u_i^2 + \rho e\right) dV + \iint_S \left\{\left(\frac{1}{2}\rho u_i^2 + \rho e\right) n_j u_j - \tau_i u_i + n_j q_j\right\} dS$$

$$= \iiint_V \rho F_i u_i\, dV, \quad \text{(energy).} \tag{3.3.1c}$$

In the preceding, we are using Cartesian tensor notation and summing over repeated indices. Thus, $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{u} \cdot \mathbf{n} = u_i n_i \equiv \sum_{i=1}^{3} u_i n_i$, and so on.

Equations (3.3.1) provide five equations for the eleven quantities ρ, u_i, τ_i, q_i, e $(i = 1, 2, 3)$. To complete the system, additional relations between the flow variables must be specified.

3.3.2 One-Dimensional Ideal Gas

For one-dimensional flow, the stress τ_1 and the heat flux q_1 are given by

$$\tau_1 \equiv -p + \frac{4}{3}\mu u_x, \tag{3.3.2a}$$

$$q_1 \equiv -\lambda \theta_x, \tag{3.3.2b}$$

where

$$p = \text{pressure},$$
$$\theta = \text{temperature},$$
$$\mu = \text{coefficient of viscosity},$$
$$\lambda = \text{thermal conductivity}.$$

An *ideal gas* is one that obeys the *equation of state*

$$p = \rho R\theta, \tag{3.3.3}$$

where

$$R = \text{gas constant} \equiv C_p - C_v,$$
$$C_p = \text{specific heat at constant pressure},$$
$$C_v = \text{specific heat at constant volume}.$$

If one assumes that C_p and C_v are both constant for an ideal gas, one can show that the internal energy e is just

$$e = C_v\theta. \tag{3.3.4}$$

In all our work here and later on, we shall assume that (3.3.2)–(3.3.4) hold with constant values for μ, λ, C_p, and C_v. In particular, if body forces are negligible, the integral conservation laws of mass, momentum, and energy (3.3.1) reduce to the following form for one-dimensional flow:

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho dx + \rho u \Big|_{x=x_1}^{x=x_2} = 0, \tag{3.3.5a}$$

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho u dx + \left(\rho u^2 + p - \frac{4}{3}\mu u_x\right)\Big|_{x=x_1}^{x=x_2} = 0, \tag{3.3.5b}$$

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho\left(\frac{u^2}{2} + C_v\theta\right) dx + \left[\left(\frac{u^2}{2} + C_v\theta\right)\rho u + pu - \frac{4}{3}\mu u u_x - \lambda\theta_x\right]\Big|_{x=x_1}^{x=x_2} = 0. \tag{3.3.5c}$$

Equations (3.3.5) combined with the equation of state (3.3.3) are four relations governing the four dependent variables ρ, u, p, and θ.

For smooth soutions (that is, if ρ, u, p, θ are continuous and have continuous first partial derivatives with respect to x and t) the preceding conservation laws reduce to the differential conservation relations (divergence relations)

$$\rho_t + (\rho u)_x = 0, \tag{3.3.6a}$$

$$(\rho u)_t + (\rho u^2 + p - \frac{4}{3}\mu u_x)_x = 0, \tag{3.3.6b}$$

$$\left(\frac{1}{2}\rho u^2 + \rho C_v\theta\right)_t + \left[\left(\frac{u^2}{2} + C_v\theta\right)\rho u + pu - \frac{4}{3}\mu u u_x - \lambda\theta_x\right]_x = 0. \tag{3.3.6c}$$

Using (3.3.6a), (3.3.6b) simplifies to

$$\rho u_t + \rho u u_x + p_x - \frac{4}{3}\mu u_{xx} = 0. \tag{3.3.7a}$$

Using (3.3.6a) and (3.3.7a), (3.3.6c) simplifies to

$$\rho C_v(\theta_t + u\theta_x) + (p - \frac{4}{3}\mu u_x)u_x - \lambda\theta_{xx} = 0. \tag{3.3.7b}$$

Thus, (3.3.3), (3.3.6a), (3.3.7a), and (3.3.7b) provide one algebraic equation plus three partial differential equations governing the four unknowns ρ, u, p, θ.

3.3.3 Signaling Problem for One-Dimensional Flow

We wish to study the signaling problem of generating a disturbance in a semi-infinite region that is initially at rest. For the sake of simplicity, assume that the disturbance is modeled by a piston moving to the right or left according to

$$X = L_0 x_p\left(\frac{t}{T_0}\right), \tag{3.3.8a}$$

and having a prescribed temperature given by

$$\theta(L_0 x_p, T) = \theta_1\theta_p\left(\frac{t}{T_0}\right), \tag{3.3.8b}$$

where L_0 and T_0 are characteristic length and time scales, respectively, associated with the piston displacement, as sketched in Figure 3.10. The characteristic temperature of the piston surface is θ_1, and x_p, θ_p are dimensionless functions.

Suppose that the ambient properties of the gas are $p_0, \rho_0, \theta_0, \lambda_0, \mu, C_p, C_v$, and $u_0 = 0$. We introduce dimensionless variables denoted by an asterisk as follows:

$$u^* \equiv \frac{u}{a_0}, \quad p^* \equiv \frac{p}{p_0}, \quad \theta^* \equiv \frac{\theta}{\theta_0}, \quad \rho^* \equiv \frac{\rho}{\rho_0}, \tag{3.3.9}$$

$$x^* \equiv \frac{x}{a_0 T_0}, \quad t^* \equiv \frac{t}{T_0}, \tag{3.3.10}$$

where a_0 is the ambient speed of sound defined by

$$a_0 \equiv \left(\frac{\gamma p_0}{\rho_0}\right)^{1/2}, \tag{3.3.11}$$

and γ is the ratio of specific heats

$$\gamma \equiv \frac{C_p}{C_v}. \tag{3.3.12}$$

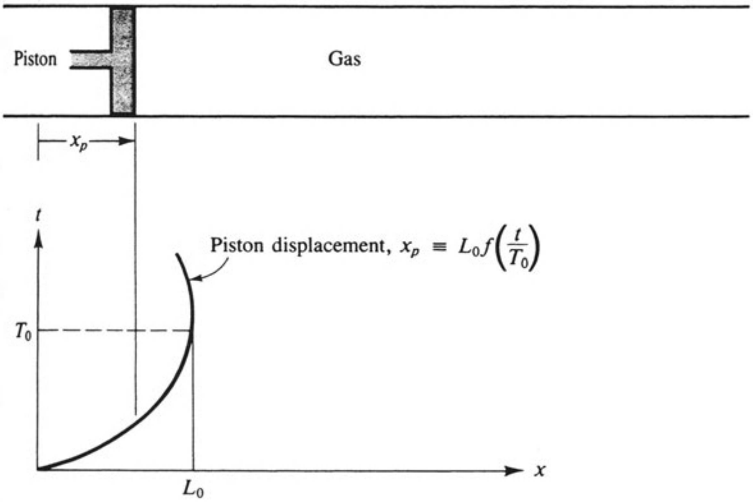

FIGURE 3.10. Piston displacement

The dimensionless version of (3.3.6a), (3.3.7a), and (3.3.7b) becomes (dropping asterisks for simplicity of notation)

$$\rho_t + (\rho u)_x = 0, \quad (3.3.13a)$$

$$\rho u_t + \rho u u_x + \frac{p_x}{\gamma} - \frac{4}{3\,\mathrm{Re}} u_{xx} = 0, \quad (3.3.13b)$$

$$\rho\theta_t + \rho u\theta_x + \left[(\gamma - 1)p - \frac{4\gamma(\gamma - 1)}{3\,\mathrm{Re}} u_x\right] u_x - \frac{\gamma}{\mathrm{Re}\,\mathrm{Pr}} \theta_{xx} = 0, \quad (3.3.13c)$$

and involves the three dimensionless parameters γ, Re, and Pr, where γ is defined in (3.3.12) and

$$\mathrm{Re} \equiv \frac{\rho_0 a_0^2 T_0}{\mu} = \text{Reynolds number}, \quad (3.3.14)$$

$$\mathrm{Pr} \equiv \frac{\mu C_p}{\lambda} = \text{Prandtl number}. \quad (3.3.15)$$

In addition, the equation of state, (3.3.3), becomes

$$p = \rho\theta. \quad (3.3.16)$$

The boundary conditions at the piston are

$$u(\epsilon x_p(t), t) = \epsilon \frac{dx_p}{dt}, \quad t > 0, \quad (3.3.17)$$

$$\theta(\epsilon x_p(t), t) = \beta\theta_p(t), \quad t > 0, \tag{3.3.18}$$

where $\epsilon \equiv L_0/a_0T_0$ and $\beta \equiv \theta_1/\theta_0$. Thus, ϵ is the ratio of the piston characteristic speed (L_0/T_0) to the ambient speed of sound a_0.

3.3.4 Inviscid, Non-Heat-Conducting Gas; Analogy with Shallow-Water Waves

For an inviscid ($\mu = 0$), non-heat-conducting ($\lambda = 0$) gas, we have Re $\to \infty$, Pr $\to \infty$, and the system (3.3.13) reduces to

$$\rho_t + (\rho u)_x = 0, \tag{3.3.19a}$$

$$\rho u_t + \rho u u_x + \frac{p_x}{\gamma} = 0, \tag{3.3.19b}$$

$$\rho\theta_t + \rho u\theta_x + (\gamma - 1)p u_x = 0. \tag{3.3.19c}$$

As discussed on p. 156 of [42], (3.3.19c) is equivalent to

$$\left(\frac{p}{\rho^\gamma}\right)_t + u\left(\frac{p}{\rho^\gamma}\right)_x = 0 \tag{3.3.20}$$

if all flow variables have continuous derivatives (Problem 3.3.1). Thus, for smooth solutions, conservation of energy implies that p/ρ^γ remains constant along particle paths. These are paths defined by $(dx/dt) = u(x, t)$. Since the entropy S is a function of p/ρ^γ, we conclude that S is constant along particle paths; such a flow is called *adiabatic*. See also Section 5.3.4 and Section 7.5.1.

If in addition, p/ρ^γ is initially constant when the gas is at rest (as is the case for our signaling problem where $p/\rho^\gamma = 1$ and $t = 0$), it will remain constant as long as no discontinuities occur in the flow. This is called *isentropic* flow.

Using $p = \rho^\gamma$ to eliminate p from (3.3.19b) gives

$$u_t + uu_x + \rho^{\gamma-2}\rho_x = 0. \tag{3.3.21}$$

The two equations (3.3.19a) and (3.3.21) are exact analogues of equations (3.2.12) for shallow-water waves if we identify u in both cases, set $h = \rho$, and take $\gamma = 2$. Or equivalently, using the dimensionless local speed of sound a defined by

$$a^2 \equiv \frac{1}{\gamma}\frac{\partial p}{\partial \rho} = \rho^{\gamma-1} \tag{3.3.22}$$

instead of ρ gives the alternative form

$$a_t + ua_x + \frac{\gamma - 1}{2}au_x = 0 \tag{3.3.23a}$$

for mass conservation and

$$u_t + uu_x + \frac{2}{\gamma - 1}aa_x = 0 \tag{3.3.23b}$$

for momentum conservation. Now, comparing these with (3.2.12), we identify u in both cases, set $a = \sqrt{h}$, and take $\gamma = 2$ to obtain the formal analogy.

Unfortunately, $\gamma = 1.4$ for air, so it is not possible to have an accurate quantitative analogue for compressible flow using a hydraulic model. This question is discussed further in Section 5.3.4.

3.3.5 Small-Disturbance Theory in One-Dimensional Flow (Signaling Problem)

Assume that $\epsilon \ll 1$, that is, the characteristic speed (L_0/T_0) associated with the piston displacement is very small compared with the ambient speed of sound a_0. Assume further that the piston temperature that is prescribed does not differ much from the ambient temperature. This means that θ_p is of the form

$$\theta_p = 1 + \nu\overline{\theta}_p(t),$$

where ν is a small dimensionless parameter of order ϵ, say $\nu = c\epsilon$ (c = constant).

Clearly, if $\epsilon = 0$, the solution must be the ambient state: $u = 0$, $p = \rho = \theta = 1$. Thus, we rescale our dependent variables as follows:

$$u(x, t; \epsilon) = \epsilon\overline{u}(x, t; \epsilon), \tag{3.3.24a}$$

$$p(x, t; \epsilon) = 1 + \epsilon\overline{p}(x, t; \epsilon), \tag{3.3.24b}$$

$$\rho(x, t; \epsilon) = 1 + \epsilon\overline{\rho}(x, t; \epsilon), \tag{3.3.24c}$$

$$\theta(x, t; \epsilon) = 1 + \epsilon\overline{\theta}(x, t; \epsilon). \tag{3.3.24d}$$

We then obtain the following system corresponding to (3.3.13):

$$\overline{\rho}_t + \overline{u}_x = -\epsilon(\overline{\rho}\,\overline{u})_x, \tag{3.3.25a}$$

$$\overline{u}_t + \frac{1}{\gamma}\overline{p}_x - \frac{4}{3\,\mathrm{Re}}\overline{u}_{xx} = -\epsilon(\overline{\rho}\,\overline{u}_t + \overline{u}\,\overline{u}_x) - \epsilon^2\overline{u}\,\overline{u}_x, \tag{3.3.25b}$$

$$\overline{\theta}_t + (\gamma - 1)\overline{u}_x - \frac{\gamma}{\mathrm{Re\,Pr}}\overline{\theta}_{xx} = \left[\frac{4\gamma(\gamma-1)}{3\,\mathrm{Re}}\overline{u}_x^2 - \overline{\rho}\,\overline{\theta}_t - \overline{u}\,\overline{\theta}_x - (\gamma - 1)\overline{p}\,\overline{u}_x\right] - \epsilon^2\overline{\rho}\,\overline{u}\overline{\theta}_x. \tag{3.3.25c}$$

The equation of state (3.3.16) becomes

$$\overline{p} - (\overline{\rho} + \overline{\theta}) = \epsilon\overline{\rho}\overline{\theta}. \tag{3.3.25d}$$

The boundary conditions (3.3.17)–(3.3.18) at the piston surface for $t > 0$ reduce to

$$\overline{u}(\epsilon x_p(t), t; \epsilon) = \frac{dx_p}{dt} = \text{prescribed}, \quad t > 0, \tag{3.3.26a}$$

$$\overline{\theta}(\epsilon x_p(t), t; \epsilon) = c\overline{\theta}_p(t) = \text{prescribed}, \quad t > 0. \tag{3.3.26b}$$

The initial conditions are simply $\overline{u} = \overline{p} = \overline{\rho} = \overline{\theta} = 0$ at $t = 0$.

A perturbation solution for ϵ small can be derived in principle. The leading approximation satisfies the linear system that results by setting $\epsilon = 0$ in (3.3.25); the boundary conditions are given by (3.2.26) evaluated at $x = 0$. This linear problem may be solved using Laplace transforms. For a discussion, see [35].

In the limit Re $\to \infty$, Pr $\to \infty$, we have the following linear problem to leading order after we use (3.3.25d) to eliminate $\overline{p}$:

$$\overline{\rho}_t + \overline{u}_x = 0, \tag{3.3.27a}$$

$$\overline{u}_t + \frac{1}{\gamma}(\overline{\rho}_x + \overline{\theta}_x) = 0, \tag{3.3.27b}$$

$$\overline{\theta}_t + (\gamma - 1)\overline{u}_x = 0. \tag{3.3.27c}$$

If we use (3.3.27a) to express $\overline{u}_x$ in terms of $\overline{\rho}_t$ in (3.3.27c), then integrating the result gives

$$\overline{\theta} - (\gamma - 1)\overline{\rho} = 0. \tag{3.3.28}$$

The function of x that arises on the right-hand side of (3.3.28) must be zero because at $\overline{t} = 0$ we have $\overline{\theta} = \overline{\rho} = 0$. This defines $\overline{\theta}$ once $\overline{\rho}$ is known. We can also define $\overline{p}$ in terms of $\overline{\rho}$ using (3.3.25d) (with $\epsilon = 0$):

$$\overline{p} = \overline{\rho} + \overline{\theta} = \gamma\overline{\rho}. \tag{3.3.29}$$

The system governing $\overline{\rho}$ and $\overline{u}$ that results from (3.3.25) is then ($\epsilon = 0$)

$$\overline{\rho}_t + \overline{u}_x = 0, \tag{3.3.30a}$$

$$\overline{u}_t + \overline{\rho}_x = 0. \tag{3.3.30b}$$

If we now eliminate $\overline{\rho}$, we obtain the wave equation

$$\overline{u}_{tt} - \overline{u}_{xx} = 0, \tag{3.3.31a}$$

subject to the boundary condition

$$\overline{u}(0, t) = \frac{dx_p}{dt} \equiv h(t). \tag{3.3.31b}$$

The initial condition

$$\overline{u}(x, 0) = 0 \tag{3.3.32a}$$

is supplemented by the second condition

$$\overline{u}_t(x, 0) = 0, \tag{3.3.32b}$$

which follows from (3.3.30b) evaluated at $t = 0$, since $\overline{\rho}(x, 0) = 0$ implies $\overline{\rho}_x(x, 0) = 0$.

3.3.6 Inviscid, Non-Heat-Conducting Flow in Three Dimensions

Upon neglecting viscosity, heat conduction, and body forces and using dimensionless variables as in (3.3.9)–(3.3.10) we obtain the following system without reference to any coordinate system from (3.3.1) for smooth solutions:

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \quad \text{(mass)} \tag{3.3.33a}$$

$$(\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \circ \mathbf{u}) + \frac{1}{\gamma} \operatorname{grad} p = 0, \quad \text{(momentum)} \tag{3.3.33b}$$

$$\left(\frac{p}{\rho^\gamma}\right)_t + \mathbf{u} \cdot \operatorname{grad}\left(\frac{p}{\rho^\gamma}\right) = 0, \quad \text{(energy).} \tag{3.3.33c}$$

Here $\rho \mathbf{u} \circ \mathbf{u}$ is the flow of momentum tensor, expressed as the dyadic product of $\rho \mathbf{u}$ with $\mathbf{u}$. The dyadic product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined without reference to any coordinate system by

$$(\mathbf{a} \circ \mathbf{b})\mathbf{v} = \mathbf{a}(\mathbf{b} \cdot \mathbf{v}), \quad \text{for any } \mathbf{v}. \tag{3.3.34}$$

It then follows that

$$\operatorname{div}(\mathbf{a} \circ \mathbf{b}) = (\operatorname{grad} \mathbf{a})\mathbf{b} + (\operatorname{div} \mathbf{b})\mathbf{a}. \tag{3.3.35}$$

In particular, for Cartesian tensors, the ith component of $\operatorname{div}(\mathbf{a} \circ \mathbf{b})$ is $\partial(a_i b_k)/\partial x_k$.

If (p/ρ^γ) is initially constant everywhere, (3.3.33c) implies that p/ρ^γ is a constant for all time,

$$\frac{p(x, y, z, t)}{\rho^\gamma(x, y, z, t)} = 1, \tag{3.3.36}$$

and this constant equals one because we have normalized p and ρ using the ambient initial values. Note that (3.3.36) is true only for smooth solutions. In Chapters 5 and 7 we discuss the situation when discontinuities (shocks) arise.

We use (3.3.33a) to eliminate ρ_t from (3.3.33b) to obtain

$$\rho \mathbf{u}_t - \mathbf{u} \operatorname{div}(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \circ \mathbf{u}) + \frac{1}{\gamma} \operatorname{grad} p = 0. \tag{3.3.37}$$

Using vector identities and dividing by ρ gives

$$\mathbf{u}_t + (\operatorname{curl} \mathbf{u}) \times \mathbf{u} + \frac{1}{2} \operatorname{grad}(\mathbf{u} \cdot \mathbf{u}) + \frac{1}{\gamma \rho} \operatorname{grad} p = 0. \tag{3.3.38}$$

We now take the curl of (3.3.38) and denote $\operatorname{curl} \mathbf{u} = \omega$, the vorticity, to obtain

$$\omega_t + \operatorname{curl}(\omega \times \mathbf{u}) = 0. \tag{3.3.39}$$

Thus, since $\omega(x, y, z, 0) = 0$, (3.3.39) implies that $\omega \equiv 0$, and therefore $\mathbf{u}$ is the gradient of a scalar potential ϕ:

$$\mathbf{u} = \operatorname{grad} \phi(x, y, z, t). \tag{3.3.40}$$

Such a flow is called irrotational.

We use (3.3.36) to express $\operatorname{grad} p$ in terms of ρ,

$$\operatorname{grad} p = \gamma \rho^{\gamma - 1} \operatorname{grad} \rho, \tag{3.3.41}$$

and (3.3.38) becomes

$$\operatorname{grad}\left[\phi_t + \frac{1}{2}(\operatorname{grad}\phi)^2 + \frac{1}{\gamma-1}\rho^{\gamma-1}\right] = 0. \tag{3.3.42}$$

Therefore,

$$\phi_t + \frac{1}{2}(\operatorname{grad}\phi)^2 + \frac{1}{\gamma-1}\rho^{\gamma-1} = \frac{1}{\gamma-1}. \tag{3.3.43a}$$

The function of t that arises from integrating (3.3.42) is equal to $(\gamma - 1)^{-1}$ because $\phi \to 0$ and $\rho \to 1$ at infinity. Using (3.3.40), equation (3.3.3a) for mass conservation becomes

$$\rho_t + \rho\Delta\phi + \operatorname{grad}\rho \cdot \operatorname{grad}\phi = 0, \tag{3.3.43b}$$

where $\Delta \equiv \operatorname{div}\operatorname{grad}$.

(i) *Weakly nonlinear acoustics*

Consider a signaling or initial-value problem where we introduce a small disturbance to the ambient state $\mathbf{u} = 0$, $p = \rho = 1$. The small disturbance theory for such a flow is called acoustics. We can proceed as in the one-dimensional case to define a small parameter ϵ that measures the amplitude of the disturbance, and rescale variables as follows:

$$\mathbf{u} = \epsilon\bar{\mathbf{u}}, \quad \phi = \epsilon\bar{\phi}, \quad \rho = 1 + \epsilon\bar{\rho}. \tag{3.3.44}$$

Using the rescaled expressions for ϕ and ρ in (3.3.43a), expanding, and then dividing by ϵ gives

$$\bar{\phi}_t + \bar{\rho} = -\frac{\epsilon}{2}\left[(\operatorname{grad}\bar{\phi})^2 + (\gamma-2)\bar{\rho}^2\right] + O(\epsilon^2). \tag{3.3.45}$$

Solving this expression for $\bar{\rho}$ in terms of $\bar{\phi}$ correct to $O(\epsilon)$ gives

$$\bar{\rho} = -\bar{\phi}_t - \epsilon\left[\frac{1}{2}(\operatorname{grad}\bar{\phi})^2 + \frac{\gamma-2}{2}\bar{\phi}_t^2\right] + O(\epsilon^2). \tag{3.3.46}$$

Therefore,

$$\bar{\rho}_t = -\bar{\phi}_{tt} - \epsilon\left[\operatorname{grad}\bar{\phi}\cdot\operatorname{grad}\bar{\phi}_t + (\gamma-2)\bar{\phi}_t\bar{\phi}_{tt}\right] + O(\epsilon^2). \tag{3.3.47}$$

The rescaled version of (3.3.43b) is

$$\bar{\rho}_t + \Delta\bar{\phi} = -\epsilon(\operatorname{grad}\bar{\rho}\cdot\operatorname{grad}\bar{\phi} + \bar{\rho}\Delta\bar{\phi}). \tag{3.3.48}$$

Using (3.3.46) for $\bar{\rho}$ and (3.3.47) for $\bar{\rho}_t$ gives

$$-\bar{\phi}_{tt} + \Delta\bar{\phi} = \epsilon\Big\{2\operatorname{grad}\bar{\phi}\cdot\operatorname{grad}\bar{\phi}_t + \bar{\phi}_t\left[(\gamma-2)\bar{\phi}_{tt} + \Delta\bar{\phi}\right]\Big\} + O(\epsilon^2). \tag{3.3.49a}$$

Since $\Delta\overline{\phi} = \overline{\phi}_{tt} + O(\epsilon)$, the right-hand side of (3.3.49a) simplifies further, and we obtain

$$-\overline{\phi}_{tt} + \Delta\overline{\phi} = \epsilon\left[2 \operatorname{grad}\overline{\phi}\cdot\operatorname{grad}\overline{\phi}_t + (\gamma - 1)\overline{\phi}_t\overline{\phi}_{tt}\right] + O(\epsilon^2). \qquad (3.3.49b)$$

Thus, the leading approximation (for $\epsilon \ll 1$) for $\overline{\phi}$ is the linear three-dimensional wave equation

$$\Delta\overline{\phi} - \overline{\phi}_{tt} = 0. \qquad (3.3.50)$$

(ii) *Steady flow over a thin body*

An important problem in aerodynamics is the calculation of the irrotational flow over a slender body that moves with constant speed through air at rest. If the body is rigid, the boundary condition of zero normal velocity at the body surface (cf. (2.4.16) is most conveniently expressed in time-independent form in a coordinate system attached to the body.

The Galilean transformation to a body-fixed coordinate system x^*, y^*, z^* that moves with dimensionless speed (Mach number) M in the negative x-direction is illustrated in Figure 3.11. This transformation is defined by

$$x^* = x + Mt, \quad y^* = y, \quad z^* = z, \quad t^* = t,$$

$$\rho^*(x^*, y^*, z^*, t^*) = \rho(x + Mt, y, z, t),$$
$$\phi(x + Mt, y, z, t) = \phi^*(x^*, y^*, z^*, t^*) - Mx^*. \qquad (3.3.51)$$

As expected, the transformation (3.3.51) leaves the basic system (3.3.43b) and (3.3.42) invariant. Integrating the transformed (3.3.42) and noting that $\phi^* = Mx^*$ and $\rho = 1$ at infinity in the moving frame gives

$$\phi^*_{t^*} + \frac{1}{2}(\operatorname{grad}^*\phi^*)^2 + \frac{1}{\gamma - 1}\rho^{*^{\gamma-1}} = \frac{1}{\gamma - 1} + \frac{M^2}{2}, \qquad (3.3.52)$$

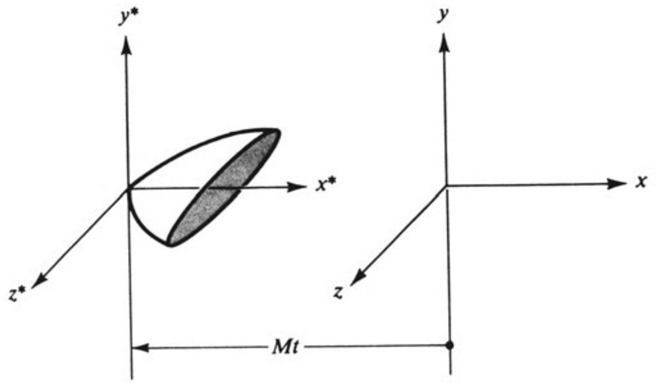

FIGURE 3.11. Moving coordinate sysetm x^*, y^*, z^*

which differs from (3.3.43a) only by the added constant $M^2/2$ on the right-hand side. Mass conservation, (3.3.43b), has the invariant form

$$\rho^*_{t^*} + \rho^* \Delta^* \phi^* + \text{grad}^* \rho^* \cdot \text{grad}^* \phi^* = 0, \tag{3.3.53}$$

where

$$\text{grad}^* \equiv \mathbf{i}\frac{\partial}{\partial x^*} + \mathbf{j}\frac{\partial}{\partial y^*} + \mathbf{k}\frac{\partial}{\partial z^*}, \tag{3.3.54a}$$

$$\Delta^* \equiv \frac{\partial^2}{\partial x^{*2}} + \frac{\partial^2}{\partial y^{*2}} + \frac{\partial^2}{\partial z^{*2}} \tag{3.3.54b}$$

in the Cartesian moving frame.

The boundary condition on the surface of a body defined by

$$y^* = \epsilon F(x^*, z^*) \tag{3.3.55}$$

(see Figure 3.12) is that the normal component of velocity vanishes. The flow velocity is

$$\mathbf{u}^* = \text{grad}\, \phi^*, \tag{3.3.56a}$$

and

$$\mathbf{n} = -\epsilon F_{x^*}\mathbf{i} + \mathbf{j} - \epsilon F_{z^*}\mathbf{k} \tag{3.3.56b}$$

is a normal to the surface. Therefore, setting $\mathbf{u}^* \cdot \mathbf{n} = 0$ gives

$$\phi^*_{y^*}(x^*, \epsilon F, z^*) = \phi^*_{x^*}(x^*, \epsilon F, z^*)F_{x^*} + \phi^*_{z^*}(x^*, \epsilon F, z^*)F_{z^*}. \tag{3.3.57}$$

For $\epsilon \to 0$, the body does not disturb the flow, and we have $\phi^* = Mx^*$. We rescale our problem to reflect this fact for ϵ small and set

$$\phi^* = Mx^* + \epsilon\varphi(x^*, y^*, z^*; \epsilon). \tag{3.3.58}$$

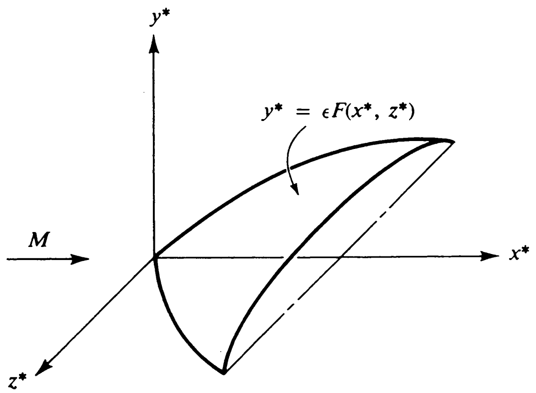

FIGURE 3.12. Slender body in a uniform flow

Substituting (3.3.58) for ϕ^* into (3.3.52)–(3.3.53), expanding, and retaining the leading nonlinear terms gives

$$(1 - M^2)\varphi_{x^*x^*} + \varphi_{y^*y^*} + \varphi_{z^*z^*} = \epsilon M[2\,\text{grad}^*\,\varphi \cdot \text{grad}^*\,\varphi_{x^*} + M^2(\gamma - 1)\varphi_{x^*}\varphi_{x^*x^*}] + O(\epsilon^2). \tag{3.3.59}$$

The details are essentially the same as those leading to (3.3.49a) and are left as an exercise (Problem 3.3.3). The boundary condition (3.3.57) becomes

$$\epsilon\varphi_{y^*}(x^*, \epsilon F, z^*; \epsilon) = [M + \epsilon\varphi_{x^*}(x^*, \epsilon F, z^*; \epsilon)]\epsilon F_{x^*} + \epsilon^2\varphi_{z^*}(x^*, \epsilon F, z^*; \epsilon)F_{z^*}. \tag{3.3.60}$$

The linear part ($\epsilon = 0$) of (3.3.59) is a two-dimensional wave equation for $M > 1$ (supersonic flow),

$$\Delta_2\varphi - \varphi_{\bar{t}\,\bar{t}} = 0, \tag{3.3.61}$$

where

$$\Delta_2 \equiv \frac{\partial^2}{\partial\bar{y}^2} + \frac{\partial^2}{\partial\bar{z}^2}, \quad \bar{y} = y^*, \quad \bar{z} = z^*, \tag{3.3.62}$$

and $\bar{t}$ is the timelike variable

$$\bar{t} = x^*/\sqrt{M^2 - 1}. \tag{3.3.63}$$

Thus, the boundary condition $\varphi_{y^*}(x^*, 0, z^*; \epsilon) = MF_{x^*}(x^*, z^*)$ that results from (3.3.60) to leading order shows that we have to solve a signaling problem for (3.3.61). The "initial" condition is $\varphi^* \equiv 0$ if $x^* < 0$. If the body is two-dimensional ($y^* = \epsilon F(z^*)$), (3.3.61) reduces to a one-dimensional wave equation ($\partial/\partial z^* \equiv 0$), and the associated signaling problem is discussed in Section 3.5. and Problem 4.2.2. Some aspects of the two-dimensional wave equation (3.3.61) are discussed in Section 3.9. and Problem 3.9.1.

For $M < 0$ (subsonic flow), the linear part of (3.3.59) is a three-dimensional Laplace equation in the coordinates $\bar{x} = x^*/\sqrt{1 - M^2}$, $\bar{y} = y^*$, $\bar{z} = z^*$. This equation is to be solved in the domain exterior to the body subject to the boundary condition $\varphi_{y^*}(x^*, 0, z^*) = MF_{x^*}(x^*, z^*)$ on the surface and $\varphi^* = 0$ at infinity.

Problems

3.3.1 Consider the system (3.3.13) for a one-dimensional ideal gas. Let D denote the operator

$$D \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}. \tag{3.3.64}$$

Thus, (3.3.13a) and (3.3.13c) may be written in the form

$$D(\rho) + u\rho_x = 0, \tag{3.3.65a}$$

$$\rho D(\theta) + (\gamma - 1)pu_x = \frac{4\gamma(\gamma - 1)}{3\,\text{Re}}u_x^2 + \frac{\gamma}{\text{Re}\,\text{Pr}}\theta_{xx} \equiv Q. \tag{3.3.65b}$$

Express p in terms of ρ and θ using (3.3.16), and then use (3.3.13a) to show that

$$D\left(\frac{p}{\rho^\gamma}\right) = \rho^{1-\gamma} D(\theta) - (1-\gamma)\rho^{-\gamma} p u_x. \tag{3.3.66}$$

Substitute the expression for $D(\theta)$ that results from the energy equation (3.3.13c) into (3.3.66) to obtain

$$D\left(\frac{p}{\rho^\gamma}\right) = \rho^{-\gamma} Q. \tag{3.3.67}$$

Thus, if Re $\to \infty$, $Q \to 0$ and energy conservation implies

$$D\left(\frac{p}{\rho^\gamma}\right) = 0. \tag{3.3.68}$$

3.3.2 In an infinite tube, consider two different gases at rest separated by a diaphragm at $x = 0$. The gas in $x > 0$ has properties λ_1, μ_1, C_{p_1}, C_{v_1}, all constants, and the gas in $x < 0$ has the constants λ_2, μ_2, C_{p_2}, C_{v_2}. Initially, both gases are at rest and have the same temperature θ_0, but the pressure p_2 in $x < 0$ is slightly larger than p_1, the pressure in $x > 0$. At $t = 0$, the diaphragm is suddenly removed. Formulate a perturbation expansion in terms of the small parameter $(p_2 - p_1)/p_1$ and derive the linearized equations analogous to (3.3.27) and initial conditions on the perturbation velocities, temperatures, pressures, and densities. Assume that the two gases do not mix and that the interface that was at $x = 0$ at time $t = 0$ always separates the two gases. It will be shown in Chapter 5 that the jump conditions at this interface are (see (5.3.34))

$$\frac{d\xi}{dt} = u(\xi^+, t) = u(\xi^-, t), \tag{3.3.69a}$$

$$p(\xi^+, t) - \frac{4}{3}\mu_1 u_x(\xi^+, t) = p(\xi^-, t) - \frac{4}{3}\mu_2 u_x(\xi^-, t), \tag{3.3.69b}$$

$$\lambda_1 \theta_x(\xi^+, t) = \lambda_2 \theta_x(\xi^-, t), \tag{3.3.69c}$$

where $x = \xi(t)$ is the location of the interface.

Use these results to derive the appropriate jump conditions for the perturbation quantities. Use Laplace transforms to show that you have a well-posed problem and can determine u, p, θ, and ρ as functions of x and t for all x and all $t \geq 0$.

3.3.3a. Work out the details leading to (3.3.59) using (3.3.58) in the system (3.3.52)–(3.3.53).

b. Show that for two-dimensional flows $(\partial/\partial z^* = 0)$, (3.3.59) reduces to

$$(1 - M^2)\varphi_{x^*x^*} + \varphi_{y^*y^*} = \epsilon M \left\{[2 + (\gamma - 1)M^2]\varphi_{x^*}\varphi_{x^*x^*} + 2\varphi_{y^*}\varphi_{x^*y^*}\right\} + O(\epsilon^2). \tag{3.3.70}$$

This is the same result correct to $O(\epsilon)$ as (6.2.315) of [26] once the notation is reconciled. In particular, $x^* \to x$, $y^* \to y$, $\varphi \to M\phi$. The perturbation solution is outlined in pp. 612–613 of [26]. The solution of the mathematically analogous signaling problem for shallow-water flow is discussed in Section 6.2.4 of [26].

3.4 The One-Dimensional Problem on $-\infty < x < \infty$

In this section we parallel the discussion of Sections 1.2–1.3 to study the general initial-value problem for the wave equation on the infinite interval

$$u_{tt} - u_{xx} = p(x, t), \quad -\infty < x < \infty, \; t \geq 0, \tag{3.4.1a}$$

$$u(x, 0^+) = f(x), \; u_t(x, 0^+) = h(x). \tag{3.4.1b}$$

In Section 3.1 we showed that (3.4.1a) is the leading-order equation governing the lateral vibrations of a uniform string with a vertical loading $p(x, t)$ (see (3.1.14b) with $r = \overline{\alpha} = 1, \overline{v} = u, \epsilon = 0$). With $p(x, t) \equiv 0$, (3.4.1a) also models the leading-order axial vibrations of this string if we rescale x and t appropriately (see (3.1.14a) with $r = f'(\sigma_0) = 1, \overline{u} = u, \epsilon = 0$). Two other applications where (3.4.1a) with $p \equiv 0$ arises were discussed in Sections 3.2–3.3. We saw that this wave equation governs both the velocity and free surface perturbations in shallow-water flow; and for small amplitude disturbances in a compressible gas, it models the velocity or density perturbations. We shall appeal to these physical models to better interpret the results that we derive in this and subsequent sections.

3.4.1 Fundamental Solution

As in Chapter 1, we begin our study with the derivation of the fundamental solution—that is,

$$u_{tt} - u_{xx} = \delta(x)\delta(t), \quad -\infty < x < \infty, \quad t \geq 0, \tag{3.4.2a}$$

$$u(x, 0^-) = 0, \tag{3.4.2b}$$

$$u_t(x, 0^-) = 0. \tag{3.4.2c}$$

A crude justification for imposing two initial conditions (rather than one, as in the diffusion equation) is to argue that (3.4.2a) is of second order in t and therefore requires two conditions in order to define a solution uniquely. Using physical reasoning, for example, for the vibrating string, we would argue that in order to define the state of a dynamical system, we must initially specify both the *displacement* and the *velocity*. When we study the solution of (3.4.1), or more general hyperbolic equations in Chapter 4, we shall develop more systematic criteria for determining what constitutes a "well-posed" problem.

In solving (3.4.2), it is convenient to consider the equivalent homogeneous equation

$$u_{tt} - u_{xx} = 0 \tag{3.4.3a}$$

on $-\infty < x < \infty$, valid for $t > 0$, with the following initial conditions imposed at $t = 0^+$:

$$u(x, 0^+) = 0, \tag{3.4.3b}$$

$$u_t(x, 0^+) = \delta(x). \tag{3.4.3c}$$

To see that (3.4.2) and (3.4.3) are equivalent, we integrate (3.4.2a) with respect to t from $t = 0^-$ to $t = 0^+$ and use the initial conditions (3.4.2b)–(3.4.2c) to derive the corresponding initial conditions (3.4.3b)–(3.4.3c).

Equation (3.4.3a) is exceptional (among linear second-order partial differential equations) in the sense that every solution can be derived in the *D'Alembert form*

$$u(x, t) = \phi(x + t) + \psi(x - t) \tag{3.4.4}$$

for appropriate functions ϕ and ψ. To see this, we first transform variables x, $t \to \zeta, \sigma$, where $\zeta \equiv x + t$ and $\sigma \equiv x - t$, and regard $u(x, t) = u((\zeta + \sigma)/2, (\zeta - \sigma)/2) \equiv U(\zeta, \sigma)$. Equation (3.4.3a) transforms to $U_{\zeta\sigma} = 0$. Therefore, integrating once with respect to σ implies that U_ζ is a function of ζ alone, say $U_\zeta = \phi'(\zeta)$, and integrating this with respect to ζ gives (3.4.4).

To determine the functions ϕ and ψ, we impose the initial conditions. Equation (3.4.3b) implies that $\phi(x) + \psi(x) = 0$ for all x. Therefore, $\psi = -\phi$, and (3.4.4) now reads

$$u(x, t) = \phi(x + t) - \phi(x - t). \tag{3.4.5}$$

The second initial condition, (3.4.3c), applied to (3.4.5) gives $\delta(x) = 2\phi'(x)$. Therefore, $\phi(x) = \frac{1}{2}H(x)$, where H is the Heaviside function (see (A.1.14)), and the solution of (3.4.3) or (3.4.2) is

$$u(x, t) = \frac{1}{2}[H(x + t) - H(x - t)]. \tag{3.4.6}$$

This represents a uniform front of height $u = \frac{1}{2}$ propagating with constant unit speed in the $+x$ and $-x$ directions, as sketched in Figure 3.13. To show this, we subdivide the xt-plane into the three regions as indicated, and note that in region (1), $x + t < 0, x - t < 0$. Therefore, $H(x + t) = H(x - t) = 0$, and hence $u = 0$. In region (2), $x + t > 0, x - t < 0$. Therefore, $H(x + t) = 1$, $H(x - t) = 0$, and hence, $u = \frac{1}{2}$. Finally, in region (3), $x + t > 0, x - t > 0$. Therefore, $H(x + t) = H(x - t) = 1$, and $u = 0$. The triangular domain (2) is called the *zone of influence* of the source at (0, 0).

In contrast with the diffusion equation of Chapter 1, (3.4.2) has a distinct "disturbance wave front," which propagates with unit speed (for our dimensionless formulation) and separates the disturbed and undisturbed regions.

The result (3.4.6) also follows less directly using Laplace transforms with respect to t or Fourier transforms with respect to x (Problem 3.4.1). For a physical discussion of (3.4.6) in terms of water waves, see Problem 3.4.2.

The fundamental solution for an arbitrary source location $x = \xi$ and switch-on time $t = \tau$ is obtained from (3.4.6) by translation. Thus, the solution of

$$u_{tt} - u_{xx} = \delta(x - \xi)\delta(t - \tau), \quad -\infty < x < \infty, \ t \geq \tau, \tag{3.4.7a}$$

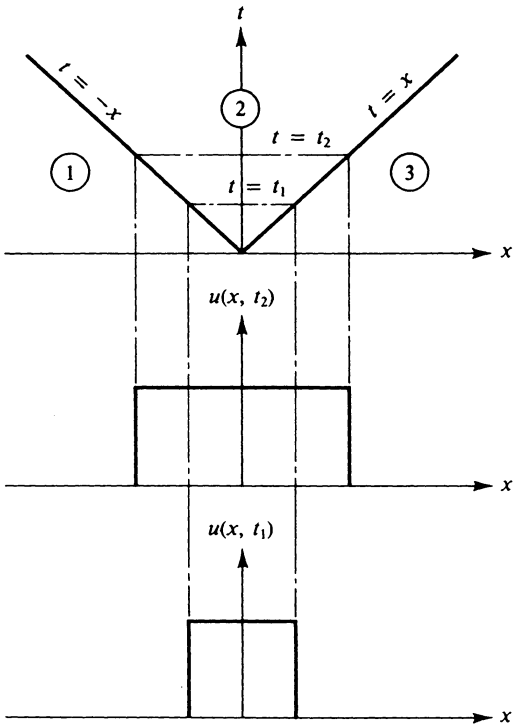

FIGURE 3.13. Fundamental solution

with ξ = constant, τ = constant, and initial conditions

$$u(x, \tau^-) = u_t(x, \tau^-) = 0, \tag{3.4.7b}$$

is

$$F(x - \xi, t - \tau) = \frac{1}{2}[H(x - \xi + t - \tau) - H(x - \xi - t + \tau)]. \tag{3.4.8}$$

3.4.2 General Initial-Value Problem on $-\infty < x < \infty$

The general initial-value problem is given by (3.4.1). Linearity allows us to split this up into the following three problems:

$$p(x, t) = \text{prescribed}; \; u(x, 0^-) = u_t(x, 0^-) = 0, \tag{3.4.9a}$$

$$p(x, t) = 0; \; u(x, 0^+) = 0; \; u_t(x, 0^+) = h(x), \tag{3.4.9b}$$

$$p(x, t) = 0; \; u(x, 0^+) = f(x); \; u_t(x, 0^+) = 0. \tag{3.4.9c}$$

The sum of the three solutions solves (3.4.1).

The solution for (3.4.9a) follows from (3.4.8) by superposition, and we obtain

$$u(x,t) = \int_0^t d\tau \int_{-\infty}^{\infty} F(x-\xi,\ t-\tau)p(\xi,\tau)d\xi. \tag{3.4.10}$$

Now, for any *fixed* point $P = (x,t)$, consider the triangular domain D bounded by the ξ-axis and the two straight lines

$$\xi = x + (t-\tau),\ \ \xi = x - (t-\tau), \tag{3.4.11}$$

as sketched in Figure 3.14. In (3.4.10) the integration with respect to ξ occurs for a fixed τ, as shown in Figure 3.14. But the expression for F vanishes whenever $\xi > x+t-\tau$ and $\xi < x-(t-\tau)$, and in the interval $x-(t-\tau) < \xi < x+(t-\tau)$, the value of F equals $\frac{1}{2}$, according to (3.4.8). Therefore, (3.4.10) reduces to

$$u(x,t) = \frac{1}{2}\int_0^t d\tau \int_{x-(t-\tau)}^{x+(t-\tau)} p(\xi,\tau)d\xi. \tag{3.4.12}$$

Thus, only the values of $p(x,t)$ defined in the triangular domain D can influence the value of u at P. The domain D is called the *domain of dependence* of the point P.

To solve (3.4.9b), we note that it is equivalent to

$$u_{tt} - u_{xx} = \delta(t)h(x), \tag{3.4.13a}$$

$$u(x,0^-) = 0,\ u_t(x,0^-) = 0, \tag{3.4.13b}$$

which is a special case of (3.4.9a) with $p(x,t) = \delta(t)h(x)$. After changing the order of integration, the solution given by (3.4.12) for this value of p is

$$u(x,t) = \frac{1}{2}\int_{\xi=x-t}^{x} h(\xi)\int_{\tau=0^-}^{\xi-x+t}\delta(\tau)d\tau\, d\xi + \frac{1}{2}\int_{\xi=x}^{x+t} h(\xi)\int_{\tau=0^-}^{x+t-\xi}\delta(\tau)d\tau\, d\xi,$$

or

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} h(\xi)d\xi. \tag{3.4.14}$$

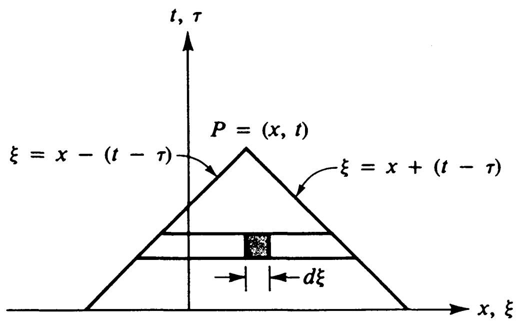

FIGURE 3.14. Integration domain for (3.4.10)

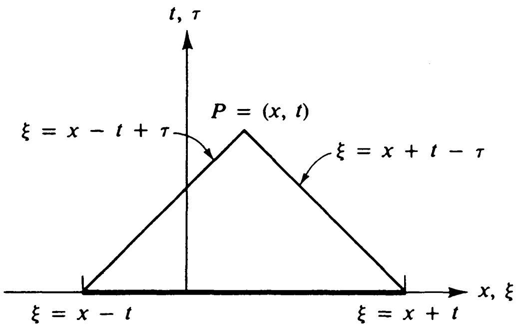

FIGURE 3.15. Domain of dependence of P

Thus, only the portion of initial data that lies in the domain of dependence of the point (x, t) influences the solution there. Information outside the interval $x - t \leq \xi \leq x + t$ does not affect the solution at (x, t) (see Figure 3.15). In particular, two initial-value problems for which h coincides on some interval $x_1 \leq x \leq x_2$, $t = 0$, but not outside, will be identical for all x, t in the triangular domain $t + x_1 \leq x \leq x_2 - t$.

Finally, (3.4.9c) may be reduced to (3.4.9b) by introducing the following transformation of dependent variable $u \to v$:

$$v(x, t) \equiv \int_0^t u(x, \tau)d\tau. \tag{3.4.15}$$

If $u(x, t)$ is a solution of (3.4.9c), then

$$\begin{aligned} v_t &= u(x, t), \quad v_{tt} = u_t(x, t), \\ v_{xx} &= \int_0^t u_{xx}(x, \tau)d\tau = \int_0^t u_{tt}(x, \tau)d\tau = u_t(x, t) - u_t(x, 0) \\ &= u_t(x, t) = v_{tt}. \end{aligned} \tag{3.4.16}$$

Therefore, $v(x, t)$ satisfies

$$v_{tt} - v_{xx} = 0, \tag{3.4.17a}$$

$$v(x, 0^+) = 0, \; v_t(x, 0^+) = u(x, 0^+) = f(x). \tag{3.4.17b}$$

It then follows from (3.4.14) that $v(x, t)$ is given by

$$v(x, t) = \frac{1}{2}\int_{x-t}^{x+t} f(\xi)d\xi,$$

and $u(x, t)$ is

$$u(x, t) = v_t(x, t) = \frac{1}{2}[f(x + t) + f(x - t)]. \tag{3.4.18}$$

Thus, the initial disturbance splits up into two identical *half-scale* shapes that propagate to the left and right undistorted with constant speed unity, as shown in Figure 3.16.

To describe the solution of the general initial-value problem, we combine the results in (3.4.12), (3.4.14), and (3.4.18). This is called D'Alembert's solution.

3.4.3 An Example

To illustrate a particular application of the preceding results, let us study the propagation of two initial discontinuities in the surface height and speed of shallow water, as sketched in Figure 3.17. In this chapter, we shall consider only the small-amplitude case, as exhibited by the occurrence of the small parameter ϵ in the initial conditions. The quasilinear problem $[\epsilon = O(1)]$ is discussed in Chapters 5, 7, and 8.

We expand u and h in (3.2.12) using (3.2.23) and (3.2.28),

$$u(x, t; \epsilon) = \epsilon u_1(x, t) + \dots, \tag{3.4.19a}$$

$$h(x, t; \epsilon) = 1 + \epsilon h_1(x, t) + \dots, \tag{3.4.19b}$$

and obtain the linear equations (3.2.30) for u_1 and h_1, that is,

$$h_{1_t} + u_{1_x} = 0, \tag{3.4.20a}$$

$$u_{1_t} + h_{1_x} = 0. \tag{3.4.20b}$$

The initial conditions indicated in Figure 3.17 translate to

$$u_1(x, 0) = \tilde{u}(x) \equiv \begin{cases} \tilde{u}_1 = \text{constant} > 0 & \text{if } -\infty < x < 0, \\ 0 & \text{if } 0 < x < 1, \\ \tilde{u}_2 = \text{constant} < 0 & \text{if } 1 < x < \infty, \end{cases} \tag{3.4.21a}$$

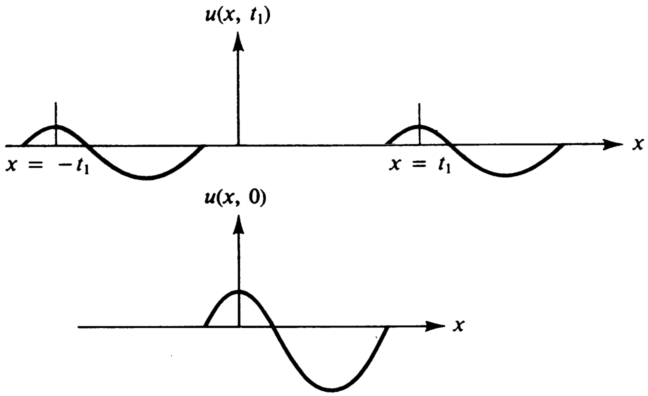

FIGURE 3.16. Initial disturbance splitting into two half-scale waves

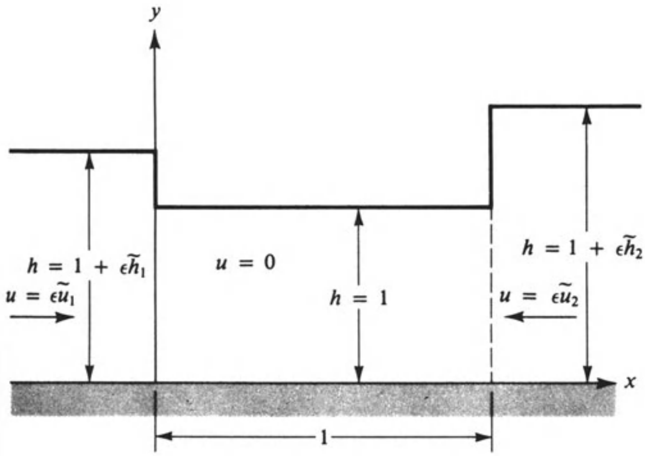

FIGURE 3.17. Initial state

$$h_1(x,0) = \tilde{h}(x) \equiv \begin{cases} \tilde{h}_1 = \text{constant} > 0 & \text{if } -\infty < x < 0, \\ 0 & \text{if } 0 < x < 1, \\ \tilde{h}_2 = \text{constant} > \tilde{h}_1 & \text{if } 1 < x < \infty. \end{cases} \quad (3.4.21b)$$

We have a number of options for calculating the solution. One is to exploit the fact that the solution must depend on the so-called characteristic coordinates [see (3.4.5)]

$$\zeta = x + t, \ \sigma = x - t. \quad (3.4.22)$$

Thus, transforming (3.4.20) to the ζ, σ variables should simplify the calculations. This is a special case of the *method of characteristics* that we discuss in general in Chapters 4 and 7; this method of solution is particularly well suited to solving coupled systems of first order, as in (3.4.20). A second approach is to eliminate u_1 or h_1 from (3.4.20) and solve the resulting wave equation using the results of the previous section. These two approaches are illustrated next. One could also use Laplace transforms with respect to t or Fourier transforms with respect to x in (3.4.20), but the details are not worked out here.

(i) *Method of characteristics*

Using (3.4.22), we obtain the transformation rules

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \sigma}, \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \sigma}.$$

Therefore, letting $h_1(x,t) = h_1((\zeta+\sigma)/2, (\zeta-\sigma)/2) \equiv H(\zeta,\sigma)$ and $u_1(x,t) = u_1((\zeta+\sigma)/2, (\zeta-\sigma)/2) \equiv U(\zeta,\sigma)$, we find that (3.4.20) transforms to

$$\overline{H}_\zeta - \overline{H}_\sigma + \overline{U}_\zeta + \overline{U}_\sigma = 0, \quad (3.4.23a)$$

$$\overline{H}_\zeta + \overline{H}_\sigma + \overline{U}_\zeta - \overline{U}_\sigma = 0. \tag{3.4.23b}$$

Adding and subtracting gives

$$(\overline{H} + \overline{U})_\zeta = 0, \ (\overline{H} - \overline{U})_\sigma = 0, \tag{3.4.24}$$

and we conclude that

$$\overline{H} + \overline{U} = F(\sigma), \ \overline{H} - \overline{U} = G(\zeta) \tag{3.4.25}$$

for functions F and G to be specified. Solving for $\overline{H}$ and $\overline{U}$ gives

$$\overline{H}(\zeta, \sigma) = \frac{1}{2}[F(\sigma) + G(\zeta)], \ \overline{U}(\zeta, \sigma) = \frac{1}{2}[F(\sigma) - G(\zeta)]. \tag{3.4.26}$$

Using the initial conditions (3.4.21) defines F and G in the form

$$F(\sigma) = \tilde{h}(\sigma) + \tilde{u}(\sigma), \ G(\zeta) = \tilde{h}(\zeta) - \tilde{u}(\zeta). \tag{3.4.27}$$

Therefore, the solution for $\overline{H}$ and $\overline{U}$ is

$$\overline{H}(\zeta, \sigma) = \frac{1}{2}\{\tilde{h}(\sigma) + \tilde{u}(\sigma) + \tilde{h}(\zeta) - \tilde{u}(\zeta)\}, \tag{3.4.28a}$$

$$\overline{U}(\zeta, \sigma) = \frac{1}{2}\{\tilde{h}(\sigma) + \tilde{u}(\sigma) - \tilde{h}(\zeta) + \tilde{u}(\zeta)\}. \tag{3.4.28b}$$

Since the functions $\tilde{h}(x)$ and $\tilde{u}(x)$ change value at $x = 0$ and $x = 1$, we must focus on the four characteristic lines $\sigma = 0$, $\sigma = 1$, $\zeta = 0$, and $\zeta = 1$ at which the functions $\tilde{h}(\sigma)$, $\tilde{h}(\zeta)$, $\tilde{u}(\sigma)$, and $\tilde{u}(\zeta)$ will also switch values. The four characteristic lines subdivide the upper half-xt-plane into the six regions sketched in Figure 3.18.

The solution (3.4.28) takes on the following values in the six regions:

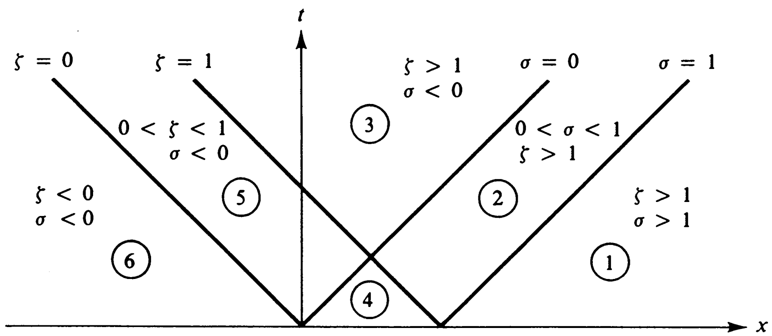

FIGURE 3.18. Characteristics emanating from $x = 0, t = 0$ and $x = 1, t = 0$

(1): $\zeta > 1, \sigma > 1$

$$\overline{H} = \frac{1}{2}\{\tilde{h}_2 + \tilde{u}_2 + \tilde{h}_2 - \tilde{u}_2\} = \tilde{h}_2,$$

$$\overline{U} = \frac{1}{2}\{\tilde{h}_2 + \tilde{u}_2 - \tilde{h}_2 + \tilde{u}_2\} = \tilde{u}_2.$$

(2): $\zeta > 1, 0 < \sigma < 1$

$$\overline{H} = \frac{1}{2}\{0 + 0 + \tilde{h}_2 - \tilde{u}_2\} = \frac{1}{2}(\tilde{h}_2 - \tilde{u}_2),$$

$$\overline{U} = \frac{1}{2}\{0 + 0 - \tilde{h}_2 + \tilde{u}_2\} = \frac{1}{2}(\tilde{u}_2 - \tilde{h}_2).$$

(3): $\zeta > 1, \sigma < 0$

$$\overline{H} = \frac{1}{2}\{\tilde{h}_1 + \tilde{u}_1 + \tilde{h}_2 - \tilde{u}_2\},$$

$$\overline{U} = \frac{1}{2}\{\tilde{h}_1 + \tilde{u}_1 - \tilde{h}_2 + \tilde{u}_2\}.$$

(4): $0 < \zeta < 1, 0 < \sigma < 1$

$$\overline{H} = \frac{1}{2}\{0 + 0 + 0 - 0\} = 0,$$

$$\overline{U} = \frac{1}{2}\{0 + 0 - 0 + 0\} = 0.$$

(5): $0 < \zeta < 1, \sigma < 0$

$$\overline{H} = \frac{1}{2}\{\tilde{h}_1 + \tilde{u}_1 + 0 - 0\} = \frac{1}{2}(\tilde{h}_1 + \tilde{u}_1),$$

$$\overline{U} = \frac{1}{2}\{\tilde{h}_1 + \tilde{u}_1 - 0 + 0\} = \frac{1}{2}(\tilde{h}_1 + \tilde{u}_1).$$

(6): $\zeta < 0, \sigma < 0$

$$\overline{H} = \frac{1}{2}\{\tilde{h}_1 + \tilde{u}_1 + \tilde{h}_1 - \tilde{u}_1\} = \tilde{h}_1,$$

$$\overline{U} = \frac{1}{2}\{\tilde{h}_1 + \tilde{u}_1 - \tilde{h}_1 + \tilde{u}_1\} = \tilde{u}_1.$$

These results are summarized in Figure 3.18.

(ii) *D'Alembert solution*

We eliminate u_1 from (3.4.20) and obtain

$$h_{1_{tt}} - h_{1_{xx}} = 0 \tag{3.4.29}$$

subject to (3.4.21b). In order to derive the initial condition on h_{1_t}, we evaluate (3.4.20a) at $t = 0$ and use the derivative of (3.4.21a) to calculate $u_{1_x}(x, 0)$. Since we are dealing with a discontinuous function, we interpret the derivative in the symbolic sense. Thus, if $f(x)$ is a function that is defined and has a continuous derivative everywhere except at $x = x_0$, then the symbolic derivative $f_s'(x)$ is defined as (see (A.1.18))

$$f_s'(x) \equiv f'(x) + [f(x_0^+) - f(x_0^-)]\delta(x - x_0). \tag{3.4.30}$$

Therefore,

$$h_{1_t}(x, 0) = -u_{1_x}(x, 0) = -\tilde{u}_s'(x), \tag{3.4.31}$$

and since $\tilde{u}'(x) = 0$ if $x \neq 0$ and $x \neq 1$, we have, according to (3.4.21),

$$h_{1_t}(x, 0) = \delta(x)\tilde{u}_1 - \delta(x - 1)\tilde{u}_2. \tag{3.4.32}$$

The solution of (3.4.29) subject to (3.4.21b) and (3.4.32) is then given by combining (3.4.14) and (3.4.18), that is,

$$h_1(x, t) = \frac{1}{2}\{\tilde{h}(x+t)+\tilde{h}(x-t)\}+\frac{1}{2}\int_{x-t}^{x+t}\{\delta(\xi)\tilde{u}_1-\delta(\xi-1)\tilde{u}_2\}d\xi. \tag{3.4.33}$$

It is left as an exercise (Problem 3.4.3) to show that

$$u_1(x, t) = \frac{1}{2}\{\tilde{u}(x+t)+\tilde{u}(x-t)\}+\frac{1}{2}\int_{x-t}^{x+t}[\delta(\xi)\tilde{h}_1-\delta(\xi-1)\tilde{h}_2]d\xi, \tag{3.4.34}$$

and that the results (3.4.33) and (3.4.34) reduce to the expressions summarized in Figure 3.19.

Let us view our results for arbitrary choices of the four initial constants $\tilde{u}_1$, $\tilde{u}_2$, $\tilde{h}_1$, and $\tilde{h}_2$. We see that at time $t = 0^+$, outward propagating disturbances arise at $x = 0$ and $x = 1$ in addition to the initial inward propagating disturbances there. In particular, the initial discontinuity at $x = 1$ splits up into two waves, one

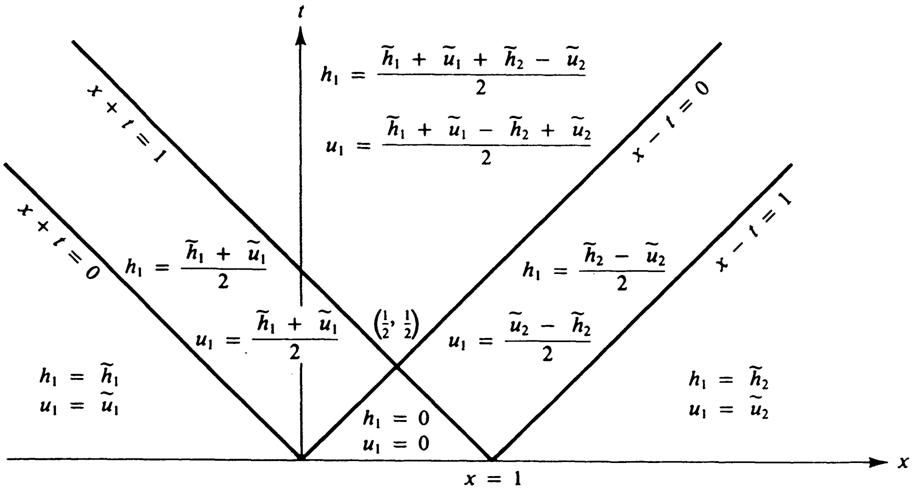

FIGURE 3.19. Solution of (3.4.20)–(3.4.21)

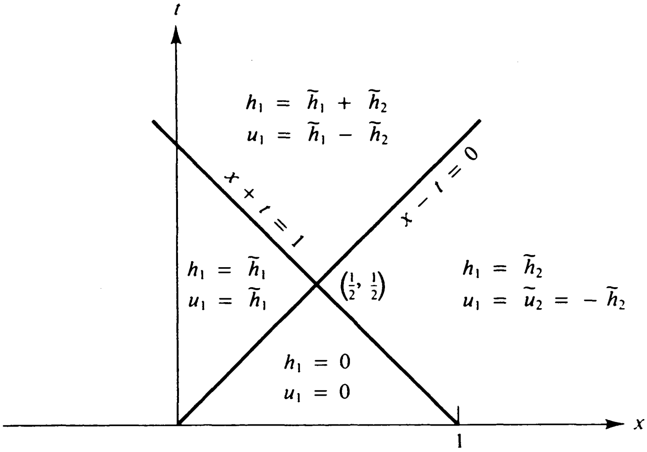

FIGURE 3.20. Initial discontinuities for traveling waves

propagating to the right along $x - t = 1$ and a second propagating to the left along $x + t = 1$. A similar situation occurs for the initial discontinuity at $x = 0$. Thus, arbitrary values of these four constants do not produce traveling waves that consist of the translation to the left of the initial discontinuity at $x = 1$ and a translation to the right of the initial discontinuity at $x = 0$. To achieve this steady configuration, we must require the solutions in regions (1) and (2) to coincide and the solutions in (5) and (6) to coincide. This will be true only if we choose

$$\tilde{u}_2 \equiv -\tilde{h}_2, \; \tilde{u}_1 \equiv \tilde{h}_1. \tag{3.4.35}$$

In this case, the solution in region (3) is given by

$$h_1 = \tilde{h}_1 + \tilde{h}_2, \; u_1 = \tilde{h}_1 - \tilde{h}_2, \tag{3.4.36}$$

and consists of the initial discontinuities alone, as sketched in Figure 3.20.

We see that in this linear problem, discontinuities propagate unchanged along the characteristics σ = constant and ζ = constant. The propagation of discontinuities for linear hyperbolic problems is explored more fully in Sections 4.2.3 and 4.3.6. The situation is significantly more complicated for quasilinear problems where discontinuities (shocks) still occur but propagate along certain curves that are not characteristics. Moreover, these shocks do interact, unlike the situation described in Figures 3.19 and 3.20. See Section 5.3 and Problem 5.3.11.

Problems

3.4.1. Solve (3.4.2) using Laplace transforms with respect to t. Repeat the derivation using Fourier transforms with respect to x.

3.4.2. The result that the fundamental solution of the wave equation,

$$h_{1_{tt}} - h_{1_{xx}} = \delta(x)\delta(t), \quad -\infty < x < \infty, \quad 0 \leq t, \tag{3.4.37a}$$

$$h_1(x, 0^-) = 0, \quad h_{1_t}(x, 0^-) = 0, \tag{3.4.37b}$$

is [see (3.4.6)]

$$h_1(x, t) = \frac{1}{2}[H(x+t) - H(x-t)] \tag{3.4.38}$$

may appear, at first glance, to be counterintuitive if ϵh_1 is interpreted as the surface height perturbation for shallow-water waves [see (3.2.23b), (3.2.28b), and (3.2.34)]. We may ask how mass, momentum, or energy can be conserved if the surface above equilibrium equals $\epsilon/2$ over the interval $-t < x < t$, which grows linearly in time.

a. Show that the problem (3.4.37) actually corresponds to the system of two first-order equations (3.2.30) with initial conditions

$$u_1(x, 0^+) = -H(x), \tag{3.4.39a}$$

$$h_1(x, 0^+) = 0. \tag{3.4.39b}$$

According to (3.4.39), the entire body of water over $x > 0$ is initially moving to the left with speed ϵ, and this may provide the mechanism for a linearly expanding interval of water of above-equilibrium surface height.

b. Solve (3.2.30) subject to (3.4.39) by the method of characteristics to obtain

$$h_1(x, t) = \frac{1}{2}[H(x+t) - H(x-t)], \tag{3.4.40a}$$

$$u_1(x, t) = -\frac{1}{2}[H(x+t) + H(x-t)]. \tag{3.4.40b}$$

Thus, (3.4.40a) confirms the result (3.4.38), and (3.4.40b) shows that the speed of the disturbed water is $-\epsilon/2$ over the interval $-t < x < t$.

c. Argue physically how the semi-infinite reservoir of water of height 1 and speed $-\epsilon$ to the right of $x = t$ "feeds" the increase in height over $-t < x < t$ because the speed of the water drops by a factor of two in this interval. Use a more careful argument and demonstrate that mass, momentum, and energy are indeed conserved over any fixed interval $x_1 \leq x \leq x_2$ for the solution (3.4.40).

3.4.3. Derive the result (3.4.34) and show that the expressions for u_1 and h_1 given in (3.4.33)–(3.4.34) are the same as those obtained by the method of characteristics as summarized in Figure 3.19.

3.4.4a For the problem discussed in Section 3.4.3, modify the initial conditions (3.4.21) so the result represents flow for $x \leq \frac{1}{2}$ with a solid vertical wall located at $x = \frac{1}{2}$.

b. Calculate the solution for u_1 and h_1 for all $t \geq 0$ and $x \leq \frac{1}{2}$.
c. Given $\tilde{u}_1$, for what value of $\tilde{h}_1$ will there be only one wave propagating initially to the right over $x \leq \frac{1}{2}$? Summarize the solution in (b) for this case.

3.5 Initial- and Boundary-Value Problems on $0 \leq x < \infty$

We can appeal to symmetry arguments, as in Chapters 1 and 2, to construct Green's functions for various homogeneous boundary-value problems. These, in turn, may be used to solve both homogeneous and inhomogeneous problems on the semi-infinite interval $0 \leq x < \infty$.

3.5.1 Green's Function of the First Kind

Green's function of the first kind solves the problem

$$u_{tt} - u_{xx} = \delta(x - \xi)\delta(t - \tau), \quad 0 \leq x, \ \tau \leq t, \tag{3.5.1a}$$

with zero initial conditions at $t = \tau$,

$$u(x, \tau^-) = u_t(x, \tau^-) = 0, \tag{3.5.1b}$$

and zero boundary value for u at $x = 0$,

$$u(0, t) = 0, \quad t > \tau. \tag{3.5.1c}$$

Here ξ and τ are arbitrary positive constants.

It is clear by symmetry that the solution is (see (1.4.2))

$$G_1(x, \xi, t - \tau) \equiv F(x - \xi, t - \tau) - F(x + \xi, t - \tau), \tag{3.5.2}$$

where F is defined by (3.4.8).

Since the two sources are switched on at $t = \tau$, the value of G_1 in region (1), that is, $t < \tau$, is equal to zero for all x (see Figure 3.21). Now, for a *fixed source location* (ξ, τ) and image sources location $(-\xi, \tau)$, consider the value that G_1 takes on at different points in the xt-plane. This value depends on whether the point (x, t) is in the zone of influence of the primary source alone, the image source alone, both, or neither. Thus, if (x, t) is in region (2), (7), or (6), it is outside the zone of influence of either the primary or image source, and $G_1 = 0$. In region (5), $G_1 = \frac{1}{2}$ because (x, t) is influenced only by the primary source, and in region (3), $G_1 = -\frac{1}{2}$ because (x, t) is influenced only by the image source. Finally, in region (4), (x, t) is influenced by both sources, and its value is zero, since the two contributions cancel.

It is also useful to look at (3.5.2) with (x, t) *fixed* for different values of ξ and τ to determine the domain of dependence of (x, t)—the set of points (ξ, τ) in the plane at which a primary source [with a mirror source located at $(-\xi, \tau)$] will result in a nonvanishing value of G_1 at (x, t). And, since (x, t) is in the first quadrant, the value of G_1 will equal $\frac{1}{2}$.

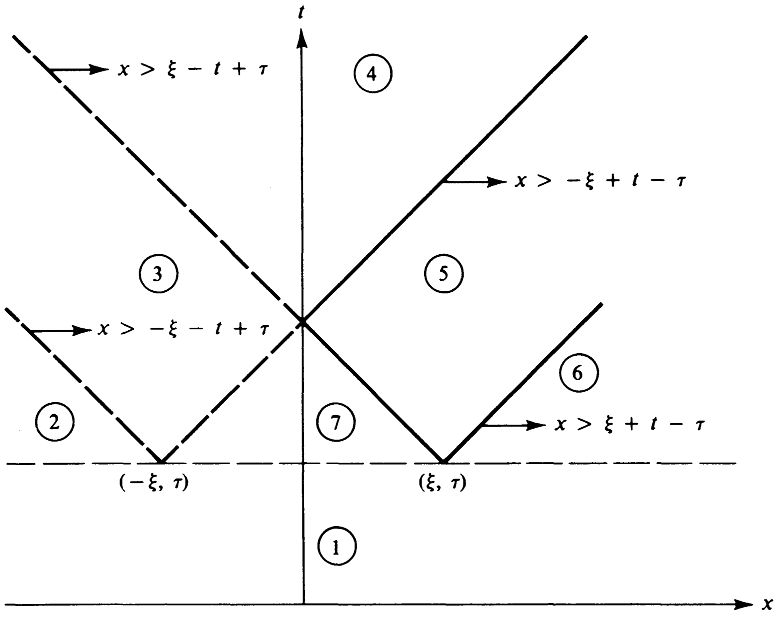

FIGURE 3.21. Zones of influence for Green's function

We distinguish two cases, $x < t$ and $x > t$, as sketched in Figure 3.22. Keeping in mind that (x, t) is a fixed point and that we are studying varying values of ξ and τ, we locate the four critical lines $\xi = x + t - \tau, \xi = -x + t - \tau, \xi = x - t + \tau$, and $\xi = -x - t + \tau$, at which the arguments of the two Heaviside functions in (3.5.2) equal zero. It then follows that the domain of dependence of (x, t) is the region bounded by the straight lines through $PQRSP$ for $x < t$. For $x > t$, this region is the triangle $PRSP$.

In summary, for a fixed P, the value of G_1 equals $\frac{1}{2}$ only if the primary source is located somewhere inside the shaded domain of dependence of P. Otherwise, $G_1 = 0$.

3.5.2 *Homogeneous Boundary Condition, Nonzero Initial Conditions*

As in (1.4.5), we can use (3.5.2) to solve

$$u_{tt} - u_{xx} = p(x, t), \ 0 \le x, \ 0 \le t, \tag{3.5.3a}$$

$$u(x, 0^-) = u_t(x, 0^-) = 0, \tag{3.5.3b}$$

$$u(0, t) = 0, \ t > 0, \tag{3.5.3c}$$

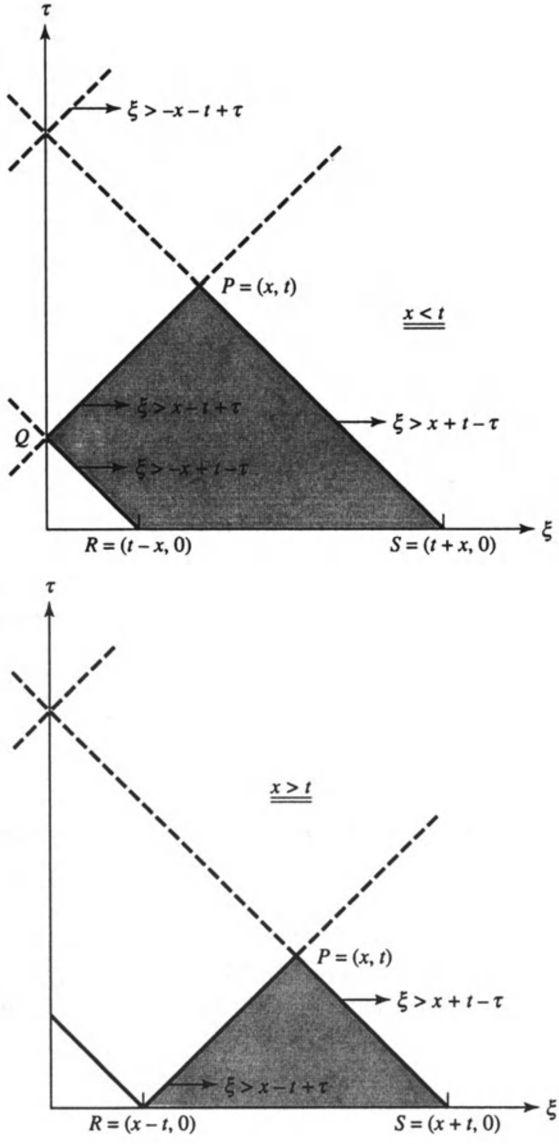

FIGURE 3.22. Domain of dependence for Green's function

by superposition in the form

$$u(x, t) = \int_0^t d\tau \int_0^\infty p(\xi, \tau)G_1(x, \xi, t - \tau)d\xi. \qquad (3.5.4)$$

For a fixed (x, t), G_1, regarded as a function of ξ and τ, vanishes outside the domain of dependence of (x, t) and equals $\frac{1}{2}$ inside. Therefore, (3.5.4) has the more explicit form

$$u(x, t) = \frac{1}{2}\int_{\tau=0}^{t}\left[\int_{\xi=x-t+\tau}^{x+t-\tau} p(\xi, \tau)d\xi\right]d\tau, \ t < x, \qquad (3.5.5a)$$

$$u(x, t) = \frac{1}{2}\int_{\tau=0}^{t-x}\left[\int_{\xi=-x+t-\tau}^{x+t-\tau} p(\xi, \tau)d\xi\right]d\tau$$
$$+ \frac{1}{2}\int_{\tau=t-x}^{t}\left[\int_{\xi=x-t+\tau}^{x+t-\tau} p(\xi, \tau)d\xi\right]d\tau, \ t > x. \qquad (3.5.5b)$$

Now, suppose we wish to solve the homogeneous wave equation with homogeneous boundary condition on the left but with a prescribed value of u_t at $t = 0$, that is,

$$u_{tt} - u_{xx} = 0, \ 0 \le x, \ 0 \le t, \qquad (3.5.6a)$$
$$u(x, 0^+) = 0, \qquad (3.5.6b)$$
$$u_t(x, 0^+) = h(x), \qquad (3.5.6c)$$
$$u(0, t) = 0, \ t > 0. \qquad (3.5.6d)$$

We convert this to the equivalent inhomogeneous equation

$$u_{tt} - u_{xx} = \delta(t)h(x), \ 0 \le x, \ 0 \le t, \qquad (3.5.7a)$$

with zero initial conditions,

$$u(x, 0^-) = u_t(x, 0^-) = 0, \qquad (3.5.7b)$$

and the same zero-boundary condition (3.5.6d).

Therefore, we can use the results in (3.5.5) with $p = \delta(t)h(x)$ to obtain

$$u(x, t) = \frac{1}{2}\int_{x-t}^{x+t} h(\xi)d\xi, \ t < x, \qquad (3.5.8a)$$

$$u(x, t) = \frac{1}{2}\int_{-x+t}^{x+t} h(\xi)d\xi, \ t > x. \qquad (3.5.8b)$$

The result (3.5.8a) for $t < x$ is the same as that in (3.4.14) for the problem on the infinite interval. This is because the boundary $x = 0$ is outside the domain of dependence of the point (x, t) for $t < x$. In other words, the observer is unaware of the boundary for times $t < x$ because reflected disturbances from $x = 0$ have not arrived yet. The result (3.5.8b), which was obtained formally from (3.5.5b), and in which the second integral of (3.5.5b) gives no contribution, can also be deduced directly using the following symmetry arguments.

To solve (3.5.6), consider the equivalent initial-value problem on the *entire axis*, $-\infty < x < \infty$, where the initial velocity $u_t(x, 0^+)$ is now defined by

$$u_t(x, 0^+) = h^*(x) = \begin{cases} h(x) & \text{if } x > 0, \\ -h(-x) & \text{if } x < 0. \end{cases} \tag{3.5.9}$$

Thus, we extend the definition of the given initial velocity $h(x)$, available on $0 \le x < \infty$, to the negative axis in such a way as to ensure a zero value of u at the origin.

The solution of this "extended initial-value problem" if $t > x$ is, according to (3.4.14),

$$\begin{aligned} u(x,t) &= \frac{1}{2}\int_{x-t}^{x+t} h^*(\xi)d\xi = \frac{1}{2}\int_{x-t}^{0} h^*(\xi)d\xi + \frac{1}{2}\int_{0}^{x+t} h^*(\xi)d\xi \\ &= \frac{1}{2}\int_{x-t}^{0} -h(-\xi)d\xi + \frac{1}{2}\int_{0}^{x+t} h(\xi)d\xi \\ &= \frac{1}{2}\int_{-x+t}^{0} h(\xi)d\xi + \frac{1}{2}\int_{0}^{x+t} h(\xi)d\xi = \frac{1}{2}\int_{-x+t}^{x+t} h(\xi)d\xi, \end{aligned} \tag{3.5.10}$$

in agreement with (3.5.8b).

To complete our listing of solutions of the homogeneous boundary-value problem, we need to consider

$$u_{tt} - u_{xx} = 0, \quad 0 \le x, \quad 0 \le t, \tag{3.5.11a}$$

$$u(x, 0^+) = f(x), \tag{3.5.11b}$$

$$u_t(x, 0^+) = 0, \tag{3.5.11c}$$

$$u(0, t) = 0, \quad t > 0. \tag{3.5.11d}$$

As in our discussion of the initial-value problem on the infinite x-axis, we introduce the transformation (3.4.15)

$$v(x,t) \equiv \int_0^t u(x,\tau)d\tau, \tag{3.5.12}$$

to obtain

$$v_{tt} - v_{xx} = 0, \tag{3.5.13a}$$

$$v(x, 0^+) = 0, \tag{3.5.13b}$$

$$v_t(x, 0^+) = f(x), \tag{3.5.13c}$$

and the boundary condition at $x = 0$ is still zero, because

$$v(0,t) = \int_0^t u(0,\tau)d\tau = 0. \tag{3.5.13d}$$

Therefore, the solution for v is given by (3.5.8) with h replaced by f, and taking the partial derivative of this result with respect to t gives u; that is,

$$u(x,t) = \frac{1}{2}[f(x+t) + f(x-t)], \quad t < x, \tag{3.5.14a}$$

$$u(x,t) = \frac{1}{2}[f(x+t) - f(-x+t)], \ t > x. \tag{3.5.14b}$$

Here again, the idea of an "image" initial condition,

$$u(x,0) = f^*(x) = \begin{cases} f(x) & \text{if } x > 0, \\ -f(-x) & \text{if } x < 0, \end{cases}$$

used in (3.4.18) gives the desired result. We have, according to (3.4.18),

$$\begin{aligned} u(x,t) &= \frac{1}{2}[f^*(x+t) + f^*(x-t)] \\ &= \frac{1}{2}[f(x+t) + f(x-t)] \quad \text{if } t < x, \end{aligned} \tag{3.5.15a}$$

and

$$u(x,t) = \frac{1}{2}[f(x+t) - f(-x+t)] \quad \text{if } t > x, \tag{3.5.15b}$$

in agreement with (3.5.14).

3.5.3 Inhomogeneous Boundary Condition $u(0,t) = g(t)$

We wish to solve the problem [see the linearized wavemaker problem (3.2.40)–(3.2.42) or the linearized piston problem in acoustics defined by (3.3.31)–(3.3.32)]

$$u_{tt} - u_{xx} = 0, \ 0 \le x, \ 0 \le t, \tag{3.5.16a}$$

with zero initial conditions

$$u(x,0^+) = 0, \ u_t(x,0^+) = 0, \tag{3.5.16b}$$

and a prescribed boundary condition

$$u(0,t) = g(t), \ t > 0. \tag{3.5.16c}$$

We introduce the homogenizing transformation $w(x,t) \equiv u(x,t) - g(t)$, as in (1.4.13), then solve the problem for w using the results in the previous section. We find that $w(x,t)$ obeys

$$w_{tt} - w_{xx} = -\ddot{g}(t), \tag{3.5.17a}$$

$$w(x,0^+) = -g(0^+), \tag{3.5.17b}$$

$$w_t(x,0^+) = -\dot{g}(0^+), \tag{3.5.17c}$$

$$w(0,t) = 0, \ t > 0. \tag{3.5.17d}$$

For $t < x$, using the results in (3.5.5a) with $p = -\ddot{g}$, (3.5.8a) with $h = -\dot{g}(0^+)$, and (3.5.14a) with $f = -g(0^+)$, gives

$$w(x,t) = -\frac{1}{2}\int_{0^+}^{t}\int_{x-t+\tau}^{x+t-\tau} \ddot{g}(\tau)d\xi\, d\tau - \frac{1}{2}\int_{x-t}^{x+t} \dot{g}(0^+)d\xi - g(0^+), \ t < x. \tag{3.5.18}$$

Evaluating the various integrals gives $w(x,t) = -g(t)$, or $u(x,t) = 0$ if $t < x$. For $t > x$, we have

$$w(x,t) = -\frac{1}{2}\int_{0^+}^{t-x}\int_{-x+t-\tau}^{x+t-\tau} \ddot{g}(\tau)d\xi\, d\tau$$
$$-\frac{1}{2}\int_{t-x}^{t}\int_{x-t+\tau}^{x+t-\tau} \ddot{g}(\tau)d\xi\, d\tau - \frac{1}{2}\int_{-x+t}^{x+t} \dot{g}(0^+)d\xi,\ t > x. \quad (3.5.19)$$

Evaluating the integrals in (3.5.19) results in

$$w(x,t) = -g(t) + g(t-x),\ t > x.$$

Therefore,

$$u(x,t) = \begin{cases} 0 & \text{if } t < x, \\ g(t-x) & \text{if } t > x. \end{cases} \quad (3.5.20)$$

This result can also be derived using Laplace transforms (Problem 3.5.1).

As discussed in Section 1.4.7, we can use Green's functions to solve the general linear boundary-value problem after an appropriate transformation of the dependent variable. This idea is further illustrated in Problem 3.5.2 for the one-dimensional wave equation.

3.5.4 Signaling Problem for Shallow-Water Waves

We illustrate the results of Sections 3.5.1–3.5.3 by studying in detail the solution of the signaling problem for water waves as formulated in Section 3.2.9. We have shown that the leading-order problem satisfies [see (3.2.30), (3.2.41), and (3.2.42)]

$$h_{1_t} + u_{1_x} = 0, \quad (3.5.21a)$$
$$u_{1_t} + h_{1_x} = 0, \quad (3.5.21b)$$

with zero initial conditions

$$u_1(x,0) = 0,\ \ h_1(x,0) = 0, \quad (3.5.21c)$$

and the wavemaker boundary condition

$$u_1(0,t) = \dot{s}(t),\ \ t > 0. \quad (3.5.21d)$$

The problem to $O(\epsilon^2)$ is governed by [see (3.2.31), (3.2.45), and (3.2.47)]

$$h_{2_t} + u_{2_x} = -(u_1 h_1)_x, \quad (3.5.22a)$$
$$u_{2_t} + h_{2_x} = -\frac{1}{2}(u_1^2)_x, \quad (3.5.22b)$$

with zero initial conditions

$$u_2(x,0) = 0,\ \ h_2(x,0) = 0, \quad (3.5.22c)$$

and the wavemaker boundary condition

$$u_2(0,t) = -u_{1_x}(0,t)s(t). \quad (3.5.22d)$$

As in Section 3.4.3, we shall solve these signaling problems first directly using the method of characteristics and then in terms of Green's function for the wave equations for u_1 and u_2 that result when h_1 and h_2 are eliminated from the governing equations.

(i) *Method of characteristics*

For the signaling problem, it is convenient to change the sign of σ in (3.4.22), and we introduce the two characteristic variables

$$\zeta = t + x, \quad \mu = t - x, \tag{3.5.23}$$

which imply the following transformations for the x and t derivatives:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \mu}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \mu}. \tag{3.5.24}$$

Denoting $u_i(x,t) = u_i((\zeta - \mu)/2, (\zeta + \mu)/2) \equiv U_i(\zeta, \mu)$ and $h_i(x,t) = h_i((\zeta - \mu)/2, (\zeta + \mu)/2) \equiv H_i(\zeta, \mu)$ for $i = 1, 2$, we find that (3.5.21a)–(3.5.21b) imply that U_1 and H_1 satisfy

$$(H_1 + U_1)_\zeta = 0, \quad (H_1 - U_1)_\mu = 0. \tag{3.5.25}$$

The characteristic lines ζ = constant and μ = constant are sketched in Figure 3.23, and we are interested in the domain $x > 0, t > 0$ $(\zeta - \mu > 0, \zeta + \mu > 0)$. Let us denote the subdomain $x > t, t > 0$ $(\mu < 0, \zeta > 0)$ by (1), and the subdomain $x < t, t > 0$ $(\mu > 0, \zeta < 0)$ by (2).

The initial conditions (3.5.21c) imply that

$$u_1(\zeta, 0) = H_1(\zeta, -\zeta) = 0, \quad \zeta \geq 0, \tag{3.5.26}$$

and the boundary condition (3.5.21d) becomes

$$U_1(\mu, \mu) = \dot{s}(\mu), \quad \mu > 0. \tag{3.5.27}$$

The solution of (3.5.25) is simply

$$H_1 + U_1 = F_1(\mu), \tag{3.5.28a}$$

$$H_1 - U_1 = G_1(\zeta), \tag{3.5.28b}$$

where the functions F_1 and G_1 are to be determined by the initial and boundary conditions. Applying the initial conditions (3.5.26) first, we find that

$$F_1(\mu) = 0 \quad \text{if } \mu \leq 0, \tag{3.5.29a}$$

$$G_1(\zeta) = 0 \quad \text{if } \zeta \geq 0. \tag{3.5.29b}$$

Therefore, (3.5.28) reduces to $H_1 + U_1 = 0$ in (1) and $H_1 - U_1 = 0$ in (1) and (2). In particular, the preceding implies that

$$U_1(\zeta, \mu) = H_1(\zeta, \mu) = 0 \text{ in (1)} \tag{3.5.30a}$$

and that

$$H_1(\zeta, \mu) = U_1(\zeta, \mu) \text{ in (2)}. \tag{3.5.30b}$$

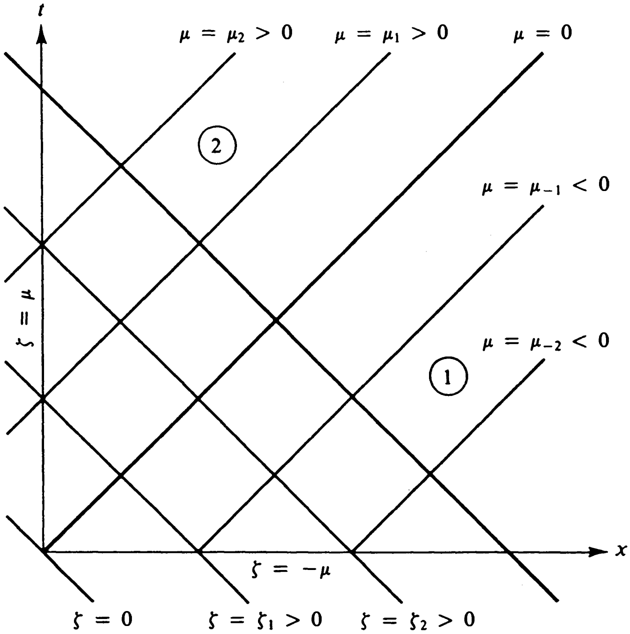

FIGURE 3.23. Characteristic lines

To complete the solution in (2), we need to calculate $F_1(\mu)$ there for $\mu > 0$. First we set $H_1 = U_1$ in (3.5.28a) to obtain

$$2U_1(\zeta, \mu) = F_1(\mu) \text{ in (2).} \tag{3.5.31}$$

Applying boundary condition (3.5.27) gives

$$2\dot{s}(\mu) = F_1(\mu) \quad \text{for } \mu > 0, \tag{3.5.32}$$

and this defines F_1 for $\mu > 0$. Therefore, (3.5.31) gives

$$U_1(\zeta, \mu) = \dot{s}(\mu) \text{ in (2),} \tag{3.5.33a}$$

and (3.5.30b) gives

$$H_1(\zeta, \mu) = \dot{s}(\mu) \text{ in (2).} \tag{3.5.33b}$$

The solution for $u_1(x, t)$ and $h_1(x, t)$ is therefore

$$u_1(x, t) = h_1(x, t) = \begin{cases} 0 & \text{if } t \geq x, \\ \dot{s}(t - x) & \text{if } t > x. \end{cases} \tag{3.5.34}$$

We shall use the same procedure for deriving U_2 and H_2, but now the inhomogeneous terms in (3.5.22) complicate the calculations slightly. These equations imply that

$$2(H_2 + U_2)_\zeta = -\left(U_1H_1 + \frac{U_1^2}{2}\right)_\zeta + \left(U_1H_1 + \frac{U_1^2}{2}\right)_\mu, \qquad (3.5.35a)$$

$$2(H_2 - U_2)_\mu = -\left(U_1H_1 - \frac{U_1^2}{2}\right)_\zeta + \left(U_1H_1 - \frac{U_1^2}{2}\right)_\mu. \qquad (3.5.35b)$$

Substituting the known solution for U_1 and H_1 gives

$$2(H_2 + U_2)_\zeta = 3\dot{s}(\mu)\ddot{s}(\mu), \qquad (3.5.36a)$$

$$2(H_2 - U_2)_\mu = \dot{s}(\mu)\ddot{s}(\mu), \qquad (3.5.36b)$$

and these equations are valid in both (1) and (2) as long as we regard

$$\dot{s}(\mu) \equiv 0 \quad \text{if } \mu \leq 0. \qquad (3.5.36c)$$

Henceforth (3.5.36c) is tacitly assumed in our calculations and results. The initial conditions (3.5.22c) transform to

$$U_2(\zeta, -\zeta) = H_2(\zeta, -\zeta) = 0, \ \zeta \geq 0, \qquad (3.5.37)$$

and when the solution (3.5.33a) is used, the boundary condition (3.5.22d) becomes

$$U_2(\mu, \mu) = \ddot{s}(\mu)s(\mu) \quad \text{for } \mu > 0. \qquad (3.5.38)$$

We integrate (3.5.36b) with respect to μ to obtain

$$H_2 - U_2 = \frac{1}{4}\dot{s}^2(\mu) + G_2(\zeta), \qquad (3.5.39a)$$

where G_2 is as yet unspecified. In view of the initial conditions (3.5.37), we must have $G_2(\zeta) \equiv 0$. Therefore, one relation between U_2 and H_2 that is valid in the entire domain of interest is

$$H_2(\zeta, \mu) - U_2(\zeta, \mu) = \frac{1}{4}\dot{s}^2(\mu). \qquad (3.5.39b)$$

We next integrate (3.5.36a) to obtain

$$H_2(\zeta, \mu) + U_2(\zeta, \mu) = \frac{3}{2}\dot{s}(\mu)\ddot{s}(\mu)\zeta + F_2(\mu). \qquad (3.5.40)$$

In preparation for evaluating the unknown function $F_2(\mu)$, we use (3.5.39b) to eliminate H_2 from (3.5.40). This gives

$$2U_2(\zeta, \mu) = -\frac{1}{4}\dot{s}^2(\mu) + \frac{3}{2}\dot{s}(\mu)\ddot{s}(\mu)\zeta + F_2(\mu). \qquad (3.5.41)$$

Imposing the boundary condition (3.5.38) on (3.5.41) defines $F_2(\mu)$ to be

$$F_2(\mu) = \frac{1}{4}\dot{s}^2(\mu) - \frac{3}{2}\dot{s}(\mu)\ddot{s}(\mu)\mu + 2\ddot{s}(\mu)s(\mu). \qquad (3.5.42)$$

Now (3.5.41) gives

$$U_2(\zeta, \mu) = \frac{3}{4}\dot{s}(\mu)\ddot{s}(\mu)(\zeta - \mu) + \ddot{s}(\mu)s(\mu), \qquad (3.5.43a)$$

and (3.5.39b) gives

$$H_2(\zeta, \mu) = \frac{1}{4}\dot{s}^2(\mu) + \frac{3}{4}\dot{s}(\mu)\ddot{s}(\mu)(\zeta - \mu) + \ddot{s}(\mu)s(\mu). \qquad (3.5.43b)$$

This completes the solution of the $O(\epsilon^2)$ problem, which has the following form in terms of the physical variables:

$$u_2(x,t) = \begin{cases} 0, & \text{if } t \leq x, \\ \frac{3}{2}\dot{s}(t-x)\ddot{s}(t-x)x + \ddot{s}(t-x)s(t-x) & \text{if } t > x, \end{cases} \qquad (3.5.44a)$$

$$h_2(x,t) = \begin{cases} 0 & \text{if } t \leq x, \\ \frac{1}{4}\dot{s}^2(t-x) + \frac{3}{2}\dot{s}(t-x)\ddot{s}(t-x)x & \\ \quad +\ddot{s}(t-x)s(t-x) & \text{if } t > x. \end{cases} \qquad (3.5.44b)$$

Suppose the wavemaker has a periodic motion—say $s(t) \equiv \sin \omega t$ with $\omega =$ constant. We see from (3.5.44) that each of our results for u or h to $O(\epsilon^2)$ contains the term $\frac{3}{2}\epsilon^2[x\dot{s}(t-x)\ddot{s}(t-x)] = -\frac{3}{2}\epsilon^2[x\omega^2 \cos\omega(t-x)\sin\omega(t-x)]$, which has an amplitude equal to $\frac{3}{2}\epsilon^2 x\omega^3$. Thus, no matter how small ϵ is, this term, which is nominally $O(\epsilon^2)$, will grow to be of order ϵ if x is as large as $O(\epsilon^{-1})$. This result is not only physically inconsistent with a periodic boundary disturbance, but it also violates the implicit ordering of terms assumed in the expansion (3.2.28). Thus, the solution we have calculated is not valid in the "far field"; it is reasonable only as long as $x = O(1)$. An expansion procedure that remains valid in the far field for the signaling problem is discussed in Section 6.2.4 of [26].

(ii) *Green's function*

Let us now rederive the preceding results using the Green's function approach discussed in Sections 3.5.1–3.5.3. As pointed out in Section 3.2.9, eliminating h_1 gives the wave equation (3.2.40) for u_1, the initial conditions (3.2.41), and the boundary condition (3.2.42). This is just the problem discussed in Section 3.5.3 with $g = \dot{s}$, and in fact the result (3.5.20) agrees with (3.5.34) for $u_1(x,t)$. To compute h_1, we substitute the preceding value of u_1 in (3.2.43) and ensure that the possible discontinuity in u_1 and u_{1_x} at $x = t$ is taken into account by writing [see the discussion following (3.4.29)]

$$u_1(x,t) = \dot{s}(t-x)H(t-x), \qquad (3.5.45a)$$

where H is the Heaviside function. The x-derivative must then be written as

$$u_{1_x}(x,t) = -\ddot{s}(t-x)H(t-x) - \dot{s}(t-x)\delta(t-x). \qquad (3.5.45b)$$

Equation (3.2.43) becomes

$$\begin{aligned} h_1(x,t) &= \int_{\tau=0}^{t} \ddot{s}(\tau - x)H(\tau - x)d\tau + \int_{\tau=0}^{t} \dot{s}(\tau - x)\delta(\tau - x)d\tau \\ &= \begin{cases} 0 & \text{if } t \leq x, \\ \int_x^t \ddot{s}(\tau - x)d\tau + \dot{s}(0^+) = \dot{s}(t-x) & \text{if } t > x \end{cases} \\ &= \dot{s}(t-x)H(t-x), \end{aligned} \qquad (3.5.46)$$

which is also in agreement with our earlier result.

Consider now the $O(\epsilon^2)$ terms governed by (3.5.22). If we substitute the expressions (3.5.45b) and (3.5.46) for u_1 and h_1 into the right-hand sides of (3.5.22a)–(3.5.22b), we have

$$h_{2_t} + u_{2_x} = 2\dot{s}(t-x)\ddot{s}(t-x)H(t-x) + \dot{s}^2(t-x)\delta(t-x), \qquad (3.5.47a)$$

$$u_{2_t} + h_{2_x} = \dot{s}(t-x)\ddot{s}(t-x)H(t-x) + \frac{1}{2}\dot{s}^2(t-x)\delta(t-x). \qquad (3.5.47b)$$

To eliminate h_2, we differentiate (3.5.47b) with respect to t and subtract from this the derivative of (3.5.47a) with respect to x to obtain

$$u_{2_{tt}} - u_{2_{xx}} = p(t-x), \qquad (3.5.48)$$

where

$$p(t-x) \equiv 3H(t-x)\frac{d}{dt}[\dot{s}(t-x)\ddot{s}(t-x)] + 6\dot{s}(t-x)\ddot{s}(t-x)\delta(t-x) + \frac{3}{2}\dot{s}^2(t-x)\dot{\delta}(t-x). \qquad (3.5.49)$$

The initial conditions are

$$u_2(x,0) = u_{2_t}(x,0) = 0, \qquad (3.5.50)$$

and the boundary condition (3.5.22d) becomes

$$u_2(0,t) = \ddot{s}(t)s(t), \ t > 0. \qquad (3.5.51)$$

The solution of (3.5.48) with $p = 0$ and subject to the given boundary and initial conditions is again a special case of (3.5.20) with $g = \ddot{s}s$. Therefore, this part of the problem contributes the last term in (3.5.44a) for the solution of u_2. To compute the contribution of the inhomogeneous term p, it is convenient to change the variables of integration from ξ, τ to $\alpha = \tau + \xi$, $\beta = \tau - \xi$ in the general formula (3.5.5b) in order to exploit the fact that $p \equiv 0$ if $t < x$. The domain of integration is the rectangle bounded by the straight lines $\beta = 0$, $\beta = t - x$, $\alpha = t - x$, $\alpha = t + x$, as shown in Figure 3.24; this is derived from the upper Figure 3.22 by deleting the shaded region below the line $\tau = \xi$. Noting that the absolute value of the Jacobian of the transformation $(\tau, \xi) \to (\alpha, \beta)$ is $\frac{1}{2}$, (3.5.5b) takes the form

$$\begin{aligned} u(x,t) &= \frac{1}{4}\int_{\alpha=t-x}^{t+x} d\alpha \int_{\beta=0}^{t-x} p(\beta)d\beta \\ &= \frac{1}{4}\int_{\alpha=t-x}^{t+x} 3\dot{s}(t-x)\ddot{s}(t-x)d\alpha \\ &= \frac{3}{2}\dot{s}(t-x)\ddot{s}(t-x)x. \end{aligned} \qquad (3.5.52)$$

In the integration of $p(\beta)$, we have made use of the identity (see (A.1.17))

$$\int_{-\epsilon}^{\epsilon} f(\beta)\dot{\delta}(\beta)d\beta = -\dot{f}(0), \ \epsilon > 0, \qquad (3.5.53)$$

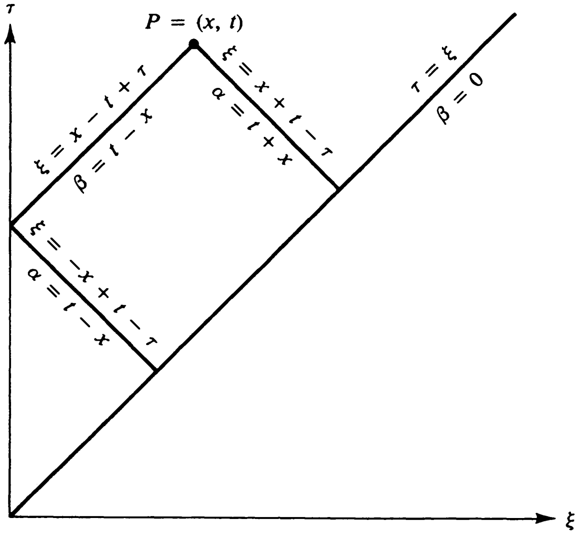

FIGURE 3.24. Domain of integration for (3.5.52)

which follows easily after integration by parts.

We note again that result (3.5.52) is in agreement with the first term in the right-hand side of (3.5.44a). The solution for h_2 follows by quadrature from (3.5.22); this calculation is left as an exercise.

The reader now has a choice of which approach to use for a signaling problem. We could use Laplace transforms with respect to t, as illustrated for the $O(\epsilon)$ solution in Problem 3.5.1. The advantage of this approach is that the solution for the transform is very easy to compute, but in general some ingenuity may be needed to invert the result and derive the explicit dependence of the solution on x, t. The method of characteristics has the advantage of a systematic and direct procedure for computing the solution. Moreover, in those cases where the equations in characteristic form cannot be solved explicitly, we have a convenient formulation for a numerical solution. This is discussed in Sections 4.2.2 and 4.3.6 for the general linear second-order problem. Finally, the result in terms of Green's function provides a compact integral expression for the solution. As we have seen, one must proceed with care to calculate an explicit result from such an expression. Also, Green's function may not be available for a more complicated problem, e.g., for a wave equation with variable coefficients, in which case a numerical solution may be the only option available.

3.5.5 A Second Example: Solution with a Fixed Interface; Reflected and Transmitted Waves

To fix ideas, consider the problem of transverse vibrations of an infinite string, half of which ($X > 0$) has a density ρ_1 and the other half ($X < 0$) has density ρ_2. The string is initially at rest with tension $\tau_0 =$ constant and is set in motion by applying a concentrated force P_0 at $T = 0$, $X/L_0 = \xi > 0$. Therefore, using the dimensionless variables in (3.1.5)–(3.1.6), and denoting $\overline{v} = \alpha_0 u(x, t) + O(\epsilon)$ in (3.1.14b) gives

$$u_{tt} - c^2 u_{xx} = \delta(x - \xi)\delta(t) \tag{3.5.54a}$$

correct to $O(\epsilon)$, where

$$c^2 = \begin{cases} c_1^2 = \rho_0/\rho_1 & \text{if } x > 0, \\ c_2^2 = \rho_0/\rho_2 & \text{if } x < 0. \end{cases} \tag{3.5.54b}$$

Note that we may choose $\rho_0 = \rho_1$ or $\rho_0 = \rho_2$ to obtain $c_1 = 1$ or $c_2 = 1$, respectively. The initial conditions are $u(x, 0^-) = u_t(x, 0^-) = 0$.

The vertical component of the tension in the deflected string at any point X and time T is $\tau_0 V_X\left(\frac{X}{L_0}, \frac{T}{T_0}\right)$, and it must be continuous everywhere, including at the interface $X = 0$. Obviously, the deflection at $X = 0$ must also be continuous. So, for this physical model, we have the following interface conditions:

$$u(0^+, t) = u(0^-, t), \tag{3.5.55a}$$

$$u_x(0^+, t) = u_x(0^-, t). \tag{3.5.55b}$$

To solve this problem, we shall use the idea of images in the right and left extended domains, as discussed in Problem 1.4.6b.

First, note that until the disturbance initiated at $t = 0$, $x = \xi$ reaches the interface $x = 0$, we must have exactly the same response as in an infinite string with $c = c_1$ throughout. This solution u_1 is just the fundamental solution (3.4.8) with $\tau \to 0$, $t \to c_1 t$, $F \to c_1 u_1$:

$$u_1(x, t) = \frac{1}{2c_1}[H(x - \xi + c_1 t) - H(x - \xi - c_1 t)]. \tag{3.5.56}$$

Now, this disturbance (a uniform wave of amplitude $1/2c_1$ spreading at speed c_1 to the left and right) propagates unchanged until the leftward-moving front reaches $x = 0$ at time $t = \xi/c_1$, as shown in Figure 3.25. At this point, we expect the interface to introduce a "reflected" disturbance propagating to the right and to allow a "transmitted" disturbance to move to the left. Since the density is constant in each of the half-spaces, the reflected and transmitted disturbances must be confined to the zone of influence of the point $P = (0, \xi/c_1)$. In the $x > 0$ portion of the zone of influence of P, we denote the solution by u_2, and in the $x < 0$ portion, we denote it by u_3.

Next, we postulate that u_2 consists of the primary disturbance u_1 defined by (3.5.56) plus a "reflected wave." In view of the region in which this reflected wave travels, we may regard it to be due to an image source of unknown strength A

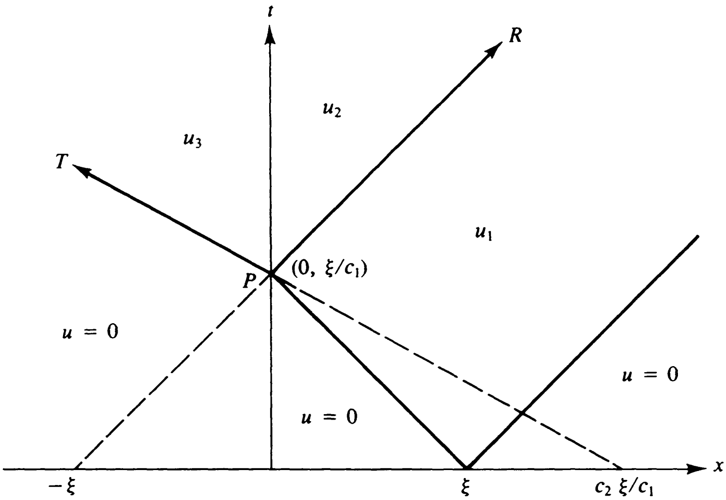

FIGURE 3.25. Primary and image sources

turned on at $t = 0$ and $x = \xi_2 = -\xi < 0$, in an infinite medium with $c = c_1$ throughout. The location of this image source is a priori obvious here (in contrast to the situation in Problem 1.4.6b) because we know that its disturbance must produce the front PR in $x > 0$, and this front propagates with speed c_1. Thus,

$$u_2(x, t) = u_1(x, t) + \frac{A}{2c_1}[H(x + \xi + c_1 t) - H(x + \xi - c_1 t)]. \quad (3.5.57)$$

Finally, we assume that the transmitted wave u_3 in $x < 0$ my be regarded as the disturbance due to an image source of unknown strength B switched on at $t = 0$ and $x = \xi_1 = c_2\xi/c_1 > 0$, in a medium with constant speed c_2 throughout. Again, the location of the image source is obvious a priori by extrapolating the front PT to the right with speed c_2. The situation depicted in Figure 3.24 corresponds to the case $(c_2/c_1) > 1(\rho_1 > \rho_2)$. The expression for u_3 is therefore given by

$$u_3(x, t) = \frac{B}{2c_2}[H(x - \xi_1 + c_2 t) - H(x - \xi_1 - c_2 t)]. \quad (3.5.58)$$

We determine A and B using the two interface conditions. Using (3.5.57)–(3.5.58) in (3.5.55a) gives

$$\frac{1}{2c_1} + \frac{A}{2c_1} = \frac{B}{2c_2}. \quad (3.5.59)$$

The evaluation of (3.5.55b), for the derivatives, requires more care. Differentiating (3.5.57) with respect to x and letting $x \to 0^+$ gives

$$u_{2_x}(0^+, t) = \frac{1}{2c_1}[\delta(-\xi + c_1 t) - \delta(-\xi - c_1 t)] + \frac{A}{2c_1}[\delta(\xi + c_1 t) - \delta(\xi - c_1 t)]. \tag{3.5.60}$$

Now, for $t > \xi/c_1$, the arguments of each of the four delta functions in (3.5.60) are nonzero, so we can obtain useful information only in the limit as $t \downarrow \xi/c_1$. In view of the fact that $\delta(-\xi + c_1 t) = \delta(\xi - c_1 t)$, we obtain

$$u_{2_x}(0^+, t) \to \frac{1 - A}{2c_1}\delta(\xi - c_1 t) \quad \text{as } t \to \left(\frac{\xi}{c_1}\right)^+. \tag{3.5.61}$$

A similar calculation for u_3 shows that

$$u_{3_x}(0^+, t) \to \frac{Bc_1}{2c_2^2}\delta(\xi - c_1 t) \quad \text{as } t \to \left(\frac{\xi}{c_1}\right)^+. \tag{3.5.62}$$

The second interface condition, (3.5.55b), then gives

$$\frac{1 - A}{2c_1} = \frac{Bc_1}{2c_2^2}. \tag{3.5.63}$$

Solving (3.5.59) and (3.5.63) for A and B gives

$$A = \frac{c_2 - c_1}{c_1 + c_2}; \quad B = \frac{2c_2^2}{c_1(c_1 + c_2)}. \tag{3.5.64}$$

We verify that the limiting cases $c_1 = c_2$ (no interface) and $c_2 = 0$ (no deflection at $x = 0$) are correctly contained in our results. In particular, for $c_1 = c_2$, we have $A = 0, B = 1$; that is, $u_3 = u_2 = u_1 = 1/2c_1$; there is no reflected wave, and the transmitted wave is just the primary wave. If $c_2 = 0$, we have $A = -1$, $B = 0$, and we see that $u_3 = u_2 = 0$, whereas $u_1 = 1/2c_1$. This is just Green's function $G_1(x, \xi, c_1 t)/c_1$ derived in (3.5.2). In Problem 3.5.3 these ideas are developed for a signaling problem with a periodic boundary condition.

3.5.6 Green's Function of the Second Kind

Green's function of the second kind solves the problem

$$u_{tt} - u_{xx} = \delta(x - \xi)\delta(t - \tau), \quad 0 \le x, \ \tau \le t, \tag{3.5.65a}$$

with zero initial conditions at $t = \tau$

$$u(x, \tau^-) = u_t(x, \tau^-) = 0, \tag{3.5.65b}$$

and a zero boundary value for u_x at $x = 0$

$$u_x(0, t) = 0; \quad t > \tau. \tag{3.5.65c}$$

Here again, $\xi = \text{constant} > 0$, and $\tau = \text{constant}$.

By symmetry, we need to add the solutions due to a positive unit source at $x = \xi, t = \tau$ and an image unit source that is also positive at $x = -\xi, t = \tau$. The result is

$$G_2(x, \xi, t - \tau) \equiv F(x - \xi, t - \tau) + F(x + \xi, t - \tau), \tag{3.5.66}$$

where F is defined by (3.4.8). The zone of influence of the source point (ξ, τ) is the same as the one for G_1 sketched in Figure 3.21. Now, however, $G_2 = 1$ in the $x > 0$ part of region (4). To solve boundary-value problems with $u_x = 0$ or u_x prescribed at $x = 0$, we follow the procedures discussed in Sections 3.5.2 and 3.5.3 (see Problem 3.5.4).

Problems

3.5.1a. Use Laplace transforms with respect to t to solve (3.5.16). Distinguish carefully the two cases $x > t$ and $x < t$.
Hint: Note that the Laplace transform of $\delta(t - x)$ is e^{-sx}, and use the convolution theorem (A.2.44) to invert your result.

b. A slight variation of the signaling problem discussed in Section 3.2.8 has the wavemaker speed proportional to the instantaneous local surface perturbation. The boundary condition that replaces (3.2.42) for the linearized problem is now

$$u_1(0, t) = ch_1(0, t), \quad t > 0, \tag{3.5.67}$$

where c is a constant. Assume that $c \neq 1$.

Use Laplace transforms with respect to t to solve the signaling problem governed by (3.2.30), the boundary condition (3.5.67), and the initial conditions

$$u_1(x, 0) = 0, \tag{3.5.68a}$$

$$h_1(x, 0) = a \sin x, \tag{3.5.68b}$$

with a = constant. Once you have studied Section 3.5.4, discuss why the case $c = 1$ is ill-posed.

3.5.2 Consider the general linear boundary-value problem

$$u_{tt} - u_{xx} = 0, \quad 0 \leq x < \infty, \quad t \geq 0, \tag{3.5.69a}$$

$$u_x(0, t) + bu(0, t) = c(t), \quad t > 0, \tag{3.5.69b}$$

$$u(x, 0) = u_t(x, 0) = 0, \tag{3.5.69c}$$

where b is a constant and $c(t)$ is a prescribed analytic function of t with $c(0) = 0$, and $c(t) = 0$ if $t < 0$.

a. Transform the dependent variable $u \to v$ using

$$v(x, t) \equiv u_x(x, t) + bu(x, t), \tag{3.5.70}$$

and show that if u satisfies (3.5.69), then v is governed by

$$v_{tt} - v_{xx} = 0, \quad 0 \le x < \infty, \quad t \ge 0, \tag{3.5.71a}$$

$$v(0,t) = c(t), \tag{3.5.71b}$$

$$v(x,0) = v_t(x,0) = 0. \tag{3.5.71c}$$

Therefore, according to (3.5.20), the solution of (3.5.71) for v is

$$v(x,t) = \begin{cases} 0 & \text{if } t < x, \\ c(t-x) & \text{if } t > x. \end{cases} \tag{3.5.72}$$

b. Using (3.5.72) for v, solve (3.5.70) for u in the form

$$u(x,t) = \alpha(t)e^{-bx} + e^{-bx}\int_t^x c(t-\xi)e^{b\xi}\,d\xi, \tag{3.5.73}$$

where α is as yet unspecified.

c. Substitute (3.5.73) into (3.5.69a) and use the initial conditions (3.5.69c) to prove that $\alpha(t) \equiv 0$. Therefore, the desired solution is

$$u(x,t) = e^{-bx}\int_t^x c(t-\xi)e^{b\xi}\,d\xi. \tag{3.5.74}$$

d. Derive (3.5.74) directly from (3.5.69) using Laplace transformation with respect to t.

3.5.3 Consider the signaling problem on $0 \le x < \infty$ for the wave equation

$$u_{tt} - c^2 u_{xx} = 0, \tag{3.5.75}$$

where $c \equiv c_1 = \text{constant} > 0$ on $0 \le x < 1$, and $c \equiv c_2 = \text{constant} > c_1$ on $1 < x < \infty$. The initial conditions are $u(x,0) = 0$, $u_t(x,0) = 0$, and the boundary condition at $x = 0$ is the periodic signal $u(0,t) = A \sin \omega t$ for $t > 0$, where A and ω are given constants. Derive the solution in terms of appropriate primary, reflected, and transmitted waves, using the ideas in Section 3.5.5 and the interface conditions (3.5.55).

3.5.4 Use Green's function G_2, defined by (3.5.66), or symmetry to solve

$$u_{tt} - u_{xx} = p(x,t), \quad 0 \le x, \quad 0 \le t, \tag{3.5.76a}$$

$$u(x,0) = f(x), \quad u_t(x,0) = h(x), \tag{3.5.76b}$$

$$u_x(0,t) = k(t), \quad t > 0. \tag{3.5.76c}$$

3.5.5 In this problem we study the one-dimensional wave equation over the time-dependent domain D: $\alpha t \le x < \infty$, $t \ge 0$, where the constant α is restricted to $0 < \alpha < 1$.

a. Calculate Green's function of the first kind; that is, solve

$$u_{tt} - u_{xx} = \delta(x-\xi)\delta(t-\tau) \tag{3.5.77a}$$

in D for a constant $\xi > 0$ such that $\alpha\tau < \xi$. The initial conditions are

$$u(x,0^-) = u_t(x,0^-) = 0, \tag{3.5.77b}$$

and the boundary condition is

$$u(\alpha t, t) = 0. \tag{3.5.77c}$$

Hint: Introduce the transformation

$$\bar{x} = x - \alpha t, \ \bar{t} = t \tag{3.5.78}$$

so that the left boundary is at the fixed point $\bar{x} = 0$ in the new frame, and solve the resulting problem by Laplace transforms with respect to $\bar{t}$.
Note: The inversion integral can be evaluated explicitly.

b. Use the result in (a) and follow the procedure in Section 3.5.3 to solve the inhomogeneous boundary-value problem

$$u_{tt} - u_{xx} = 0, \tag{3.5.79a}$$

$$u(x, 0) = u_t(x, 0) = 0, \tag{3.5.79b}$$

$$u(\alpha t, t) = \begin{cases} 1 & \text{if } 0 < t < T, \\ 0 & \text{if } T < t, \end{cases} \tag{3.5.79c}$$

in D, where T is a positive constant.

c. Now calculate the solution of (3.5.79) by first transforming it to the $\bar{x}$, $\bar{t}$ variables defined by (3.5.78) and then using Laplace transforms to solve the resulting problem.

3.6 Initial- and Boundary-Value Problems on $0 \leq x \leq 1$

As in Section 1.5, we derive Green's function by introducing an infinite array of sources located along the x-axis in such a way as to satisfy homogeneous boundary conditions at the endpoints of a finite interval.

3.6.1 Green's Function of the First Kind on $0 \leq x \leq 1$

As in Chapter 1, we have four Green's functions on the unit interval $0 \leq x \leq 1$, depending on whether $u = 0$ or $u_x = 0$ at each end. The formulas are identical to those in (1.5.2) and Problem 1.5.2 with regard to the dependence on the fundamental solution and are therefore not repeated (see Problem 3.6.1). Of course, in the present case F is defined by (3.4.8).

Consider, for example, G_1, which solves

$$u_{tt} - u_{xx} = \delta(x - \xi)\delta(t - \tau), \quad 0 \leq x \leq 1, \quad \tau \leq t, \tag{3.6.1}$$

with $0 < \xi = \text{constant} < 1$ and $\tau = \text{constant}$. The initial conditions are $u(x, \tau^-) = u_t(x, \tau^-) = 0$, and the boundary conditions are $u(0, t) = u(1, t) = 0$ for $t > \tau$. Henceforth, a problem for which u is specified at both ends will be called a boundary-value problem of the *first kind.*

As in (1.5.2), we have

$$G_1(x, \xi, t-\tau) = \sum_{n=-\infty}^{\infty} \{F(x-2n-\xi, t-\tau) - F(x-2n+\xi, t-\tau)\}, \quad (3.6.2)$$

where F is defined by (3.4.8). It is also easy to show that Green's function satisfies the symmetry property $G_1(x, \xi, t-\tau) = G_1(\xi, x, t-\tau)$ here also. If we now deploy a triangular zone of influence for each source in (3.6.2), as discussed in Section 3.4.1, we obtain the pattern sketched in Figure 3.26.

Notice that unlike the situation in Chapter 1, where the effect of the sources in the far field was negligible initially, here *any source can be exactly ignored* until the first signal from it arrives, but then this signal is unattenuated and cannot be ignored, no matter how far the source is. Fortunately, we also notice that the pattern of disturbances in $0 \leq x \leq t$ *is an odd periodic function of* $t - \tau$ with period 2; thus, we need only define the value of G_1 in the time interval $0 \leq t - \tau \leq 1$ to have it for any other time.

Next, we illustrate the application of these results to boundary-value problems of the first kind. Other applications involving G_2, G_3, G_4, and various associated boundary-value problems are not catalogued. An example is outlined in Problem 3.6.1b.

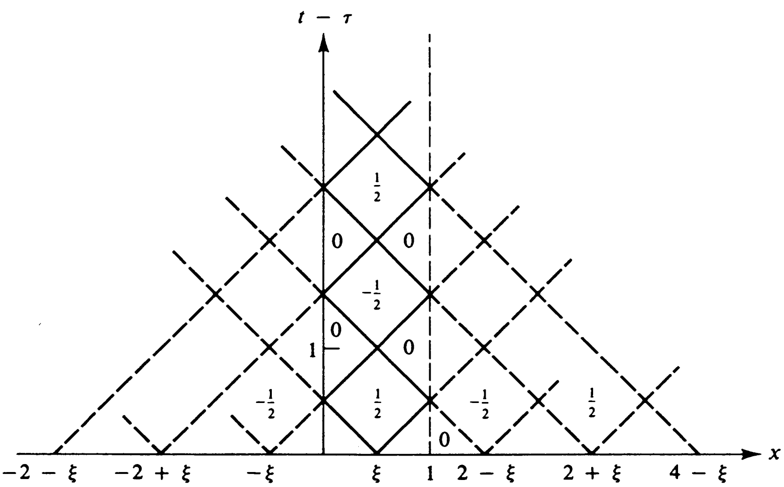

FIGURE 3.26. Sources and their zones of influence for G_1

3.6.2 The Inhomogeneous Problem, Nonzero Initial Conditions

First, we consider the inhomogeneous wave equation

$$u_{tt} - u_{xx} = p(x,t), \quad 0 \le x \le 1, \quad 0 \le t, \tag{3.6.3a}$$

with zero initial conditions

$$u(x,0^-) = u_t(x,0^-) = 0 \tag{3.6.3b}$$

and homogeneous boundary conditions of the first kind:

$$u(0,t) = u(1,t) = 0, \quad t > 0. \tag{3.6.3c}$$

The solution follows immediately by superposition in terms of Green's function G_1 of (3.6.2) in the form

$$u(x,t) = \int_0^t d\tau \left[\int_0^1 G_1(x,\xi,t-\tau)p(\xi,\tau)d\xi \right]. \tag{3.6.4}$$

As in Section 3.5.2, we can use this result to also solve the homogeneous wave equation with homogeneous boundary conditions of the first kind but a nonzero initial condition. For example, note that

$$u_{tt} - u_{xx} = 0, \quad 0 \le x \le 1, \quad 0 \le t, \tag{3.6.5a}$$

$$u(x,0^+) = 0, \quad u_t(x,0^+) = h(x), \tag{3.6.5b}$$

$$u(0,t) = u(1,t) = 0, \quad t > 0, \tag{3.6.5c}$$

is equivalent to

$$u_{tt} - u_{xx} = \delta(t)h(x), \quad 0 \le x \le 1, \quad 0 \le t, \tag{3.6.6a}$$

$$u(x,0^-) = u_t(x,0^-) = 0, \tag{3.6.6b}$$

$$u(0,t) = u(1,t) = 0, \quad t > 0. \tag{3.6.6c}$$

Therefore, using (3.6.4) with $p = \delta(t)h(x)$, the solution can be written in the form

$$u(x,t) = \int_0^1 h(\xi)G_1(x,\xi,t)d\xi. \tag{3.6.7}$$

As in the case of the diffusion equation, we can derive a second representation of the solution using separation of variables. In fact (see Problem 3.6.2a), this is obtained in the form

$$u(x,t) = \int_0^1 h(\xi)K(x,\xi,t)d\xi, \tag{3.6.8}$$

where

$$K(x,\xi,t) \equiv \sum_{n=1}^{\infty} \frac{\sin n\pi t}{n\pi}[\cos n\pi(x-\xi) - \cos n\pi(x+\xi)]. \tag{3.6.9}$$

It is easy to show that for any fixed x and ξ, K is just the Fourier series in $\sin n\pi t$ of the odd periodic function of t defined by the rectangular pattern for G_1 in the xt-plane.

A third representation of the solution of (3.6.5) may be derived using Laplace transforms (Problem 3.6.2b). Just as in the case of the diffusion equation (Problems 1.5.4–1.5.5), when the inversion integral is approximated for s *large*, it reduces to (3.6.7).

In summary, we see that if we take account of all the sources that contribute to the value of u at a given time t, (3.6.7) is *exact*. The infinite series (3.6.9) converges to G_1 everywhere except at the discontinuities on the edges of the rectangles where G_1 changes value. Here the infinite series converges to one-half the value of the jump. Truncating the series (3.6.9) will result in an approximation that is valid for t *large*, and the truncated series exactly satisfies the boundary conditions at $x = 0$ and $x = 1$.

The other possibility for a nonzero initial condition has

$$u_{tt} - u_{xx} = 0, \quad 0 \le x \le 1, \quad 0 \le t, \tag{3.6.10a}$$

$$u(x, 0^+) = f(x), \quad u_t(x, 0^+) = 0, \tag{3.6.10b}$$

$$u(0, t) = u(1, t) = 0, \quad t > 0. \tag{3.6.10c}$$

As in Section 3.5.2, we change variables to $v(x, t)$ defined by (3.5.12), and note that $v(x, t)$ obeys

$$v_{tt} - v_{xx} = 0, \tag{3.6.11a}$$

$$v(x, 0^+) = 0, \quad v_t(x, 0^+) = f(x), \tag{3.6.11b}$$

$$v(0, t) = v(1, t) = 0, \quad t > 0. \tag{3.6.11c}$$

Therefore, $v(x, t)$ is defined by (3.6.7) with $h = f$, and u is given by v_t; that is,

$$u(x, t) = \int_0^1 f(\xi) \frac{\partial G_1}{\partial t}(x, \xi, t) d\xi. \tag{3.6.12}$$

Since G_1 is a series of Heaviside functions, the expressions in (3.6.7) and (3.6.12) may be developed further. One may also derive these results by appropriate extensions of the initial data on the entire x-axis (Problem 3.6.3).

3.6.3 Inhomogeneous Boundary Conditions

Consider now the case where $u(0, t)$ and $u(1, t)$ are prescribed functions of time. We have

$$u_{tt} - u_{xx} = 0, \tag{3.6.13a}$$

$$u(x, 0^+) = u_t(x, 0^+) = 0, \tag{3.6.13b}$$

$$u(0, t) = g(t), \quad t > 0, \tag{3.6.13c}$$

$$u(1, t) = l(t), \quad t > 0. \tag{3.6.13d}$$

We pick the simple linear (in x) transformation to a new dependent variable $w(x, t)$ that satisfies zero boundary conditions at the endpoints:

$$w(x, t) \equiv u(x, t) + [g(t) - l(t)]x - g(t). \tag{3.6.14}$$

If $u(x, t)$ solves (3.6.13), then $w(x, t)$ must satisfy

$$w_{tt} - w_{xx} = (\ddot{g} - \ddot{l})x - \ddot{g} \equiv p(x, t), \tag{3.6.15a}$$

$$w(x, 0) = [g(0) - l(0)]x - g(0) \equiv f(x), \tag{3.6.15b}$$

$$w_t(x, 0) = [\dot{g}(0) - \dot{l}(0)]x - \dot{g}(0) \equiv h(x), \tag{3.6.15c}$$

$$w(0, t) = w(1, t) = 0, \quad t > 0. \tag{3.6.15d}$$

Therefore, the solution for $w(x, t)$ is obtained by combining (3.6.4), (3.6.7), and (3.6.12) for the expressions p, f, and h defined in (3.6.15a)–(3.6.15c). Having defined w, u follows from (3.6.14).

3.6.4 Uniqueness of the General Initial- and Boundary-Value Prolem of the First Kind

The most general initial- and boundary-value problem of the first kind on the unit interval combines (3.6.3), (3.6.5), (3.6.10), and (3.6.13), that is,

$$u_{tt} - u_{xx} = p(x, t), \quad 0 \leq x \leq 1, \quad 0 \leq t, \tag{3.6.16a}$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = h(x), \tag{3.6.16b}$$

$$u(0, t) = g(t), \quad u(1, t) = l(t), \quad t > 0. \tag{3.6.16c}$$

We shall now prove that (3.6.16) has a unique solution. As a preliminary step in the proof, we show that the "energy" associated with *any* solution of the homogeneous wave equation with homogeneous boundary conditions is constant. More precisely, let $\phi(x, t)$ denote any solution of

$$\phi_{tt} - \phi_{xx} = 0 \tag{3.6.17a}$$

satisfying the homogeneous boundary conditions

$$\phi(0, t) = \phi(1, t) = 0, \quad t > 0. \tag{3.6.17b}$$

To calculate the total energy, we proceed as in dynamics and multiply (3.6.17a) by the "velocity" ϕ_t and integrate with respect to x over $0 \leq x \leq 1$. This gives

$$\int_0^1 [\phi_t \phi_{tt} - \phi_{xx}\phi_t] dx = 0. \tag{3.6.18}$$

The first term in (3.6.18) can be written as

$$\int_0^1 \phi_t \phi_{tt} dx = \frac{d}{dt} \int_0^1 \frac{\phi_t^2}{2} dx, \tag{3.6.19a}$$

and is just the time rate of change of the kinetic energy. Integrating the second term in (3.6.18) by parts gives

$$-\int_0^1 \phi_t \phi_{xx} dx = -\phi_x \phi_t \Big|_0^1 + \int_0^1 \phi_x \phi_{xt} dx.$$

But $\phi(0, t) = \phi(1, t) = 0$; therefore, $\phi_t(0, t) = \phi_t(1, t) = 0$, and we obtain

$$-\int_0^1 \phi_t \phi_{xx} dx = \int_0^1 \phi_x \phi_{xt} dx = \frac{d}{dt} \int_0^1 \frac{\phi_x^2}{2} dx, \tag{3.6.19b}$$

which is just the time rate of change of the potential energy.

Therefore, we have shown that along any solution $\phi(x, t)$ of (3.6.17), the total energy, defined as

$$E(t) \equiv \frac{1}{2} \int_0^1 (\phi_t^2 + \phi_x^2) dx = E(0), \tag{3.6.20}$$

is actually a constant equal to its initial value. The corresponding result for a nonlinear wave equation is discussed in Problem 3.6.7.

Now return to problem (3.6.16), and assume that there are *two solutions*, $u_1(x, t)$ and $u_2(x, t)$, which satisfy all the conditions (3.6.16). Denote $\phi(x, t) \equiv u_1(x, t) - u_2(x, t)$. Clearly, $\phi(x, t)$ satisfies (3.6.17); therefore, E = constant = $E(0)$. But since $\phi(x, 0) = \phi_t(x, 0) = 0$, we also have $E(0) = 0$. Therefore, the integral in (3.6.20) vanishes. Now, the integrand in (3.6.20) is nonnegative, so we conclude that $\phi_t(x, t) \equiv 0$ and $\phi_x(x, t) \equiv 0$. This means that $\phi(x, t)$ is a constant, and we may evaluate this constant at any (x, t), say $(x, 0)$, where $\phi = 0$. Therefore, $\phi(x, t) \equiv 0$, or $u_1 = u_2$.

Problems

3.6.1a Calculate and sketch the remaining three Green's functions [in addition to (3.6.2)] for the wave equation on $0 \le x \le 1$. Denote, as in Problem 1.5.2,

$$G_2(x, \xi, t) : G_2(0, \xi, t) = G_{2_x}(1, \xi, t) = 0, \tag{3.6.21}$$

$$G_3(x, \xi, t) : G_{3_x}(0, \xi, t) = G_3(1, \xi, t) = 0, \tag{3.6.22}$$

$$G_4(x, \xi, t) : G_{4_x}(0, \xi, t) = G_{4_x}(1, \xi, t) = 0. \tag{3.6.23}$$

b. Use G_3 after transformation to homogeneous boundary conditions to solve

$$u_{tt} - u_{xx} = 0, \quad 0 \le x \le 1, \quad 0 \le t, \tag{3.6.24a}$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = 0, \tag{3.6.24b}$$

$$u_x(0, t) = g(t), \quad u(1, t) = l(t), \quad t > 0. \tag{3.6.24c}$$

3.6.2a Use separation of variables to derive (3.6.8), and show that (3.6.9) is just the Fourier series of G_1 as defined by (3.6.2).

b. Solve (3.6.5) using Laplace transforms with respect to time. For $h = 1$, show that if the inversion integral is approximated for s large, one obtains the same result as (3.6.7).

3.6.3 Evaluate (3.6.12) for the case

$$f(x) = \begin{cases} 1 + x & \text{on } 0 \leq x \leq \frac{1}{3}, \\ 2(1 - x) & \text{on } \frac{1}{3} \leq x \leq 1. \end{cases} \tag{3.6.25}$$

Then show that the same result follows from (3.4.18) by extending the definition of $f(x)$ over the entire x-axis in a manner that is appropriate for the boundary conditions (3.6.10d).

3.6.4 To what extent can we carry out the uniqueness proof of Section 3.6.4 to the case where $u(x, t)$ obeys (3.6.16a)–(3.6.16b) and the following three possible pairs of inhomogeneous boundary conditions?

$$u_x(0, t) = \text{prescribed}, \quad u_x(1, t) = \text{prescribed}, \tag{3.6.26}$$

$$u_x(0, t) = \text{prescribed}, \quad u(1, t) = \text{prescribed}, \tag{3.6.27}$$

$$u(0, t) = \text{prescribed}, \quad u_x(1, t) = \text{prescribed}. \tag{3.6.28}$$

3.6.5 Consider shallow-water waves in a finite basin generated by an initial discontinuity of the surface height. The dimensionless problem for the perturbation quantities (see Section 3.2 and drop subscripts) is

$$h_t + u_x = 0, \quad u_t + h_x = 0, \quad 0 \leq x \leq \pi, \quad 0 \leq t. \tag{3.6.29a}$$

The initial conditions are

$$u(x, 0) = 0, \tag{3.6.29b}$$

$$h(x, 0) = \begin{cases} \tilde{h} = \text{constant} & \text{if } 0 \leq x < \xi, \\ 0 & \text{if } \xi < x \leq \pi. \end{cases} \tag{3.6.29c}$$

The boundary conditions are $u(0, t) = u(\pi, t) = 0$ for $t > 0$. Solve the wave equation for u using separation of variables and using Green's functions.

3.6.6 Consider shallow-water waves in a finite basin where the flow is initially at rest and the left wall is set in small-amplitude motion at $t = 0$. The dimensionless problem is governed by (3.6.29a), with the initial condition $u(x, 0) = h(x, 0) = 0$ and the boundary conditions

$$u(0, t) = g(t), \quad u(\pi, t) = 0, \quad t > 0. \tag{3.6.30}$$

Homogenize the boundary conditions for the wave equation for u and solve the resulting problem using separation of variables. Specialize to the case $g(t) = \sin \omega t$ and exhibit the resonant growth of the amplitude of waves if ω is an integer.

3.6.7 Consider the nonlinear wave equation

$$u_{tt} - u_{xx} + F(u) = 0, \quad 0 \leq x \leq \pi, \quad 0 \leq t, \tag{3.6.31a}$$

where F is a given function with $F(0) = 0$, $F(-u) = -F(u)$, and $F > 0$ if $u > 0$. The boundary conditions are

$$u(0, t) = u(\pi, t) = 0, \quad t > 0, \tag{3.6.31b}$$

and the initial conditions are

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x). \tag{3.6.31c}$$

a. Multiply (3.6.31a) by u_t and integrate the result with respect to x over $0 \leq x \leq \pi$ to derive the energy integral

$$\begin{aligned} &\int_0^\pi \left[\frac{1}{2}(u_t^2 + u_x^2) + G(u)\right] dx \\ &= \int_0^\pi \left[\frac{1}{2}g^2(x) + \frac{1}{2}f'^2(x) + G(g(x))\right] dx \\ &\equiv E = \text{constant}, \end{aligned} \tag{3.6.32}$$

where $G(u)$ is the potential defined by $dG/du = F(u)$.

b. Assume a solution of (3.6.31) in the form of a series of the eigenfunctions of the linear problem, that is,

$$u(x, t) = \sum_{n=1}^{\infty} q_n(t) \sin nx. \tag{3.6.33}$$

Substituting the series into the wave equation (3.6.31a), show that the $q_n(t)$ obey a system of *coupled* nonlinear oscillator equations of the form

$$\frac{d^2 q_n}{dt^2} + n^2 q_n + F_n(q_1, q_2, \ldots) = 0, \quad n = 1, 2, \ldots. \tag{3.6.34}$$

Express the F_n in (3.6.34) as the Fourier coefficients of F, and show in particular that if $F = u$, then $F_n = q_n$. What is F_n if $F = u + \epsilon u^2$ and ϵ is a constant?

3.7 Effect of Lower-Derivative Terms

In many applications, the linearized wave equation that we derive in the limit of small disturbances involves terms proportional to u, u_x, and u_t. It will be shown in Chapter 4 that

$$u_{tt} - u_{xx} + au_x + bu_t + cu = 0 \tag{3.7.1}$$

is a general form (see (4.1.17)), where a, b, and c are functions of x and t. Here we restrict attention to the case where a, b, and c are constants.

For example, with $a = b = 0, c > 0$, we could interpret (3.7.1) as the equation of a vibrating string on an elastic support (see (3.1.15)). With $a = c = 0$ and $b > 0$, (3.7.1) is the "telegraph equation," and u gives the voltage along an electric transmission line in appropriate dimensionless variables. More generally, consider the oscillations of a chain or cable suspended from one end in the vertical direction. Here, the equilibrium tension in the cable is due to its weight and increases linearly with height from the value zero at the free end. Thus, the basic wave equation for the transverse motion is

$$u_{tt} - (xu_x)_x = 0. \tag{3.7.2}$$

3.7.1 Transformation to D'Alembert Form; Removal of Lower-Derivative Terms

We introduce the characteristic variables (see (3.5.23))

$$\zeta \equiv t + x, \ \mu = -\sigma \equiv t - x \tag{3.7.3a}$$

and denote

$$u(x, t) = u\left(\frac{\zeta - \mu}{2}, \frac{\zeta + \mu}{2}\right) \equiv U(\zeta, \mu). \tag{3.7.3b}$$

We then calculate

$$u_x = U_\zeta - U_\mu, \ u_t = U_\zeta + U_\mu, \tag{3.7.4a}$$

$$u_{xx} = U_{\zeta\zeta} - 2U_{\zeta\mu} + U_{\mu\mu}, \ u_{tt} = U_{\zeta\zeta} + 2U_{\zeta\mu} + U_{\mu\mu}. \tag{3.7.4b}$$

Therefore, after dividing by 4, (3.7.1) transforms to

$$U_{\zeta\mu} + \frac{(b+a)}{4}U_\zeta + \frac{(b-a)}{4}U_\mu + \frac{c}{4}U = 0. \tag{3.7.5}$$

We can remove the terms depending on U_ζ and U_μ by transforming the dependent variable $U \to W$ according to

$$U(\zeta, \mu) \equiv W(\zeta, \mu)\exp\left[-\frac{(b-a)\zeta}{4} - \frac{(b+a)\mu}{4}\right]. \tag{3.7.6}$$

If U obeys (3.7.5), it is easily seen that W obeys

$$W_{\zeta\mu} + \lambda W = 0, \tag{3.7.7a}$$

where λ is the constant

$$\lambda = \frac{a^2 - b^2 + 4c}{16}. \tag{3.7.7b}$$

Note that if $\lambda \neq 0$, we no longer have the simple D'Alembert solution (3.4.4). However, the canonical form (3.7.7) is still easier to solve than the original equation, (3.7.1). In Chapter 4 we shall derive the equation corresponding to (3.7.5) when a, b, and c are functions of x and t (see 4.1.15).

3.7.2 Fundamental Solution; Stability

The fundamental solution of (3.7.7a), with the proper source strength to correspond to a unit source $\delta(x)\delta(t)$ applied to the right-hand side of (3.7.1), can be derived by similarity (Problem 3.7.1). The result for $\lambda > 0$ is

$$W(\zeta, \mu) = \begin{cases} \frac{1}{2} J_0(2\sqrt{\lambda\zeta\mu}) & \text{if } \zeta > 0, \mu > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{3.7.8a}$$

where J_0 is the Bessel function of the first kind of order zero. For $\lambda < 0$, we have

$$W(\zeta, \mu) = \begin{cases} \frac{1}{2} I_0(2\sqrt{-\lambda\zeta\mu}) & \text{if } \zeta > 0, \mu > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{3.7.8b}$$

where I_0 is the modified Bessel function of the first kind of order zero. Having defined W, we obtain u from (3.7.3) and (3.7.6). Thus, if $t < |x|$, $u = 0$, and if $t > |x|$, we have

$$u = \begin{cases} \rho(x, t) J_0(2\sqrt{\lambda(t^2 - x^2)}) & \text{if } \lambda > 0, \\ \rho(x, t) I_0(2\sqrt{-\lambda(t^2 - x^2)}) & \text{if } \lambda < 0, \end{cases} \tag{3.7.9a}$$

where

$$\rho(x, t) = \frac{1}{2} \exp\left(\frac{ax - bt}{2}\right). \tag{3.7.9b}$$

Since $\lambda = 0$ if $a = b = c = 0$ and $J_0(0) = I_0(0) = 1$, we recover the result in (3.4.6) for this special case. For arbitrary values of a, b, and c, the behavior of the solution in the far field ($|x|$ and t large) depends on the behavior of J_0 and I_0 and the relative importance of these compared with $\rho(x, t)$.

Using standard tables (for example, (9.2.1) and (9.7.1) of [3]), we find that for z real,

$$J_0(z) = \left(\frac{2}{\pi z}\right)^{1/2} \left[\cos\left(z - \frac{\pi}{4}\right) + O(z^{-1})\right] \quad \text{as } z \to \infty, \tag{3.7.10a}$$

$$I_0(z) = \frac{e^z}{(2\pi z)^{1/2}} [1 + O(z^{-1})] \quad \text{as } z \to \infty. \tag{3.7.10b}$$

Consider first the limit $t \to \infty$, x fixed. We have

$$2\sqrt{\pm\lambda(t^2 - x^2)} = 2t\sqrt{\pm\lambda} + O(t^{-1}).$$

Therefore, in this limit we have

$$J_0(2\sqrt{\lambda(t^2 - x^2)}) = \frac{1}{\sqrt{\pi t\sqrt{\lambda}}} \cos\left(2t\sqrt{\lambda} - \frac{\pi}{4}\right) + \dots, \tag{3.7.11a}$$

$$I_0(2\sqrt{-\lambda(t^2 - x^2)}) = \frac{1}{2\sqrt{\pi t\sqrt{-\lambda}}} \exp(2t\sqrt{-\lambda}) + \dots. \tag{3.7.11b}$$

Using (3.7.9a) we obtain the following behavior for u as $t \to \infty$, x fixed:

$$u(x, t) = \frac{1}{2}\sqrt{\frac{1}{\pi + \sqrt{\lambda}}} \exp\left(\frac{ax - bt}{2}\right) \cos\left(2t\sqrt{\lambda} - \frac{\pi}{4}\right) + \dots \tag{3.7.12a}$$

if $\lambda > 0$, and

$$u(x,t) = \frac{1}{4\sqrt{\pi t\sqrt{-\lambda}}} \exp\left[\frac{ax + \left(\sqrt{b^2 - a^2 - 4c} - b\right)t}{2}\right] + \dots \quad (3.7.12b)$$

if $\lambda < 0$. We also have the exact result

$$u(x,t) = \frac{1}{2}\exp\left(\frac{ax - bt}{2}\right) \text{ if } \lambda = 0. \quad (3.7.12c)$$

It follows from (3.7.12a) and the definition (3.7.7b) of λ that for $a^2 - b^2 - 4c \geq 0$, the fundamental solution is bounded if $b \geq 0$ and unbounded if $b < 0$. For $a^2 - b^2 + 4c < 0$, (3.7.12b) implies that the fundamental solution is bounded if $b - \sqrt{b^2 - a^2 - 4c} \geq 0$ and unbounded if $b - \sqrt{b^2 - a^2 - 4c} < 0$. In this latter case, because $0 \leq \sqrt{b^2 - a^2 - 4c} \leq b^2$, the fundamental solution is again bounded if $b \geq 0$ and unbounded if $b < 0$. Thus, for fixed x, a necessary and sufficient condition for a bounded fundamental solution as $t \to \infty$ is to have

$$b \geq 0. \quad (3.7.13)$$

It is more meaningful to consider the behavior of the fundamental solution as both $|x|$ and t become large at the same rate, i.e., as $t \to \infty$ either along the $\mu = \mu_0 =$ constant characteristics or the $\zeta = \zeta_0 =$ constant characteristics. To evaluate the first limit ($\zeta \to \infty$, $\mu = \mu_0 =$ constant), we set $x = t - \mu_0$ in the fundamental solution and let $t \to \infty$. In the second limit ($\mu \to \infty$, $\zeta = \zeta_0 =$ constant), we set $x = \zeta_0 - t$ and let $t \to \infty$.

We now have

$$2\sqrt{\pm\lambda(t^2 - x^2)} = \begin{cases} 2\sqrt{\pm\lambda(2\mu_0 t - \mu_0^2)} & \text{if } \mu = \mu_0, \\ 2\sqrt{\pm\lambda(2\zeta_0 t - \zeta_0^2)} & \text{if } \zeta = \zeta_0 \end{cases}$$
$$= \begin{cases} (2^{3/2})(\pm\lambda\mu_0 t)^{1/2} + \dots & \text{as } t \to \infty, \mu = \mu_0, \\ (2^{3/2})(\pm\lambda\zeta_0 t)^{1/2} + \dots & \text{as } t \to \infty, \zeta = \zeta_0 \end{cases} \quad (3.7.14)$$

and

$$\rho(x,t) = \begin{cases} \frac{e^{-a\mu_0}}{2}\exp\left[\frac{(a-b)t}{2}\right] & \text{as } t \to \infty, \mu = \mu_0, \\ \frac{e^{a\zeta_0}}{2}\exp\left[-\frac{(a+b)t}{2}\right] & \text{as } t \to \infty, \zeta = \zeta_0. \end{cases} \quad (3.7.15)$$

Using (3.7.14) in (3.7.10) shows that J_0 decays like $t^{-1/4}$, whereas I_0 grows like $t^{-1/4}\exp[2^{3/2}(-\lambda\mu_0 t)^{1/2}]$ or $t^{-1/4}\exp[2^{3/2}(-\lambda\zeta_0 t)^{1/2}]$. Thus, the behavior of J_0 or I_0 does not contribute to the stability of the fundamental solution, because the linear exponential behavior in ρ dominates. We see from (3.7.15) that as $t \to \infty$, $\mu = \mu_0$, the solution is bounded if $(b - a) \geq 0$ and unbounded if $(b - a) < 0$. As $t \to \infty$, $\zeta = \zeta_0$, the solution is bounded if $b + a \geq 0$ and unbounded if $b + a < 0$. These two conditions are equivalent to the necessary and sufficient condition

$$b \geq |a| \quad (3.7.16)$$

for boundedness of the fundamental solution.

In Section 4.2.3 we shall rederive this result from the requirement that an initial discontinuity in u_x and u_t remain bounded.

We must be on guard against using the linear problem for cases where the stability condition is violated. In such cases, either the physical model leading to (3.7.1) is inaccurate, or the small disturbance assumption leading to (3.2.7) is not valid.

3.7.3 Green's Functions; Initial- and Boundary-Value Problems

Starting with the fundamental solution (3.7.9), we can proceed as in Sections 3.4–3.6 to derive solutions on the infinite, semi-infinite, and bounded intervals. We shall not catalog the results here but merely point out that all statements regarding domains of validity, zones of influence, and so on are still true in the general case, since the disturbance due to a source at ξ, τ propagates along the straight characteristics $x = \xi - (t - \tau)$ and $x = \xi + (t - \tau)$, as before. The only difference in our formulas will be the more complicated expression (3.7.9) for the fundamental solution. Two specific examples are outlined in Problems 3.7.2 and 3.7.3. In addition, since there is no loss of generality in setting $a = b = 0$, and $c = 1$ in (3.7.1) (see Problem 3.8.1), we will study this special case in detail in the next section.

Problems

3.7.1 Derive (3.7.8) using similarity. Be careful to show that the arbitrary constant that arises in solving (3.7.7) is $\frac{1}{2}$ by requiring the result to satisfy (3.7.1) with $\delta(x)\delta(t)$ on the right-hand side.

3.7.2 Work out the results analogous to (3.4.12), (3.4.14), and (3.4.18) for the general one-dimensional wave equation (3.7.1) for the case $\lambda > 0$. Note that the fundamental solution may be written as the expression in (3.7.9) multiplied by $[H(x+t) - H(x-t)]$.

3.7.3 Consider the signaling problem for the telegraph equation:

$$u_{tt} - u_{xx} + u_t = 0, \ 0 \le x, \ 0 \le t, \tag{3.7.17a}$$

$$u(x, 0^+) = u_t(x, 0^+) = 0, \tag{3.7.17b}$$

$$u(0, t) = g(t), \ t > 0. \tag{3.7.17c}$$

a. Take Laplace transforms with respect to t; then use the convolution theorem to derive the integral representation

$$u(x,t) = e^{-x/2} g(t - x) + \frac{x}{2} \int_x^t \frac{e^{-\tau/2} g(t-\tau)}{(\tau^2 - x^2)^{1/2}} I_1\left(\frac{1}{2}\sqrt{\tau^2 - x^2}\right) d\tau \tag{3.7.18}$$

if $t > x$, and $u = 0$ if $t < x$, where I_1 is the modified Bessel function of the first kind of order one.

b. Calculate Green's function of the first kind analogous to (3.5.2) for this case; then follow the approach in Section 3.5.3 and derive a result analogous to (3.5.19). Show that this result reduces to the expression (3.7.18). *Hint*: Show that the Laplace transform of your result using Green's function is the same as the Laplace transform of your result in (a).

3.8 Dispersive Waves on $-\infty < x < \infty$

If λ, as defined in (3.7.7b), is positive, (3.7.1) may be transformed to the simple expression (see Problem 3.8.1)

$$u_{tt} - u_{xx} + u = 0, \tag{3.8.1}$$

which we study in this section.

3.8.1 Uniform Waves (Traveling Waves)

Although we study the example of (3.8.1), many of the ideas discussed in this section apply to more general, possibly nonlinear, equations. For a given partial differential equation in x and t, uniform waves are bounded solutions of the form

$$u \equiv U(\theta), \ \theta \equiv kx - \omega(k)t, \tag{3.8.2}$$

for a given constant k. At this stage, the two functions U of θ, and ω of k, have not been specified. Geometrically, such a solution represents the uniform translation of the initial shape $u = U(kx)$ to the right (if $\omega > 0$). Since θ is a linear function of x and t, the translation speed $c(k)$, which is called the *phase speed*, remains constant and represents the speed with which each phase of the "wave" U moves. As all points on U move with the same constant speed, we have a "uniform wave" in the sense that the initial waveform is not distorted as it travels (see Figure 3.27). To calculate the value of $c(k)$, consider a fixed phase A on the wave. Suppose that this phase corresponds to $u = u_0$ at $x = x_0$ and $t = 0$; that is, $u_0 \equiv U(kx_0)$. A short time Δt later, the wave shape is defined by $u = U(kx - \omega(k)\Delta t)$. Assume that the phase A has now moved to the point $x_0 + \Delta x$. Therefore, $u_0 = U(kx_0 + k\Delta x - \omega(k)\Delta t) = U(kx_0)$. This means that we must have $k\Delta x - \omega(k)\Delta t = 0$; that is, the phase speed is

$$c(k) \equiv \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{\omega(k)}{k}. \tag{3.8.3}$$

If a wave equation—or, for that matter, any partial differential equation in the two variables (x, t)—admits a uniform wave solution, substitution of the form of solution (3.8.2) into the governing partial differential equation defines U and ω (see Problems 3.8.3–3.8.6 for other examples). In our case, substitution of (3.8.2)

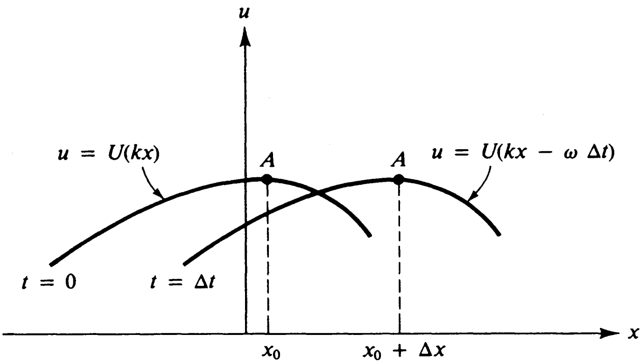

FIGURE 3.27. Translation of a uniform wave at time Δt

into (3.8.1) gives

$$(\omega^2 - k^2)\frac{d^2U}{d\theta^2} + U = 0. \tag{3.8.4}$$

Thus, bounded solutions of the form (3.8.2) are possible on the *infinite interval* only if $\omega^2 - k^2 > 0$, say $\omega^2 - k^2 \equiv \lambda^2$, in which case U has the form

$$U = \alpha \sin\left(\frac{kx \mp \sqrt{k^2 + \lambda^2}t}{\lambda} + \beta\right). \tag{3.8.5a}$$

Here α and β are arbitrary constants defining the amplitude and phase shift. Since k and λ can be chosen arbitrarily and U depends only on the ratio k/λ, it is convenient to normalize the preceding result by setting $\lambda = 1$. Thus,

$$U = \alpha \sin(kx \mp \sqrt{k^2 + 1}t + \beta). \tag{3.8.5b}$$

The resulting relations for $\omega(k)$ and $c(k)$ are

$$\omega(k) = \pm(k^2 + 1)^{1/2}, \tag{3.8.6a}$$

$$c(k) = \frac{\omega(k)}{k} = \frac{\pm(1 + k^2)^{1/2}}{k}. \tag{3.8.6b}$$

The relation (3.8.6a) defining $\omega(k)$ is called the *dispersion* relation, and solutions of the form (3.8.5) are called *dispersive waves* for reasons we shall discuss later on. Observe that if we apply the preceding ideas to the wave equation $u_{tt} - u_{xx} = 0$, the result corresponding to (3.8.4) will be $(\omega^2 - k^2)U'' = 0$. Setting $U'' = 0$ gives only unbounded solutions (or trivial ones, U = constant). Therefore, we must take $\omega = \pm k$. In this case, U is *arbitrary*, and we recover the D'Alembert form (3.4.4). The phase speed is ± 1, independent of k; such a wave is called *nondispersive*.

Another important property of the uniform dispersive wave (3.8.5) is that is is periodic. In particular, given the *wave number* k, the wave defined by (3.8.5b) is periodic in x and t. Thus, for any fixed time t_0, u is a peirodic function of x. This period, called the *wavelength*, is $L \equiv 2\pi/k$. Conversely, for a fixed $x = x_0$, U is a periodic function of time, and this peirod is $T \equiv 2\pi/(1 + k^2)^{1/2}$.

It is not necessary that a uniform wave be periodic. A very important class of uniform waves that arise in nonlinear problems are the so-called solitary waves described by functions U that tend to zero as $|\theta| \to \infty$. An example is outlined in Problem 3.8.4. Uniform waves also arise in nonlinear diffusion equations as outlined for a simple example in Problem 3.8.6. An important feature of the present linear problem is the absence of any amplitude dependence in the dispersion relation. This is no longer true for nonlinear waves; for example, see pp. 486–489 of [42] for a discussion of the nonlinear counterpart of (3.8.1). See also Problem 3.8.5.

Note that (3.8.5) is an exact solution of (3.8.1) for the *special initial conditions*

$$u(x, 0) = \alpha \sin(kx + \beta), \tag{3.8.7a}$$

$$u_t(x, 0) = -(1 + k^2)^{1/2} \alpha \cos(kx + \beta). \tag{3.8.7b}$$

It is important to keep in mind that for a given sinusoidal initial "deflection" (3.8.7a), we must also specify a particular initial "velocity" (3.8.7b) to produce the uniform wave (3.8.5) propagating to the right.

Because of linearity, we can add any number of such waves to obtain a "discrete wave train" propagating to the right:

$$u(x, t) = \sum_{i=1}^{N} \alpha_i \sin(k_i x - \sqrt{1 + k_i^2} t + \beta_i), \tag{3.8.8}$$

which also solves (3.8.1). Corresponding solutions for waves propagating to the left can also be added, and linearity ensures that these waves do not interact. Thus, if each term in the preceding series solves (3.8.1), the sum is also a solution. The corresponding statement is not true for uniform periodic waves associated with a nonlinear problem.

It is now natural to ask what role, if any, the uniform periodic waves play in the solution of a general initial-value problem.

3.8.2 General Initial-Value Problem

We shall show here that the waves (3.8.5b) are the fundamental building blocks in constructing the general solution of (3.8.1). Consider the wave equation for the general initial conditions

$$u(x, 0^+) = f(x), \; u_t(x, 0^+) = h(x), \tag{3.8.9}$$

on the infinite x-axis. Using Fourier transforms, we calculate the solution in the form

$$u(x, t) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ \left[\bar{f}(k) - i \frac{\bar{h}(k)}{(1 + k^2)^{1/2}} \right] e^{-i\theta^+(x,t,k)} \right.$$

$$+ \left[\overline{f}(k) + i \frac{\overline{h}(k)}{(1+k^2)^{1/2}} \right] e^{-i\theta^-(x,t,k)} \Bigg\} dk, \quad (3.8.10a)$$

where $\overline{f}$ and $\overline{h}$ are the Fourier transforms of $f(x)$ and $h(x)$, respectively (see (A.2.23)),

$$\overline{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ikx} dx = \overline{f}_1(k) + i\overline{f}_2(k), \quad (3.8.10b)$$

$$\overline{h}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{ikx} dx = \overline{h}_1(k) + i\overline{h}_2(k), \quad (3.8.10c)$$

and we have introduced the notation

$$\theta^+ = kx - \sqrt{1+k^2}t, \quad \theta^- = kx + \sqrt{1+k^2}t. \quad (3.8.10d)$$

Recognizing that $\overline{f}_1$, $\overline{h}_1$ are even in k, that $\overline{f}_2$, $\overline{h}_2$ are odd in k, and that $\theta^+(x,t,-k) = -\theta^-(x,t,k)$, we verify that the imaginary part of (3.8.10a) is zero, and we may express the solution in the form

$$u(x,t) = \int_{-\infty}^{\infty} \alpha^+(k) \sin[\theta^+(x,t,k) + \beta^+(k)] dk$$
$$+ \int_{-\infty}^{\infty} \alpha^-(k) \sin[\theta^-(x,t,k) + \beta^-(k)] dk, \quad (3.8.11)$$

where

$$\alpha^+(k) = \frac{1}{2\sqrt{2\pi}} \left\{ \left[\overline{f}_1 + \frac{\overline{h}_2}{(1+k^2)^{1/2}} \right]^2 + \left[\overline{f}_2 - \frac{\overline{h}_1}{(1+k^2)^{1/2}} \right]^2 \right\}^{1/2},$$

$$\alpha^-(k) = \frac{1}{2\sqrt{2\pi}} \left\{ \left[\overline{f}_1 - \frac{\overline{h}_2}{(1+k^2)^{1/2}} \right]^2 + \left[\overline{f}_2 + \frac{\overline{h}_1}{(1+k^2)^{1/2}} \right]^2 \right\}^{1/2},$$

$$\beta^+(k) = \tan^{-1} \left[\frac{(1+k^2)^{1/2}\overline{f}_1 + \overline{h}_2}{(1+k^2)^{1/2}\overline{f}_2 - \overline{h}_1} \right],$$

$$\beta^-(k) = \tan^{-1} \left[\frac{(1+h^2)^{1/2}\overline{f}_1 - \overline{h}_2}{(1+k^2)^{1/2}\overline{f}_2 + \overline{h}_1} \right].$$

The form (3.8.11) of the solution has the following simple interpretation. For a fixed value of k, the first integrand in (3.8.11) consists of a rightgoing uniform wave of the type (3.8.5b), and the second integrand is its leftgoing counterpart. The amplitudes α^+, α^- and phase shifts β^+, β^- are given functions of k as defined above. The solution (3.8.11a), expressed as an integral over all k, is thus a *continuous superposition* of all these uniform waves. This superposition is continuous, in contrast to the discrete superposition in (3.8.8), where the k_i take on distinct values.

For the special case of the fundamental solution of (3.8.1), we have $f(x) = 0$, $h(x) = \delta(x)$, and (3.8.10a) reduces to

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi i}\int_{-\infty}^{\infty} \frac{e^{-i\theta^+} - e^{-i\theta^-}}{\sqrt{1+k^2}}\,dk \\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{1}{\sqrt{1+k^2}}[\sin\theta^- - \sin\theta^+]dk. \end{aligned} \tag{3.8.12a}$$

This integral can be evaluated exactly to give

$$u(x,t) = \begin{cases} \frac{1}{2}J_0(\sqrt{t^2 - x^2}) & \text{if } t > |x|, \\ 0 & \text{if } t < |x|, \end{cases} \tag{3.8.12b}$$

in agreement with the result (3.7.9) for $\lambda > 0$ calculated by similarity, since for $a = b = 0$ and $c = 1$, (3.7.7b) gives $\lambda = \frac{1}{4}$.

3.8.3 Group Speed

In contrast to the special case (3.8.8) of discrete waves, the general initial-value problem involves a continuous superposition of waves with all values of k as described in (3.8.10) or (3.8.12). This type of superposition involves an interesting kinematic behavior, which we now study in some detail.

To fix ideas, consider the behavior of the rightgoing *wave packet* consisting of the sum of the uniform wave with wave number k_0:

$$\overline{u}^+(x,t,k_0) \equiv \alpha^+(k_0)\sin[k_0 x - \omega(k_0)t + \beta^+(k_0)], \tag{3.8.13a}$$

and its neighbor having wave number $k_0 + \Delta k$:

$$\begin{aligned} \overline{u}^+(x,t,k_0+\Delta k) &= [\alpha^+(k_0) + O(\Delta k)]\sin\{k_0 x - \omega(k_0)t \\ &\quad + \beta^+(k_0) + \Delta k[x - v(k_0)t + \gamma^+(k_0)]\} + O(\Delta k^2), \end{aligned} \tag{3.8.14a}$$

where

$$v(k) \equiv \frac{d\omega}{dk} = \frac{k}{\sqrt{1+k^2}}, \quad \gamma^+(k) = \frac{d\beta^+}{dk}. \tag{3.8.14b}$$

Thus, denoting the wave packet (normalized by dividing by $\alpha^+(k_0)$) by u, i.e.,

$$u(x,t,k_0,\Delta k) \equiv \frac{1}{\alpha^+(k_0)}[\overline{u}^+(x,t,k_0) + \overline{u}^+(x,t,k_0+\Delta k)], \tag{3.8.15}$$

and using the trigonometric identity for the sum of two sines, we obtain

$$\begin{aligned} u(x,t,k_0,\Delta k) &= 2\cos\left\{\frac{\Delta k}{2}\left[x - v(k_0)t + \gamma^+(k_0)\right]\right\} \\ &\quad \times \sin\{k_0[x - c(k_0)t] + \beta^+(k_0)\} + O(\Delta k). \end{aligned} \tag{3.8.16}$$

Since each of the two terms on the right-hand side of (3.8.15) is an exact solution of (3.8.1), the sum given in (3.8.16) is also an exact solution. The form of (3.8.16) exhibits the familiar phenomenon of beats, wherein the short wave defined by

$\sin\{k_0[x - c(k_0)t] + \beta^+(k_0)\}$ has an amplitude $2\cos\{\frac{\Delta k}{2}[x - v(k_0)t + \gamma^+(k_0)]\}$, which modulates the oscillations over the long wavelength $2\pi/\Delta k$.

Figure 3.28 illustrates the behavior in (3.8.16) (with $\gamma^+ = 0$) for a fixed time t over one wave length of the amplitude modulation, and the result may be extended over all x by periodicity. In the limit $\Delta k \to dk$, we refer to the result in (3.8.16) as the *packet of waves* in the interval $k_0 \leq k \leq k_0 + dk$ of wave space.

If we evaluate the result (3.8.16) at time $t + \Delta t$, we see that the portion of the packet contained in the x-interval,

$$x_1(t) \equiv -\frac{\pi}{\Delta k} + vt + \gamma^+ \leq x \leq \frac{\pi}{\Delta k} + vt + \gamma^+ \equiv x_2(t), \tag{3.8.17}$$

will have translated *unchanged* to the right a distance $v\Delta t$ if we neglect terms in u having amplitude equal to $O(\Delta t)$. Thus, the packet of waves contained in the envelope $\pm 2\cos(\Delta k/2)(x - vt + \gamma^+)$ moves with the speed $v(k_0)$ of the envelope, and this is called the *group speed* of the waves with wave number k_0.

In addition to this kinematic description of the group speed, we can, in this example, derive a dynamical description based on the propagation of the average energy contained in the packet. Because of periodicity, we need consider only the solution in the interval $x_1(t) \leq x \leq x_2(t)$.

As discussed in Section 3.6.4 and Problem 3.6.7, the wave equation

$$u_{tt} - u_{xx} + F(u) = 0 \tag{3.8.18}$$

has the energy conservation law $E(t) = \text{constant}$, where

$$E(t) \equiv \frac{1}{2}\int_{x_1}^{x_2} (u_t^2 + u_x^2 + 2G(u))dx, \tag{3.8.19}$$

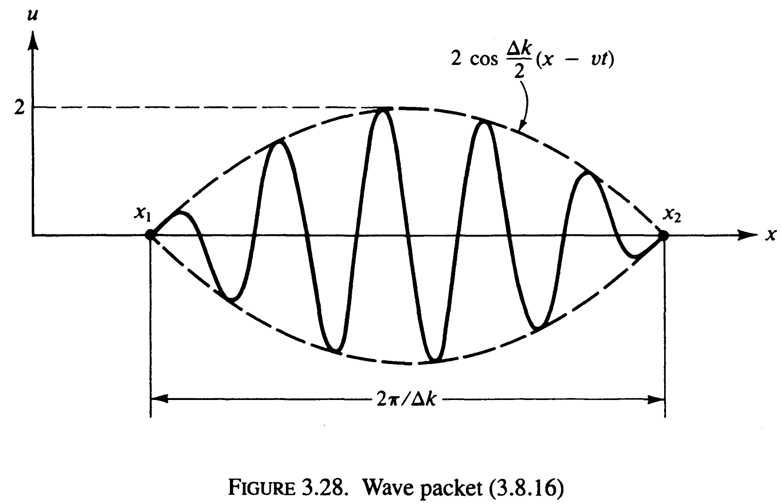

FIGURE 3.28. Wave packet (3.8.16)

and $dG/du \equiv F(u)$, *as long as* x_1 *and* x_2 *are fixed points at which the solution of (3.8.18) vanishes.*

In the case of the solution (3.8.16), $u(x_1(t), t) = u(x_2(t), t) = 0$, but the points x_1 *and* x_2 *move uniformly to the right* with the group speed $v(k_0)$. What happens to $E(t)$ as defined by (3.8.19) in this case? We shall show next that $E(t)$ is still constant.

We have

$$E(t) \equiv \frac{1}{2}\int_{x_1(t)}^{x_2(t)} (u_t^2 + u_x^2 + u^2)dx, \tag{3.8.20}$$

where $x_1(t)$ and $x_2(t)$ are defined in (3.8.17).

Differentiating (3.8.20) and noting that $\dot{x}_1 = \dot{x}_2 = v$ gives

$$\begin{aligned}\frac{dE}{dt} &= \int_{x_1(t)}^{x_2(t)} (u_t u_{tt} + u_x u_{xt} + uu_t)dx \\ &+ \frac{v}{2}[u_t^2(x,t) + u_x^2(x,t) + u^2(x,t)]_{x=x_1(t)}^{x=x_2(t)}.\end{aligned} \tag{3.8.21}$$

Integrating the second term inside the integral by parts shows that

$$\int_{x_1(t)}^{x_2(t)} u_x u_{xt} dx = u_x u_t\Big|_{x=x_1(t)}^{x=x_2(t)} - \int_{x_1(t)}^{x_2(t)} u_t u_{xx} dx.$$

Therefore, (3.8.21) is equal to

$$\frac{dE}{dt} = \left[\frac{v}{2}(u_t^2 + u_x^2 + u^2) + u_x u_t\right]_{x=x_1(t)}^{x=x_2(t)}, \tag{3.8.22}$$

because what remains in the integrand is identically equal to zero, since u satisfies (3.8.1).

To facilitate the evaluation of the right-hand side of (3.8.22) at the two endpoints, let us write u in the form

$$u(x,t) = 2\cos\phi(x,t)\sin\psi(x,t) + O(\Delta k), \tag{3.8.23}$$

where (cf. (3.8.16))

$$\phi(x,t) = \frac{\Delta k}{2}(x - vt + \gamma^+), \tag{3.8.24a}$$

$$\psi(x,t) = k_0(x - ct) + \beta^+. \tag{3.8.24b}$$

Since $\cos\phi(x_1(t), t) = \cos\phi(x_2(t), t) = 0$, the only terms that contribute to (3.8.22) are

$$\frac{dE}{dt} = [2v(\phi_t^2 + \phi_x^2) + 4\phi_x\phi_t]\sin^2\phi\sin^2\psi\Big|_{x_1}^{x_2} = O(\Delta k), \tag{3.8.25}$$

because $\phi_t = -v\Delta k/2$ and $\phi_x = \Delta k/2$. Therefore, since the expression (3.8.23) for u is correct to $O(\Delta k)$, we conclude that $dE/dt = 0$ correct to $O(\Delta k)$ also.

The preceding calculation indicates that as the packet of waves $k_0 \leq k \leq k_0 + dk$ moves to the right, the average energy in this packet remains constant.

3.8.4 Dispersion

Next, we consider the behavior of solutions of (3.8.1) for large times. The case of a sum of discrete waves is not interesting because each wave in the sum evolves unchanged from the point of view of an observer moving with the phase speed of that wave. So, we turn our attention to a result involving a continuous superposition, as given in general by (3.8.10).

This result may be analyzed asymptotically for $t \to \infty$, $r \equiv x/t$ fixed, using the method of stationary phase (for example, see pp. 272–275 of [8]). It is equally instructive and more transparent to examine the result (3.8.12b) for the fundamental solution, since this is given explicitly in terms of J_0.

We see, in fact, that result (3.8.12b) may be interpreted as the continuous superposition of the right and left propagating waves of *all wave numbers*, as defined in (3.8.12a). This latter messy expression eventually sorts itself out to give the asymptotic form [see (3.7.10a)]

$$u = \frac{1}{[2\pi(t^2 - x^2)^{1/2}]^{1/2}} \cos\left[(t^2 - x^2)^{1/2} - \frac{\pi}{4}\right] + \cdots. \tag{3.8.26a}$$

Notice that this result is not valid near the wave front $t \approx |x|$.

A little algebra shows that (3.8.26a) can also be written in the form

$$u(x, t) = -\frac{1}{[2\pi t(1 - r^2)^{1/2}]^{1/2}} \sin\left[Kx - \Omega t - \frac{\pi}{4}\right] + \ldots, \tag{3.8.26b}$$

where

$$K(r) \equiv \frac{r}{(1 - r^2)^{1/2}}, \quad r \equiv \frac{x}{t}, \quad 0 \leq r < 1, \tag{3.8.27a}$$

$$\Omega(K) \equiv [1 + K^2]^{1/2} = \frac{1}{(1 - r^2)^{1/2}}. \tag{3.8.27b}$$

Thus, for a fixed r, (3.8.26b) describes a *uniform wave* with constant wave number K, frequency Ω, and phase shift $-\pi/4$. The amplitude, which is given by the factor multiplying the sine function, decays like $t^{-1/2}$, and this feature is consistent with energy conservation. We can interpret the preceding behavior geometrically and say that an observer moving at the constant speed $v = r$ sees only a wave of one wave number, $K(r)$. In fact, v *is the group speed* of the K-wave because

$$\frac{d\Omega}{dK} = \frac{K}{(1 + K^2)^{1/2}} = r. \tag{3.8.28}$$

We could also interpret the result (3.8.26b) to mean that for t large, wave numbers and frequencies are locally propagated with the associated group speed.

Dispersion refers to the fact that eventually waves of different wave numbers separate (disperse) as they propagate with their group velocity.

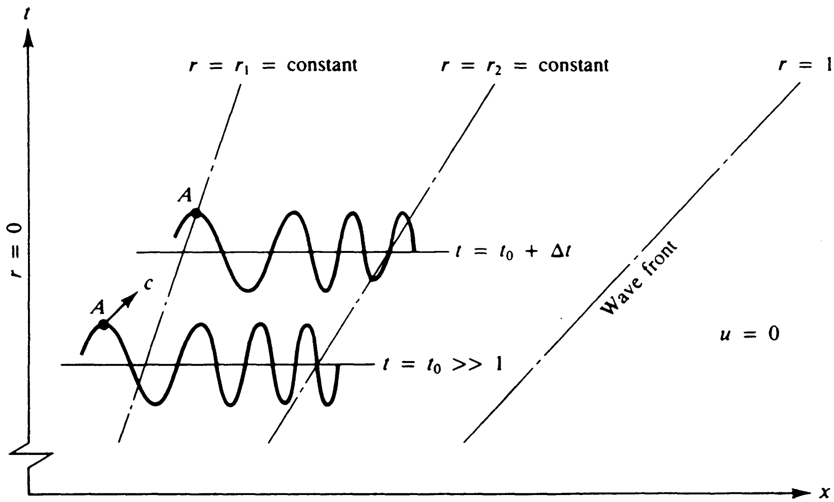

FIGURE 3.29. Dispersion for $t \to \infty$

We also note that at large times, a stationary observer sees progressively longer waves arriving from the left, whereas the observer moving with speed r and focusing on a given phase sees this phase moving slowly to the right (since the local phase speed $c(K)$ is larger than r in this example).

The reason we claim that K, and hence Ω and c, change slowly with x, is that for a large fixed t, we have

$$\frac{\partial c}{\partial x} = \frac{\partial c}{\partial K}\frac{\partial K}{\partial x}; \quad \frac{\partial \Omega}{\partial x} = \frac{\partial \Omega}{\partial K}\frac{\partial K}{\partial x}; \ldots$$

Now $\partial K/\partial x$ is small if $t \to \infty$ with r fixed because

$$\frac{\partial K}{\partial x} = \frac{r_x}{(1-r^2)^{3/2}} = \frac{1}{t}\frac{1}{(1-r^2)^{3/2}} = O(t^{-1}). \tag{3.8.29}$$

Therefore, c_x and Ω_x are $O(t^{-1})$ as $t \to \infty$ with r fixed; that is, *they vary slowly with x*. Figure 3.29 illustrates these features for the present example.

In concluding this section, we reiterate that the preceding ideas are not restricted to the linear wave equation (3.8.1); they apply to a variety of other linear and nonlinear partial differential equations. The reader is referred to Chapter 11 of [42] for a discussion of general linear dispersive waves. Selected nonlinear dispersive wave problems are also analyzed there.

Problems

3.8.1a. Show that the transformation $u \to \overline{w}, x \to \overline{x}, t \to \overline{t}$ defined by

$$\overline{w}(\overline{x}, \overline{t}) = \exp\left(\frac{b\overline{t} - a\overline{x}}{4\sqrt{\lambda}}\right) u\left(\frac{\overline{x}}{2\sqrt{\lambda}}, \frac{\overline{t}}{2\sqrt{\lambda}}\right) \tag{3.8.30}$$

takes (3.7.1) to $\overline{w}_{\overline{t}\,\overline{t}} - \overline{w}_{\overline{x}\,\overline{x}} + \overline{w} = 0$ for $\lambda > 0$.

b. Consider (3.7.1) with a, b, and c as functions of x and t. Show that a necessary and sufficient condition to be able to transform (3.7.1) in this case to the form $w_{tt} - w_{xx} + \lambda(x, t)w = 0$ is

$$a_t + b_x = 0. \tag{3.8.31}$$

Calculate $\lambda(x, t)$ and $r(x, t)$ in the transformation $u = w(x, t)r(x, t)$.

3.8.2 Consider the following initial-value problem for (3.8.1) with periodic initial data

$$u_{tt} - u_{xx} + c^2 u = 0, \quad -\infty < x < \infty, \quad 0 \leq t, \tag{3.8.32a}$$

$$u(x, 0) = f(x) = \sum_{n=0}^{\infty} f_n \cos nx, \quad u_t(x, 0) = 0, \tag{3.8.32b}$$

where c is a constant and the Fourier coefficients f_n are given by the standard formula.

a. Show that the solution has the following Fourier series:

$$u(x, t) = \sum_{n=0}^{\infty} f_n \cos \sqrt{n^2 + c^2 t} \cos nx. \tag{3.8.33}$$

b. Use the fundamental solution (3.7.9) to derive the following integral representation of the solution:

$$u(x, t) = \frac{1}{2} f(x + t) + \frac{1}{2} f(x - t) + \frac{1}{2} \int_{x-t}^{x+t} f(\xi) \frac{\partial}{\partial t} \left[J_0(c\sqrt{t^2 - (x - \xi)^2})\right] d\xi. \tag{3.8.34}$$

c. Prove the identity

$$(\cos\sqrt{n^2 + c^2 t} - \cos nt) \cos nx = \frac{1}{2} \int_{x-t}^{x+t} (\cos n\xi) \frac{\partial}{\partial t} \left[J_0(c\sqrt{t^2 - (x - \xi)^2}\right] d\xi \tag{3.8.35}$$

to show that the results (3.8.33) and (3.8.34) agree.

3.8.3 In appropriate dimensionless variables, the equation describing the transverse motion of a beam on an elastic support is

$$u_{xxxx} + u_{tt} + u = 0. \tag{3.8.36}$$

a. Assume a uniform wave solution of the form (3.8.2) on $-\infty < x < \infty$ and show that U is bounded only if $\omega^4 - 4k^4 > 0$. Denote

$$\omega^4 - 4k^4 = 4\mu^4 = \text{constant}, \ \omega = 2^{1/2}(k^4 + \mu^4)^{1/4}, \qquad (3.8.37)$$

and show that U is a linear combination of the four periodic uniform waves

$$U(\theta) = \alpha_1^+ \sin(\theta_1^+ + \beta_1^+) + \alpha_1^- \sin(\theta_1^- + \beta_1^-) + \alpha_2^+ \sin(\theta_2^+ + \beta_2^+) + \alpha_2^- \sin(\theta_2^- + \beta_2^-), \qquad (3.8.38)$$

where $\alpha_1^\pm$, $\beta_1^\pm$ are arbitrary constants and

$$\theta_i^\pm = \lambda_i(kx \mp \omega t), \ i = 1, 2, \qquad (3.8.39a)$$

$$\lambda_1 = \frac{1}{k^2}[(k^4 + \mu^4)^{1/2} + \mu^2]^{1/2}, \ \lambda_2 = \frac{1}{k^2}[(k^4 + \mu^4)^{1/2} - \mu^2]^{1/2}. \qquad (3.8.39b)$$

b. Denote $\kappa_1 = \lambda_1 k$, $\kappa_2 = \lambda_2 k$ and show that

$$\theta_i^\pm = \kappa_i x \mp \sqrt{1 + \kappa_i^4}\,t, \ i = 1, 2. \qquad (3.8.40)$$

c. Solve the general initial-value problem for (3.8.36) on $-\infty < x < \infty$ with

$$u(x, 0) = f(x), \ u_t(x, 0) = g(x) \qquad (3.8.41)$$

using Fourier transforms, in the form (see (3.8.10a))

$$u(x, t) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ \left(\overline{f} - i\frac{\overline{g}}{\sqrt{1 + k^4}} \right) e^{-i(kx - \sqrt{1+k^4}t)} + \left(\overline{f} + i\frac{\overline{g}}{\sqrt{1 + k^4}} \right) e^{-i(kx + \sqrt{1+k^4}t)} \right\} dk, \qquad (3.8.42)$$

where $\overline{f}$ and $\overline{g}$ are the Fourier transforms of f and g, respectively. Thus, the general solution (3.8.42) is a continuous superposition of uniform periodic waves (3.8.38).

3.8.4 Consider the Korteweg–de Vries equation

$$u_t + (1 + \frac{3}{2}u)u_x + \frac{\delta^2}{6}u_{xxx} = 0, \qquad (3.8.43)$$

where δ is a constant.

a. For the linearized problem (ignoring the uu_x term), derive the following periodic uniform wave solution

$$u(x, t) = \alpha \sin\left[kx - k\left(1 - \frac{\delta^2}{6}k^2\right)t + \beta\right] + c, \qquad (3.8.44)$$

where α, β, and c are arbitrary constants.

b. Use Fourier transforms to derive the integral representation of the solution of the linearized equation for the initial condition $u(x, 0) = f(x)$. Show that your result corresponds to a continuous superposition of the uniform periodic waves (3.8.44). Specialize your result to the case $f(x) = \delta(x)$ and express this integral in terms of the Airy function (see (10.4.32) of [3]). Use the asymptotic behavior for the Airy function (see (10.4.59)–(10.4.60) of [3]) to study the solution as $t \to \infty$.

c. Now consider the full equation (3.8.43). Set $w = u + \frac{2}{3}$ and calculate a solitary uniform wave solution for w. A solitary wave has w and w_x equal to zero at $x = \pm\infty$ and all t. Express this solitary wave solution in the form

$$w = w_0 W(kx - \omega(k)t), \tag{3.8.45}$$

where w_0 is a constant chosen such that the maximum value of W equals 1. Show that

$$w = w_0 \operatorname{sech}^2\left[\frac{\sqrt{3w_0}}{2\delta}\left(x - \frac{w_0}{2}t\right)\right], \tag{3.8.46}$$

where $w_0 = 4k^2\delta^2/3$. Thus, the phase speed equals $w_0/2$. Expressing the phase speed in terms of k gives $c(k) = 2k^2\delta^2/3$, and the dispersion relation is $\omega(k) = 2k^3\delta^2/3$. Thus, the group speed is $(d\omega/dk) = 2k^2\delta^2$.

For a discussion of the periodic uniform wave solutions of (3.8.43), the reader is referred of Section 2.3 of [14].

3.8.5 Consider the nonlinear wave equation

$$u_{tt} - u_{xx} + u + \epsilon u^3 = 0, \tag{3.8.47}$$

where ϵ is a positive constant.

a. Show that the assumption of a uniform wave, as in (3.8.2), leads to the nonlinear problem for U

$$(\omega^2 - k^2)U'' + U + \epsilon U^3 = 0, \tag{3.8.48}$$

which has periodic solutions for $\omega^2 - k^2 > 0$ defined by the Jacobian elliptic function sn. See Section 4.1.1 of [26], and Section 16 of [3].

b. To calculate the dispersion relation, note that (3.8.48) implies the energy conservation equation

$$(\omega^2 - k^2)\frac{U'^2}{2} + \frac{U^2}{2} + \epsilon\frac{U^4}{4} = E = \text{constant}. \tag{3.8.49}$$

Therefore, periodic solutions in θ correspond to closed curves in the (U, U') phase-plane for any given $E > 0$. If we normalize the period in θ to be 2π and indicate the closed contour for $E =$ constant by C, we have

$$(\omega^2 - k^2)^{1/2}\oint_C \frac{dU}{\left[2E - U^2 - \epsilon U^4/2\right]^{1/2}} = 2\pi, \tag{3.8.50}$$

where the proper sign for the square root must be used, depending on where the integration occurs. Show that for $\epsilon \to 0$, (3.8.50) reduces to the dispersion relation (3.8.6a).

For $\epsilon \neq 0$, (3.8.50) gives a relation linking ω, k, and E. Thus, in general,

$$\omega = \omega(k, E; \epsilon), \tag{3.8.51}$$

and it is only in the limit $\epsilon \to 0$ that the dispersion relation is independent of E (or the amplitude α).

c. For $0 < \epsilon \ll 1$, show that the dispersion relation becomes

$$\omega(k, E; \epsilon) = (1 + k^2)^{1/2} \left[1 + \frac{3E}{4(1 + k^2)} \epsilon + O(\epsilon^2) \right]. \tag{3.8.52}$$

3.8.6 Fisher's equation is the following nonlinear reaction–diffusion equation:

$$u_t - u_{xx} - u(1 - u) = 0. \tag{3.8.53}$$

a. Assume a uniform wave solution in the form

$$u(x, t) \equiv U(\zeta), \ \zeta \equiv x - ct, \tag{3.8.54}$$

where c is a positive constant, and derive the ordinary differential equation for $U(\zeta)$.

b. Study the differential equation for U in the phase plane (U, U'). Locate and identify the nature of the singular points. How does the behavior of the solution depend on c near these singular points?

c. Use the results in (b) to show that there exists a solution $U(\zeta)$ satisfying

$$U(-\infty) = 1, \ U(\infty) = 0. \tag{3.8.55}$$

Sketch this uniform solution in the (U, ζ)-plane for representative values of c.

3.9 The Three-Dimensional Wave Equation; Acoustics

In Section 3.3.6 we showed that the velocity potential in the three-dimensional flow due to small disturbances in an initially ambient, inviscid, non-heat-conducting gas obeys the dimensionless wave equation

$$\Delta\phi - \phi_{tt} = 0 \tag{3.9.1}$$

to leading order, where we have omitted the overbar for simplicity (see (3.3.50)). Recall that the velocity $\mathbf{u}$ is given by [see (3.3.40)]

$$\mathbf{u} = \operatorname{grad} \phi, \tag{3.9.2}$$

and the density perturbation $\overline{\rho}$ measured from its ambient value is related to ϕ_t according to (see (3.3.45))

$$\overline{\rho} = -\phi_t. \tag{3.9.3}$$

Equation (3.3.36) then implies that the pressure perturbation $\overline{p}$ is given by

$$\overline{p} = -\gamma\phi_t. \tag{3.9.4}$$

Thus, knowing ϕ defines all pertinent flow quantities to leading order.

3.9.1 Fundamental Solution

The fundamental solution satisfies

$$\Delta\phi - \phi_{tt} = \delta(x)\delta(y)\delta(z)\delta(t) \tag{3.9.5a}$$

in the infinite domain with zero initial conditions

$$\phi(x, y, z, 0^-) = \phi_t(x, y, z, 0^-) = 0. \tag{3.9.5b}$$

For the interpretation of (3.9.5a) as the velocity potential in acoustics, the right-hand side represents a *positive unit source of mass*, because to leading order, mass conservation with a unit source at the origin and $t = 0$ is given by (see (3.3.48))

$$\overline{\rho}_t + \Delta\phi = \delta(x)\delta(y)\delta(z)\delta(t). \tag{3.9.6}$$

When the leading term in (3.3.47) is used to eliminate $\overline{\rho}_t$, (3.9.6) gives (3.9.5a).

Since the disturbance is spherically symmetric in space, ϕ depends only on the radial distance r from the origin and the time t, and we need consider only

$$\phi_{rr} + \frac{2}{r}\phi_r - \phi_{tt} = \delta_3(r)\delta(t), \tag{3.9.7a}$$

$$\phi(r, 0^-) = \phi_t(r, 0^-) = 0, \tag{3.9.7b}$$

where $\delta_3(r)$ is the three-dimensional delta function defined in (1.6.9b).

Consider (3.9.7a) with zero right-hand side, which can be written in the form

$$(r\phi)_{rr} - (r\phi)_{tt} = 0. \tag{3.9.8}$$

Therefore, $r\phi$ has the general D'Alembert form [see (3.4.4)]

$$r\phi = v(t - r) + w(t + r), \tag{3.9.9}$$

where v and w are arbitrary functions of their respective arguments. Since w represents an incoming disturbance, it is not appropriate for the case of (3.9.7), and we discard it. (In a homogeneous unbounded domain, there is no mechanism for generating reflected disturbances traveling toward the origin when a source is turned on at the origin.)

Thus, ϕ is in the form

$$\phi(r, t) = \frac{v(t - r)}{r}. \tag{3.9.10}$$

To determine v, we integrate (3.9.7a) over the interior of a sphere of radius $\epsilon \ll 1$ centered at the origin. This gives

$$\iiint_{G_\epsilon} [\Delta\phi - \phi_{tt}]dV = \delta(t), \tag{3.9.11}$$

where G_ϵ denotes the interior of the ϵ-sphere. Using Gauss' theorem [as in (2.3.7)] to express the volume integral of $\Delta\phi$ in terms of ϕ_r on the boundary gives

$$\iiint_{G_\epsilon} \Delta\phi dV = \iint_{\Gamma_\epsilon} \phi_r dA = \iint_{\Gamma_\epsilon} \left[-\frac{v(t-r)}{r^2} - \frac{v'(t-r)}{r} \right] dA,$$

where Γ_ϵ is the surface and dA is the element of area of the ϵ-sphere. In the limit $\epsilon \to 0$, the first term of the surface integral will contribute and the second term will not because $dA = O(\epsilon^2)$. Using spherical polar coordinates (θ = colatitude, ψ = longitude) we obtain

$$\lim_{\epsilon\to 0} \iiint_{G_\epsilon} \Delta\phi dV = -\lim_{\epsilon\to 0} \int_{\theta=0}^{\pi} \int_{\psi=0}^{2\pi} v(t-\epsilon) \sin\theta d\theta d\psi = -4\pi v(t).$$

Also,

$$\lim_{\epsilon\to 0} \iiint_{G_\epsilon} \phi_{tt} dV = \frac{\partial^2}{\partial t^2} \lim_{\epsilon\to 0} \iiint_{G_\epsilon} \phi dV = 0.$$

Therefore, (3.9.11) reduces to $-4\pi v(t) = \delta(t)$, or $v(t) = -\delta(t)/4\pi$, and the solution of (3.9.7) is

$$\phi(r,t) = -\frac{1}{4\pi} \frac{\delta(t-r)}{r}. \tag{3.9.12}$$

More generally, the solution of

$$\Delta\phi - \phi_{tt} = \delta(x-\xi)\delta(y-\eta)\delta(z-\zeta)\delta(t-\tau), \tag{3.9.13a}$$

$$\phi(x,y,z,\tau^-) = \phi_t(x,y,z,\tau^-) = 0, \tag{3.9.13b}$$

is

$$F(x-\xi, y-\eta, z-\zeta, t-\tau) \equiv -\frac{1}{4\pi} \frac{\delta(t-\tau-r_{PQ})}{r_{PQ}}, \tag{3.9.14}$$

where $P = (x, y, z)$, $Q = (\xi, \eta, \zeta)$, and

$$r_{PQ}^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2. \tag{3.9.15}$$

It is instructive to interpret the result in (3.9.12) in terms of the pressure disturbance $\tilde{p} = -\gamma\phi_t = \gamma\delta'(t-r)/4\pi r$, which results form turning on a unit mass source at the origin for an instant at time $t = 0$. If we regard $\delta'(s)$ qualitatively as the function sketched at the top of Figure 3.30, an observer at the fixed point $r = r_0 > 0$ receives the pressure signal indicated in the lower figure as a function of time. Thus, the pressure disturbance remains zero up until time $t = r_0$, when the disturbance emitted at the origin and $t = 0$ arrives. This disturbance is qualitatively a rapid rise in the pressure, followed by a rapid drop, followed by a return to a null value. This so-called N-wave is typical of acoustic disturbances (see Sec. 3.9.4i) as well as more energetic disturbances such as sonic booms or explosions.

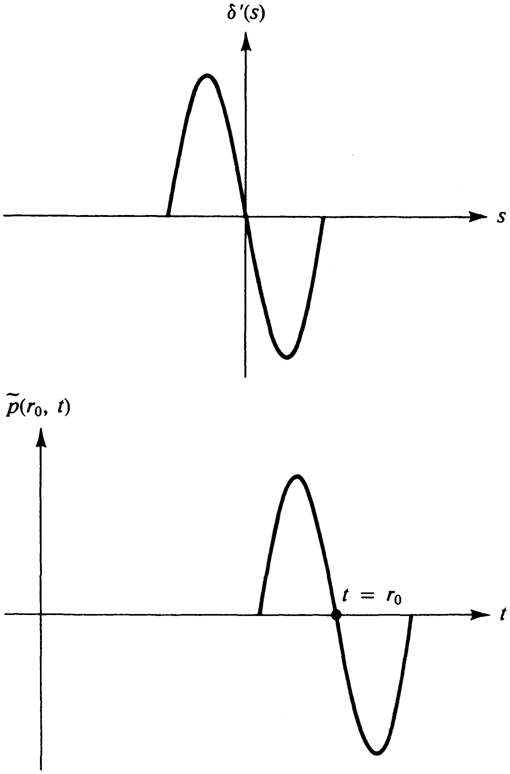

FIGURE 3.30. Pressure disturbance as a function of time

3.9.2 Arbitrary Source Distribution

Consider the problem

$$\Delta\phi - \phi_{tt} = S(x, y, z, t), \tag{3.9.16a}$$

$$\phi \equiv 0, \quad S \equiv 0 \quad \text{if } t < 0, \tag{3.9.16b}$$

which in acoustics corresponds to a prescribed spatial and temporal distribution of mass sources of strength S turned on at $t = 0$. Using the expression (3.9.14) for the fundamental solution and superposition, we have

$$\phi(x, y, z, t) = -\frac{1}{4\pi} \int_{\tau=0}^{t} d\tau \iiint_{-\infty}^{\infty} S(\xi, \eta, \zeta, \tau) \frac{\delta(t - \tau - r_{PQ})}{r_{PQ}} d\xi \, d\eta \, d\zeta. \tag{3.9.17}$$

Changing variables from τ to $\sigma \equiv t - \tau - r_{PQ}$ and performing the $\tau(\sigma)$ integration first gives

$$\phi(x, y, z, t) = -\frac{1}{4\pi} \iiint\limits_{r_{PQ} \le t} \frac{S(\xi, \eta, \zeta, t - r_{PQ})}{r_{PQ}} \, d\xi \, d\eta \, d\zeta. \tag{3.9.18}$$

Note that for a fixed $P = (x, y, z)$ and a fixed t, the integration variables $Q = (\xi, \eta, \zeta)$ range only over the *interior* of the sphere $r_{PQ} = t$ centered at P. This result is called the *retarded potential* because of the delay effect in the time dependence in S.

As a special case, let

$$S = \delta(x)\delta(y)\delta(z)f(t), \tag{3.9.19}$$

corresponding to a *point source* at the origin having a time-varying strength f. This is an idealization in acoustics of a point source of sound, such as a speaker at the origin. Using (3.9.19) for S in (3.9.18) and performing the integrations gives

$$\phi(x, y, z, t) = -\frac{1}{4\pi} \frac{f(t - r)}{r}, \tag{3.9.20}$$

where $r^2 = x^2 + y^2 + z^2$. Thus, the given signal is received a distance r away from the source in a form that is *undistorted*, $f(t) \to f(t - r)$; rather, it is merely *attenuated* like r^{-1}. In particular, the pressure disturbance created by the source at $r = 0$ arrives undistorted but weaker some distance away. Thus, the receiver (ear) has a relatively simple task of interpreting the signal, all of which is fortunate for us. Mathematically, this feature is a consequence of the presence of the delta function in the fundamental solution, which ensures that a given disturbance propagates along a *distinct* front corresponding to the vanishing of the argument of the delta function.

This situation is no longer true in two dimensions. Consider, for example, the signal due to a two-dimensional point source at $x = y = 0$. This is just a *line* source of variable strength $f(t)$ along the z-axis in three dimensions, i.e.,

$$S = \delta(x)\delta(y)f(t). \tag{3.9.21}$$

In this case (3.9.18) reduces to (Problem 3.9.1a)

$$\phi(x, y, t) = -\frac{1}{2\pi} \int_0^{t-r} \frac{f(\tau)d\tau}{\sqrt{(t - \tau)^2 - r^2}}, \tag{3.9.22}$$

where $r^2 = x^2 + y^2$. Thus, an "integrated" version of the signal arrives at the observer location r, and this is physically obvious, since each point on the z-axis (and its mirror image along $-z$) is at a different distance from the observer, as shown in Figure 3.31. Hence, signals that arrive at $r = r_0$ at a given time $t = t_0$ were broadcast at different times. In particular, at the time $t = t_0$, the disturbance that arrives at P is made up of the signal sent from 0 at time $t = t_0 - r_0$, the signal sent from A and A' at $t = t_0 - r_1$, the signals sent from B and B' at $t = t_0 - r_2$, and so on.

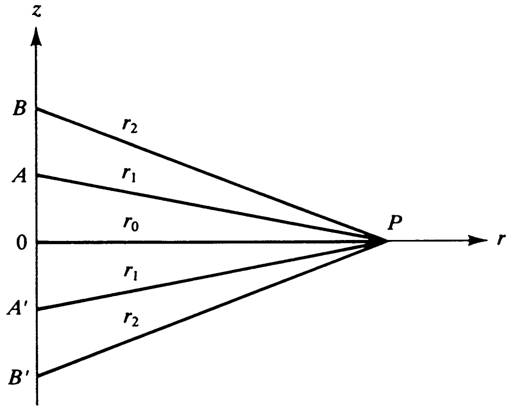

FIGURE 3.31. Geometry for a two-dimensional source

As a further specialization of (3.9.22), set $f(t) = \delta(t)$, which gives the fundamental solution for the two-dimensional wave equation as

$$\phi = -\frac{1}{2\pi}\frac{H(t-r)}{\sqrt{t^2-r^2}}, \quad r^2 = x^2 + y^2, \tag{3.9.23}$$

where H is the Heaviside function [see also Problems (3.9.1b)–(3.9.1d)].

Let us return now to the result (3.9.18) and interpret the potential ϕ as a superposition of certain *averages* evaluated over different concentric spherical shells surrounding P. To do this, we define $\langle h\rangle_\rho$ to be the average value of a given function $h(\xi, \eta, \zeta)$ evaluated at the fixed point $P = (x, y, z)$ with respect to the sphere of radius ρ centered at P. Thus,

$$\langle h\rangle_\rho \equiv \frac{1}{4\pi\rho^2}\iint_{r_{PQ}=\rho} h(Q)dA_Q. \tag{3.9.24}$$

More explicitly, if we introduce the local spherical polar coordinate system at P, defined by $\xi - x = \rho\sin\theta\cos\psi$, $\eta - y = \rho\sin\theta\sin\psi$, $\zeta - z = \rho\cos\theta$, then (3.9.24) becomes

$$\langle h\rangle_\rho = \frac{1}{4\pi\rho^2}\int_{\theta=0}^{\pi}\int_{\psi=0}^{2\pi} h(x+\rho\sin\theta\cos\psi,\ y+\rho\sin\theta\sin\psi,$$
$$z+\rho\cos\theta)\rho^2\sin\theta\, d\psi\, d\theta. \tag{3.9.25}$$

Thus, for a given h and a given point $P = (x, y, z)$, $\langle h\rangle_\rho$ is a function of x, y, z, and ρ. The notation $\langle h\rangle_\rho$ should not be misinterpreted as denoting the partial derivative of $\langle h\rangle$ with respect to ρ.

Now, if we write (3.9.17) using τ, ρ, θ, and ψ as integration variables, we have

$$\phi(x, y, z, t) = -\frac{1}{4\pi}\int_{\tau=0}^{t} d\tau \int_{\rho=0}^{\infty} \frac{\delta(t-\tau-\rho)}{\rho} d\rho$$

$$\times \left\{ \int_{\theta=0}^{\pi}\int_{\psi=0}^{2\pi} S(x+\rho\sin\theta\cos\psi\; y+\rho\sin\theta\sin\psi, z+\rho\cos\theta, \tau)\rho^2 \sin\theta\, d\psi\, d\theta \right\}, \tag{3.9.26}$$

which is just

$$\phi(x, y, z, t) = -\int_{\tau=0}^{t} \langle \rho S \rangle_{t-\tau} d\tau. \tag{3.9.27}$$

Thus, for a fixed $P = (x, y, z)$ and time t, as τ increases from 0 to t, the contribution to ϕ consists of the *sum of the averages of* $-\rho S$ on spheres centered at P and having decreasing radii equal to $t - \tau$. The maximal radius is, of course, $\rho = t$.

3.9.3 Initial-Value Problems for the Homogeneous Equation

Consider the initial-value problem

$$\Delta\phi - \phi_{tt} = 0, \tag{3.9.28a}$$

$$\phi(x, y, z, 0^+) = 0, \;\; \phi_t(x, y, z, 0^+) = f(x, y, z). \tag{3.9.28b}$$

In acoustics, this would correspond to a prescribed initial pressure perturbation $\tilde{p}$ equal to $-\gamma f$ [see (3.9.4)]. The preceding is equivalent to the inhomogeneous equation

$$\Delta\phi - \phi_{tt} = -\delta(t) f(x, y, z) \tag{3.9.29a}$$

with zero initial conditions

$$\phi(x, y, z, 0^-) = \phi_t(x, y, z, 0^-) = 0. \tag{3.9.29b}$$

Thus, we have a special case of (3.9.27) with $S = -\delta(t) f(x, y, z)$, and the solution is

$$\phi(x, y, z, t) = t\langle f(\xi, \eta, \zeta)\rangle_t. \tag{3.9.30}$$

The other initial-value problem has

$$\Delta\phi - \phi_{tt} = 0, \tag{3.9.31a}$$

$$\phi(x, y, z, 0^+) = g(x, y, z), \;\; \phi_t(x, y, z, 0^+) = 0. \tag{3.9.31b}$$

We can reduce this to the form (3.9.28) by introducing the new dependent variable Φ defined by [see (3.4.15)]

$$\Phi(x, y, z, t) \equiv \int_0^t \phi(x, y, z, \tau) d\tau. \tag{3.9.32}$$

Now $\partial\Phi/\partial t = \phi$, and $\partial^2\Phi/\partial t^2 = \partial\phi/\partial t$. Thus, Φ satisfies the initial conditions

$$\Phi_t(x, y, z, 0^+) = \phi(x, y, z, 0^+) = g(x, y, z), \quad (3.9.33a)$$

$$\Phi(z, y, z, 0^+) = 0. \quad (3.9.33b)$$

Also, Φ obeys the wave equation because

$$\Delta\Phi = \int_0^t \Delta\phi d\tau = \int_0^t \phi_{tt} d\tau = \phi_t(x, y, z, t) - \phi_t(x, y, z, 0^+)$$

$$= \phi_t = \frac{\partial^2\Phi}{\partial t^2}.$$

Hence, using (3.9.30), we have

$$\Phi(x, y, z, t) = t\langle g\rangle_t,$$

or

$$\phi(x, y, z, t) = \frac{\partial}{\partial t}\{t\langle g\rangle_t\}. \quad (3.9.34)$$

3.9.4 Examples

In this section we outline two examples of applications of the preceding results.

(i) *The bursting balloon*

A spherical balloon is inflated to a pressure $p = p_1 > p_0$ and radius L, as shown in Figure 3.32(a). At time $t = 0$, the balloon bursts. What is the pressure disturbance as a function of time that is felt at a distance $R > L$ measured from the center of the balloon? The case $R < L$ is considered in Problem 3.9.4.

If $\epsilon \equiv (p_1 - p_0)/p_0 \ll 1$, it is appropriate to use the linear theory of this section. Since $\tilde{p} = -\gamma\phi_t$, the dimensionless formulation (using L to normalize

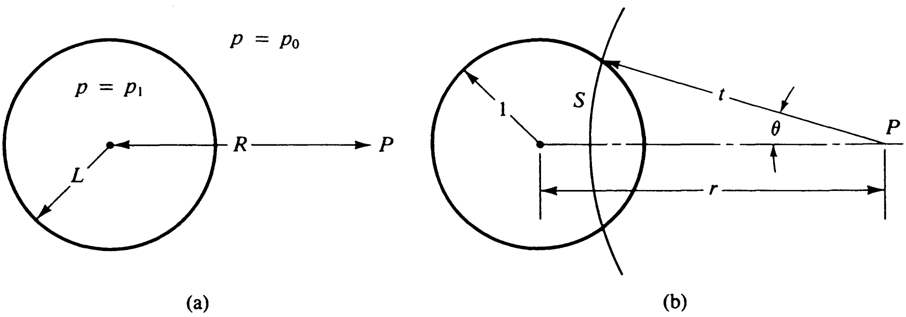

FIGURE 3.32. Bursting balloon

lengths, L/a_0 to normalize time, and La_0 to normalize the velocity potential) is given by (3.9.28) with spherical symmetry ($\phi(r, t)$) and with

$$f(r) = \begin{cases} -1/\gamma & \text{if } r < 1, \\ 0 & \text{if } r > 1. \end{cases} \tag{3.9.35}$$

The solution for ϕ is given by (3.9.30), and we need to calculate the area of S, the portion of the sphere of radius t (centered at P) that lies inside the balloon [see Figure 3.32(b)]. Using trigonometric identities, it is easily seen that $\cos\theta = (t^2 + r^2 - 1)/2rt$. Therefore, the area of S is

$$A \equiv 2\pi t^2 \int_0^\theta \sin\theta' d\theta' = \frac{\pi t}{r}[1 - (r - t)^2],$$

and we find that

$$\langle f \rangle_t = \frac{A\left(-\dfrac{1}{\gamma}\right)}{4\pi t^2} = -\frac{1 - (r - t)^2}{4\gamma r t} \tag{3.9.36}$$

if $r - 1 < t < r + 1$ and that $\langle f \rangle_t$ is zero otherwise.

Equation (3.9.30) for the disturbance potential becomes

$$\phi(r, t) = \begin{cases} -\dfrac{1}{4\gamma}\left[\dfrac{(1 - (r - t)^2}{r}\right] & \text{if } r - 1 < t < r + 1, \\ 0 & \text{otherwise,} \end{cases} \tag{3.9.37}$$

and the overpressure $\epsilon\tilde{p}$ is defined by

$$\tilde{p} = -\gamma\phi_t(r, t) = \begin{cases} \dfrac{r - t}{2r} & \text{if } r - 1 < t < r + 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.9.38}$$

Because of spherical symmetry, this problem can also be solved directly using the general solution (3.9.9) (see Problem 3.9.4).

Figure 3.33 shows a sketch of $\tilde{p}$ as a function of time at the *fixed position* r. We find a symmetric N-wave consisting of a sudden rise in overpressure to the value $1/2r$, followed by a linear drop to the value $-1/2r$ and a sudden rise to the zero level.

Now, suppose there is a vertical wall at a distance $r_2 > r$ from the origin. What is the reflected noise? The boundary condition to be satisfied at $r = r_2$ is $\phi_r(r_2, t) = 0$, that is, no flow normal to the wall at the wall. We can satisfy this boundary condition by introducing an image balloon at a distance $2r_2 - r$ to the right of the observer and bursting it at $t = 0$. The resulting reflected noise is then obtained by replacing r in (3.9.38) by $(2r_2 - r)$. Thus,

$$\tilde{p}_{\text{reflected}} = \begin{cases} \dfrac{2r_2 - r - t}{4r_2 - 2r} & \text{if } 2r_2 - r - 1 < t < 2r_2 - r + 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.9.39}$$

Of course, if t satisfies both inequalities in (3.9.38) and (3.9.39), we must also add the primary source contribution of (3.9.38).

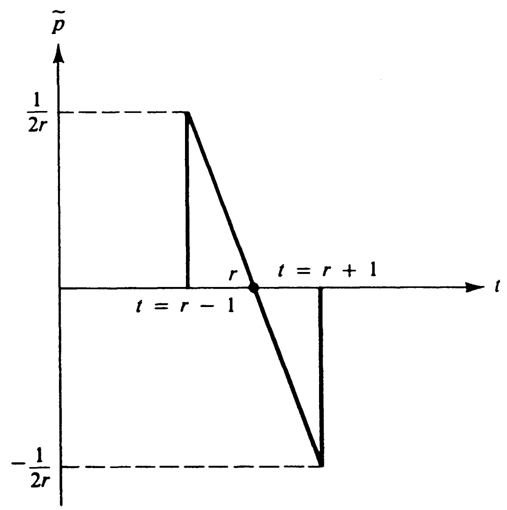

FIGURE 3.33. N-wave for a bursting balloon

(ii) *Source distribution over the plane*

Consider the initial- and boundary-value problem

$$\Delta\phi - \phi_{tt} = 0 \quad \text{on } 0 \le y \tag{3.9.40a}$$

with initial conditions

$$\phi(x, y, z, 0^+) = \phi_t(x, y, z, 0^+) = 0 \tag{3.9.40b}$$

and boundary conditions

$$\phi_y(x, 0^+, z, t) = h(x, z, t), \quad t > 0, \tag{3.9.40c}$$

$$\phi(x, \infty, z, t) = 0. \tag{3.9.40d}$$

It is clear from symmetry that we can replace (3.9.40) by the problem on $-\infty < y < \infty$ by appending the boundary conditions

$$\phi_y(x, 0^-, z, t) = -h(x, z, t), \quad t > 0, \tag{3.9.40e}$$

$$\phi(x, -\infty, z, t) = 0. \tag{3.9.40f}$$

Let us attempt to solve (3.9.40) by introducing a source sheet of strength/unit area S equal to $2h(x, z, t)$ on the $y = 0$ plane; that is, we claim that the initial-value problem

$$\Delta\phi - \phi_{tt} = 2\delta(y)h(x, z, t), \quad -\infty < y < \infty, \tag{3.9.41a}$$

$$\phi(x, y, z, 0^+) = \phi_t(x, y, z, 0^+) = 0, \tag{3.9.41b}$$

with sources as indicated on the $y = 0$ plane, is equivalent to (3.9.40).

It is clear that the requirement $S = 2\delta(y)h$ is *necessary* in order that (3.9.41a) produce the correct boundary conditions at $y = 0^+$ and $y = 0^-$. This follows by integrating (3.9.41a) from $y = 0^-$ to $y = 0^+$ to obtain

$$\phi_y(x, 0^+, z, t) - \phi_y(x, 0^-, z, t) = 2h(x, z, t), \tag{3.9.42}$$

after noting that ϕ_{xx}, ϕ_{zz}, and ϕ_{tt} are continuous at $y = 0$ and hence do not contribute to (3.9.42). The above is *not enough*; we must also show that the solution of (3.9.41) tends to the *individual* limits (3.9.40c) and (3.9.40e).

The solution of (3.9.41) is a special case of (3.9.18) with $S = 2\delta(y)h$. Therefore,

$$\phi(x, y, z, t) = -\frac{1}{2\pi} \iiint\limits_{r_{PQ} \le t} \frac{\delta(\eta)h(\xi, \zeta, t - r_{PQ})}{r_{PQ}} \, d\xi \, d\eta \, d\zeta, \tag{3.9.43a}$$

and the delta function restricts the domain of integration to the intersection of the sphere $r_{PQ} \le t$ with the plane $\eta = 0$. Thus,

$$\phi(x, y, z, t) = -\frac{1}{2\pi} \iint\limits_{r_{PQ_0} \le t} \frac{h(\xi, \zeta, t - r_{PQ_0})}{r_{PQ_0}} \, d\xi \, d\zeta, \tag{3.9.43b}$$

where $r_{PQ_0}^2 = (x - \xi)^2 + y^2 + (z - \zeta)^2$.

As shown in Figure 3.34, the sphere of radius t centered at $P = (x, y, z)$ intersects with the $\eta = 0$ plane to form the circular disk D centered at $\xi = x$, $\eta = 0$, $\zeta = z$ and having radius $\sqrt{t^2 - y^2}$. The integral in (3.9.43b) is evaluated over D. Therefore, it is convenient to introduce the polar coordinates in the plane of D defined by $\xi - x = \rho \cos\theta$, $\zeta - z = \rho \sin\theta$.

The integral (3.9.43b) then becomes

$$\phi(x, y, z, t) =$$
$$-\frac{1}{2\pi} \int_{\rho=0}^{\sqrt{t^2-y^2}} \int_{\theta=0}^{2\pi} \frac{h(x + \rho\cos\theta, z + \rho\sin\theta, t - \sqrt{\rho^2 + y^2})}{\sqrt{\rho^2 + y^2}} \rho \, d\theta \, d\rho. \tag{3.9.44}$$

The derivative of this expression with respect to y leads to an improper integral with respect to ρ as $y \to 0^+$. This situation is entirely analogous to the one discussed in Section 2.4.4 for a surface distribution of sources for Laplace's equation. Therefore, as in (2.4.33), we split the integration with respect to ρ into a contribution over the small disk of radius $\rho = \epsilon$ plus the contribution over the annulus $\epsilon \le \rho \le \sqrt{t^2 - y^2}$. Denoting these contributions to ϕ by ϕ_ϵ and ϕ_a, we have

$$\phi_\epsilon(x, y, z, t; \epsilon) = -\frac{1}{2\pi} \int_{\rho=0}^{\epsilon} \int_{\theta=0}^{2\pi} \frac{h\rho \, d\theta \, d\rho}{(\rho^2 + y^2)^{1/2}}, \tag{3.9.45a}$$

$$\phi_a(x, y, z, t; \epsilon) = -\frac{1}{2\pi} \int_{\rho=\epsilon}^{\sqrt{t^2-y^2}} \int_{\theta=0}^{2\pi} \frac{h\rho \, d\theta \, d\rho}{(\rho^2 + y^2)^{1/2}}, \tag{3.9.45b}$$

where the arguments of h are as indicated in (3.9.44), and $\phi = \phi_\epsilon + \phi_a$.

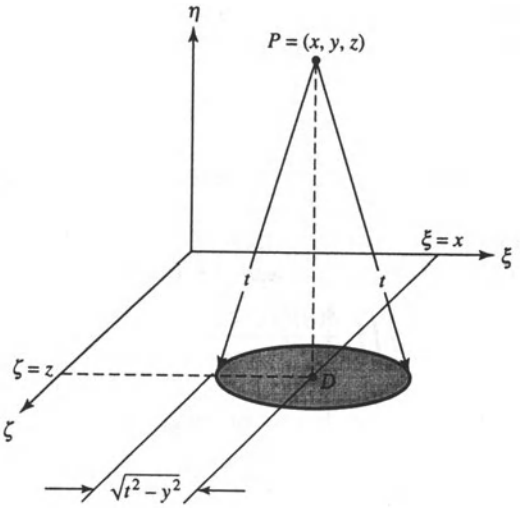

FIGURE 3.34. Integration domain for (3.9.44)

We define the limit as $y \to 0^+$ as in (2.4.26) with y and z interchanged. Now, it is easily seen that $\partial\phi_a/\partial y \to 0$ as $\epsilon \to 0$, as long as $(y/\epsilon) \to 0$. To evaluate $\partial\phi_\epsilon/\partial y$ as ϵ and y tend to zero, we approximate ϕ_ϵ for ϵ small and then take the derivative of the result. We have

$$\begin{aligned}
\phi_\epsilon(x, y, z, t; \epsilon) &= -\frac{1}{2\pi}\int_{\rho=0}^{\epsilon}\int_{\theta=0}^{2\pi}\frac{[h(x, z, t-y) + O(\rho)]\rho\, d\theta\, d\rho}{(\rho^2 + y^2)^{1/2}} \\
&= -h(x, z, t-y)\int_{\rho=0}^{\epsilon}\frac{\rho\, d\rho}{(\rho^2+y^2)^{1/2}} + O\left(\int_0^{\epsilon}\frac{\rho^2 d\rho}{(\rho^2+y^2)^{1/2}}\right) \\
&= -h(x, z, t-y)[\sqrt{\epsilon^2 + y^2} - y] + O(\epsilon y) \quad \text{as } y \to 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial\phi_\epsilon}{\partial y}(x, y, z, t; \epsilon) = {}& h_t(x, z, t-y)[(\epsilon^2 + y^2)^{1/2} - y] \\
& - h(x, z, t-y)[y(\epsilon^2 + y^2)^{-1/2} - 1] + O(\epsilon).
\end{aligned}$$

In the limit $\epsilon \to 0$, $(y/\epsilon) \to 0^+$, we obtain

$$\lim_{\substack{\epsilon\to 0\\ (y/\epsilon)\to 0^+}} \frac{\partial\phi_\epsilon}{\partial y}(x, y, z, t; \epsilon) = h(x, z, t).$$

Thus,

$$\phi_y(x, 0^+, z, t) = \lim_{\substack{\epsilon\to 0\\ (y/\epsilon)\to 0^+}} \frac{\partial\phi_\epsilon}{\partial y}(x, y, z, t; \epsilon) = h(x, z, t). \tag{3.9.46}$$

Similarly, $\phi_y(x, 0^-, z, t) = -h(x, z, t)$, and this completes the proof that (3.9.40c) and (3.9.40e) are satisfied. Therefore, the solution is given by (3.4.44).

Problems

3.9.1 This problem concerns various aspects of the two-dimensional wave equation.

a. Work out the details leading from (3.9.18), in which S is given by (3.9.21), to obtain (3.9.22). Next, use similarity arguments to derive (3.9.23) directly for the two-dimensional wave equation.

b. Show that $\Phi(r, s)$, the Laplace transform of the fundamental solution of the two-dimensional wave equation, is given by

$$\Phi(r, s) = -\frac{1}{2\pi} K_0(sr), \tag{3.9.47}$$

where $r^2 = x^2 + y^2$, s is the Laplace transform variable, and K_0 is the modified Bessel function of the second kind of order zero. Use tables of Laplace transforms to show that the inversion formula for (3.9.47) gives (3.9.23).

c. Now use Fourier transforms with respect to x and y to show that $\overline{\phi}(k_1, k_2, t)$, the double Fourier transform of the fundamental solution of the two-dimensional wave equation, is given by

$$\overline{\phi}(k_1, k_2, t) = -\frac{1}{2\pi} \frac{\sin kt}{k}, \quad k^2 = (k_1^2 + k_2^2). \tag{3.9.48}$$

Then introduce polar coordinates in the inversion integral and show that it simplifies to

$$\phi(r, t) = -\frac{1}{2\pi} \int_0^\infty J_0(kr) \sin kt\, dt \tag{3.9.49a}$$

when you use the integral representation

$$J_0(kr) \equiv \frac{1}{\pi} \int_0^\pi \cos(kr \sin\theta) d\theta \tag{3.9.49b}$$

for the Bessel function of the first kind of order zero. Use integral tables to show that (3.9.49a) gives (3.9.23).

d. The Hankel transform of a function $f(r)$ is denoted by $f^*(\omega)$ and is defined as

$$f^*(\omega) \equiv \int_0^\infty J_0(\omega r) f(r) r\, dr, \tag{3.9.50a}$$

whenever the integral exists. Show that

$$f(r) = \int_0^\infty J_0(\omega r) f^*(\omega) d\omega \tag{3.9.50b}$$

is the corresponding inversion integral. Use (3.9.50a) to show that the Hankel transform of the fundamental solution of the two-dimensional wave equation is

$$\phi^*(\omega, t) = -\frac{1}{2\pi\omega} \sin \omega t. \tag{3.9.51}$$

Therefore, the inversion integral (3.9.50b) directly gives the previously obtained result (3.9.49b).

e. Construct Green's function for the two-dimensional wave equation in the corner domain $x \geq 0$, $y \geq 0$ with zero initial and boundary values; that is, solve

$$u_{tt} - u_{xx} - u_{yy} = \delta(x - \xi)\delta(y - \eta)\delta(t - \tau), \tag{3.9.51a}$$

$$u(x, y, \tau^-) = u_t(x, y, \tau^-) = 0, \tag{3.9.51b}$$

$$u(0, y, t) = u(x, 0, t) = 0, \ t > \tau, \tag{3.9.51c}$$

where ξ, η, and τ are positive constants.

Use this result to obtain an integral representation for the solution of

$$u_{tt} - u_{xx} - u_{yy} = p(x, t), \tag{3.9.52a}$$

$$u(x, y, 0) = f(x, y), \ u_t(x, y, 0) = g(x, y), \tag{3.9.52b}$$

$$u(x, 0, t) = u(0, y, t) = 0, \ t > 0, \tag{3.9.52c}$$

on $x \geq 0, t \geq 0$.

Hint: Consider three separate problems as in Section 3.5.2 for the one-dimensional problem, and introduce a transformation analogous to (3.5.12) to reduce the problem for $f \neq 0, g = 0$ to one with $f = 0, g \neq 0$.

f. Now consider the case of inhomogeneous boundary conditions and solve

$$u_{tt} - u_{xx} - u_{yy} = 0, \tag{3.9.53a}$$

$$u(x, y, 0) = u_t(x, y, 0) = 0, \tag{3.9.53b}$$

$$u(x, 0, t) = k(x, t), \ u(0, y, t) = \ell(y, t), \ t > 0, \tag{3.9.53c}$$

on $x \geq 0, y \geq 0$.

Hint: Transform $u(x, y, t)$ to a new dependent variable $w(x, y, t)$ defined in the form

$$u(x, y, t) \equiv w(x, y, t) + \alpha(x, y, t) \tag{3.9.54}$$

for some α such that $w(x, 0, t) = w(0, y, t) = 0$. One possible choice for α is

$$\alpha(x, y, t) = \frac{k(x, t)x}{(x^2 + y^2)^{1/2}} + \frac{\ell(y, t)y}{(x^2 + y^2)^{1/2}}. \tag{3.9.55}$$

g. Use symmetry arguments to construct Green's function for the two-dimensional wave equation in the strip $-\infty < x < \infty, 0 \leq y \leq 1$

with zero boundary values; that is, solve

$$u_{tt} - u_{xx} - u_{yy} = \delta(x - \xi)\delta(y - \eta)\delta(t - \tau), \tag{3.9.56a}$$

$$u(x, y, \tau^-) = u_t(x, y, \tau^-) = 0, \tag{3.9.56b}$$

$$u(x, 0, t) = u(x, 1, t) = 0, \; t > \tau. \tag{3.9.56c}$$

As in (e), use this result to calculate solutions with arbitrarily prescribed initial and boundary values.

3.9.2 Consider the potential ϕ due to a unit mass source moving with constant speed M along the x-axis. Thus, ϕ obeys

$$\Delta\phi - \phi_{tt} = \delta(x + Mt)\delta(y)\delta(z). \tag{3.9.57}$$

a. For $M < 1$, use the superposition integral (3.9.17) with the limits $-\infty$ to ∞ on τ to show that

$$\phi(x, y, z, t) = -\frac{1}{4\pi} \frac{1}{[(x + Mt)^2 + (1 - M^2)(y^2 + z^2)]^{1/2}}. \tag{3.9.58}$$

Rederive this result as the fundamental solution of Laplace's equation in appropriate stretched variables.

b. For $M > 1$, introduce the Galilean transformation $x^* = Mt + x$, $y^* = y$, $z^* = z$, $\phi^* = \phi$, and then use (3.9.23) to solve the resulting problem and obtain

$$\phi(x, y, z, t) = \frac{-1}{2\pi} \frac{H[x + Mt - \sqrt{(M^2 - 1)(y^2 + z^2)}]}{[(x + Mt)^2 - (M^2 - 1)(y^1 + z^2)]^{1/2}}. \tag{3.9.59}$$

3.9.3 Consider the signaling problem for acoustic disturbances down a semi-infinite wave-guide with a square cross section. The wave-guide occupies $0 \le x \le \pi, 0 \le y \le \pi, 0 \le z < \infty$, and the boundary condition on the walls is the usual one of zero normal velocity there. The flow is initially at rest, and at $t = 0$ we start sending a given sinusoidal signal at the left end. So, the velocity potential satisfies

$$\Delta\phi - \phi_{tt} = 0, \; 0 \le x \le \pi, \; 0 \le y \le \pi, \; 0 \le z, \; 0 \le t, \tag{3.9.60a}$$

$$\phi(x, y, z, 0) = \phi_t(x, y, z, 0) = 0, \tag{3.9.60b}$$

$$\phi_x(0, y, z, t) = \phi_x(\pi, y, z, t) = 0, \tag{3.9.60c}$$

$$\phi_y(x, 0, z, t) = \phi_y(x, \pi, z, t) = 0, \tag{3.9.60d}$$

$$\phi_z(x, y, 0, t) = A(x, y) \sin \omega t, \; t > 0, \tag{3.9.60e}$$

where $A(x, y)$ is a prescribed function and ω is a constant.

a. Use separation of variables with respect to the x and y dependence and Laplace transforms with respect to t to show that

$$\phi(x, y, z, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \psi_{mn}(z, t) \cos mx \cos ny, \tag{3.9.61}$$

where

$$\psi_{mn}(z, t) = -A_{mn} \int_z^t J_0(\sqrt{(m^2 + n^2)(\tau^2 - z^2)}) \sin \omega(t - \tau) d\tau \tag{3.9.62}$$

if $t > z$, and $\psi_{mn} = 0$ if $t < z$. Here J_0 is the Bessel function of the first kind of order zero and

$$A_{mn} = \frac{4}{\pi^2} \int_0^\pi \int_0^\pi A(x, y) \cos mx \cos ny \, dx \, dy \tag{3.9.63a}$$

if $m \neq 0,\ n \neq 0$, and

$$A_{m0} = \frac{2}{\pi^2} \int_0^\pi \int_0^\pi A(x, y) \cos mx \, dx \, dy, \tag{3.9.63b}$$

$$A_{0n} = \frac{2}{\pi^2} \int_0^\pi \int_0^\pi A(x, y) \cos ny \, dx \, dy, \tag{3.9.63c}$$

$$A_{00} = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi A(x, y) dx \, dy. \tag{3.9.63d}$$

b. For $t \to \infty$, z fixed, assume that each mode ψ_{mn} has the form

$$\psi_{mn}(z, t) = \alpha_{mn} \cos \omega \left(t - \frac{z}{c_{mn}} \right). \tag{3.9.64}$$

Show by substitution into the equation governing ψ_{mn} that the constants c_{mn}, α_{mn} are given by

$$c_{mn} = \frac{\omega}{[\omega^2 - (m^2 + n^2)]^{1/2}}, \tag{3.9.65a}$$

$$\alpha_{mn} = \frac{A_{mn}}{[\omega^2 - (m^2 + n^2)]^{1/2}}. \tag{3.9.65b}$$

Comment on the nonexistence of such solutions if ω is smaller than the "cutoff" frequency $\omega_c \equiv \sqrt{m^2 + n^2}$ associated with the given mode. Comment on the relationship of your result with the solution in (a).

3.9.4 Consider the balloon problem discussed in Section 3.9.4i. The solution for ϕ is spherically symmetric. Therefore, the velocity potential depends only on r and t and has the form [see (3.9.9)]

$$\phi(r, t) = \frac{1}{r} \phi_1(r - t) + \frac{1}{r} \phi_2(r + t). \tag{3.9.66}$$

The initial conditions are

$$\phi(r, 0) = 0, \tag{3.9.67a}$$

$$\phi_t(r, 0) = \begin{cases} -\dfrac{1}{\gamma} & \text{if } r < 1, \\ 0 & \text{if } r > 1. \end{cases} \tag{3.9.67b}$$

Show that equations (3.9.67) determine ϕ_1 and ϕ_2 for positive values of their arguments. Since the argument of ϕ_1 may be negative, we need one more condition. This is obtained by recalling that the homogeneous wave equation (3.9.1) corresponds to *zero mass sources*. In particular, since there is no source at the origin, we must have

$$\lim_{r \to 0} r^2 \phi_r(r, t) = 0, \ t \geq 0. \tag{3.9.68}$$

Thus, the radial velocity cannot grow at a rate faster than r^{-2} as $r \to 0$. Show that (3.9.68) determines ϕ_1 for negative values of its argument in terms of ϕ_2 for positive argument, and that ϕ_1 and ϕ_2 are given by

$$\phi_1(z) = \begin{cases} \frac{1}{4\gamma}(z^2 - 1) & \text{if } -1 < z < 1, \\ 0 & \text{if } |z| > 1, \end{cases} \tag{3.9.69a}$$

$$\phi_2(z) = \begin{cases} -\frac{1}{4\gamma}(z^2 - 1) & \text{if } 0 < z < 1, \\ 0 & \text{if } 1 < z. \end{cases} \tag{3.9.69b}$$

Verify that using (3.9.69) in (3.9.66) gives (3.9.37) if $r > 1$. Calculate the solution for $r < 1$ and sketch the variation of $\tilde{p}$ with time for this case.

3.9.5 A point source of mass with time-dependent strength $f(t)$ is located at the point $x = 0$, $y = 1$, $z = 0$. The source strength $f(t)$ is zero if $t < 0$, and $f(t)$ is prescribed for $t > 0$. Use linear acoustics to calculate the velocity potential in the half-space $y \geq 0$ for the case where there is a rigid boundary at $y = 0$. Thus, the component of the flow velocity normal to the boundary vanishes. Calculate the pressure perturbation on $y = 0^+$.

3.9.6 The equations governing shallow-water flow with axial symmetry over a flat horizontal bottom are (see (3.2.57))

$$u_t + (hu)_r + \frac{1}{r} hu = 0, \tag{3.9.70a}$$

$$u_t + uu_r + h_r = 0, \tag{3.9.70b}$$

where h is the free surface height and u is the radial speed.

a. Consider the initial-value problem on $0 \leq r < \infty$ defined by

$$h(r, 0; \epsilon) = \begin{cases} 1 & \text{if } r > 1, \\ 1 + \epsilon & \text{if } r < 1, \end{cases} \quad u(r, 0; \epsilon) = 0, \tag{3.9.71}$$

where $0 < \epsilon \ll 1$. Assume an expansion for u and h in the form

$$h(r, t; \epsilon) = 1 + \epsilon h_1(r, t) + O(\epsilon), \tag{3.9.72a}$$

$$u(r, t; \epsilon) = \epsilon u_1(r, t) + O(\epsilon), \tag{3.9.72b}$$

and show that h_1 and u_1 obey

$$h_{1_t} + u_{1_r} + \frac{1}{r} u_1 = 0, \ u_{1_t} + h_{1_r} = 0, \tag{3.9.73a}$$

$$h_1(r, 0) = \begin{cases} 0 & \text{if } r > 1, \\ 1 & \text{if } r < 1, \end{cases} \quad u_1(r, 0) = 0. \tag{3.9.73b}$$

Thus, h_1 obeys the two-dimensional axisymmetric wave equation

$$h_{1_{tt}} - h_{1_{rr}} - \frac{1}{r} h_1 = 0 \tag{3.9.74}$$

with $h_1(r, 0)$ as given in (3.9.73b), and $h_{1_t}(r, 0) = 0$.

b. Let

$$v(r, t) = \int_0^t h_1(r, \tau) d\tau \tag{3.9.75}$$

and show that v obeys

$$v_{tt} - v_{rr} - \frac{1}{r} v_r = 0, \tag{3.9.76a}$$

$$v(r, 0) = 0, \ v_t(r, 0) = \begin{cases} 0 & \text{if } r > 1, \\ 1 & \text{if } r < 1 \end{cases} \equiv \tilde{v}. \tag{3.9.76b}$$

Use the fundamental solution (3.9.23) to express the solution of (3.9.76) in the form

$$v(r, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{H(t - r_{PQ}) \tilde{v}(\xi, \eta)}{\sqrt{t^2 - r_{PQ}^2}} d\xi \, d\eta, \tag{3.9.77}$$

where $r_{PQ}^2 = (x - \xi)^2 + (y - \eta)^2$ and $\tilde{v}$ is defined in (3.9.76b).

c. For a fixed point $P = r > 0$, introduce polar coordinates centered at P and show that for $0 \leq t \leq 1$, $h_1 = v_t$ has the following values obtained from (3.9.77):

$$h_1 = 0 \quad \text{if } r > t + 1, \ h_1 = 1 \quad \text{if } 0 \leq r \leq 1 - t, \tag{3.9.78a}$$

$$h_1 = -\frac{1}{\pi} \int_{r-1}^{t} \frac{t(\rho^2 + 1 - r^2)}{\rho \sqrt{(t^2 - \rho^2)[4r^2\rho^2 - (r^2 + \rho^2 - 1)^2]}} d\rho \tag{3.9.78b}$$

if $1 \leq r \leq 1 + t$,

$$h_1 = 1 - \frac{1}{\pi} \int_0^{\theta_0} \frac{t}{\sqrt{t^2 - \rho_1^2(\theta)}} d\theta \quad \text{if } 1 - t \leq r \leq 1, \tag{3.9.78c}$$

where

$$\theta_0 = \cos^{-1} \left(\frac{1 - r^2 - t^2}{2rt} \right), \ \rho_1(\theta) = -r \cos\theta + \sqrt{1 - r^2 \sin^2\theta}. \tag{3.9.79}$$

Derive the corresponding expressions for h_1 for $1 < t < 2$ and for $t \geq 2$.

4

Linear Second-Order Equations with Two Independent Variables

In this chapter we first consider the general linear second-order partial differential equation in two independent variables x, y and show that, depending upon the values of the coefficients multiplying the second derivatives, it can be transformed to one of the three canonical forms discussed in Chapters 1–3. In the remainder of the chapter we concentrate on the hyperbolic case, for which an appropriate choice of independent variables ξ, η leads to a transformed equation where the only remaining second-derivative term is $\partial^2 u/\partial\xi\partial\eta$. We also study the corresponding problem for a pair of linear first-order partial differential equations. The quasilinear problem, for which the coefficients also involve the dependent variable, is discussed in Chapter 5 for a single first-order equation, and in Chapter 7 for systems of first order.

4.1 Classification and Transformation to Canonical Form

4.1.1 A General Transformation of Variables

The general linear second-order equation in the two independent variables x, y is

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g, \tag{4.1.1}$$

where a, b, c, d, e, f, and g are prescribed functions of x and y. We denote the linear differential operator by L and write (4.1.1) as

$$L(u) + fu = g, \tag{4.1.2}$$

where

$$L \equiv a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x\partial y} + c\frac{\partial^2}{\partial y^2} + d\frac{\partial}{\partial x} + e\frac{\partial}{\partial y}. \tag{4.1.3}$$

Suppose that (ξ, η), where ξ and η are given by

$$\xi = \phi(x, y), \ \ \eta = \psi(x, y), \tag{4.1.4}$$

is an *arbitrary* curvilinear coordinate system that we wish to use instead of the original system (x, y). A necessary condition on the functions ϕ and ψ to ensure that (4.1.4) is a coordinate transformation—that is, that for every point (x, y) there corresponds a unique point (ξ, η) and vice versa—is that the Jacobian

$$J(x, y) \equiv \phi_x \psi_y - \psi_x \phi_y \tag{4.1.5}$$

does not vanish identically in the domain of interest. Let $U(\xi, \eta)$ denote the dependent variable regarded as a function of the new independent variables (ξ, η); that is,

$$u(x, y) \equiv U(\phi(x, y), \psi(x, y)). \tag{4.1.6}$$

We now calculate

$$\begin{aligned}
u_x &= U_\xi \phi_x + U_\eta \psi_x, \quad u_y = U_\xi \phi_y + U_\eta \psi_y, \\
u_{xx} &= U_{\xi\xi}\phi_x^2 + 2U_{\xi\eta}\phi_x\psi_x + U_{\eta\eta}\psi_x^2 + U_\xi \phi_{xx} + U_\eta \psi_{xx}, \\
u_{yy} &= U_{\xi\xi}\phi_y^2 + 2U_{\xi\eta}\phi_y\psi_y + U_{\eta\eta}\psi_y^2 + U_\xi \phi_{yy} + U_\eta \psi_{yy}, \\
u_{xy} &= U_{\xi\xi}\phi_x\phi_y + U_{\xi\eta}(\phi_x\psi_y + \psi_x\phi_y) + U_{\eta\eta}\psi_x\psi_y + U_\xi \phi_{xy} + U_\eta \psi_{xy}.
\end{aligned}$$

Therefore, (4.1.2) transforms to

$$M(U) + FU = G, \tag{4.1.7}$$

where the linear operator M has the form

$$M \equiv A\frac{\partial^2}{\partial \xi^2} + 2B\frac{\partial^2}{\partial \xi \partial \eta} + C\frac{\partial^2}{\partial \eta^2} + D\frac{\partial}{\partial \xi} + E\frac{\partial}{\partial \eta}. \tag{4.1.8}$$

Here A, B, and C are the following quadratic forms:

$$B(\xi, \eta) \equiv a\phi_x\psi_x + b(\phi_x\psi_y + \psi_x\phi_y) + c\phi_y\psi_y, \tag{4.1.9a}$$

$$A(\xi, \eta) \equiv a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2, \tag{4.1.9b}$$

$$C(\xi, \eta) \equiv a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2. \tag{4.1.9c}$$

The functions D and E are given by [see (4.1.3)]

$$D(\xi, \eta) = L(\phi), \ E(\xi, \eta) = L(\psi) \tag{4.1.10a}$$

and

$$F(\xi, \eta) = f, \ G(\xi, \eta) = g. \tag{4.1.10b}$$

Note that to compute A, B, or C we must evaluate the functions of x and y defined by the right-hand sides of (4.1.9) and then express x and y in terms of ξ, η. This latter step is always possible for a coordinate transformation, since $J \neq 0$. In deriving D and E in (4.1.10a), we must evaluate the second-order differential operator L for the given functions $\phi(x, y)$ and $\psi(x, y)$. The resulting functions of x and y are then expressed in terms of ξ and η. We also express x and y in terms of ξ, η in the right-hand sides of (4.1.10b) to obtain F and G.

A general property of the transformation (4.1.4) is that the sign of $b^2 - ac$ is the same as the sign of $(B^2 - AC)$ because it follows from (4.1.9) that

$$\Delta \equiv b^2 - ac = \frac{B^2 - AC}{J^2}. \tag{4.1.11}$$

We shall see next that depending on whether $\Delta > 0$, $\Delta < 0$, or $\Delta = 0$, we can choose a corresponding particular transformation (4.1.4) that reduces (4.1.7) to a simple canonical form.

4.1.2 The Hyperbolic Problem, $\Delta > 0$; $A = C = 0$

Recalling the D'Alembert form of the wave equation, (3.7.5), it is natural to seek the conditions for which one is able to reduce (4.1.7) to this simple form with regard to the dependence on the second derivatives. Clearly, if we can find functions ϕ and ψ such that $A = C = 0$ and $B \neq 0$, we can divide (4.1.7) by B to obtain the desired result.

We see from (4.1.9b) and (4.1.9c) that the equation governing ϕ that results from setting $A = 0$ is the same as the equation governing ψ in order to have $C = 0$. So, it is sufficient to consider either one of these equations, say,

$$a(x, y)\phi_x^2 + 2b(x, y)\phi_x\phi_y + c(x, y)\phi_y^2 = 0. \tag{4.1.12}$$

Now, this nonlinear first-order partial differential equation for ϕ is less formidable than it looks. Consider a level curve $\phi(x, y) = \xi_0 =$ constant on any solution surface $\xi = \phi(x, y)$ of (4.1.12) as sketched in Figure 4.1.

On this level curve, the slope $y' \equiv (dy/dx)$ is given by $y' = -\phi_x(x, y)/\phi_y(x, y)$ if $\phi_y \neq 0$. Assuming temporarily that $\phi_y \neq 0$, we divide (4.1.12) by ϕ_y to obtain

$$a\frac{\phi_x^2}{\phi_y^2} + 2b\frac{\phi_x}{\phi_y} + c = 0, \tag{4.1.13a}$$

or

$$ay'^2 - 2by' + c = 0. \tag{4.1.13b}$$

Solving the quadratic expression (4.1.13b) for y' gives a real result only if $\Delta \geq 0$. We defer discussion of the case $\Delta = 0$ to Section 4.1.3 and note here that a real transformation of variables that renders $A = C = 0$ does not exist if $\Delta < 0$. In the case $\Delta > 0$, we have two real and distinct roots:

$$y' = \frac{b + \sqrt{\Delta}}{a}, \quad y' = \frac{b - \sqrt{\Delta}}{a}. \tag{4.1.14}$$

The solution of the two ordinary differential equations (4.1.14) defines two distinct families of level curves, as shown next. Since the solution of the equation $C = 0$ gives the same pair of families, we may identify the curves $\phi =$ constant with one sign for $\sqrt{\Delta}$, say the plus sign, and let the curves $\psi =$ constant correspond to the minus sign in (4.1.14).

Thus, the canonical form

$$U_{\xi\eta} = \tilde{D}U_\xi + \tilde{E}U_\eta + \tilde{F}U + \tilde{G}, \tag{4.1.15}$$

where

$$\tilde{D} = -\frac{D}{2B}, \quad \tilde{E} = -\frac{E}{2B}, \quad \tilde{F} = -\frac{F}{2B}, \quad \tilde{G} = \frac{G}{2B}, \tag{4.1.16}$$

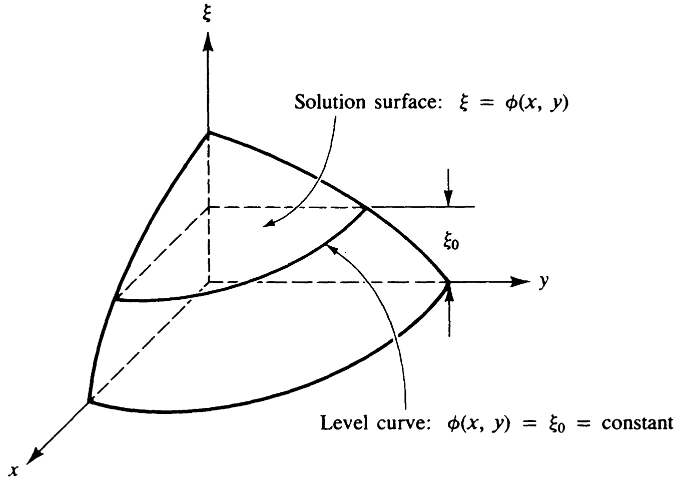

FIGURE 4.1. Level curve $\phi(x, y) = \xi$ = constant

is indeed possible if $\Delta > 0$.

We now verify that the two families of curves ϕ = constant and ψ = constant so defined do indeed represent a coordinate transformation, that is, $J \neq 0$. If we divide (4.1.5) by $\phi_y \psi_y \neq 0$ (recall that we have assumed that neither ϕ_y nor ψ_y vanishes), we obtain

$$\frac{J}{\phi_y \psi_y} = \frac{\phi_x}{\phi_y} - \frac{\psi_x}{\psi_y} = -\frac{b}{a} - \frac{\sqrt{b^2 - ac}}{a} + \frac{b}{a} - \frac{\sqrt{b^2 - ac}}{a}$$
$$= -\frac{2\sqrt{b^2 - ac}}{a} \neq 0,$$

and hence, $J \neq 0$. The two families of curves ϕ = constant and ψ = constant are called *characteristic curves* [see (3.4.22)].

If $\phi_y \equiv 0$, then (4.1.12) implies that $\phi_x \equiv 0$ also, and we have the trivial solution ϕ = constant, which always satisfies (4.1.2). So, we exclude this case. If $a = 0$, we may interchange the roles of x and y so that the divisor in (4.1.14) is c. If $c = 0$ also, we need not transform (4.1.2), because it is already in canonical form. Also note that $B \equiv 0$ implies that $\Delta \equiv 0$, as can be seen by substituting (4.1.14) for ϕ_x/ϕ_y and ψ_x/ψ_y in (4.1.9a). Therefore, $B \neq 0$, and division by $2B$ is not troublesome in (4.1.16).

We have shown that if $\Delta > 0$, we can always define a transformation of variables (4.1.4) so that the equation governing U is in the canonical form (4.1.15). Depending upon the functions $a(x, y)$, $b(x, y)$, and $c(x, y)$, there may or may not exist a domain $\mathcal{D}$ in the xy-plane in which $\Delta > 0$. If $\mathcal{D}$ exists, (4.1.2) is said to be hyperbolic in $\mathcal{D}$. Notice that the existence of $\mathcal{D}$ depends only on the coefficients a, b, and c of the second-derivative terms and not on the coefficients d, e, f, and g.

Moreover, this property is independent of the choice of variables used to express (4.1.2) since the sign of Δ is coordinate-invariant.

We can also transform (4.1.15) to the alternative canonical form [see (3.7.1)]

$$V_{XX} - V_{YY} = D^* V_X + E^* V_Y + F^* V + G^* \tag{4.1.17}$$

by introducing the transformation

$$X = \xi + \eta, \quad Y = \xi - \eta, \quad V(X, Y) = U\left(\frac{X+Y}{2}, \frac{X-Y}{2}\right).$$

We obtain

$$D^* = \tilde{D} + \tilde{E}, \quad E^* = \tilde{D} - \tilde{E}, \quad F^* = \tilde{F}, \quad G^* = \tilde{G}. \tag{4.1.18}$$

In general, it is not possible to simplify (4.1.15) or (4.1.17) further. The exception occurs when $\tilde{D}$ and $\tilde{E}$ satisfy the condition $\tilde{D}_\xi = \tilde{E}_\eta$, in which case we can also eliminate the first-derivative terms (see Problem 3.8.1b and 4.1.2). In particular, the condition $\tilde{D}_\xi = \tilde{E}_\eta$ holds trivially if D and E are constant.

(i) *Example: Hyperbolic equation with constant coefficients*

The general hyperbolic equation with *constant* coefficients has a, b, c, d, e, and f all constant in (4.1.1) with $\Delta = b^2 - ac > 0$. We obtain

$$y' = \frac{b \pm \sqrt{\Delta}}{a} = \alpha_1, \ \alpha_2, \tag{4.1.19}$$

where α_1 and α_2 are unequal constants. Therefore, the characteristics are the straight lines $\xi =$ constant and $\eta =$ constant defined by

$$\xi = y - \alpha_1 x, \quad \eta = y - \alpha_2 x. \tag{4.1.20}$$

Thus, $\phi_x = -\alpha_1, \phi_y = 1, \psi_x = -\alpha_2, \psi_y = 1$, and $\phi_{xx} = \phi_{xy} = \phi_{yy} = \psi_{xx} = \psi_{xy} = \psi_{yy} = 0$. Using (4.1.9a) and (4.1.10a), we compute

$$2B = \frac{4(ac - b^2)}{a}, \quad D = -\alpha_1 d + e, \quad E = -\alpha_2 d + e. \tag{4.1.21}$$

Therefore, U obeys

$$U_{\xi\eta} = \tilde{D} U_\xi + \tilde{E} U_\eta + \tilde{F} U + \tilde{G} \tag{4.1.22}$$

with

$$\tilde{D} = -\frac{Da}{4(ac - b^2)}, \quad \tilde{E} = -\frac{Ea}{4(ac - b^2)},$$
$$\tilde{F} = -\frac{fa}{4(ac - b^2)}, \quad \tilde{G} = \frac{ga}{4(ac - b^2)}. \tag{4.1.23}$$

(ii) *The Tricomi equation*

An important equation in the linearized theory of transonic aerodynamics is the *Tricomi* equation (for example, see Section 3.5 of [10]):

$$y u_{xx} - u_{yy} = 0. \tag{4.1.24}$$

Here $a = y$, $b = 0$, $c = -1$, and $d = e = f = g = 0$. Thus, $\Delta = y$, and (4.1.24) is hyperbolic for $y > 0$. The characteristic curves satisfy

$$y' = \pm y^{-1/2}, \quad y > 0. \tag{4.1.25}$$

We compute

$$\xi = \frac{2}{3}y^{3/2} - x, \quad \eta = \frac{2}{3}y^{3/2} + x. \tag{4.1.26}$$

The curves ξ = constant and η = constant are a pair of one-parameter families in $y > 0$ that end up with a cusp on the x-axis, as shown in Figure 4.2.

To derive the transformed equation in terms of the ξ, η variables we note that $\phi_x = -1$, $\psi_x = 1$, $\phi_{xx} = \psi_{xx} = \phi_{xy} = \psi_{xy} = 0$, $\phi_y = \psi_y = y^{1/2}$, $\phi_{yy} = \psi_{yy} = 1/2y^{1/2}$. Therefore, $B = -2y$, $\tilde{D} = \tilde{E} = -1/8y^{3/2} = -1/6(\xi + \eta)$. Substituting the expressions for $\tilde{D}$ and $\tilde{E}$ into (4.1.15) gives

$$U_{\xi\eta} + \frac{1}{6(\xi + \eta)}(U_\xi + U_\eta) = 0. \tag{4.1.27}$$

4.1.3 The Parabolic Problem, $\Delta = 0$; $C = 0$

In this case, the conditions, $A = 0$ and $C = 0$ define the *same* single family of characteristics satisfying

$$y' = \frac{b}{a}. \tag{4.1.28}$$

Suppose that we set $C = 0$. Then the solution of (4.1.28) defines the family $\eta = \psi(x, y)$ = constant, and we may choose the family $\xi = \phi(x, y)$ = constant *arbitrarily* as long as $J \neq 0$; that is, $A \not\equiv 0$.

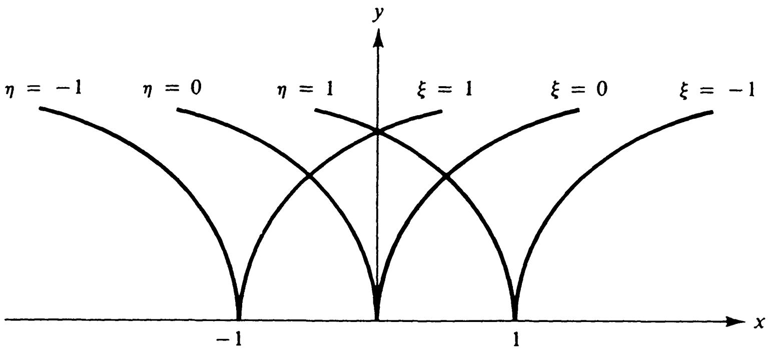

FIGURE 4.2. Characteristics of the Tricomi equation

Since $\Delta = 0$ also implies $B^2 - AC = 0$, we have $B = 0$, and the canonical form for U is

$$U_{\xi\xi} = \overline{D}U_\xi + \overline{E}U_\eta + \overline{F}U + \overline{G}, \qquad (4.1.29)$$

where

$$\overline{D} = -\frac{D}{A}, \quad \overline{E} = -\frac{E}{A}, \quad \overline{F} = -\frac{F}{A}, \quad \overline{G} = \frac{G}{A}. \qquad (4.1.30)$$

Of course, it is also possible to set $A = 0, C \neq 0$ and obtain the canonical form

$$U_{\eta\eta} = D^* U_\xi + E^* U_\eta + F^* U + G^*, \qquad (4.1.31)$$

where the starred coefficients are obtained from the coefficients in (4.1.7) by dividing by C.

For the special case where the linear equation (4.1.2) has constant coefficients, $\Delta = 0$ corresponds to $ac = b^2$. We have $y' = b/a$, that is, $\eta = y - bx/a$, and we may choose $\xi = x$, as this results in $J = 1$ for all x, y. We then obtain $A = a$, $B = C = 0$, and the canonical form (4.1.29) is

$$U_{\xi\xi} = -\frac{d}{a}U_\xi + \left(\frac{db}{a^2} - \frac{e}{a}\right)U_\eta - \frac{f}{a}U + \frac{g}{a}. \qquad (4.1.32)$$

In general, for given functions $a(x, y)$, $b(x, y)$, and $c(x, y)$, the condition $\Delta(x, y) = 0$ may either be satisfied identically in some domain $\mathcal{D}$ (as for the case where $c(x, y) \equiv b^2(x, y)/a(x, y)$ in $\mathcal{D}$), or $\Delta = 0$ may be true only on some curve $\mathcal{C}$ (for example, for the Tricomi equation, the curve $\mathcal{C}$ is the x-axis), or $\Delta(x, y) = 0$ may have no real solution. If $\Delta(x, y)$ has a real solution in some domain $\mathcal{D}$ or along some curve $\mathcal{C}$, we say that (4.1.2) is parabolic in $\mathcal{D}$ or on $\mathcal{C}$.

4.1.4 The Elliptic Problem, $\Delta < 0$; $B = 0$, $A = C$

In this case, we cannot satisfy (4.1.14) anywhere for real functions $\phi(x, y)$ and $\psi(x, y)$, and therefore $A \neq 0, C \neq 0$. So, the only alternative for simplification is to eliminate the mixed partial derivative $U_{\xi\eta}$ in (4.1.7) by setting $B = 0$. Then requiring $A = C$ allows the remaining second-derivative terms to reduce to the Laplacian when (4.1.7) is divided by A.

Thus, we need to solve the following two coupled equations for ϕ and ψ that result from (4.1.9) when we set $B = 0$ and $A - C = 0$:

$$a(\phi_x^2 - \psi_x^2) + 2b(\phi_x\phi_y - \psi_x\psi_y) + c(\phi_y^2 - \psi_y^2) = 0, \qquad (4.1.33a)$$

$$a\phi_x\psi_x + b(\phi_x\psi_y + \phi_y\psi_x) + c\phi_y\psi_y = 0. \qquad (4.1.33b)$$

If we multiply (4.1.33b) by $2i$ and add this to (4.1.33a), we obtain the complex version of (4.1.12) in terms of the complex variable $\zeta = \xi + i\eta$; that is,

$$a\zeta_x^2 + 2b\zeta_x\zeta_y + c\zeta_y^2 = 0, \qquad (4.1.34)$$

or, equivalently,

$$\frac{\phi_x + i\psi_x}{\phi_y + i\psi_y} = -\frac{b + i\sqrt{ac - b^2}}{a}. \tag{4.1.35}$$

The real and imaginary parts of (4.1.35) give

$$\phi_x = \frac{c\psi_y + b\psi_x}{(ac - b^2)^{1/2}}, \quad \phi_y = -\frac{b\psi_y + a\psi_x}{(ac - b^2)^{1/2}}. \tag{4.1.36}$$

These two first-order equations for ϕ and ψ are called the *Beltrami* equations, and *any* solution $\phi(x, y)$, $\psi(x, y)$ with $J \not\equiv 0$ defines a transformation to the canonical form

$$U_{\xi\xi} + U_{\eta\eta} = \text{lower-derivative terms.} \tag{4.1.37}$$

Although we may eliminate one of the dependent variables in favor of the other from (4.1.36), the resulting second-order equation is in general more complicated and harder to solve than the original system (4.1.36). For example, eliminating ϕ gives

$$\left[\frac{a\psi_x + b\psi_y}{(ac - b^2)^{1/2}}\right]_x + \left[\frac{c\psi_y + b\psi_x}{(ac - b^2)^{1/2}}\right]_y = 0, \tag{4.1.38}$$

and we note that in general, solving this for ψ is not any easier than solving the system (4.1.36).

If a, b, and c are analytic, we can construct solutions of the Beltrami equations by solving (cf. (4.1.14))

$$\frac{dy}{dx} = \frac{b + i\sqrt{ac - b^2}}{a} \tag{4.1.39}$$

in the complex plane. These ideas are discussed in [18] and are not pursured further here. Actually, a general solution of the Beltrami equations is not needed in order to implement the transformation to the canonical form (4.1.37); any solution that satisfies the requirement $J \not\equiv 0$ will do. These ideas are illustrated next for the special case of constant coefficients. See also Problem 4.1.3.

If (4.1.1) has constant coefficients, the general solution of (4.1.36) is linear in x and y and is easily derived. To see this, assume a solution of the form

$$\phi = \alpha x + \beta y, \quad \psi = \gamma x + \delta y, \tag{4.1.40}$$

with α, β, γ, and δ constant. Substitution into (4.1.36) gives the following two relations linking the four constants:

$$\alpha = \tilde{c}\delta + \tilde{b}\gamma, \quad \beta = -(\tilde{b}\delta + \tilde{a}\gamma), \tag{4.1.41}$$

where a tilde over a constant indicates that the constant is divided by $(ac - b^2)^{1/2}$. Thus, for any choice of δ, γ, we have α, β defined. For simplicity, take $\delta = 0$, $\gamma = 1$ to obtain $\alpha = \tilde{b}$ and $\beta = -\tilde{a}$ and the transformation

$$\xi = \tilde{b}x - \tilde{a}y, \quad \eta = x. \tag{4.1.42}$$

This results in the canonical form

$$U_{\xi\xi} + U_{\eta\eta} + \frac{db - ea}{a(ac - b^2)^{1/2}} U_\xi + \frac{d}{a} U_\eta + \frac{f}{a} U = \frac{g}{a}, \qquad (4.1.43)$$

where division by a leads to no difficulties because $\Delta < 0$ implies that $a \neq 0$.

We can again remove the first-derivative terms U_ξ and U_η by the transformation of dependent variable $U \to W$ defined by

$$U(\xi, \eta) \equiv W(\xi, \eta) \exp\left[-\frac{db - ea}{2a(ac - b^2)^{1/2}} \xi - \frac{d}{2a} \eta \right]. \qquad (4.1.44)$$

A straightforward calculation shows that W obeys

$$W_{\xi\xi} + W_{\eta\eta} + \left[\frac{f}{a} + \frac{2dbe - e^2 a - dc^2}{4a(ac - b^2)} \right] W$$

$$= \frac{g}{a} \exp\left[\frac{db - ea}{2a(ac - b^2)^{1/2}} \xi + \frac{d}{2a} \eta \right]. \qquad (4.1.45)$$

In this section we have restricted attention to the case of two independent variables. In general, it is not possible to reduce a second-order equation in more than two independent variables to a simple canonical form (see, for example, Chapter 3, Section 2, of [18]).

Problems

4.1.1. Classify (4.1.1) for the case $a = \alpha x^m y^n$, $b = 0$, $c = \beta x^r y^s$, with integer values of m, n, r, s and constants α, β. Derive the characteristics and the canonical form (4.1.15) for the hyperbolic case.

4.1.2. Consider the transformation of dependent variable $U \to W$ in (4.1.15) defined by

$$U(\xi, \eta) = W(\xi, \eta) R(\xi, \eta) \qquad (4.1.46)$$

for an arbitrary function $R(\xi, \eta)$. Show that the partial differential equation that results from (4.1.15) for W will be free of W_ξ and W_η if R satisfies

$$R_\eta - \tilde{D} R = 0, \quad R_\xi - \tilde{E} R = 0. \qquad (4.1.47)$$

Prove that a necessary and sufficient condition for this system to have a solution is that

$$\tilde{D}_\xi = \tilde{E}_\eta, \qquad (4.1.48)$$

in which case

$$R(\xi, \eta) = \exp\left\{ \int_{\xi_0}^{\xi} \tilde{E}(\sigma, \eta) d\sigma + \int_{\eta_0}^{\eta} \tilde{D}(\xi, s) ds \right\}, \qquad (4.1.49)$$

where we have normalized the solution so that $R(\xi_0, \eta_0) = 1$. Show also that W satisfies

$$W_{\xi\eta} + (\tilde{D}_\xi - \tilde{E}\tilde{D} - \tilde{F})W = \frac{\tilde{G}}{R}. \qquad (4.1.50)$$

4.1.3. Consider the Tricomi equation (4.1.24) in the $y < 0$ half-plane. Show that the Beltrami equations (4.1.36) for this case are

$$\phi_x = -\frac{1}{(-y)^{1/2}}\psi_y, \quad \phi_y = (-y)^{1/2}\psi_x. \qquad (4.1.51)$$

Change variables from (x, y) to $(\overline{x}, \overline{y})$ defined by $\overline{x} = x, \overline{y} = \frac{2}{3}(-y)^{3/2}$ and show that equations (4.1.51) reduce to the Cauchy–Riemann equations:

$$\overline{\phi}_{\overline{x}} = \overline{\psi}_{\overline{y}}, \quad \overline{\phi}_{\overline{y}} = -\overline{\psi}_{\overline{x}}, \qquad (4.1.52)$$

where $\overline{\phi}(x, \frac{2}{3}(-y)^{3/2}) \equiv \phi(x, y)$, $\overline{\psi}(x, \frac{2}{3}(-y)^{3/2}) \equiv \psi(x, y)$. A simple solution of (4.1.52) is $\overline{\phi} = \overline{y}$, $\overline{\psi} = -\overline{x}$. Show that the resulting transformation $(x, y) \to (\xi, \eta)$ is given by

$$\xi = \frac{2}{3}(-y)^{3/2}, \quad \eta = -x, \qquad (4.1.53)$$

and that the Tricomi equation transforms to the canonical form

$$U_{\xi\xi} + U_{\eta\eta} + \frac{1}{3\xi}U_\xi = 0, \qquad (4.1.54)$$

where $U(\frac{2}{3}(-y)^{3/2}, -x) \equiv u(x, y)$.

4.2 The General Hyperbolic Equation

4.2.1 *The Role of Characteristics*

One interpretation of the characteristic curves $\phi(x, y) = \xi = $ constant and $\psi(x, y) = \eta = $ constant, defined by (4.1.19) with $\Delta > 0$, is that this pair of one-parameter families of curves defines a coordinate transformation of (4.1.1) to the canonical form (4.1.15). In this section we give two other interpretations that are equally significant for these curves.

(i) *The Cauchy problem*

Let $\phi(x, y) = \xi_0 = $ constant define a curve C in the xy-plane. The Cauchy problem for (4.1.1) consists in the solution of this equation subject to prescribed values of u and $\partial u/\partial n$ on C. As usual, $\partial/\partial n$ indicates the directional derivative in the direction normal to C. It is also understood that we have specified on which side of C we wish to solve (4.1.1); hence the unit normal $\mathbf{n}$ to C is taken to point into the domain of interest $\mathcal{D}$ (see Figure 4.3). For example, if y denotes time, the solution must evolve in the direction of increasing y.

Now consider a curvilinear coordinate system (ξ, η) defined by

$$\xi = \phi(x, y), \quad \eta = \psi(x, y), \tag{4.2.1}$$

where $\phi(x, y) = \xi_0$ is the same function as the one defining $\mathcal{C}$, and the curves $\eta = \text{constant}$ are chosen to be noncollinear to the curves $\xi = \text{constant}$; that is, $\phi_x \psi_y - \psi_x \phi_y \neq 0$ in $\mathcal{D}$.

We saw in Section 4.1.1 that the governing equation (4.1.1) transforms to (4.1.7):

$$A(\xi, \eta)U_{\xi\xi} + 2B(\xi, \eta)U_{\xi\eta} + C(\xi, \eta)U_{\eta\eta} + D(\xi, \eta)U_{\xi} + E(\xi, \eta)U_{\eta} + F(\xi, \eta)U = G(\xi, \eta), \tag{4.2.2}$$

where $A, \ldots, G$ are defined in terms of the given ϕ, ψ and the original coefficients $a, \ldots, g$ in (4.1.9)–(4.1.10).

If u and $\partial u/\partial n$ are prescribed along $\mathcal{C}$, it means that we have

$$U(\xi_0, \eta) \equiv \alpha(\eta), \quad U_{\xi}(\xi_0, \eta) \equiv \beta(\eta), \tag{4.2.3}$$

for prescribed functions $\alpha(\eta)$ and $\beta(\eta)$. This information is called *Cauchy data* on $\mathcal{C}$.

We now ask whether it is possible to use the Cauchy data (4.2.3) in conjunction with the partial differential equation (4.2.2) to define $U(\xi, \eta)$ on some neighboring curve $\xi_1 = \xi_0 + \Delta\xi = \text{constant}$.

Using a Taylor series, we obtain

$$U(\xi_1, \eta) = U(\xi_0, \eta) + U_{\xi}(\xi_0, \eta)\Delta\xi + U_{\xi\xi}(\xi_0, \eta)\frac{\Delta\xi^2}{2} + \ldots, \tag{4.2.4}$$

and we know $U(\xi_0, \eta)$, $U_{\xi}(\xi_0, \eta)$. Therefore, to compute the Taylor series to $O(\Delta\xi^2)$, all we need is $U_{\xi\xi}(\xi_0, \eta)$.

Clearly, we can obtain $U_{\xi\xi}$ from (4.2.2) in terms of known quantities *as long as the curve $\mathcal{C}$ is not characteristic*. In fact, if $\mathcal{C}$ is not characteristic, then $A \neq 0$, and

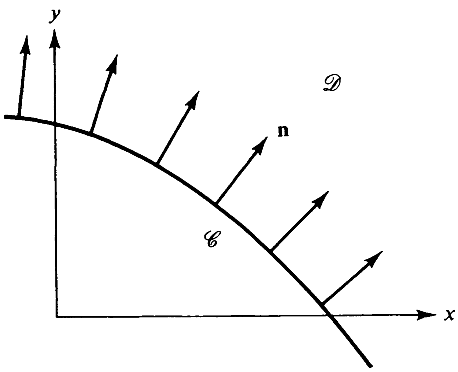

FIGURE 4.3. Normals along initial curve

we have

$$U_{\xi\xi}(\xi,\eta) = -\frac{2B}{A}U_{\xi\eta} - \frac{C}{A}U_{\eta\eta} - \frac{D}{A}U_{\xi} - \frac{E}{A}U_{\eta} - \frac{F}{A}U + \frac{G}{A}. \quad (4.2.5)$$

Each of the coefficients on the right-hand side of (4.2.5) is known and can therefore be evaluated at $\xi = \xi_0$. Moreover, on $\xi = \xi_0$, U and U_ξ are the known functions of η given by the Cauchy data in the form (4.2.3). To compute U_η, $U_{\xi\eta}$, and $U_{\eta\eta}$ on $\xi = \xi_0$, we also use (4.2.3) and obtain $U_\eta(\xi_0,\eta) = \alpha'(\eta)$, $U_{\eta\eta}(\xi_0,\eta) = \alpha''(\eta)$, $U_{\xi\eta}(\xi_0,\eta) = \beta'(\eta)$.

This shows that we can evaluate $U_{\xi\xi}(\xi_0,\eta)$. To evaluate each higher derivative $U_{\xi\xi\xi}$ and so on, we differentiate (4.2.5) with respect to ξ, solve for $U_{\xi\xi\xi}(\xi,\eta)$, and evaluate $U_{\xi\xi\xi}(\xi,\eta)$ along $\mathcal{C}$ in terms of known quantities there. This is always possible as long as the coefficients $A, B, \ldots$ are analytic and $A \neq 0$. The formal theorem ensuring that this construction generates a unique solution as long as $a, \ldots, g$ are analytic is attributed to Cauchy and Kowalewski. (For a proof, see Chapter 1 of [18]). This construction is not restricted to any given type (hyperbolic, elliptic, or parabolic) of equation; however, it is not very useful as a practical solution technique [see the discussion in Section 4.2.2v)]. In the next section we outline a method based on the behavior of solutions along characteristic curves to calculate U numerically for hyperbolic equations.

Returning to the exceptional case where the curve $\mathcal{C}$ is characteristic, we see that one cannot use Cauchy data in this case to extend these data to a neighboring curve. In fact, if $\mathcal{C}$ is a characteristic $\xi = \xi_0 =$ constant, we *cannot even specify α and β arbitrarily on it* and expect the result to be part of a solution of (4.2.2). With the curves $\xi =$ constant and $\eta =$ constant as characteristics of (4.1.1), any solution must satisfy (4.1.15) everywhere. In particular, along the characteristic $\xi = \xi_0 =$ constant, (4.1.15) reduces to the following consistency condition governing the functions $\alpha(\eta)$ and $\beta(\eta)$:

$$2B(\xi_0,\eta)\beta'(\eta) + D(\xi_0,\eta)\beta(\eta) + E(\xi_0,\eta)\alpha'(\eta) + F(\xi_0,\eta)\alpha(\eta) = G(\xi_0,\eta). \quad (4.2.6)$$

Given $\alpha(\eta)$, this is a first-order differential equation that determines $\beta(\eta)$ to within an arbitrary constant or vice versa. Therefore, $\alpha(\eta)$ and $\beta(\eta)$ cannot be specified arbitrarily on $\xi = \xi_0$.

(ii) *Characteristics as carriers of discontinuities in the second derivative*

Regard (4.1.4) as a coordinate transformation $(x, y, u) \to (\xi, \eta, U)$. Thus, $U(\xi,\eta)$ satisfies (4.2.2) if $u(x, y)$ satisfies (4.1.1). Let the curve $\mathcal{C}_0$ correspond to $\phi(x, y) = \xi_0$ for a fixed constant ξ_0. Assume that on either side of $\mathcal{C}_0$ we have solved (4.2.2) and that this solution has U, U_ξ, U_η, $U_{\xi\xi}$, $U_{\xi\eta}$, $U_{\eta\eta}$ continuous for all $\xi \neq \xi_0$ and all η. For future reference, we call such a solution a *strict* solution of (4.1.1) or (4.2.2).

Now suppose that on $\mathcal{C}_0$, the functions U, U_ξ, U_η, $U_{\xi\eta}$, and $U_{\eta\eta}$ are also continuous but $U_{\xi\xi}$ *is not*. Thus, strictly speaking, (4.1.1) and (4.2.2) are not satisfied on $\mathcal{C}_0$ because $U_{\xi\xi}$ is not defined there. Can we choose $\mathcal{C}_0$ in such a way that (4.1.1) is also satisfied there even with $U_{\xi\xi}(\xi_0^+, \eta) \neq U_{\xi\xi}(\xi_0^-, \eta)$? To answer this question, we evaluate (4.2.2) on either side of $\xi = \xi_0$ and subtract the two resulting expressions. Since all the terms except $U_{\xi\xi}$ are continuous, we are left with

$$A(\xi_0, \eta)[U_{\xi\xi}(\xi_0^+, \eta) - U_{\xi\xi}(\xi_0^-, \eta)] = 0.$$

Therefore, (4.2.2) is satisfied on $\xi = \xi_0$ either trivially, if $U_{\xi\xi}$ is continuous there, or if $A(\xi_0, \eta) = 0$, that is, if $\xi = \xi_0$ is the characteristic curve in the xy-plane, defined by the first equation in (4.1.14). This shows that within the framework of a *strict solution everywhere*, the characteristics $\xi = \xi_0 =$ constant are loci of possible discontinuity in $U_{\xi\xi}$. Similarly, the characteristics $\eta = \eta_0 =$ constant are loci of possible discontinuity in $U_{\eta\eta}$.

Geometrically, we may interpret this result to imply that we can join two solutions smoothly along a characteristic. For example, in Figure 4.4 we show a solution surface $S_0 + S_1$, which branches smoothly into the surface $S_0 + S_2$. The branching is smooth in the sense that U, U_ξ, U_η, $U_{\xi\eta}$, and $U_{\eta\eta}$ are continuous everywhere, including along $\xi = \xi_0$. On $\xi = \xi_0^+$, the value of $U_{\xi\xi}$ is different for S_1 and S_2. One or both of these values may also differ from $U_{\xi\xi}$ on $\xi = \xi_0^-$, that is, for S_0. Similarly, two solutions may be joined smoothly at a characteristic $\eta = \eta_0$ where the values of $U_{\eta\eta}$ differ.

Actually, once the jump in the value of $U_{\xi\xi}$ is specified at some point on $\mathcal{C}_0$, the propagation of this jump along $\mathcal{C}_0$ is determined by (4.2.2). To derive the equation governing the propagation of this jump, we first take the partial derivative of (4.2.2) with respect to ξ to obtain

$$\begin{aligned} &AU_{\xi\xi\xi} + A_\xi U_{\xi\xi} + 2BU_{\xi\xi\eta} + 2B_\xi U_{\xi\eta} + CU_{\xi\eta\eta} + C_\xi U_{\eta\eta} + DU_{\xi\xi} + D_\xi U_\xi \\ &\quad + EU_{\xi\eta} + E_\xi U_\eta + FU_\xi + F_\xi U - G_\xi = 0. \end{aligned} \tag{4.2.7}$$

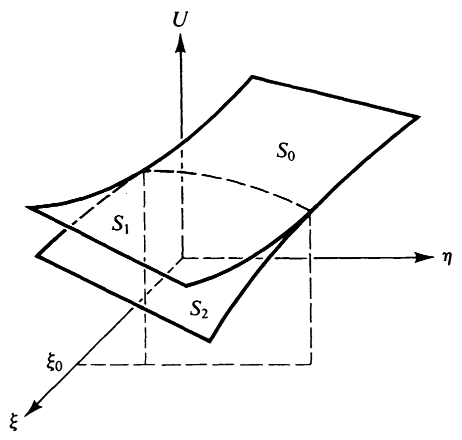

FIGURE 4.4. Smooth branching of solutions at a characteristic

Now we evaluate (4.2.7) on either side of $\mathcal{C}_0$, that is, at ξ_0^+, and ξ_0^-, and subtract the two resulting expressions. If we denote the jump in $U_{\xi\xi}$ by

$$\rho(\xi_0, \eta) \equiv U_{\xi\xi}(\xi_0^+, \eta) - U_{\xi\xi}(\xi_0^-, \eta) \tag{4.2.8}$$

and note that $A(\xi_0, \eta) = 0$ while $U_{\xi\eta}$, $U_{\eta\eta}$, U_ξ, U_η, and U are continuous on $\xi = \xi_0$, we obtain the linear first-order ordinary differential equation

$$\frac{\partial \rho}{\partial \eta} = -R(\xi_0, \eta)\rho, \tag{4.2.9a}$$

where

$$R(\xi_0, \eta) \equiv \frac{D(\xi_0, \eta) + A_\xi(\xi_0, \eta)}{2B(\xi_0, \eta)}. \tag{4.2.9b}$$

Thus, once ρ is prescribed at some point $\xi = \xi_0$, $\eta = \eta_0$ on $\mathcal{C}_0$, the solution of (4.2.9a) determines ρ everywhere along $\mathcal{C}_0$ in the form

$$\rho(\xi_0, \eta) = \rho(\xi_0, \eta_0) \exp\left[-\int_{\eta_0}^{\eta} R(\xi_0, \sigma) d\sigma\right]. \tag{4.2.10}$$

This result shows that ρ is identically equal to zero if it vanishes initially, and that if $\rho(\xi_0, \eta_0) \neq 0$, then ρ is never equal to zero.

Similarly, discontinuities in $U_{\eta\eta}$ propagate along characteristics $\eta =$ constant according to

$$\lambda(\xi, \eta_0) = \lambda(\xi_0, \eta_0) \exp\left[-\int_{\xi_0}^{\xi} S(\sigma, \eta_0) d\sigma\right], \tag{4.2.11}$$

where λ now denotes the jump in $U_{\eta\eta}$:

$$\lambda(\xi, \eta_0) \equiv U_{\eta\eta}(\xi, \eta_0^+) - U_{\eta\eta}(\xi, \eta_0^-), \tag{4.2.12a}$$

and

$$S(\xi, \eta_0) \equiv \frac{E(\xi, \eta_0) + C_\eta(\xi, \eta_0)}{2B(\xi, \eta_0)}. \tag{4.2.12b}$$

In summary, we have seen three alternative interpretations of the characteristic curves of (4.1.1) for the hyperbolic case:

1. They are curves that transform (4.1.1) to the canonical form (4.1.15).
2. They are curves on which Cauchy data do not specify a unique solution in a neighborhood.
3. They are curves along which a strict solution of (4.1.1) may have a discontinuity in the second derivative normal to the curve. Thus, two solutions may be joined smoothly along a characteristic.

4.2.2 Solution of Hyperbolic Equations in Terms of Characteristics

We showed in Section 4.1.2 that the general partial differential equation (4.1.1) can be reduced to the following form if it is hyperbolic [see (4.1.15)]

$$U_{\xi\eta} = \tilde{D}U_\xi + \tilde{E}U_\eta + \tilde{F}U + \tilde{G}. \tag{4.2.13}$$

Given the coefficients d, e, f, and g, which are functions of x and y, we can explicitly derive the functions $\tilde{D}$, $\tilde{E}$, $\tilde{F}$, and $\tilde{G}$ of ξ and η from (4.1.16). As pointed out earlier, in general it is not possible to simplify (4.2.13) further.

If we define

$$U_\xi(\xi, \eta) \equiv P(\xi, \eta), \quad U_\eta(\xi, \eta) \equiv Q(\xi, \eta), \tag{4.2.14}$$

we can interpret (4.2.13) as an equation governing the propagation of P along the ξ = constant characteristics, or an equation governing the propagation of Q along the η = constant characteristics. In fact, (4.2.13) is just

$$P_\eta = \tilde{D}P + \tilde{E}Q + \tilde{F}U + \tilde{G} \equiv \tilde{H}, \tag{4.2.15a}$$

or

$$Q_\xi = \tilde{D}P + \tilde{E}Q + \tilde{F}U + \tilde{G} \equiv \tilde{H}. \tag{4.2.15b}$$

In this section we formulate a solution procedure based on the characteristic form (4.2.15) and discuss how we may implement this procedure numerically.

(i) *Cauchy data on a spacelike arc*

Let $\mathcal{C}_0$ be a smooth noncharacteristic ($\mathcal{C}_0$ is neither ξ = constant nor η = constant) curve defined in the parametric form

$$\xi = \xi^*(\tau), \quad \eta = \eta^*(\tau), \tag{4.2.16}$$

where the parameter τ varies monotonically along $\mathcal{C}_0$. Thus, the functions ξ^* and η^* as well as $\dot{\xi}^*$ and $\dot{\eta}^*$ are continuous.

We denote by $\mathcal{D}$ the domain on the side of $\mathcal{C}_0$ over which (4.2.13) is to be solved and let $\mathbf{n}$ denote the unit normal *into* $\mathcal{D}$ and $\boldsymbol{\tau}$ the unit tangent in the direction of increasing τ. See any of the cases sketched in Figure 4.5. Here the characteristics are horizontal and vertical lines in terms of a Cartesian (ξ, η) frame, and with no loss of generality, we take the origin of this frame somewhere on the curve $\mathcal{C}_0$.

We distinguish two possible types of noncharacteristic curves, denoted by *spacelike* and *timelike*. A spacelike arc $\mathcal{S}_0$ has two characteristics, either emerging from every point on it into $\mathcal{D}$ as in Figure 4.5a, or entering every point on it from $\mathcal{D}$, as in Figure 4.5b. Thus, the two components of $\mathbf{n}$ in the ξ and η directions are either both positive or both negative for a spacelike arc. On a timelike arc $\mathcal{T}_0$, the components of $\mathbf{n}$ have different signs, and only one family of characteristics emerges from $\mathcal{T}_0$ into $\mathcal{D}$. These may be the η = constant characteristics, as in Figure 4.5c, or the ξ = constant characteristics, as in Figure 4.5d. In the preceding, the terms *entering* and *emerging* are associated with the directions of increasing ξ or η.

The terms spacelike and timelike originate from the interpretation of (4.1.1) as the wave equation:

$$u_{xx} - u_{tt} = \text{lower-derivative terms,} \tag{4.2.17}$$

where x is a distance and t is the time. In characteristic form, (4.2.17) becomes [see (4.1.14)]

$$U_{\xi\eta} = \text{lower-derivative terms,} \tag{4.2.18}$$

with $\xi = t + x$ and $\eta = t - x$. Now, if we want to solve (4.2.17) for $t \geq 0$, $-\infty < x < \infty$, we specify u and u_t on the x-axis, a spacelike curve. This maps to the straight line $\mathcal{S}_0$: $\eta = -\xi$, which may also be defined parametrically as $\xi = \tau$, $\eta = -\tau$, $-\infty < \tau < \infty$. The domain $t \geq 0$ maps to $\mathcal{D}$: $\xi + \eta \geq 0$ above $\mathcal{S}_0$ as in Figure 4.5a, and $\mathbf{n} = (1/\sqrt{2}, 1/\sqrt{2})$. Therefore, the x-axis is indeed a spacelike arc according to our definition. Notice that with the choice $\xi = -t - x$ and $\eta = -t + x$, (4.2.17) also transforms to (4.2.18), but now the domain $t \geq 0$ is below $\mathcal{S}_0$ as in Figure 4.5b. More generally, any curve $x = f(t)$ with $|\dot{f}(t)| > 1$ is spacelike. Conversely, if $|\dot{f}| < 1$, the curve is timelike. In particular, the vertical line $x = c =$ constant in the xt-plane (which is the time axis if $c = 0$) is timelike for the solution domain on either side.

Consider now Cauchy data on a spacelike arc $\mathcal{S}_0$ with $\mathcal{D}$ to the right of $\mathcal{S}_0$, as in Figure 4.5a. We are given U and $\partial U/\partial n$ on $\mathcal{S}_0$ and can therefore express these parametrically in terms of τ; that is,

$$U(\xi^*(\tau), \eta^*(\tau)) \equiv U^*(\tau) = \text{given,} \tag{4.2.19a}$$

$$\frac{\partial U}{\partial n}(\xi^*(\tau), \eta^*(\tau)) \equiv V^*(\tau) = \text{given.} \tag{4.2.19b}$$

For the time being, let us assume that $U^*(\tau)$, $\dot{U}^*(\tau)$, and $V^*(\tau)$ are continuous on $\mathcal{S}_0$. We can then derive P and Q as continuous functions of τ on $\mathcal{S}_0$ as follows. Let us define

$$P^*(\tau) \equiv \frac{\partial U}{\partial \xi}(\xi^*(\tau), \eta^*(\tau)), \quad Q^*(\tau) \equiv \frac{\partial U}{\partial \eta}(\xi^*(\tau), \eta^*(\tau)). \tag{4.2.20}$$

Differentiating (4.2.19a) with respect to τ and using (4.2.20) gives

$$P^*\dot{\xi}^* + Q^*\dot{\eta}^* = \dot{U}^*, \tag{4.2.21}$$

a linear relation linking P^* and Q^*. A second relation between P^* and Q^* is just (4.2.19b), in which we express $\mathbf{n}$ in terms of $\dot{\xi}^*$ and $\dot{\eta}^*$. For the spacelike arc in Figure 4.5a, $\dot{\eta}^* > 0$ and $\dot{\xi}^* < 0$; hence

$$\mathbf{n} = \left(\frac{\dot{\eta}^*}{\dot{s}}, \frac{-\dot{\xi}^*}{\dot{s}}\right), \quad \dot{s} \equiv (\dot{\xi}^{*^2} + \dot{\eta}^{*^2})^{1/2}. \tag{4.2.22}$$

Therefore, (4.2.19b) becomes

$$\frac{\partial U}{\partial n} \equiv \text{grad}\, U \cdot \mathbf{n} = P^*\frac{\dot{\eta}^*}{\dot{s}} - Q^*\frac{\dot{\xi}^*}{\dot{s}} = V^*. \tag{4.2.23}$$

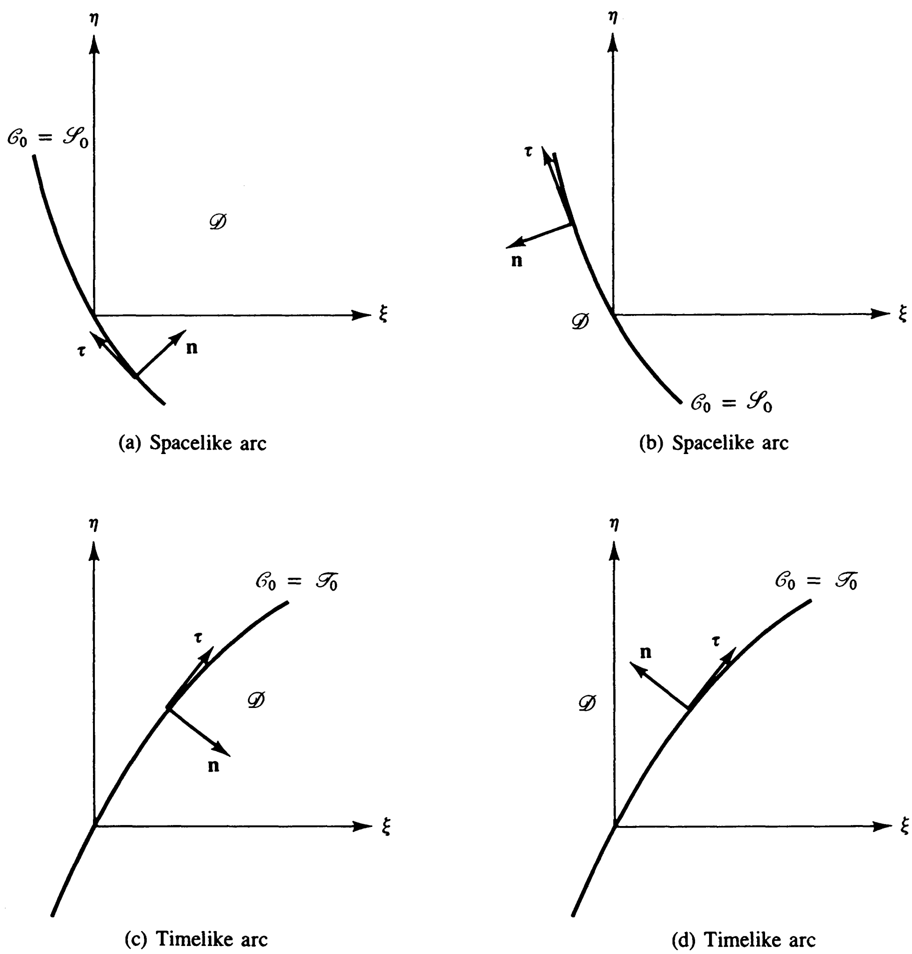

FIGURE 4.5. Spacelike and timelike arcs

Equations (4.2.21) and (4.2.23) are linearly independent algebraic equations linking $\dot{U}^*$, V^* to P^*, Q^*. Solving these gives

$$P^* = \frac{\dot{\xi}^*\dot{U}^* + \dot{\eta}^*\dot{s}V^*}{\dot{s}^2}, \quad Q^* = -\frac{\dot{\xi}^*\dot{s}V^* - \dot{\eta}^*\dot{U}^*}{\dot{s}^2}. \tag{4.2.24}$$

Thus, knowing U^* and V^* on a given spacelike arc, we compute P^* and Q^* there using (4.2.24). The formulas for the situation in Figure 4.5b are obtained from the preceding by replacing $\dot{s}$ by $-\dot{s}$.

(ii) *Cauchy problem; the numerical method of characteristics*

We are now in a position to extend the initial data U, P, Q given on $\mathcal{S}_0$ to a neighboring curve using (4.2.14) and (4.2.15) as rules for the propagation of P, Q in the characteristic directions. This is a general version of the initial-value problem for the wave equation discussed in Section 3.4. Again referring to the case of Figure 4.5a, we subdivide $\mathcal{D}$ into a rectangular grid, with a variable grid spacing (as might be dictated by the rate of change of the initial data) as shown in Figure 4.6. Denote the values of U, P, Q at each gridpoint by the associated subscript. Thus, let

$$U_{i,j} \equiv U(\xi_i, \eta_j), \quad P_{i,j} \equiv P(\xi_i, \eta_j), \quad Q_{i,j} \equiv Q(\xi_i, \eta_j). \tag{4.2.25}$$

Let the horizontal distance between the points (i, j) and $(i + 1, j)$ be denoted by ℓ_i, and the vertical distance between the points (i, j) and $(i, j + 1)$ by h_j; that is,

$$\ell_i \equiv \xi_{i+1} - \xi_i, \quad h_j \equiv \eta_{j+1} - \eta_j. \tag{4.2.26}$$

Assume that U, P, and Q are known at the two adjacent points $A = (i - 1, j)$ and $B = (i, j - 1)$, and we want to compute U, P, and Q at the point $C = (i, j)$ (see Figure 4.6).

The definition of P in (4.2.14) gives the following forward difference approximation for U in the ξ-direction,

$$U_{i,j}^{(\xi)} = U_{i-1,j} + \ell_{i-1} P_{i-1,j}, \tag{4.2.27a}$$

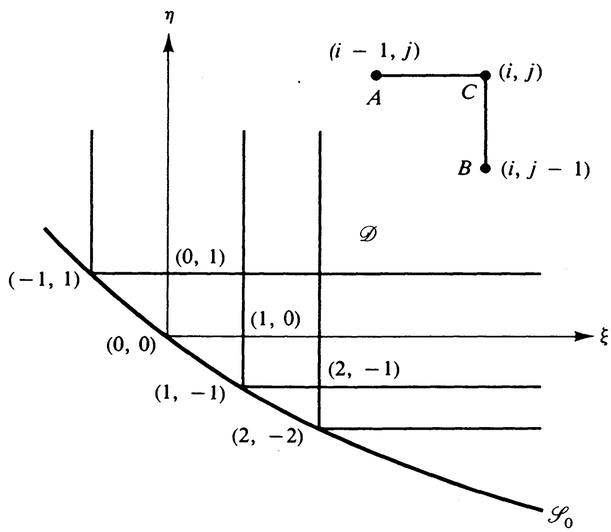

FIGURE 4.6. Characteristic grid

and the definition of Q gives the forward difference in the η-direction,

$$U_{i,j}^{(\eta)} = U_{i,j-1} + h_{j-1} Q_{i,j-1}. \tag{4.2.27b}$$

Each of these expressions defines $U_{i,j}$ in terms of known quantities, and one approach is to use the weighted average to calculate $U_{i,j}$:

$$\begin{aligned} U_{i,j} &= \frac{U_{i,j}^{(\xi)} h_{j-1} + U_{i,j}^{(\eta)} \ell_{i-1}}{h_{j-1} + \ell_{i-1}} \\ &= \frac{h_{j-1} U_{i-1,j} + \ell_{i-1} U_{i,j-1} + \ell_{i-1} h_{j-1} (P_{i+1}, + Q_{i,j-1})}{\ell_{i-1} + h_{j-1}}. \end{aligned} \tag{4.2.28}$$

To compute $P_{i,j}$, we use the forward difference of (4.2.15a):

$$P_{i,j} = P_{i,j-1} + h_{j-1} \tilde{H}_{i,j-1}. \tag{4.2.29a}$$

Similarly, we obtain $Q_{i,j}$ from (4.2.15b):

$$Q_{i,j} = Q_{i-1,j} + \ell_{i-1} \tilde{H}_{i-1,j}. \tag{4.2.29b}$$

Equations (4.2.28)–(4.2.29) define U, P, and Q at the point C in terms of known values at the points A and B, and this process can be repeated to generate the solution at successive points. We see that if we use Cauchy data along a finite segment of the initial curve, say between the gridpoints $(-N, N)$ and $(M, -M)$, we are able to define the solution of (4.2.13) in the triangular domain bounded by the given segment of the initial curve and the characteristics $\eta = \eta_N =$ constant and $\xi = \xi_M =$ constant. This triangular domain is the domain of dependence of the point (ξ_M, η_N) [see the discussion following (3.4.14)]. For this construction, it is crucial to start with *U and a derivative of U in a direction that is not tangent to* $\mathcal{C}_0$ (for example, the normal derivative) in order to be able to march the values of P and Q forward. This result confirms the argument used in Section 4.2.1i based on Taylor series.

(iii) *Goursat's problem; boundary condition on a timelike arc*

Consider a characteristic arc $\mathcal{C}_0$, say $0 \leq \xi \leq \xi_F, \eta = 0$ (where ξ_F may equal ∞), and a timelike arc $\mathcal{T}_0$ over the same interval in ξ given parametrically in the form $\xi = \overline{\xi}(\tau), \eta = \overline{\eta}(\tau)$, as shown in Figure 4.7. Assume that U is prescribed on both these arcs and we wish to solve (4.2.13) in the domain $\mathcal{D}_1$. Assume also that U is continuous and has a continuous derivative along both $\mathcal{C}_0$ and $\mathcal{T}_0$. In particular, the value of U as $\xi \to 0^+$ on $\eta = 0$ is the same as the value of U as the origin is approached along $\mathcal{T}_0$. This is Goursat's problem, and we demonstrate next that specifying U along these two arcs is sufficient to define U, P, and Q at all points in $\mathcal{D}_1$ using a characteristic construction.

First, we note that specifying U on $\mathcal{C}_0$ implies that we know $P = U_\xi$ along $\mathcal{C}_0$ also. In particular, $U_{0,0}$, $P_{0,0}$, $U_{1,0}$, $P_{1,0}$, . . ., $U_{i,0}$, $P_{i,0}$, are all known. Next, we take the directional derivative of the prescribed data, $U = \overline{U}(\tau)$ on $\mathcal{T}_0$, to obtain

$$\frac{d\overline{U}}{d\tau} = \overline{P}(\tau) \frac{d\overline{\xi}}{d\tau} + \overline{Q}(\tau) \frac{d\overline{\eta}}{d\tau}, \tag{4.2.30}$$

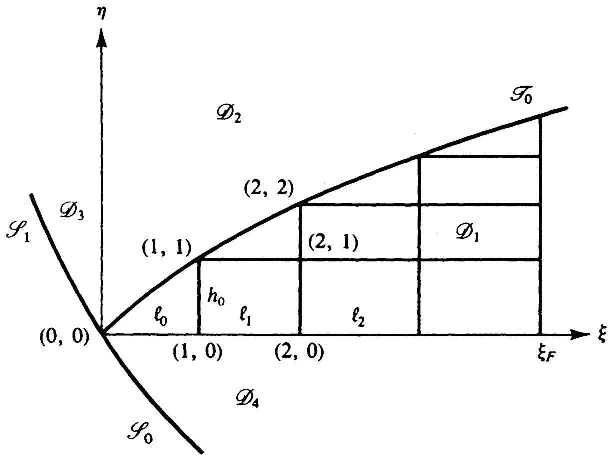

FIGURE 4.7. Characteristic grid for the Goursat problem

where $P = \overline{P}(\tau)$ and $Q = \overline{Q}(\tau)$ are the unknown values of P and Q along $\mathcal{T}_0$. Knowing $\overline{U}(\tau)$, and hence $d\overline{U}/d\tau$, as well as $d\overline{\xi}/d\tau$ and $d\overline{\eta}/d\tau$ at each point along $\mathcal{T}_0$.means that (4.2.30) provides a linear relation linking P and Q along $\mathcal{T}_0$ in the form

$$\alpha_i P_{i,i} + \beta_i Q_{i,i} = \gamma_i, \quad i = 0, 1, \ldots, \tag{4.2.31}$$

where the constants α_i, β_i, and γ_i are all known. In particular, at (0, 0), where $P_{0,0}$ is known, (4.2.31) defines $Q_{0,0} = (\gamma_0 - \alpha_0 P_{0,0})/\beta_0$. Note that $\beta_0 \neq 0$ because $\mathcal{T}_0$ is timelike and is therefore not tangent to the ξ-axis.

Consider now the points along the ξ-axis where Q is unknown. Since U and P are known all along this line, (4.2.15b) reduces to a linear first-order ordinary differential equation for Q as a function of ξ. If this differential equation subject to the given boundary value $Q_{0,0}$ cannot be evaluated explicitly, we can use the associated forward difference equation to obtain

$$Q_{i,0} = Q_{i-1,0} + \tilde{H}_{i-1,0}\ell_{i-1}, \quad i = 1, 2, \ldots. \tag{4.2.31}$$

For each $i = 1, \ldots, 2$, the right-hand side is known in terms of prescribed and previously calculated values.

To evaluate P at (1, 1) we use the forward difference of (4.2.15a)

$$P_{1,1} = P_{1,0} + \tilde{H}_{1,0}h_0, \tag{4.2.32}$$

and substituting this expression for $P_{1,1}$ into (4.2.31) for $i = 1$ defines $Q_{1,1}$. We now have U, P, and Q at the three points (0, 0), (1, 0) and (1, 1).

The solution at (2, 1) is given by (4.2.28) and (4.2.29). Once this solution has been computed, $P_{2,2}$ is obtained from the expression (see (4.2.32))

$$P_{2,2} = P_{2,1} + \tilde{H}_{2,1}h_1, \tag{4.2.33}$$

and this process can be continued to calculate U, P, and Q everywhere in $\mathcal{D}_1$.

Suppose now that we wish to solve (4.2.13) in $\mathcal{D}_1 + \mathcal{D}_4$ (see Figure 4.7) subject to Cauchy data being specified on the spacelike arc $\mathcal{S}_0$ and U being specified on the timelike arc $\mathcal{T}_0$. This is the general version of the initial- and boundary-value problem for the wave equation over the semi-infinite domain discussed in Section 3.5. Clearly, this solution is defined by the union of the solution of the Cauchy problem in $\mathcal{D}_4$ (which specifies U on the ξ-axis) followed by the solution of the Goursat problem in $\mathcal{D}_1$.

(iv) *Characteristic boundary-value problem*

The limiting case of the Goursat problem in $\mathcal{D}_1 + \mathcal{D}_2$ for which the arc $\mathcal{T}_0$ becomes the characteristic η-axis is known as the *characteristic boundary-value problem.* It is easily seen that specifying U on the ξ-axis defines P there, whereas specifying U on the η-axis defines Q there. In particular, we have both P and Q at the origin. To compute P on the η-axis, we evaluate (4.2.15a) for $\xi = 0$. The result is a linear first-order ordinary differential equation for P as a function of η. Solving this subject to the boundary condition for P at $\eta = 0$ defines P uniquely. Equivalently, we can use the difference expression corresponding to (4.2.15a) [see (4.2.32)]. Similarly, we compute Q on the ξ-axis by solving (4.2.15b) either exactly or in difference form subject to the boundary condition for Q at the origin [see (4.2.31)]. Note that in this case $\dot{\bar{\xi}} = 0$ on the η-axis, and (4.2.30) specializes to the definition of Q along the η-axis. The solution for all interior points can now be computed recursively using the scheme employed for the Cauchy problem.

(v) *Well-posedness*

Based on the examples discussed so far, we observe that our characteristic construction of the solution in each case provides an explicit demonstration that the type of boundary data imposed defines a "well-posed" problem in the sense that we have exactly enough information to calculate a unique solution. In particular, it was necessary to specify *both* U and $\partial U/\partial n$ on the spacelike arc $\mathcal{S}_0$ for the Cauchy problem; we would have been unable to solve the problem had we specified only U or $\partial U/\partial n$. In contrast, for Goursat's problem it would be inconsistent to specify both U and $\partial U/\partial n$ (or equivalently U, P, and Q subject to (4.2.30)) on the timelike arc $\mathcal{T}_0$. In doing so, note that the value of P at the (1, 1) gridpoint, for example, would depend on whether we used the partial differential equation—that is, (4.2.32)—or the prescribed data on $\mathcal{T}_0$. The most general linear boundary condition that is allowable on $\mathcal{T}_0$ for Goursat's problem is

$$a(\tau)U(\bar{\xi}(\tau), \bar{\eta}(\tau)) + b(\tau)\frac{\partial U}{\partial n}(\bar{\xi}(\tau), \bar{\eta}(\tau)) = c(\tau) \tag{4.2.34}$$

for given functions a, b, and c. This boundary condition specializes to the case that we discussed earlier if $b = 0$ and $c/a = \overline{U}$. The other limiting case, $a = 0$, is also important. For example, it corresponds physically to a boundary condition on the velocity in any interpretation where (4.2.13) is a wave equation and U is a velocity potential. The derivation of the difference scheme for the general boundary condition (4.2.34) is left as an exercise (Problem 4.2.3).

Although we have confined our discussion to a left boundary, analogous remarks apply to a right boundary and to domains contained between two timelike arcs.

In view of the role of characteristics in propagating the values of P and Q, we note that we always need to specify two independent conditions on a spacelike arc and one condition on a timelike arc. Also, as pointed out in Section 4.2.2i, we cannot specify both U and $\partial U/\partial n$ arbitrarily on a characteristic arc; the requirement that (4.2.13) hold on a characteristic arc implies that only one condition, possibly of the form (4.2.23), can be imposed there. In this case, a unique solution results only if we specify a second condition of the form (4.2.34) on an intersecting timelike arc as in the Goursat problem; Cauchy data on an intersecting spacelike arc are inconsistent. Before concluding this discussion, the following disclaimers must be made.

The primitive numerical algorithms that we have used so far (as well as similar algorithms to be used later on in this chapter and in Chapter 7) are based on *forward differencing* along characteristic coordinates. These are not necessarily the most efficient or accurate numerical schemes for computing solutions. Rather, they provide direct and concise demonstrations that solutions may be derived in a consistent manner for certain types of initial or boundary data. A discussion of sophisticated numerical solution methods is beyond the scope of this text and is not attempted.

Our arguments concerning well-posedness have all relied on our being able to construct a unique solution in some neighborhood of a given curve with given boundary data. Needless to say, these arguments do not constitute rigorous proofs; they merely ensure that in our calculation of a solution, no inconsistencies result from the given information. A broader definition of well-posedness, which we have not addressed, requires that the solution we calculate *depend continuously on the boundary data*. In this regard, let us consider a striking counterexample first proposed by Hadamard (see the discussion in Section 4.1 of [18]). This example is designed to show that a unique extension of boundary data to some neighborhood of the boundary does not necessarily imply that the solution in the extended domain depends continuously on the boundary data.

We study Laplace's equation

$$u_{xx} + u_{yy} = 0$$

in $x \geq 0$ subject to the *two* boundary conditions on the y-axis

$$u(0, y) = 0, \quad u_x(0, y) = \frac{1}{n} \sin ny, \tag{4.2.35}$$

where n is an integer. Our physical intuition suggests that there must be something wrong in prescribing Cauchy data for the Laplacian. Based on our experience with Laplace's equation, we would have been more comfortable with only one of the boundary conditions at $x = 0$ and a second boundary condition at $x = \infty$. Nevertheless, the Taylor series construction we used in Section 4.2.1 can be implemented with no difficulties because the boundary data are analytic, and we can easily construct the series in powers of x. In fact, the Taylor series can be summed to give the following result, which can also be obtained by separation of variables:

$$u(x, y) = \frac{1}{n^2} \sinh nx \sin ny. \qquad (4.2.36)$$

For any fixed $x > 0$, this result predicts oscillations in y with unbounded amplitude [since $(\sinh nx)/n^2 \to \infty$ as $n \to \infty$] and unbounded wave number n. But the Cauchy data tend to zero uniformly as $n \to \infty$. Thus, the solution does not depend continuously on the boundary data. Also, note that even if n is a fixed finite integer, this solution is unstable in the sense that $u \to \infty$ as $x \to \infty$ for any fixed y for which $\sin ny \neq 0$. There is indeed something very wrong with prescribing Cauchy data for the Laplacian.

In the absence of an exact solution, we must rely on further information—for example, physical reasoning—to argue that specifying Cauchy data on the boundary for an elliptic equation leads to an ill-posed problem.

4.2.3 Weak Solutions; Propagation of Discontinuities in P and Q; Stability

In many physical applications, the Cauchy data or boundary data are discontinuous. For example, in the acoustic approximation of the bursting balloon discussed in Section 3.9.4i, the time derivative of the velocity potential (representing the pressure perturbation) is discontinuous at the surface of the balloon, say at $r = 1$, on the axis $0 \leq r < \infty$. A discontinuity in u itself occurs naturally in a signaling problem, say over $0 \leq x < \infty$, whenever the boundary and initial values of u do not agree at $x = 0, t = 0$. Examples of this type were routinely handled in Chapter 3 using the method of characteristics. How do such discontinuities propagate into the solution domain in the general case?

Before starting this discussion, it is important to acknowledge that if a discontinuity in U, U_ξ, or U_η exists along some curve C in the "solution" domain of (4.2.13), then this equation is not satisfied on C; we have a strict solution on either side of C but not on C itself. In contrast, recall that discontinuities in $U_{\xi\xi}$ across a $\xi =$ constant characteristic or discontinuities in $U_{\eta\eta}$ across an $\eta =$ constant characteristic are perfectly allowable within the context of a strict solution.

A solution for which U, U_ξ, or U_η becomes discontinuous on a curve C is called a *weak solution*; we study such solutions in detail for quasilinear problems in Chapters 5 and 7. As we have already observed from our results in Chapter 3, discontinuities in the solution occur in linear problems only if the initial or boundary data are discontinuous, and in such cases these discontinuities propagate along characteristics. We next confirm that this observation remains true for the general linear problem (4.2.13), and we work out the details for discontinuities in the first derivative. The discussion of discontinuities in U is deferred until Section 4.3.6 because this case is best treated in terms of a system of two first-order equations. In fact, the simple examples from water waves studied in Sections 3.4.3 and 3.5.4 clearly indicate the efficiency of calculations in terms of characteristics using the system of two first-order equations.

Consider now the general problem (4.2.13), where Cauchy data have a discontinuity in P and Q at some point (ξ_0, η_0) on a spacelike arc $\mathcal{S}_0$, whereas U is continuous everywhere on $\mathcal{S}_0$. As shown in Figure 4.8, we subdivide the domain of interest into the three parts $\mathcal{D}_1$, $\mathcal{D}_0$, and $\mathcal{D}_2$ and subdivide $\mathcal{S}_0$ into $\mathcal{S}_1$ and $\mathcal{S}_2$. The solution of the Cauchy problems in $\mathcal{D}_1$ and $\mathcal{D}_2$ are strict solutions because U, P, Q are continuous on $\mathcal{S}_1$ and $\mathcal{S}_2$ separately. In particular, the values $U(\xi_0^-, \eta)$ and $U(\xi, \eta_0^-)$ are provided by these solutions.

This is as far as we can proceed within the framework of a strict solution; the solution in $\mathcal{D}_0$ cannot be calculated unless we assume that P is continuous across $\eta = \eta_0$ and Q is continuous across $\xi = \xi_0$. These are certainly plausible assumptions; in fact, they are implicit in our interpretation of (4.2.13) as a rule for the propagation of P and Q along $\xi =$ constant and $\eta =$ constant characteristics, respectively. It then follows that U is continuous across both these characteristics; that is,

$$U(\xi_0^+, \eta) = U(\xi_0^-, \eta), \quad U(\xi, \eta_0^+) = U(\xi, \eta_0^-). \tag{4.2.37}$$

The values of $U(\xi_0^+, \eta)$ and $U(\xi, \eta_0^+)$ can then be used as boundary values to solve the characteristic boundary-value problem in $\mathcal{D}_0$. Notice that the values of $U(\xi_0^+, \eta)$ and $U(\xi, \eta_0^+)$ *must not* be calculated using the averages indicated in (4.2.28), because this would introduce a spurious discontinuity in U along each

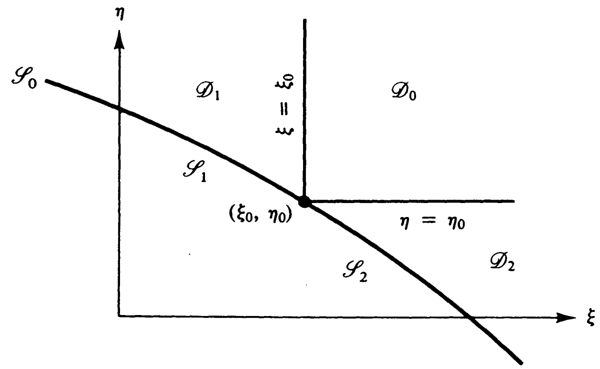

FIGURE 4.8. Initial discontinuity

of the ξ_0 and η_0 characteristics. This difficulty does not arise if the solution in $\mathcal{D}_0$ is calculated separately starting with the given characteristic boundary values (4.2.37).

The given initial discontinuities in P and Q at the point ξ_0, η_0 thus propagate along the $\xi = \xi_0$ and $\eta = \eta_0$ characteristics, respectively. These propagation rules can be derived explicitly by an analogous but more direct approach than that used in Section 4.2.1i for computing how jumps in second derivatives propagate. Consider first how the initial jump in Q propagates. We evaluate (4.2.13) on either side of $\eta = \eta_0$ and subtract the result, noting that P and U are continuous across $\eta = \eta_0$. This leads to the following first-order ordinary differential equation along $\eta = \eta_0$:

$$\frac{\partial \kappa}{\partial \xi} = \tilde{E}(\xi, \eta_0)\kappa, \tag{4.2.38a}$$

where κ denotes the jump in Q; that is,

$$\kappa(\xi, \eta_0) \equiv Q(\xi, \eta_0^+) - Q(\xi, \eta_0^-). \tag{4.2.38b}$$

Thus, we have the explicit propagation rule

$$\kappa(\xi, \eta_0) = \kappa(\xi_0, \eta_0) \exp\left[\int_{\xi_0}^{\xi} \tilde{E}(\sigma, \eta_0) d\sigma\right], \tag{4.2.39}$$

which implies, in particular, that a discontinuity $\kappa(\xi, \eta_0)$ occurs if and only if $\kappa(\xi_0, \eta_0) \neq 0$.

Similarly, the discontinuity in P,

$$\mu(\xi_0, \eta) \equiv P(\xi_0^+, \eta) - P(\xi_0^-, \eta), \tag{4.2.40}$$

propagates along the $\xi = \xi_0$ characteristic according to

$$\mu(\xi_0, \eta) = \mu(\xi_0, \eta) \exp\left[\int_{\eta_0}^{\eta} \tilde{D}(\xi_0, \sigma) d\sigma\right]. \tag{4.2.41}$$

Equations (4.2.39) and (4.2.41) may be used to characterize the behavior of the solution for large ξ and η. We argue that if an initial discontinuity in the values of P or Q at some point tends to grow as $\xi \to \infty$ or $\eta \to \infty$, then the solution is unstable. We see from (4.2.39) and (4.2.40) that a *necessary condition* for stability is that $\tilde{D} < 0$ for sufficiently large η and $\tilde{E} < 0$ for sufficiently large ξ. (For the situation sketched in Figure 4.5b, we reverse the signs.)

Consider the wave equation with constant coefficients (3.7.1). The canonical form (4.2.13) corresponds to (3.7.5) for this case with $\xi \to \zeta$, $\eta \to \mu$. Thus, $\tilde{D} = -(b+a)/4$ and $\tilde{E} = -(b-a)/4$, and the requirement $\tilde{D} < 0$, $\tilde{E} < 0$ gives the same condition (3.7.16) that we obtained earlier from the asymptotic behavior of the fundamental solution. We reiterate that (3.7.16) is only a necessary condition for stability. In the next section we study the stability of the constant-coefficient hyperbolic equation more generally.

4.2.4 Stability of the General Hyperbolic Equation with Constant Coefficients

We have seen that the general hyperbolic equation with constant coefficients can be transformed to the form (see (4.1.17))

$$u_{tt} - u_{xx} + au_x + bu_t + cu = 0, \tag{4.2.42}$$

where a, b, and c are constants. To study the stability of this equation for an arbitrary Cauchy problem on $-\infty < x < \infty$, we take its Fourier transform (see (A.2.9)) to obtain

$$\bar{u}_{tt} + b\bar{u}_t + (k^2 + iak + c)\bar{u} = 0, \tag{4.2.43}$$

where k is real. We need to solve this ordinary differential equation for $\bar{u}(k, t)$ subject to given initial conditions on $\bar{u}(k, 0)$ and $\bar{u}_t(k, 0)$. Looking for a solution $\bar{u} \sim e^{\sigma t}$ gives the following characteristic equation for the complex number σ:

$$\sigma^2 + b\sigma + k^2 + aik + c = 0. \tag{4.2.44}$$

The two roots of the quadratic (4.2.44) are

$$\sigma^+ = \sigma_1^+ + i\sigma_2^+, \quad \sigma^- = \sigma_1^- + i\sigma_2^-, \tag{4.2.45}$$

where σ_1^+, σ_1^-, σ_2^+, and σ_2^- are the real constants

$$\sigma_1^+ = \frac{1}{2}\left[-b + \sqrt{\frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{2}}\right], \tag{4.2.46a}$$

$$\sigma_1^- = \frac{1}{2}\left[-b - \sqrt{\frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{2}}\right], \tag{4.2.46b}$$

$$\sigma_2^+ = -\sigma_2^- = \sqrt{\frac{\alpha - \sqrt{\alpha^2 + \beta^2}}{2}}, \tag{4.2.46c}$$

and

$$\alpha = b^2 - 4(k^2 + c), \quad \beta = -4ak. \tag{4.2.46d}$$

Assuming that the inverse Fourier transform exists (as is the case for appropriately restricted initial data), the solution will consist of a continuous superposition of functions of k times $\exp(ikx + \sigma t)$. Therefore, it will be bounded as $t \to \infty$ if σ_1^+ and σ_1^- are nonpositive; the imaginary part of the exponents $kx + \sigma_2^+ t$ or $kx + \sigma_2^- t$ will not affect the stability of solutions. Since $\sqrt{\alpha + \sqrt{\alpha^2 + \beta^2}}$ is, by definition, positive, the requirement $\sigma_1^+ \leq 0$ implies that we must have

$$0 \leq \sqrt{\alpha + \sqrt{a^2 + \beta^2}} \leq \sqrt{2}b \tag{4.2.47}$$

for stability. We recall that (see Section 3.7.2) the condition $b > 0$ was obtained by requiring the fundamental solution to decay for a fixed x and $t \to \infty$. The second exponent σ_1^- is automatically nonpositive once (4.2.47) is imposed. Using the expressions for α and β in (4.2.46d) to simplify (4.2.47) gives the condition

$$R(k) \equiv k^2(b^2 - a^2) + b^2 c \geq 0 \tag{4.2.48}$$

to ensure that solutions are bounded as $t \to \infty$.

We note that for $|k| \to \infty$, i.e., focusing on the short-wavelength contributions to the inversion integral, boundedness of $u(x, t)$ requires $b^2 - a^2 > 0$. This, together with the condition $b \geq 0$, is exactly the condition (3.7.11) that we derived using either the fundamental solution or the rule for propagation of initial discontinuities in u_x and u_t. The other limit, $k \to 0$, corresponds to the long-wavelength contributions and requires $c > 0$.

More generally, depending upon the values of the constants a, b, and c, the parabola defined by R as a function of k has one of the four shapes given in Figure 4.9, and we have the following stability conditions:

(i) $b \geq |a|, c \geq 0$. Solutions are bounded for all k.
(ii) $b < |a|, c > 0$. Solutions are unbounded for all k.
(iii) $b \leq |a|, c \geq 0$. Solutions are bounded if $|k| \leq b\sqrt{\frac{c}{a^2-b^2}} \equiv k_c$.
(iv) $b \geq |a|, c \leq 0$. Solutions are bounded if $|k| \geq b\sqrt{\frac{c}{b^2-a^2}} \equiv k_d$.

Since for arbitrary initial data the inversion integral for $u(x, t)$ is a continuous superposition over *all* values of k, the two conditions

$$b \geq |a| \text{ and } c \geq 0 \tag{4.2.49}$$

are necessary and sufficient to ensure bounded solutions in general.

It is interesting to recall that the behavior of the fundamental solution in the far field and the rule for the propagation of initial discontinuities in the derivatives of initial data both correctly provide the first stability condition $b \geq |a|$ (see (3.7.16) and the discussion following (4.2.41)). However, neither of these criteria provides the second stability condition $c \geq 0$ because it involves long-wave ($k \approx 0$) contributions to the solution that are not excited by either of these disturbance mechanisms.

Of course, for special initial conditions it is possible to violate (4.2.49) and still have a bounded solution. For example, let $b = 1$, $a = \sqrt{2}$, $c = 1$. This corresponds to Figure 4.9c, and (4.2.48) predicts that solutions are bounded for $|k| \leq k_c = 1$. If we look for a solution of (4.2.42) with the above values of a, b, and c in the form ($k = \frac{1}{2}$)

$$u(x, t) = e^{-\sigma t} \sin \frac{1}{2}(x - vt), \tag{4.2.50}$$

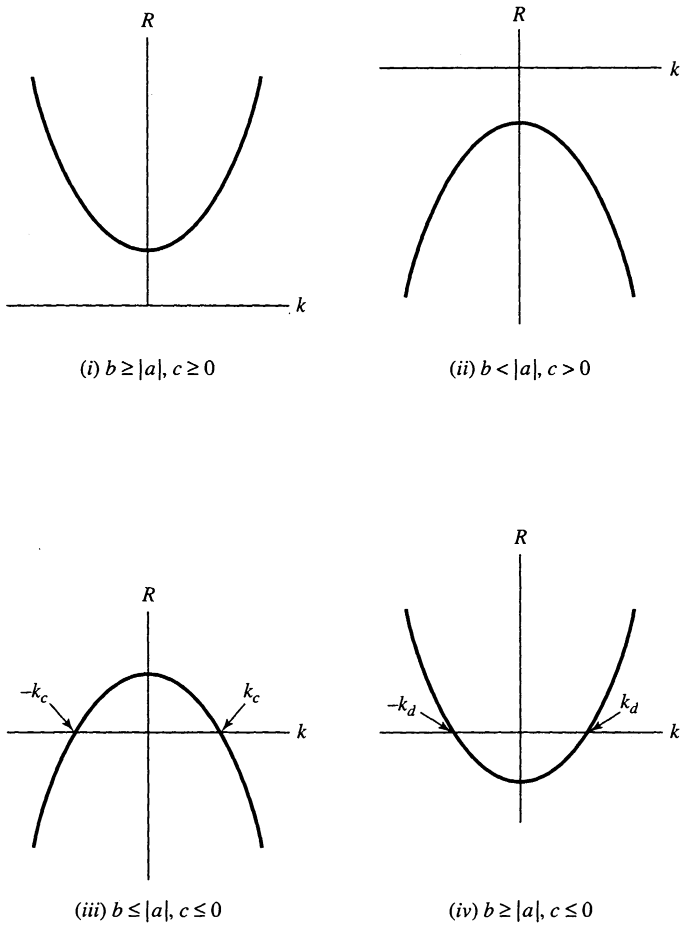

FIGURE 4.9. Stability of (4.2.42) as a function of k for different choices of a, b, and c

for unknown constants σ and v, we find that these constants must satisfy the pair of algebraic equations

$$\frac{v^2}{4} - \sigma^2 + \sigma = \frac{5}{4}, \quad \frac{v}{2} - v\sigma = \frac{1}{\sqrt{2}}. \tag{4.2.51}$$

The system (4.2.51) has two pairs of real roots,

$$\sigma = 0.16478\ldots, \quad v = 2.10938\ldots, \tag{4.2.52a}$$

$$\sigma = 0.83522\ldots, \quad v = -2.10938\ldots, \tag{4.2.52b}$$

that one can easily compute numerically. Thus, the values in (4.2.52a) show that (4.2.50) is a right-going *damped* uniform wave, and the values in (4.2.52b) give a left-going damped uniform wave. Both waves have the same phase speed (v), but the left-going wave is more strongly damped. The solution (4.2.50) corresponds to the special initial conditions

$$u(x,0) = \sin\frac{x}{2}, \quad u_t(x,0) = -\sigma\sin\frac{x}{2} - \frac{v}{2}\cos\frac{x}{2}, \tag{4.2.53}$$

and we see that a stable solution exists for $k \le k_c$ even though the first inequality in (4.2.49) is violated.

Problems

4.2.1 Consider the following Cauchy problem for the inhomogeneous dispersive wave equation (k = constant)

$$u_{tt} - u_{xx} + k^2u = p(x,t), \quad -\infty, x < \infty;\ 0 \le t, \tag{4.2.54a}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x). \tag{4.2.54b}$$

a. Introduce the characteristic coordinates

$$\xi \equiv t + x, \quad \eta \equiv t - x, \tag{4.2.55}$$

and transform (4.2.54a) to the form

$$U_{\xi\eta} = \tilde{F}U + \tilde{G}, \tag{4.2.56}$$

where $\tilde{F} \equiv -k^2/4$ and

$$U(\xi,\eta) \equiv u\left(\frac{\xi-\eta}{2}, \frac{\xi+\eta}{2}\right), \quad \tilde{G}(\xi,\eta) \equiv \frac{1}{4}p\left(\frac{\xi-\eta}{2}, \frac{\xi+\eta}{2}\right). \tag{4.2.57}$$

b. Now parameterize the initial curve $t = 0$ in the form

$$\xi = \tau, \quad \eta = -\tau, \tag{4.2.58}$$

that is, $\xi^* = \tau$, $\eta^* = -\tau$ in (4.2.16), and show that the initial conditions imply that U^* of (4.2.19a) and P^*, Q^* of (4.2.20) are given by

$$U^*(\tau) = f(\tau), \quad P^*(\tau) = \frac{g(\tau) + \dot{f}(\tau)}{2}, \quad Q^*(\tau) = \frac{g(\tau) - \dot{f}(\tau)}{2}. \tag{4.2.59}$$

c. Let

$$p(x,t) = \sin x \cos t, \quad f(x) = \sin 2x, \quad g(x) = \cos 3x. \tag{4.2.60}$$

Solve (4.2.54) numerically for these values in the interval $-\pi < x < \pi$, and $0 \le t \le 1$. Use a uniform mesh spacing equal to 0.1 in ξ and η.

d. Now assume that $f(x)$ and $f'(x)$ are continuous for all x and that $g(x)$ is also continuous everywhere except at $x = 0$, where it has a finite discontinuity ($g(0^+) - g(0^-) \equiv \rho \neq 0$). Show that the initial discontinuities in

P and Q equal $\rho/2$ and that these propagate unchanged along the $\xi = 0$ and $\eta = 0$ characteristics.

e. Suppose you wish to define the initial data at $t = 0$ in such a way that U^* and P^* as given in (4.2.59) are both continuous everywhere on the initial cuve, but Q^* has a finite discontinuity equal to κ_0 at $x = 0$. Show that we must have f continuous everywhere and that $\dot{f}$ and g are discontinuous at the origin with $\dot{f}(0^+) - \dot{f}(0^-) = g(0^-) - g(0^+) = -\kappa_0/2$. Show that the initial discontinuity in Q propagates unchanged along the $\eta = 0$ characteristic, and that U and P are continuous everywhere; the only discontinuity in Q is along the $\eta = 0$ characteristic.

f. Use the fundamental solution (see (3.7.9))

$$F(x-\xi, t-\tau) = \begin{cases} \frac{1}{2} J_0(k\sqrt{(t-\tau)^2 - (x-\xi)^2}) & \text{if } t-\tau > |x-\xi|, \\ 0 & \text{if } t-\tau < |x-\xi| \end{cases} \tag{4.2.61}$$

to derive the solution of (4.2.54) in the form

$$\begin{aligned} u(x,t) =& \frac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} p(\xi,\tau) J_0(k\sqrt{(t-\tau)^2 - (x-\xi)^2}) d\xi\, d\tau \\ &+ \frac{1}{2}\int_{x-t}^{x+t} J_0(k\sqrt{t^2 - (x-\xi)^2}) g(\xi) d\xi \\ &+ \frac{1}{2}\frac{\partial}{\partial t}\int_{x-t}^{x+t} J_0(k\sqrt{t^2 - (x-\xi)^2}) f(\xi) d\xi. \end{aligned} \tag{4.2.62}$$

Specialize this result to the expressions for p, f, and g in part (c) and verify the accuracy of your numerical results.

4.2.2 In Section 3.3.6ii we derived the equation governing steady flow over a thin body in the form (3.3.59). For two-dimensional flow, this result reduces to (3.3.70) for the perturbation velocity potential $\varphi(x^*, y^*; \epsilon)$. Consider a symmetric body having the upper surface defined by

$$y = \begin{cases} F(x^*), 0 \le x^* \le 1, F(0) = F(1) = 0, \\ 0, x^* \le 0, x^* \ge 1. \end{cases} \tag{4.2.63}$$

The boundary condition of zero normal velocity on the body surface is given by (3.3.60) for $0 \le x \le 1$, and symmetry implies that

$$\varphi_{y^*}(x^*, 0; \epsilon) = 0 \text{ if } x^* < 0 \text{ or } x^* > 1. \tag{4.2.64}$$

For supersonic flow ($M > 1$), disturbances propagate only downstream. Thus, x^* is a timelike variable, and the "initial conditions" are

$$\varphi(0, y^*; \epsilon) = \varphi_{y^*}(0, y^*; \epsilon) = 0, \ y^* > 0. \tag{4.2.65}$$

a. Expand φ for small ϵ in the form

$$\varphi(x^*, y^*; \epsilon) = M[\varphi^{(1)}(x^*, y^*) + \epsilon\varphi^{(2)}(x^*, y^*)] + O(\epsilon^2), \tag{4.2.66}$$

and show that $\varphi^{(1)}$ and $\varphi^{(2)}$ are governed by

$$(M^2 - 1)\varphi^{(1)}_{xx} - \varphi^{(1)}_{yy} = 0, \tag{4.2.67a}$$

$$(M^2 - 1)\varphi_{xx}^{(2)} - \varphi_{yy}^{(2)} = - M^2[(\gamma - 1)M^2 + 2]\varphi_x^{(1)}\varphi_{xx}^{(1)} - 2M^2\varphi_y^{(1)}\varphi_{xy}^{(1)}, \qquad (4.2.67b)$$

where we have dropped the asterisks for simplicity. Show that (3.3.60) implies the following boundary conditions on the upper surface,

$$\varphi_y^{(1)}(x, 0) = F'(x), \qquad (4.2.68a)$$

$$\varphi_y^{(1)}(x, 0) = F'(x)\varphi_x^{(1)}(x, 0) - F(x)\varphi_{yy}^{(1)}(x, 0), \qquad (4.2.68b)$$

and the upstream boundary condition becomes

$$\varphi^{(1)}(0^-, y) = \varphi^{(2)}(0^-, y) = 0. \qquad (4.2.69)$$

b. Show that the solution for $\varphi^{(1)}$ is

$$\varphi^{(1)}(x, y) = -\frac{1}{\sqrt{M^2 - 1}} F(x - \sqrt{M^2 - 1}y) \qquad (4.2.70)$$

if $0 \le x - \sqrt{M^2 - 1}y \le 1$, and $\varphi^{(1)}(x, y) \equiv 0$ otherwise. Thus, the assumed expansion (4.2.66) breaks down for transonic flow ($M \approx 1$). A discussion of the correct expansion for transonic flow is discussed in [10]. Henceforth, assume that $M - 1$ is not small.

c. Introduce the characteristic coordinates

$$\xi \equiv x - \sqrt{M^2 - 1}y, \;\; \eta = x + \sqrt{M^2 - 1}y, \qquad (4.2.71)$$

and denote $\varphi^{(1)}\left(\frac{\xi+\eta}{2}, \frac{\eta-\xi}{2\sqrt{M^2-1}}\right) \equiv \phi^{(1)}(\xi, \eta)$. Show that (4.2.67b) transforms to

$$\phi_{\xi\eta}^{(2)} = -\frac{\gamma + 1}{4}\frac{M^4}{(M^2 - 1)^2} F'(\xi)F''(\xi). \qquad (4.2.72)$$

d. Solve (4.2.72) in the form

$$\phi^{(2)}(\xi, \eta) = \frac{\gamma + 1}{8}\frac{M^4}{(M^2 - 1)^2}\eta[F'^2(0) - F'^2(\xi)] + f_2(\xi) \qquad (4.2.73)$$

if $0 < \xi < 1$, and $\phi^{(2)} \equiv 0$ if $\xi < 0$ or $\xi > 1$. Determine the function $f_2(\xi)$ using the boundary condition (4.2.68b).

Notice that the term proportional to η in $\phi^{(2)}$ contributes a term proportional to $\epsilon^2\eta$ to the expansion of the velocity potential (see (3.3.58) and (4.2.66)). Thus, $\epsilon^2\phi^{(2)}$ is of order ϵ^2, as implied by (3.3.58) and (4.2.66), only as long as η is not large; if η is as large as $O(\epsilon^{-1})$, then $\epsilon^2\phi^{(2)} = O(\epsilon)$, in violation of the assumed ordering of terms. Now, $\eta \to \infty$ with ξ fixed implies that $y \to \infty$ along the characteristic ray $\xi = $ constant. Thus, the assumed form of the perturbation expansion is correct only if $y = O(1)$; it breaks down to $O(\epsilon^2)$ if $y = O(\epsilon^{-1})$. This type of nonuniformity in the far field was also encountered in our study of shallow-water flow (see (3.5.44)). An expansion procedure that remains valid in the far field is developed in Chapter 8.

4.2.3 Generalize the forward difference scheme of Sec 4.2.2iii for solving Goursat's problem when the boundary condition on the timelike arc is given by (4.2.34). First use (4.2.23) to express $\partial U/\partial n$ in terms of P and Q along the timelike arc $\mathcal{T}_0$, and then show that this result, together with the specification of U on the characteristic arc $\mathcal{S}_0$, uniquely defines U, P, and Q everywhere in $\mathcal{D}_1$.

4.2.4 In Section 4.2.1i we saw that the following is one of the ways to define a characteristic curve $\phi(x, y) = \xi =$ constant for a hyperbolic equation in two independent variables: Given (4.1.1) and Cauchy data on $\phi(x, y) = \xi$, we cannot compute $\partial^2 U/\partial \xi^2$ on this curve. In this problem we study how this idea generalizes to the three-dimensional wave equation with a space-dependent signal speed; that is,

$$\sum_{j=1}^{3} \frac{\partial^2 v}{\partial x_j^2} - \frac{1}{c^2(x_1, x_2, x_3)} \frac{\partial^2 v}{\partial x_4^2} = 0. \tag{4.2.74}$$

Here x_1, x_2, x_3 are Cartesian coordinates, and x_4 is the time t. Assume that instead of $x_1, \ldots, x_4$, we introduce new variables $\xi_1, \ldots, \xi_4$ defined in general by the four functions

$$\phi_j(x_1, \ldots, x_4) = \xi_j = \text{constant}, \quad j = 1, \ldots, 4. \tag{4.2.75}$$

a. Denote

$$V(\phi_1(x_1, \ldots, x_4), \ldots, \phi_4(x_1, \ldots, x_4)) \equiv v(x_1, \ldots, x_4), \tag{4.2.76}$$

and show that (4.2.74) transforms to

$$\sum_{k=1}^{4}\sum_{\ell=1}^{4} \frac{\partial^2 V}{\partial \xi_k \partial \xi_\ell} \left(\sum_{j=1}^{3} \frac{\partial \phi_k}{\partial x_j} \frac{\partial \phi_\ell}{\partial x_j} - \frac{1}{c^2} \frac{\partial \phi_k}{\partial x_4} \frac{\partial \phi_\ell}{\partial x_4} \right)$$
$$+ \sum_{k=1}^{4} \frac{\partial V}{\partial \xi_k} \left(\sum_{j=1}^{3} \frac{\partial^2 \phi_k}{\partial x_j^2} - \frac{1}{c^2} \frac{\partial^2 \phi_k}{\partial x_4^2} \right) = 0. \tag{4.2.77}$$

b. Now assume that we are given Cauchy data on $\phi_4(x_1, \ldots, x_4) = 0$; that is, we are given the values of $V(\xi_1, \xi_2, \xi_3, 0)$ and $(\partial V/\partial \xi_4)(\xi_1, \xi_2, \xi_3, 0)$. It then follows that we also know $\partial V/\partial \xi_1$, $\partial V/\partial \xi_2$, $\partial V/\partial \xi_3$ as well as $\partial^2 V/\partial \xi_1 \partial \xi_4$, $\partial^2 V/\partial \xi_2 \partial \xi_4$, and $\partial^2 V/\partial \xi_3 \partial \xi_4$ on $\xi_4 = 0$. But we do not know $\partial^2 V/\partial \xi_4^2$ on $\xi_4 = 0$. Show that on $\xi_4 = 0$, (4.2.77) takes the form

$$\frac{\partial^2 V}{\partial \xi_4^2} \left[\sum_{j=1}^{3} \left(\frac{\partial \phi_4}{\partial x_j} \right)^2 - \frac{1}{c^2(x_1, x_2, x_3)} \left(\frac{\partial \phi_4}{\partial x_4} \right)^2 \right]$$
$$= \text{known terms on } \xi_4 = 0. \tag{4.2.78}$$

Thus, either the quadratic form

$$Q \equiv \sum_{j=1}^{3} \left(\frac{\partial \phi_4}{\partial x_j} \right)^2 - \frac{1}{c^2} \left(\frac{\partial \phi_4}{\partial x_4} \right)^2 \tag{4.2.79}$$

is not equal to zero and we can solve for $\partial^2 V/\partial \xi_4^2$, or $Q = 0$ and the Cauchy data cannot be specified arbitrarily on $\xi_4 = 0$. The three-dimensional manifold $\phi_4(x_1, \ldots, x_4) = 0$ that satisfies $Q = 0$ is called a characteristic manifold. If we assume a solution of $Q = 0$ in the form

$$x_4 = u(x_1, x_2, x_3), \tag{4.2.80}$$

we see that u obeys

$$\left(\frac{\partial u}{\partial x_1} \right)^2 + \left(\frac{\partial u}{\partial x_2} \right)^2 + \left(\frac{\partial u}{\partial x_3} \right)^2 = \frac{1}{c^2(x_1, x_2, x_3)}. \tag{4.2.81}$$

This is the eikonal equation that we will study in Chapter 6.

c. Parallel the development in Section 4.2.1ii to show that $Q = 0$ also represents the manifold on which the first derivatives of a solution of (4.2.77) as well as all second derivatives except $\partial^2 V/\partial \xi_4^2$ are continuous. But $\partial^2 V/\partial \xi_4^2$ may be discontinuous.

4.3 Hyperbolic Systems of Two First-Order Equations

In our discussion so far in this chapter we have concentrated on the general second-order equation (4.1.1), which is the primary mathematical model in a number of physical applications—for example, the diffusion equation for one-dimensional heat conduction, Laplace's equation for steady heat conduction in two dimensions, and the one-dimensional wave equation for the small-amplitude vibrations of a string. In other applications, the second-order form (4.1.1) is the result of eliminating one of the two dependent variables from a model that occurs naturally as a system of two first-order equations. This is the case, for example, for steady incompressible irrotational flow in two dimensions. As discussed in Chapter 2, Laplace's equation is a consequence of the two first-order equations div $\mathbf{v} = 0$, curl $\mathbf{v} = 0$, where $\mathbf{v}$ is the velocity vector. Also, in our study of shallow-water waves of small amplitude in Chapter 3, we derived the one-dimensional wave equation for the velocity or free-surface perturbations after eliminating one of these variables from the basic first-order equations for mass and momentum conservation.

In those applications where the mathematical model of the basic physical problem is in the form of a system of first-order equations, it is important to reconsider some of the results in Sections 4.1–4.2, particularly for the hyperbolic case.

4.3.1 The Perturbation of a Quasilinear System Near a Known Solution

In many physical applications, (4.1.1) governs the perturbations to a known solution of a quasilinear system of equations. We have already seen some examples of this in our discussion of acoustics, shallow-water waves, and the like, where we considered perturbations to the trivial solution of no flow. In Chapters 5 and 7 we study in what sense such quasilinear systems represent physical conservation laws. For the purposes of our discussion here, we start with a given pair of quasilinear differential equations written in vector form,

$$\mathbf{v}_t + A\mathbf{v}_x = \mathbf{f}, \tag{4.3.1}$$

where $\mathbf{v}$ and $\mathbf{f}$ are two-dimensional vectors with components

$$\mathbf{v} = (v_1, v_2), \quad \mathbf{f} = (f_1, f_2), \tag{4.3.2}$$

and A is a linear operator with (2×2) matrix components A_{ij}. We assume that the components of $\mathbf{f}$ and A are functions of v_1, v_2, x, t:

$$A_{ij} = A_{ij}(v_1, v_2, x, t), \quad f_i = f_i(v_1, v_2, x, t), \tag{4.3.3}$$

where the indices i, j may equal 1 or 2. Since the A_{ij} do not depend on the $\partial v_i/\partial x$ or $\partial v_i/\partial t$, the two components of (4.3.1) represent a quasilinear pair of first-order equations.

Typically, the pair of partial differential equations corresponding to the components of (4.3.1) arise when we simplify a pair of conservation laws. For example, the laws of mass and momentum conservation (3.2.11) for shallow water simplify to the system (3.2.12), which is a special case of (4.3.1). A similar situation in acoustics leads to the system (3.3.23). In both these examples $\mathbf{f} \equiv 0$. Two examples where $f \neq 0$ are discussed in Section 4.3.4.

In this section we derive the leading *linear* system that results from (4.3.1) when we perturb a known solution. Let

$$\mathbf{v}^{(0)}(x, t) = (v_1^{(0)}(x, t), v_2^{(0)}(x, t)) \tag{4.3.4}$$

be a known solution of (4.3.1). If $\mathbf{f} \equiv 0$, an important special case has $\mathbf{v}^{(0)}$ equal to a constant vector. A number of examples of this case were discussed in Chapter 3. A more general case, where $\mathbf{f}$ is independent of t, has $\mathbf{v}^{(0)}$ independent of t also, that is, a steady-state solution of the pair of ordinary differential equations that result from (4.3.1) if $\partial/\partial t = 0$. In the slightly more general case with $\mathbf{f}$ depending on $x - ct$, $c =$ constant, we look for a solution where $\mathbf{v}^{(0)}$ is a function of $x - ct$. This is the nonlinear version of the "uniform wave" solution considered in Section 3.8.1. In the most general case, $\mathbf{v}^{(0)}(x, t)$ may represent some particular *exact* solution of (4.3.1) [see the example of Section 4.3.4i] that we wish to perturb in the following sense.

Assume that we want to solve an initial-value problem for (4.3.1) in $-\infty < x < \infty$, $0 \leq t$ where the given initial value $\mathbf{v}(x, 0)$ is close to the one corresponding to the nominal solution (4.3.4). More precisely, assume that the initial value of $\mathbf{v}$

is given in the form

$$\mathbf{v}(x, 0; \epsilon) = \mathbf{h}(x) + \epsilon \mathbf{g}(x) \tag{4.3.5}$$

for a specified bounded vector function $\mathbf{g}(x) = (g_1, g_2)$ and a small parameter ϵ. In (4.3.5), $\mathbf{h}(x) \equiv \mathbf{v}^{(0)}(x, 0)$, the special solution (4.3.4) evaluated at $t = 0$. The perturbation idea is that the solution $\mathbf{v}(x, t; \epsilon)$ of the initial-value problem defined by (4.3.1) and (4.3.5) is close to the special solution $\mathbf{v}^{(0)}(x, t)$, and we assume that it has the form

$$\mathbf{v}(x, t; \epsilon) = \mathbf{v}^{(0)}(x, t) + \epsilon \mathbf{u}(x, t) + O(\epsilon^2). \tag{4.3.6}$$

In Chapter 8 we explore the conditions under which this "regular perturbation" idea is correct.

Assuming that the expansion (4.3.6) is correct for sufficiently small ϵ, we need to compute the expressions that result for A and $\mathbf{f}$ when (4.3.6) is substituted for $\mathbf{v}$. In most physical applications A and $\mathbf{f}$ are analytic functions of v_1 and v_2, and therefore they have the following expansions in a neighborhood of the nominal solution $\mathbf{v}^{(0)}(x, t)$:

$$\begin{aligned} A_{ij}(v_1, v_2, x, t; \epsilon) &= A_{ij}(v_1^{(0)} + \epsilon u_1, v_2^{(0)} + \epsilon u_2, x, t; \epsilon) + O(\epsilon^2) \\ &= A_{ij}(v_1^{(0)}(x, t), v_2^{(0)}(x, t), x, t; 0) \\ &\quad + \epsilon \left(\frac{\partial A_{ij}}{\partial v_1} u_1 + \frac{\partial A_{ij}}{\partial v_2} u_2 + \frac{\partial A_{ij}}{\partial \epsilon} \right) + O(\epsilon^2), \quad (4.3.7a) \\ f_i(v_1, v_2, x, t; \epsilon) &= f_i(v_1^{(0)} + \epsilon u_1, v_2^{(0)} + \epsilon u_2, x, t; \epsilon) + O(\epsilon^2) \\ &= f_i(v_1^{(0)}(x, t), v_2^{(0)}(x, t), x, t; 0) \\ &\quad + \epsilon \left(\frac{\partial f_i}{\partial v_1} u_1 + \frac{\partial f_i}{\partial v_2} u_2 + \frac{\partial f_i}{\partial \epsilon} \right) + O(\epsilon^2), \quad (4.3.7b) \end{aligned}$$

where we evaluate the arguments of the partial derivatives of A_{ij} and f_i for $v_1 = v_1^{(0)}(x, t)$, $v_2 = v_2^{(0)}(x, t)$, and $\epsilon = 0$. In (4.3.7) we have generalized (4.3.3) to allow the A_{ij} and f_i to depend also on ϵ. Thus, (4.3.4) is an exact solution of (4.3.1) for $\epsilon = 0$.

Substituting the expansions (4.3.6)–(4.3.7) into (4.3.1), ignoring terms of $O(\epsilon^2)$, and taking into account that $\mathbf{v}^{(0)}(x, t)$ is a solution of (4.3.1) results in the *linear* vector partial differential equation for $\mathbf{u}(x, t)$

$$\mathbf{u}_t + A^{(0)} \mathbf{u}_x + B\mathbf{u} = \mathbf{f}^{(1)}, \tag{4.3.8a}$$

and initial condition

$$\mathbf{u}(x, 0) = \mathbf{g}(x). \tag{4.3.8b}$$

Here we have introduced the notation $A^{(0)}$ for the linear operator with components that are functions of x, t defined by the $O(1)$ term in (4.3.7a); that is,

$$A_{ij}^{(0)}(x, t) \equiv A_{ij}(v_1^{(0)}(x, t), v_2^{(0)}(x, t), x, t; 0). \tag{4.3.9a}$$

The components of the operator B and vector $\mathbf{f}^{(1)}$ are also functions of x, t defined by

$$B_{ij}(x,t) \equiv -\frac{\partial f_i}{\partial v_j} + \sum_{\ell=1}^{2} \frac{\partial A_{i\ell}}{\partial v_j} \frac{\partial v_\ell^{(0)}}{\partial x}, \tag{4.3.9a}$$

$$f_i^{(1)} \equiv \frac{\partial f_i}{\partial \epsilon} - \sum_{\ell=1}^{2} \frac{\partial A_{i\ell}}{\partial \epsilon} \frac{\partial v_\ell^{(0)}}{\partial x}, \tag{4.3.9b}$$

where again we have set $v_1 = v_1^{(0)}(x,t)$, $v_2 = v_2^{(0)}(x,t)$, and $\epsilon = 0$ in the arguments of f_i and the partial derivatives of $A_{i\ell}$. In component form, (4.3.8a) reads

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{2} A_{ij}^{(0)}(x,t)\frac{\partial u_j}{\partial x} + \sum_{j=1}^{2} B_{ij}(x,t)u_j = f_i^{(1)}(x,t), \ i = 1, 2. \tag{4.3.10}$$

We note that if $\mathbf{f} \equiv 0$ and $\mathbf{v}^{(0)}$ is a constant, then $B \equiv 0$ and $\mathbf{f}^{(1)} \equiv 0$. In this case $A^{(0)}$ is a constant operator only if the operator A in the quasilinear problem (4.3.1) does not depend on x and t. The linearized equations for shallow-water waves [see (3.2.30)] and the linearized equations for one-dimensional acoustics [see (3.3.30)] are examples of this special case. In both these examples $\mathbf{f} \equiv 0$ and the nominal solution $\mathbf{v}^{(0)}$ is a constant vector. An example with $\mathbf{f} = 0$ and $\mathbf{v}^{(0)}(x,t)$ is discussed in Section 4.3.4i. A second example with $\mathbf{f} \neq 0$ and $\mathbf{v}^{(0)} =$ constant appears in Section 4.3.4ii.

The extension of the preceding results to higher dimensions is straightforward; in effect (4.3.10) generalizes to a system of N equations for the N components u_i. This case is discussed for the quasilinear problem (4.3.1) in Section 7.2.

The case of more independent variables is also important. For example, in three-dimensional acoustics, the system corresponding to (4.3.10) that follows from (3.3.33a), (3.3.38), and (3.3.36) is

$$\overline{\rho}_t + \operatorname{div} \overline{v} = 0, \quad \overline{\mathbf{u}}_t + \operatorname{grad} \overline{\rho} = 0, \tag{4.3.11}$$

where according to (3.3.44) we have perturbed the velocity about $\mathbf{u} = 0$ and the density about $\rho = 1$. For this special case, $\overline{\mathbf{u}} = \operatorname{grad} \overline{\phi}$, and (4.3.11) reduces to the wave equation (3.3.49) with $\epsilon = 0$. In general, such a reduction is not possible, and we discuss only the case of two independent variables.

4.3.2 Characteristics

Here again we define a characteristic curve for (4.3.10) based on whether it is possible to extend given Cauchy data on a curve $\mathcal{C}$ to a neighboring curve (see Section 4.2.1i).

Let $\mathcal{C}$ be defined in the implicit form

$$\phi(x,t) = \xi_0 = \text{constant}, \tag{4.3.12}$$

and assume that u_1 and u_2 are given on $\mathcal{C}$. Now (ϕ_x, ϕ_t) are the components of a normal to $\mathcal{C}$ and $(\phi_t, -\phi_x)$ are the components of a tangent. Therefore, knowing the u_i on $\mathcal{C}$ means that we also know the tangential derivative $(\phi_t(\partial u_i/\partial x) - \phi_x(\partial u_i/\partial t))$ there. It then follows that we can express $\partial u_i/\partial t$, say, in terms of $\partial u_i/\partial x$ on $\mathcal{C}$ in the form

$$\frac{\partial u_i}{\partial t} = -\lambda \frac{\partial u_i}{\partial x} + \text{known terms}, \tag{4.3.13a}$$

where we have assumed $\phi_x \neq 0$ and have denoted

$$\lambda \equiv -\frac{\phi_t}{\phi_x}. \tag{4.3.13b}$$

If we use (4.3.13) in (4.3.10), this reduces to the following pair of linear algebraic equations for $\partial u_1/\partial x$ and $\partial u_2/\partial x$ on $\mathcal{C}$:

$$\sum_{j=1}^{2}(A_{ij}^{(0)} - \delta_{ij}\lambda)\frac{\partial u_j}{\partial x} = \text{known terms}, \quad i = 1, 2, \tag{4.3.14}$$

where δ_{ij} is the Kronecker delta ($\delta_{ij} = 1, i = j$; $\delta_{ij} = 0, i \neq j$). The system (4.3.14) has a unique solution if the determinant of coefficients does not vanish. In this case, the given Cauchy data can be extended to a neighborhood of $\mathcal{C}$.

The vanishing of the determinant of coefficients in (4.3.14) corresponds exactly to the quadratic expression defining the eigenvalues of the $\{A_{ij}^{(0)}\}$ matrix. To see this, let $\mathbf{w}$ be a right eigenvector of $A^{(0)}$ belonging to the eigenvalue λ; that is,

$$A^{(0)}\mathbf{w} = \lambda\mathbf{w}, \quad \mathbf{w} \neq \mathbf{0}, \tag{4.3.15a}$$

or

$$(A^{(0)} - \lambda\mathbf{I})\mathbf{w} = \mathbf{0}, \quad \mathbf{w} \neq \mathbf{0}, \tag{4.3.15b}$$

where $\mathbf{I}$ is the identity matrix. Since (4.3.15b) holds for a nonzero vector $\mathbf{w}$, the determinant of $(A^{(0)} - \lambda\mathbf{I})$, which is just the determinant of coefficients in (4.3.14), must vanish. Setting $\det\{A^{(0)} - \lambda\mathbf{I}\} = 0$ gives the quadratic expression

$$\lambda^2 - (A_{11}^{(0)} + A_{22}^{(0)})\lambda + A_{11}^{(0)}A_{22}^{(0)} - A_{12}^{(0)}A_{21}^{(0)} = 0, \tag{4.3.16a}$$

which has the two solutions

$$\lambda_{1,2}(x, t) = \frac{1}{2}\{A_{11}^{(0)} + A_{22}^{(0)} \pm [(A_{11}^{(0)} - A_{22}^{(0)})^2 + 4A_{12}^{(0)}A_{21}^{(0)}]^{1/2}\}, \tag{4.3.16b}$$

in which henceforth we shall use the plus sign for λ_1 and the minus sign for λ_2.

The eigenvalues λ_i are real and distinct if

$$(A_{11}^{(0)} - A_{22}^{(0)})^2 + 4A_{12}^{(0)}A_{21}^{(0)} > 0, \tag{4.3.17}$$

and we concentrate on this case, which corresponds exactly to the hyperbolic problem for the second-order equation (4.1.1).

The characteristic curves $\phi_1(x, t) = \xi_1 =$ constant and $\phi_2(x, t) = \xi_2 =$ constant are defined by the solutions of $dx/dt = \lambda_1(x, t)$ and $dx/dt = \lambda_2(x, t)$, respectively, and since λ_1 and λ_2 are real and distinct, these two families of curves define a coordinate system in terms of the ξ_1, ξ_2 variables. Again, we note that Cauchy data along a characteristic curve do not define the solution uniquely near this curve and that for a given solution of (4.3.10), u_1 and u_2 are not independent; they must satisfy (4.3.14) with the left-hand side equal to zero.

The following definitions and results from linear algebra will be helpful in the next section as well as in Section 7.2, where we consider the corresponding quasilinear problem.

Given an n-vector in the component form $(u_1, u_2, \dots, u_n)$ and an $n \times n$ matrix with elements A_{ij}, we may regard the u_i and A_{ij} as the *components of a vector* $\mathbf{u}$ *and a linear operator* A associated respectively with some unspecified basis $\mathbf{b}_1, \dots, \mathbf{b}_n$ according to the definition

$$\mathbf{u} = \sum_{k=1}^{n} u_k \mathbf{b}_k, \tag{4.3.18a}$$

$$A\mathbf{b}_j = \sum_{k=1}^{n} A_{kj} \mathbf{b}_k, \quad \text{for each } j = 1, \dots, n. \tag{4.3.18b}$$

If the eigenvalues of the $\{A_{ij}\}$ matrix are *real and distinct*, the associated eigenvectors are linearly independent. To prove this, assume that the eigenvectors $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$ associated with $\lambda_1, \lambda_2, \dots, \lambda_n$, respectively, are linearly dependent. That is, there exist constants $c_1, c_2, \dots, c_n$ not all equal to zero such that

$$\sum_{k=1}^{n} c_k \mathbf{w}_k = \mathbf{0}. \tag{4.3.19a}$$

This means that

$$\sum_{k=1}^{n} c_k A\mathbf{w}_k = \sum_{k=1}^{n} c_k \lambda_k \mathbf{w}_k = \mathbf{0}, \tag{4.3.19b}$$

which contradicts (4.3.19a) for distinct λ_k.

4.3.3 *Transformation to Characteristic Dependent and Independent Variables*

In this section we shall proceed from the component form (4.3.10) of the governing equations and verify explicitly that a transformation to a basis of eigenvectors diagonalizes the matrix $\{A_{ij}^{(0)}\}$.

Recall [or observe directly from (4.3.15a)] that any scalar times an eigenvector is still an eigenvector. Therefore, one can specify only the ratio of components of a two-dimensional eigenvector. Let w_{i1} and w_{i2} be the first and second components, respectively, of the eigenvector $\mathbf{w}_i$ belonging to the eigenvalue λ_i. Writing (4.3.15b) in component form shows that

$$\frac{w_{i1}}{w_{i2}} = -\frac{A_{12}^{(0)}}{A_{11}^{(0)} - \lambda_i}, \quad i = 1, 2, \tag{4.3.20a}$$

and we choose

$$\mathbf{w}_i = (-A_{12}^{(0)}, A_{11}^{(0)} - \lambda_i), \quad i = 1, 2. \tag{4.3.20b}$$

Note that we may assume $A_{12}^{(0)} \neq 0$ because if $A_{12}^{(0)} = 0$ in (4.3.10), we can interchange components so that $A_{21}^{(0)}$ occurs in (4.3.20). If $A_{21}^{(0)} = 0$ also, we need not proceed further because (4.3.10) is already in characteristic form [see (4.3.25)].

(i) *Characteristic dependent variables*

Since the λ_i are distinct, the $\mathbf{w}_i$ are linearly independent and may be chosen as a new basis. Let the components of $\mathbf{u}$ with respect to the $\mathbf{w}_1$, $\mathbf{w}_2$ basis be (U_1, U_2), i.e.,

$$\mathbf{u} = \sum_{k=1}^{2} U_k \mathbf{w}_k. \tag{4.3.21}$$

Using (4.3.20), we obtain the following linear transformation linking the u_i to the U_i:

$$u_i = \sum_{j=1}^{2} W_{ij} U_j, \quad i = 1, 2, \tag{4.3.22a}$$

where the transformation matrix $\{W_{ij}\}$ is given by

$$\{W_{ij}\} \equiv \begin{pmatrix} -A_{12}^{(0)} & -A_{12}^{(0)} \\ A_{11}^{(0)} - \lambda_1 & A_{11}^{(0)} - \lambda_2 \end{pmatrix}. \tag{4.3.22b}$$

To calculate the system of equations corresponding to (4.3.10) governing the U_i, we first use (4.3.22a) to express the u_i in terms of the U_i in (4.3.10) to obtain

$$\sum_{k=1}^{2} \left(W_{\ell k} \frac{\partial U_k}{\partial t} + \frac{\partial W_{\ell k}}{\partial t} U_k \right) + \sum_{j=1}^{2} \sum_{k=1}^{2} A_{\ell j} \left(W_{jk} \frac{\partial U_k}{\partial x} + \frac{\partial W_{jk}}{\partial x} U_k \right)$$
$$+ \sum_{j=1}^{2} \sum_{k=1}^{2} B_{\ell j} W_{jk} U_k = f_\ell^{(1)}, \quad \ell = 1, 2. \tag{4.3.23}$$

Now, we multiply (4.2.23) by $V_{i\ell}$ and sum over ℓ, where $\{V_{ij}\}$ is the inverse matrix of $\{W_{ij}\}$; that is,

$$\{V_{ij}\} \equiv \frac{1}{A_{12}^{(0)}(\lambda_2 - \lambda_1)} \begin{pmatrix} A_{11}^{(0)} - \lambda_2 & A_{12}^{(0)} \\ \lambda_1 - A_{11}^{(0)} & -A_{12}^{(0)} \end{pmatrix}. \tag{4.3.24}$$

The final result takes the *diagonal characteristic form*

$$\frac{\partial U_i}{\partial t} + \lambda_i(x, t) \frac{\partial U_i}{\partial x} + \sum_{k=1}^{2} C_{ik}(x, t) U_k = F_i(x, t), \quad i = 1, 2. \tag{4.3.25}$$

Here the C_{ik} are given by

$$C_{ik} \equiv \sum_{\ell=1}^{2} V_{i\ell} \frac{\partial W_{\ell k}}{\partial t} + \sum_{\ell=1}^{2}\sum_{j=1}^{2} V_{i\ell} A_{\ell j} \frac{\partial W_{jk}}{\partial x} + \sum_{\ell=1}^{2}\sum_{j=1}^{2} V_{i\ell} B_{\ell j} W_{jk}, \quad (4.3.26a)$$

which can also be written in matrix form as the three matrix products

$$\{C\} = \{V\}\left\{\frac{\partial W}{\partial t}\right\} + \{V\}\{A\}\left\{\frac{\partial W}{\partial x}\right\} + \{V\}\{B\}\{W\}. \quad (4.3.26b)$$

The reader should also verify that the matrix product $\{V\}$ $\{A\}$ $\{W\}$ is just the diagonal matrix $\{\lambda_i \delta_{ij}\}$. The F_i are given by

$$F_i \equiv \sum_{\ell=1}^{2} V_{i\ell} F_\ell^{(1)}. \quad (4.3.26c)$$

The inverse relation to (4.3.22a) gives the U_i in terms of the u_i:

$$U_i = \sum_{j=1}^{2} V_{ij} u_j. \quad (4.3.26d)$$

(ii) *Characteristic independent variables*

The diagonal characteristic form for the differential operator in (4.3.25) becomes even more transparent once we change independent variables to ξ_1 and ξ_2. Regard the U_i, λ_i, C_{ij}, F_i in (4.3.25) as functions of ξ_1, ξ_2 without changing the notation, for the sake of simplicity. We compute

$$\frac{\partial U_1}{\partial t} = \frac{\partial U_1}{\partial \xi_1}\frac{\partial \phi_1}{\partial t} + \frac{\partial U_1}{\partial \xi_2}\frac{\partial \phi_2}{\partial t}, \quad \frac{\partial U_1}{\partial x} = \frac{\partial U_1}{\partial \xi_1}\frac{\partial \phi_1}{\partial x} + \frac{\partial U_1}{\partial \xi_2}\frac{\partial \phi_2}{\partial x}. \quad (4.3.27)$$

Therefore,

$$\begin{aligned}\frac{\partial U_1}{\partial t} + \lambda_1 \frac{\partial U_1}{\partial x} &= \left(\frac{\partial \phi_1}{\partial t} + \lambda_1 \frac{\partial \phi_1}{\partial x}\right)\frac{\partial U_1}{\partial \xi_1} + \left(\frac{\partial \phi_2}{\partial t} + \lambda_1 \frac{\partial \phi_2}{\partial x}\right)\frac{\partial U_1}{\partial \xi_2} \\ &= (\lambda_1 - \lambda_2)\frac{\partial \phi_2}{\partial x}\frac{\partial U_1}{\partial \xi_2}, \quad (4.3.28a)\end{aligned}$$

since $\lambda_i = -(\partial\phi_i/\partial t)/(\partial\phi_i/\partial x)$. Similarly, interchanging indices gives

$$\frac{\partial U_2}{\partial t} + \lambda_2 \frac{\partial U_2}{\partial x} = (\lambda_2 - \lambda_1)\frac{\partial \phi_1}{\partial x}\frac{\partial U_2}{\partial \xi_1}. \quad (4.3.28b)$$

Thus, (4.3.25) simply reduces to a definition of the directional derivative of U_1 along the $\phi_1 = \xi_1 =$ constant characteristics and of U_2 along the $\phi_2 = \xi_2 =$ constant characteristics; that is,

$$\frac{\partial U_1}{\partial \xi_2} + \sum_{k=1}^{2} \tilde{C}_{1k}(\xi_1, \xi_2) U_k = \tilde{F}_1(\xi_1, \xi_2), \quad (4.3.29a)$$

$$\frac{\partial U_2}{\partial \xi_1} + \sum_{k=1}^{2} \tilde{C}_{2k}(\xi_1, \xi_2) U_k = \tilde{F}_2(\xi_1, \xi_2), \quad (4.3.29b)$$

where

$$\tilde{C}_{1k} \equiv \frac{C_{1k}}{(\lambda_1 - \lambda_2)(\partial\phi_2/\partial x)}, \quad \tilde{C}_{2k} \equiv \frac{C_{2k}}{(\lambda_2 - \lambda_1)(\partial\phi_1/\partial x)}, \tag{4.3.30a}$$

$$\tilde{F}_1 \equiv \frac{F_1}{(\lambda_1 - \lambda_2)(\partial\phi_2/\partial x)}, \quad \tilde{F}_2 \equiv \frac{F_2}{(\lambda_2 - \lambda_1)(\partial\phi_1/\partial x)}. \tag{4.3.30b}$$

Eliminating U_1 in favor of U_2 (or vice versa) in (4.3.29) gives the second-order equation (4.2.13), which can be solved explicitly if the fundamental solution is known (as, for example, if the $\tilde{C}_{ij}$ are constants) (see Problem 4.3.4). The system (4.3.29) is essentially decoupled if one of the off-diagonal elements of the $\tilde{C}$ matrix is zero, say $\tilde{C}_{12} \equiv 0$. In this case, (4.3.29a) is an *ordinary* differential equaiton for U_1 as a function of ξ_2 along each characteristic ξ_1 = constant. The solution has the form

$$U_1(\xi_1, \xi_2) = \frac{1}{R_1(\xi_1, \xi_2)}\left[K_1(\xi_1) + \int^{\xi_2} \tilde{F}_1(\xi_1, \sigma)R_1(\xi_1, \sigma)d\sigma\right], \tag{4.3.31}$$

where K_1 is an arbitrary function and

$$R_1(\xi_1, \xi_2) = \exp\left[\int^{\xi_2} \tilde{C}_{11}(\xi_1, \sigma)d\sigma\right]. \tag{4.3.32}$$

Substituting (4.3.31) for U_1 into (4.3.29b) reduces this to the ordinary differential equation

$$\frac{\partial U_2}{\partial \xi_1} + \tilde{C}_{22}(\xi_1, \xi_2)U_2 = \tilde{F}_2 - \tilde{C}_{21}U_1 \equiv P(\xi_1, \xi_2), \tag{4.3.33}$$

where $\tilde{C}_{12}$ and P are known functions of ξ_1 and ξ_2. The solution for U_2 is given in the form

$$U_2(\xi_1, \xi_2) = \frac{1}{R_2(\xi_1, \xi_2)}\left[K_2(\xi_2) + \int^{\xi_1} P(\sigma, \xi_2)R_2(\sigma, \xi_2)d\sigma\right], \tag{4.3.34}$$

where K_2 is an arbitrary function and

$$R_2(\xi_1, \xi_2) = \exp\left[\int^{\xi_1} \tilde{C}_{22}(\sigma, \xi_2)d\sigma\right]. \tag{4.3.35}$$

The functions K_1 and K_2 will be known once initial data (for example) are specified on a spacelike arc. Two examples are discussed in Section 4.3.4. See also Problems 4.3.1–4.3.3.

In general, the system (4.3.29) cannot be solved analytically, and we have to use a numerical solution as discussed in Section 4.3.6.

4.3.4 Examples from Shallow-Water Flow

In Chapter 3 we studied several examples that lead to special cases of the linear pair of partial differential equations (4.3.10). In this section we discuss two further examples, the first to illustrate the effect of perturbation about a nontrivial exact solution, and the second to provide a physical model that gives rise to a nonzero $\mathbf{f}$ in (4.3.1).

(i) *Perturbation of the dam-breaking problem*

Consider the flow that results when a "dam" at $x = 0$ is suddenly removed. We assume that upstream of the dam ($x < 0$) the water is nearly at rest ($u \approx 0$) and the surface height is nearly constant ($h \approx 1$). For simplicity we assume that there is no water downstream. The solution for the case where there is water at rest downstream of the dam is discussed in Section 7.4.3iii. We also assume that (3.2.12) correctly models the flow. Thus, using the notation in Section 4.3.1, we wish to solve

$$\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial x} = 0, \quad \frac{\partial v_2}{\partial t} + v_2 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_2}{\partial x} = 0, \tag{4.3.36}$$

where $v_1(x, t)$ is the vertically averaged horizontal speed and $v_2(x, t)$ is the height of the free surface. (In the notation of Chapter 3, $v_1 = u$ and $v_2 = h$.)

Initial conditions corresponding to no water downstream of the dam and nearly quiescent flow upstream are (see Figure 4.10)

$$v_1(x, 0^-) = \epsilon H(-x) g_1(x), \quad v_2(x, 0^-) = H(-x)[1 + \epsilon g_2(x)], \tag{4.3.37}$$

where H is the Heaviside function and $g_1(x)$, $g_2(x)$ are arbitrarily prescribed bounded functions. The boundary condition of no flow at the dam boundary requires that $g_1(0) = 0$, and by an appropriate choice of the vertical scale, we can also set $g_2(0) = 0$ with no loss of generality. The small parameter ϵ measures the departure of the initial conditions from the quiescent state $v_1^{(0)} = 0$, $v_2^{(0)} = H(-x)$.

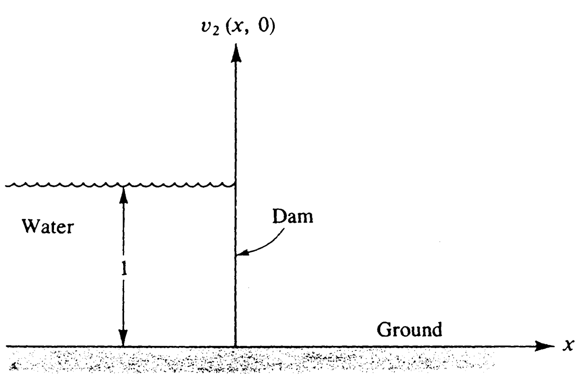

FIGURE 4.10. Initial state for the dam-breaking problem

In Section 7.4.2 we derive the following *exact* solution for the $\epsilon = 0$ problem

$$v_1^{(0)}(x,t) = \begin{cases} \dfrac{2}{3}\left(\dfrac{x}{t}+1\right), & -t < x < 2t, \\ 0, & x \le -t \text{ or } x \ge 2t, \end{cases} \tag{4.3.38a}$$

$$v_2^{(0)}(x,t) = \begin{cases} \dfrac{1}{9}\left(2-\dfrac{x}{t}\right)^2, & -t < x < 2t, \\ 1, & x \le -t, \\ 0, & x \ge 2t, \end{cases} \tag{4.3.38b}$$

which is easy to verify by direct substitution into (4.3.36). As shown in Figure 4.11a, the initial discontinuity in the height develops into the moving parabolic profile given in (4.3.38b), which spreads to the right with speed 2 and to the left into the quiescent water with speed -1. On rays $x/t = $ constant in the xt-plane, both the speed $v_1^{(0)}$ and height $v_2^{(0)}$ remain constant. This is shown in Figure 4.11b. Since the solution differs from the initial state only in the domain $-t < x < 2t, t > 0$, we shall concentrate on the perturbation solution over this domain. Consider first the domain $-t < x < 2t, t > 0$.

Using the result (4.3.38) in (4.3.9), we obtain the following expressions for the matrix components of $A^{(0)}$ and B ($\mathbf{f}^{(1)} = 0$):

$$\{A_{ij}^{(0)}(x,t)\} = \begin{pmatrix} v_1^{(0)} & 1 \\ v_2^{(0)} & v_1^{(0)} \end{pmatrix}, \quad \{B_{ij}(x,t)\} = \begin{pmatrix} \dfrac{\partial v_1^{(0)}}{\partial x} & 0 \\ \dfrac{\partial v_2^{(0)}}{\partial x} & \dfrac{\partial v_1^{(0)}}{\partial x} \end{pmatrix}. \tag{4.3.39}$$

Using the above values for the A_{ij} in (4.3.16b) gives the two characteristic speeds λ_1 and λ_2 as

$$\lambda_1 = v_1^{(0)}(x,t) + \sqrt{v_2^{(0)}(x,t)} = \frac{1}{3}\left(\frac{x}{t}+4\right), \tag{4.3.40a}$$

$$\lambda_2 = v_1^{(0)}(x,t) - \sqrt{v_2^{(0)}(x,t)} = \frac{x}{t}. \tag{4.3.40b}$$

Therefore, the eigenvectors (4.3.20b) are given by

$$\mathbf{w}_1 = \left(-1, -\sqrt{v_2^{(0)}(x,t)}\right), \quad \mathbf{w}_2 = \left(-1, \sqrt{v_2^{(0)}(x,t)}\right), \tag{4.3.41}$$

and the transformation matrix $\{W\}$ and its inverse $\{V\}$ are given by

$$\{W_{ij}\} = \begin{pmatrix} -1 & -1 \\ -\sqrt{v_2^{(0)}} & \sqrt{v_2^{(0)}} \end{pmatrix}, \quad \{V_{ij}\} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2\sqrt{v_2^{(0)}}} \\ -\frac{1}{2} & \frac{1}{2\sqrt{v_2^{(0)}}} \end{pmatrix}. \tag{4.3.42}$$

We also compute the following matrix components for $\{C\}$ using (4.3.26a):

$$\{C_{ij}\} = \begin{pmatrix} 0 & 0 \\ \dfrac{1}{3t} & \dfrac{1}{t} \end{pmatrix}. \tag{4.3.43}$$

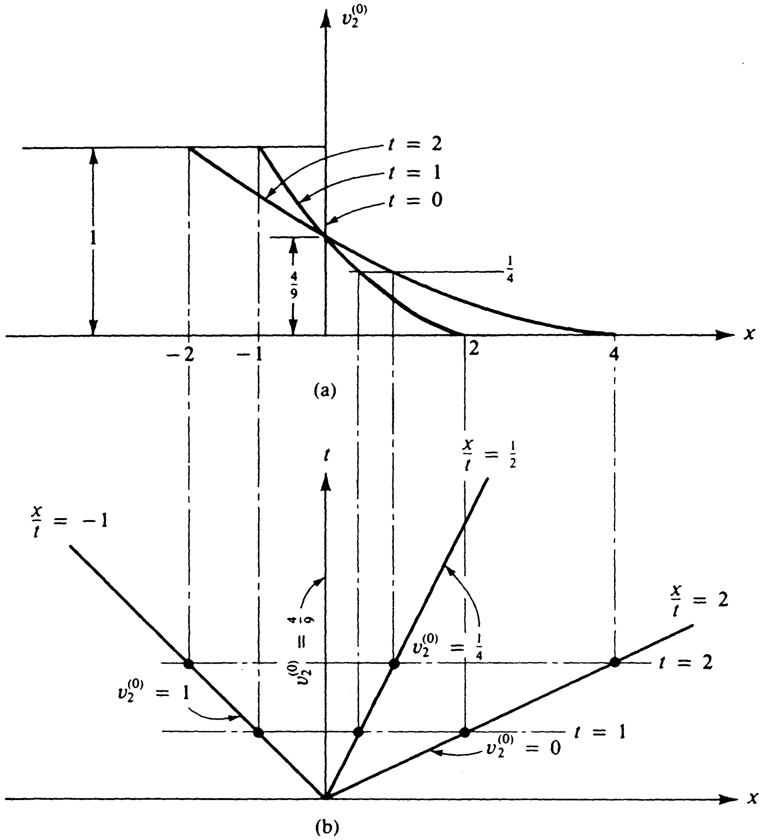

FIGURE 4.11. Exact solution for $\epsilon = 0$

Therefore, the diagonal characteristic form (4.3.25) is

$$\frac{\partial U_1}{\partial t} + \frac{1}{3}\left(\frac{x}{t} + 4\right)\frac{\partial U_1}{\partial x} = 0, \tag{4.3.44a}$$

$$\frac{\partial U_2}{\partial t} + \frac{x}{t}\frac{\partial U_2}{\partial x} + \frac{1}{3t}U_1 + \frac{1}{t}U_2 = 0, \tag{4.3.44b}$$

for the transformed variables

$$U_1 = -\frac{1}{2}u_1 - \frac{1}{2\sqrt{v_2^{(0)}}}u_2, \quad U_2 = -\frac{1}{2}u_1 + \frac{1}{2\sqrt{v_2^{(0)}}}u_2. \tag{4.3.45}$$

Notice that (4.3.44a) for U_1 is decoupled from (4.3.44b) for U_2. Thus, we can first solve for U_1 and then substitute the result in (4.3.44b) to obtain an inhomogeneous first-order equation for U_2 that can also be solved. In Section 5.2.2 we will discuss a general approach for solving such first-order equations. Here, we proceed using the characteristic independent variables ξ_1 and ξ_2 instead of x and t. In preparation for this transformation, we first solve the characteristic differential equations

$$\frac{dx}{dt} = \frac{1}{3}\left(\frac{x}{t} + 4\right), \quad \frac{dx}{dt} = \frac{x}{t}. \tag{4.3.46}$$

This gives the two families of characteristic curves

$$\phi_1(x, t) \equiv -xt^{-1/3} + 2t^{2/3} = \xi_1 = \text{constant}, \tag{4.3.47a}$$

$$\phi_2(x, t) \equiv \frac{x}{t} = \xi_2 = \text{constant}. \tag{4.3.47b}$$

In Figure 4.12 we show the two families of characteristics $\xi_1 = \text{constant}$ and $\xi_2 = \text{constant}$. The domain of interest is the triangular region (2) bounded by $\xi_2 = -1$ (that is, $x = -t$) and $\xi_1 = 0$ (that is, $x = 2t$). In this domain only the curves $\xi_1 = \text{constant} > 0$ that lie to the right of the ray $\xi_2 = -1$ are of interest. We note that the ray $x = 2t$ coincides with *both* characteristics $\xi_2 = 2$ and $\xi_1 = 0$. Thus, the coordinate transformation (4.3.47) breaks down along this ray. This does not pose any difficulties, as we shall see later on.

The solution of the linearized problem in region (3), to the right of $\xi_2 = 2$, is the trivial solution $u_1(x, t) = u_2(x, t) = 0$. In region (1), to the left of the ray $\xi_2 = -1$, we need to solve the system

$$\mathbf{u}_t + A\mathbf{u}_x = 0, \ \{A_{ij}\} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4.3.48}$$

subject to the initial conditions

$$u_1(x, 0) = g_1(x), \quad u_2(x, 0) = g_2(x). \tag{4.3.49}$$

This defines $u_1(x, t)$ and $u_2(x, t)$ in (1) (see Problem 4.3.3b), including the ray $\xi_2 = -1$. Thus, we have a characteristic boundary-value problem to solve in (2), where we know the value of U_1 on the ray $\xi_2 = -1$ and $U_2 = 0$ along the ray $\xi_1 = 0$.

To derive the equations in terms of the ξ_1 and ξ_2 variables, we solve (4.3.47) for x and t and use the result

$$x = \xi_2 \left(\frac{\xi_1}{2 - \xi_2}\right)^{3/2}, \quad t = \left(\frac{\xi_1}{2 - \xi_2}\right)^{3/2}, \tag{4.3.50}$$

in (4.3.29) to obtain

$$\frac{\partial U_1}{\partial \xi_2} = 0, \ \xi_1 \frac{\partial U_2}{\partial \xi_1} + \frac{3}{2}U_2 + \frac{1}{2}U_1 = 0. \tag{4.3.51}$$

This confirms our earlier observation that U_1 can be solved first in the form

$$U_1 = \Gamma(\xi_1). \tag{4.3.52}$$

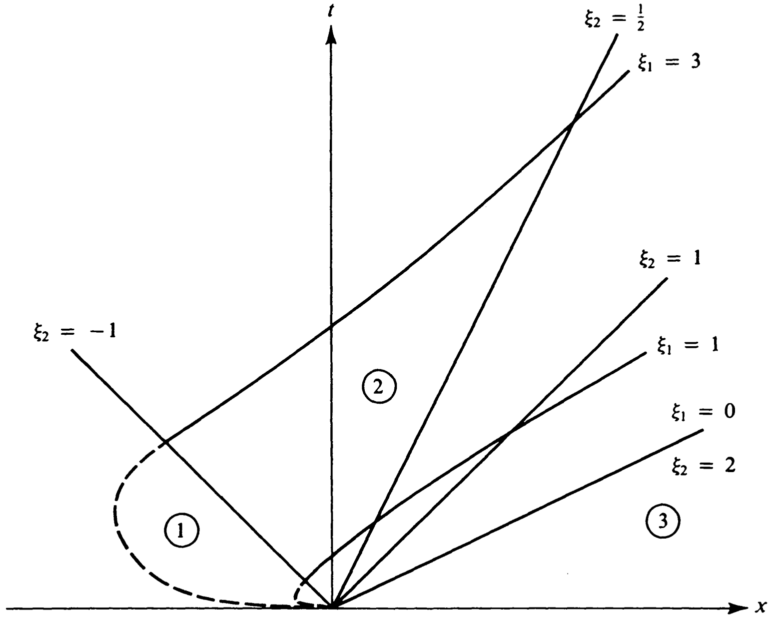

FIGURE 4.12. Characteristic curves

The second equation (4.3.51) can now be solved in the form

$$U_2 = \xi_1^{-3/2}\left[\Omega(\xi_2) - \frac{1}{2}\int_0^{\xi_1} s^{1/2}\Gamma(s)\right]ds.$$

Here Γ and Ω are arbitrary functions of their arguments that must be determined in terms of the boundary conditions on the bounding characteristics for region (2). The solution on region (1), when evaluated on $\xi_2 = -1$, defines $\Gamma(\xi_1)$. It also follows that we must set $\Omega(\xi_2) \equiv 0$ because U_2 must tend to zero as $\xi_1 \to 0$ along *any* ray $\xi_2 =$ constant with $-1 \leq \xi_2 \leq 2$. Thus,

$$U_2(\xi_1) = -\frac{1}{2\xi_1^{3/2}}\int_0^{\xi_1} s^{1/2}\Gamma(s)ds. \tag{4.3.53}$$

In effect, (4.3.53) is the solution of the ordinary differential equation in (4.3.51) for U_2 along the rays $\xi_2 =$ constant with the boundary condition $U_2(0) = 0$. In this regard, note that $U_2(0)$ is well behaved as long as $\Gamma(0)$ is a finite constant; in fact, $\Gamma(0) = 0$ (see Problem 4.3.3c), which means that $U_2(0) = 0$.

(ii) *Shallow-water flow in an inclined channel with friction*

We consider the flow of water down a channel having a rectangular cross section and inclined at a constant angle s to the horizontal, as shown in Figure 4.13. Let

G denote the domain bounded by the free surface, the bottom, and the two planes $X = X_1$ and $X = X_2$ normal to the bottom.

Mass conservation gives (see (3.2.4))

$$\frac{d}{dT}\int_{X_1}^{X_2} \rho_0 B_0 H(X,T)dX = -\rho_0 B_0 H(X,T)U(X,T)\Big|_{X=X_1}^{X=X_2}. \tag{4.3.54}$$

We are using dimensional T, X, Y, H, U variables, where T is the time, X is the distance measured along the bottom, and Y is the distance normal to it. Thus, H is the height of the free surface in the Y direction and U is the X-component of the flow velocity averaged over $0 \leq Y \leq H$. The constant density of water is ρ_0, and B_0 is the constant breadth of the channel.

To complete the problem formulation, we must relate U to H, and we use momentum conservation. As in Section 3.2, we assume hydrostatic balance in the vertical direction and ignore surface tension, but now we do account for viscous effects approximately through the introduction of a frictional force exerted on the fluid by the solid boundaries of the channel.

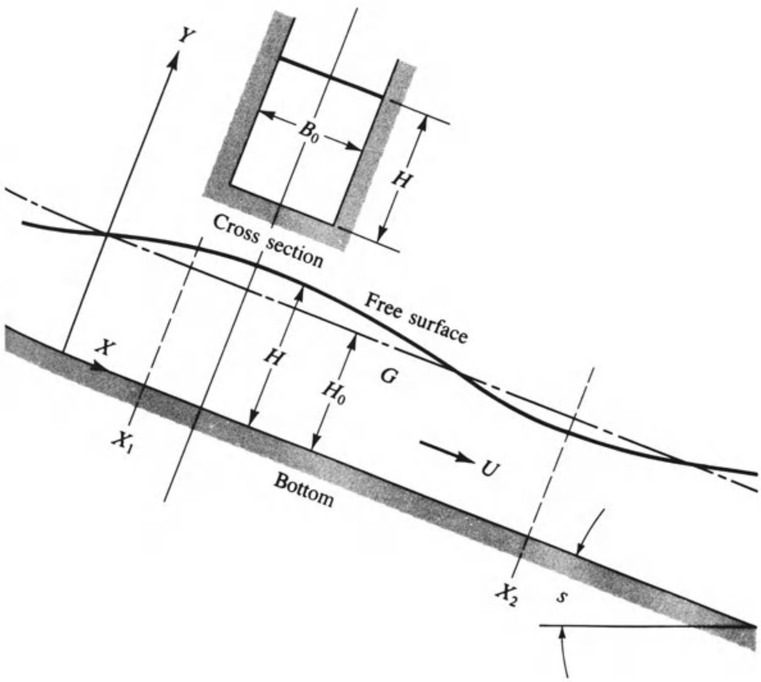

FIGURE 4.13. Shallow-water flow in an inclined channel

We identify the following contributions to the momentum balance for G in the direction parallel to the bottom:

$$\frac{d}{dT}\int_{X_1}^{X_2} \rho_0 B_0 H(X,T) U(X,T)\, dX = \left\{\begin{array}{l}\text{rate of change}\\ \text{of momentum}\\ \text{of } G,\end{array}\right. \tag{4.3.55a}$$

$$-\rho_0 B_0 U^2(X,T) H(X,T)\Big|_{X=X_1}^{X=X_2} = \left\{\begin{array}{l}\text{net inflow of}\\ \text{momentum}\\ \text{into } G,\end{array}\right. \tag{4.3.55b}$$

$$-\int_0^{H(X_2,T)} \rho_0 B_0 g(H-Y)\cos s\, dY = \left\{\begin{array}{l}\text{streamwise}\\ \text{component of}\\ \text{the pressure}\\ \text{force on } G\\ \text{at } X = X_2,\end{array}\right. \tag{4.3.55c}$$

$$\int_0^{H(X_1,T)} \rho_0 B_0 g(H-Y)\cos s\, dY = \left\{\begin{array}{l}\text{streamwise}\\ \text{component of}\\ \text{the pressure}\\ \text{force on } G\\ \text{at } X = X_1,\end{array}\right. \tag{4.3.55d}$$

$$\int_{X_1}^{X_2} \rho_0 B_0 g H(X,T)\sin s\, dX = \left\{\begin{array}{l}\text{streamwise}\\ \text{component of}\\ \text{the gravitational}\\ \text{force on } G,\end{array}\right. \tag{4.3.55e}$$

$$-\int_{X_1}^{X_2} C\rho_0[B_0 + 2H(X,T)]U^2(X,T)\, dX = \left\{\begin{array}{l}\text{frictional force}\\ \text{acting on } G\\ \text{due to solid}\\ \text{boundaries.}\end{array}\right. \tag{4.3.55f}$$

In (4.3.55f), C is a dimensionless friction coefficient, which we assume to be a constant independent of Reynolds number, and we have used the standard expression $C\rho_0 A U^2$ for the frictional force over a surface of wetted area A. For a rectangular channel, the wetted area of an element of length dX is $dA = (B_0 + 2H)dX$.

After integrating (4.3.55c) and (4.3.55d) and dividing by the factor $\rho_0 B_0$, we obtain the following integral conservation law:

$$\begin{aligned}\frac{d}{dT}\int_{X_1}^{X_2} & H(X,T)U(X,T)\, dX \\ &= -\left\{U^2(X,T)H(X,T) + \frac{1}{2}gH^2(X,T)\cos s\right\}_{X=X_1}^{X=X_2} \\ &\quad + \int_{X_1}^{X_2}\left\{gH(X,T)\sin s - C\left(1 + \frac{2H}{B_0}\right)U^2(X,T)\right\}dX.\end{aligned} \tag{4.3.56}$$

A special solution of (4.3.54) and (4.3.56) corresponds to uniform flow ($U \equiv U_0 = \text{constant}$, $H \equiv H_0 = \text{constant}$) with the frictional and gravitational forces in perfect balance; that is,

$$gH_0 \sin s = C(1 + 2\sigma)U_0^2, \tag{4.3.57}$$

where σ is the dimensionless parameter $\sigma \equiv H_0/B_0$. For a broad channel $\sigma \ll 1$. Equation (4.3.57) defines a unique U_0 for a given H_0, or the converse

$$U_0 = \sqrt{\frac{gH_0 \sin s}{C(1 + 2\sigma)}}, \quad H_0 = \frac{C(1 + 2\sigma)U_0^2}{g \sin s}. \tag{4.3.58}$$

We introduce the following dimensionless variables in terms of a length scale (L_0) and speed (V_0) that are to be specified

$$x \equiv \frac{X}{L_0}, \quad t \equiv \frac{T}{L_0/V_0}, \quad h \equiv \frac{H}{H_0}, \quad u \equiv \frac{U}{V_0}. \tag{4.3.59}$$

It is easy to see that the choice

$$L_0 = \frac{H_0}{\tan s}, \quad V_0 = (gH_0 \cos s)^{1/2}, \tag{4.3.60}$$

simplifies the notation and leads to

$$\frac{d}{dt}\int_{x_1}^{x_2} h(x,t)dx = -\{h(x,t)u(x,t)\}_{x=x_1}^{x=x_2}, \tag{4.3.61a}$$

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)h(x,t)dx = -\left\{u^2(x,t)h(x,t) + \frac{1}{2}h^2(x,t)\right\}_{x=x_1}^{x=x_2}$$

$$+\int_{x_1}^{x_2}\left\{h(x,t) - \frac{1 + 2\sigma h(x,t)}{F^2}u^2(x,t)\right\}dx. \tag{4.3.61b}$$

Here F is the dimensionless speed (Froude number) defined by

$$F^2 = \frac{U_0^2(1 + 2\sigma)}{gH_0 \cos s}, \tag{4.3.62a}$$

and (4.3.57) reduces to the condition

$$F^2 = \frac{\tan s}{C}, \tag{4.3.62b}$$

relating F, C, and s.

For smooth solutions, (4.3.61a) and (4.3.61b) give the pair of divergence relations

$$h_t + (hu)_x = 0, \tag{4.3.63a}$$

$$(uh)_t + \left(u^2 h + \frac{1}{2}h^2\right)_x = h - \frac{1 + 2\sigma h}{F^2}u^2, \tag{4.3.63b}$$

and using (4.3.63a) to eliminate h_t from (4.3.63b) gives

$$u_t + uu_x + h_x = 1 - \frac{1 + 2\sigma h}{F^2}\frac{u^2}{h}. \tag{4.3.64}$$

If σ is small, (4.3.64) reduces to

$$u_t + uu_x + h_x = 1 - \frac{u^2}{F^2 h}, \tag{4.3.65}$$

and we shall use (4.3.65) henceforth for simplicity. Note that if s is small, $\cos s \approx 1$, and we may regard h as the height of the free surface in the vertical direction. Also, according to (4.3.60), $H_0/L_0 = \tan s \ll 1$, and this is consistent with the assumption of shallow-water flow.

In the notation of Section 4.3.1, we set $h = v_1$, $u = v_2$, and regard (4.3.63a) as the first component of (4.3.1) and (4.3.65) as the second. (Note that this is the reverse of the choice made in (4.3.36).) Thus,

$$A = \begin{pmatrix} v_2 & v_1 \\ 1 & v_2 \end{pmatrix}, \quad \mathbf{f} = \left(0,\, 1 - \frac{v_2^2}{F^2 v_1}\right). \tag{4.3.66}$$

Consider now the initial conditions

$$v_1 = 1 + \epsilon g_1(x), \quad v_2 = F + \epsilon g_2(x), \tag{4.3.67}$$

where $0 < \epsilon \ll 1$ and $g_1(x)$ and $g_2(x)$ are arbitrarily prescribed bounded functions. Thus, the initial state deviates only slightly from the steady state $v_1 = 1$, $v_2 = F$ that corresponds to a perfect balance between the frictional and gravitational forces. Note that in the general case (4.3.1), such a constant steady solution is possible if $\mathbf{f}$ does not depend on x and t, and the two algebraic equations $f_1(v_1, v_2) = 0$, $f_2(v_1, v_2) = 0$ have a real physically realistic solution: $v_1 = v_1^{(0)} = \text{constant}$, $v_2 = v_2^{(0)} = \text{constant}$.

We perturb the solution about the steady state as in (4.3.6),

$$v(x, t; \epsilon) = 1 + \epsilon u_1(x, t) + O(\epsilon^2), \quad v_2(x, t; \epsilon) = F + \epsilon u_2(x, t) + O(\epsilon^2), \tag{4.3.68}$$

and calculate the following system corresponding to (4.3.8):

$$\frac{\partial u_1}{\partial t} + F\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} = 0, \tag{4.3.69a}$$

$$\frac{\partial u_2}{\partial t} + \frac{\partial u_1}{\partial x} + F\frac{\partial u_2}{\partial x} - u_1 + \frac{2}{F}u_2 = 0. \tag{4.3.69b}$$

Thus,

$$\{A_{ij}^{(0)}\} = \begin{pmatrix} F & 1 \\ 1 & F \end{pmatrix}, \quad \{B_{ij}\} = \begin{pmatrix} 0 & 0 \\ -1 & \frac{2}{F} \end{pmatrix}, \quad \mathbf{f}^{(1)} \equiv 0. \tag{4.3.70}$$

The eigenvalues of $A^{(0)}$ are $\lambda_1 = F+1$ and $\lambda_2 = F-1$. Thus, the characteristic dependent variables are

$$\xi_1 = x - (F + 1)t, \quad \xi_2 = x - (F - 1)t. \tag{4.3.71}$$

It follows from (4.3.20) that $\mathbf{w}_1 = (-1, -1)$, $\mathbf{w}_2 = (-1, 1)$. Therefore,

$$\{W_{ij}\} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \{V_{ij}\} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}, \tag{4.3.72}$$

and the characteristic dependent variables are given by $U_1 = -(u_1 + u_2)/2$ and $U_2 = (u_2 - u_1)/2$. Using the above expressions for $\{B_{ij}\}$, $\{W_{ij}\}$, and $\{V_{ij}\}$ in (4.3.26a) gives

$$\{C_{ij}\} = \begin{pmatrix} \frac{1}{F} - \frac{1}{2} & -\frac{1}{F} - \frac{1}{2} \\ -\frac{1}{F} + \frac{1}{2} & \frac{1}{F} + \frac{1}{2} \end{pmatrix}, \tag{4.3.73}$$

and this defines the diagonal characteristic form (4.3.25),

$$\frac{\partial U_1}{\partial t} + (F+1)\frac{\partial U_1}{\partial x} + \left(\frac{1}{F} - \frac{1}{2}\right)U_1 - \left(\frac{1}{F} + \frac{1}{2}\right)U_2 = 0, \tag{4.3.74a}$$

$$\frac{\partial U_2}{\partial t} + (F-1)\frac{\partial U_2}{\partial x} - \left(\frac{1}{F} - \frac{1}{2}\right)U_1 + \left(\frac{1}{F} + \frac{1}{2}\right)U_2 = 0, \tag{4.3.74b}$$

because $F_1 = F_2 = 0$.

We note that the equations for U_1 and U_2 are essentially coupled for $F \neq 2$. However, because the λ_i and C_{ij} are constants, the solution of (4.3.74) can be calculated, as we know the fundamental solution of the associated second-order hyperbolic equation. (See Problem 4.3.4.)

4.3.5 Connection with the Second-Order Equation

Here we explore the relation between system (4.3.25) and the second-order equation (4.1.1) for the hyperbolic case. To reduce (4.3.25) to a second-order equation, let us eliminate one of the variables U_1 or U_2. We exclude the case $C_{ij} = 0, i \neq j$, because we have seen that this corresponds to a decoupled pair of equations for U_1 and U_2. If $C_{21} \neq 0$, say, we can solve (4.3.25) with $i = 2$ for U_1 and substitute the result into the expressions for $i = 1$ to obtain the following second-order equation for U_2:

$$\frac{\partial^2 U_2}{\partial t^2} + (\lambda_1 + \lambda_2)\frac{\partial^2 U_2}{\partial x \partial t} + \lambda_1\lambda_2\frac{\partial^2 U_2}{\partial x^2} + d^*\frac{\partial U_2}{\partial t} + e^*\frac{\partial U_2}{\partial x} + f^*U_2 = g^*, \tag{4.3.75a}$$

where

$$d^* \equiv C_{11} + C_{22} + C_{21}\left[\left(\frac{1}{C_{21}}\right)_t + \lambda_1\left(\frac{1}{C_{21}}\right)_x\right], \tag{4.3.75b}$$

$$e^* \equiv \lambda_1 C_{22} + \lambda_2 C_{11} + C_{21}\left[\left(\frac{\lambda_2}{C_{21}}\right)_t + \lambda_1\left(\frac{\lambda_2}{C_{21}}\right)_x\right], \tag{4.3.75c}$$

$$f^* \equiv C_{11}C_{22} - C_{12}C_{21} + C_{21}\left[\left(\frac{C_{22}}{C_{21}}\right)_t + \lambda_1\left(\frac{C_{22}}{C_{21}}\right)_x\right], \qquad (4.3.75d)$$

$$g^* \equiv F_2C_{11} - F_1C_{12} + C_{21}\left[\left(\frac{F_2}{C_{21}}\right)_t + \lambda_1\left(\frac{F_2}{C_{21}}\right)_x\right]. \qquad (4.3.75e)$$

In view of the definition (4.1.14) for the characteristic slopes, we can immediately identify (4.3.75) with (4.1.1) if we divide (4.1.1) by a, and replace $x \to t$, $y \to x, d/a \to d^*, e/a \to e^*, f/a \to f^*$, and $g/a \to g^*$. Thus, given a hyperbolic system of two first-order equations, (4.3.75) defines a *unique* second-order hyperbolic equation for U_2. A corresponding second-order hyperbolic equation for U_1 can also be uniquely defined. If the λ_i and C_{ij} are constant, the bracketed expressions in (4.3.75b)–(4.3.75e) all vanish and d^*, e^*, f^* are also constant; only g^* is a function of x, t if the F_i depend on x, t. Since the fundamental solution is known for this case, we can calculate U_1 and U_2 explicitly for given Cauchy data. (See Problem 4.3.4.)

The converse problem—that is, that of associating a pair of first-order equations with a given second-order hyperbolic equation—while possible, is not unique. To see this, it suffices to consider the constant-coefficient problem

$$u_{tt} - u_{xx} + ku = p(x, t), \qquad (4.3.76)$$

which we have shown to be equivalent to the most general hyperbolic equation with constant coefficients. For the purposes of this discussion, assume that $k > 0$ and p is a given function of x, t.

We ask whether we can find a linear transformation of the dependent variable u and its derivatives u_t, u_x such that (4.3.76) is equivalent to (4.3.25). The most general linear transformation is

$$U_i = \alpha_i u_t + \beta_i u_x + \gamma_i u, \quad i = 1, 2, \qquad (4.3.77)$$

for as yet unspecified constants $\alpha_i, \beta_i, \gamma_i$. Substituting (4.3.77) into (4.3.25) shows that the following pair of equations must be satisfied identically:

$$\alpha_i u_{tt} + (\beta_i + \lambda_i\alpha_i)u_{xt} + \lambda_i\beta_i u_{xx} + \left(\gamma_i + \sum_{j=1}^{2} C_{ij}\alpha_j\right)u_t + \left(\lambda_i\gamma_i + \sum_{j=1}^{2} C_{ij}\beta_j\right)u_x + \left(\sum_{j=1}^{2} C_{ij}\gamma_j\right)u = F_i, \quad i = 1, 2, \qquad (4.3.78)$$

where $\lambda_1 = 1$ and $\lambda_2 = -1$.

Suppose that we require (4.3.78) for $i = 1$ to vanish identically and to reduce to (4.3.76) if $i = 2$. It is easily seen that we must set $\alpha_1 = 0, \beta_1 = 0, \gamma_1 = -C_{12}$, $C_{11} = 0, F_1 = 0, \alpha_2 = 1, \beta_2 = 1, \gamma_2 = 0, C_{21} = -k/C_{12}, C_{22} = 0, F_2 = 0,$ and C_{12} is arbitrary. One choice has $C_{12} = -1$, in which case the transformation

(4.3.77) becomes

$$U_1 = u, \quad U_2 = u_t + u_x, \tag{4.3.79}$$

and the system (4.3.25) appears in the form

$$\frac{\partial U_1}{\partial t} + \frac{\partial U_1}{\partial x} - U_2 = 0, \quad \frac{\partial U_2}{\partial t} - \frac{\partial U_2}{\partial x} + kU_1 = p. \tag{4.3.80}$$

Other choices of C_{12} lead to different forms for (4.3.80). A second class of transformations results from requiring that both equations (4.3.78) reduce to (4.3.76). Again, the choice of constants is not unique; one selection that accomplishes this is

$$\alpha_1 = 1, \beta_1 = -1, \gamma_1 = -\sqrt{k}, C_{11} = 0, C_{12} = \sqrt{k}, F_1 = p, \tag{4.3.81a}$$

$$\alpha_2 = 1, \beta_2 = 1, \gamma_2 = \sqrt{k}, C_{21} = -\sqrt{k}, C_{22} = 0, F_2 = p. \tag{4.3.81b}$$

The transformation (4.3.77) is now

$$U_1 = u_t - u_x - \sqrt{k}u, \quad U_2 = u_t + u_x + \sqrt{k}u, \tag{4.3.82}$$

and (4.3.25) has the following form:

$$\frac{\partial U_1}{\partial t} + \frac{\partial U_1}{\partial x} + \sqrt{k}U_2 = p, \quad \frac{\partial U_2}{\partial t} - \frac{\partial U_2}{\partial x} - \sqrt{k}U_1 = p. \tag{4.3.83}$$

Other choices of the constants are also possible, so there is no unique pair of first-order equations that we can associate with a given second-order hyperbolic equation. However, regardless of the particular choice of decomposition, the final solution for u is the same. (See Problem 4.3.5 and the example in Sec. 4.3.6iii.)

4.3.6 Numerical Solutions; Propagation of Discontinuities

In this section we outline the numerical solution of the general system (4.3.29),

$$\frac{\partial U_1}{\partial \xi_2} + \sum_{k=1}^{2} \tilde{C}_{1k}(\xi_1, \xi_2)U_k = \tilde{F}_1(\xi_1, \xi_2), \tag{4.3.84a}$$

$$\frac{\partial U_2}{\partial \xi_1} + \sum_{k=1}^{2} C_{2k}(\xi_1, \xi_2)U_k = \tilde{F}_2(\xi_1, \xi_2). \tag{4.3.84b}$$

(i) *Cauchy problem*

Here we specify U_1 and U_2 on a spacelike arc $\mathcal{S}_0$ in the $\xi_1\xi_2$-plane, as sketched in Figure 4.14.

Consider the two adjacent points $\alpha = (\xi_1^{(0)}, \xi_2^{(0)} + \Delta\xi_2), \beta = (\xi_1^{(0)} + \Delta\xi_1, \xi_2^{(0)})$ on $\mathcal{S}_0$, and the point $\gamma = (\xi_1^{(0)} + \Delta\xi_1, \xi_2^{(0)} + \Delta\xi_2)$ a small distance away. To compute U_1 at γ, we use the known values of U_1 and U_2 at β in (4.3.84a) and obtain

$$U_1(\gamma) = U_1(\beta) + \Delta\xi_2 \left[-\sum_{k=1}^{2} \tilde{C}_{1k}(\beta)U_k(\beta) + \tilde{F}_1(\beta) \right], \tag{4.3.85a}$$

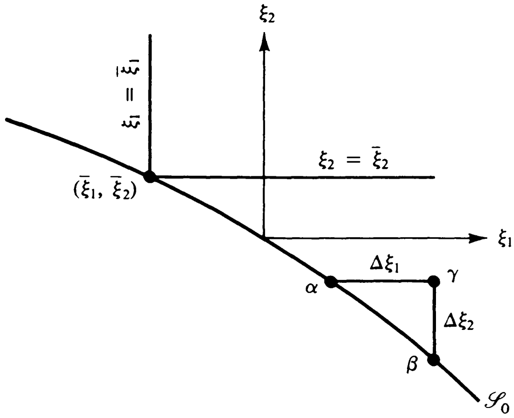

FIGURE 4.14. Propagation of initial discontinuity in U_1 and U_2

and $U_2(\gamma)$ follows from (4.3.84b):

$$U_2(\gamma) = U_2(\alpha) + \Delta\xi_1 \left[-\sum_{k=1}^{2} \tilde{C}_{2k}(\alpha)U_k(\alpha) + \tilde{F}_2(\alpha) \right]. \qquad (4.3.85b)$$

This construction is easy to implement and remains valid in the case where the initial data are discontinuous. In fact, we see that a finite discontinuity in the Cauchy data for U_2 at some point propagates along the ξ_2 = constant characteristic passing through that point—that is, in the ξ_1 direction from that point and vice versa. Such discontinuities arise naturally in many physical applications, and as in Section 4.2.3, we can calculate explicitly how they propagate. For example, let the Cauchy data for U_1 and U_2 have a finite discontinuity at the point $\xi_1 = \bar{\xi}_1, \xi_2 = \bar{\xi}_2$ on $\mathcal{S}_0$, as shown in Figure 4.14. Let us denote the jump in U_1 by μ and the jump in U_2 by κ; that is,

$$\mu(\bar{\xi}_1, \xi_2) \equiv U_1(\bar{\xi}_1^+, \xi_2) - U_1(\bar{\xi}_1^-, \xi_2), \quad \kappa(\xi_1, \bar{\xi}_2) \equiv U_2(\xi_1, \bar{\xi}_2^+) - U_2(\xi_1, \bar{\xi}_2^-). \qquad (4.3.86)$$

Now, if we evaluate (4.3.84a) on either side of the $\xi_1 = \bar{\xi}_1$ characteristic and subtract the result, we have

$$\frac{\partial\mu}{\partial\xi_2} + \tilde{C}_{11}\mu = 0, \qquad (4.3.87a)$$

because U_2 is continuous on the $\xi_1 = \bar{\xi}_1$ characteristic. Similarly, the difference of (4.3.84b) on either side of the $\xi_2 = \bar{\xi}_2$ characteristic gives

$$\frac{\partial\kappa}{\partial\xi_1} + \tilde{C}_{22}\kappa = 0. \qquad (4.3.87b)$$

Therefore, μ and κ propagate according to

$$\mu(\overline{\xi}_1, \xi_2) = \mu(\overline{\xi}_1, \overline{\xi}_2) \exp\left\{-\int_{\overline{\xi}_2}^{\xi_2} \tilde{C}_{11}(\overline{\xi}_1, \sigma)d\sigma\right\} \text{ on } \xi_1 = \overline{\xi}_1, \quad (4.3.88a)$$

$$\kappa(\xi_1, \overline{\xi}_2) = \kappa(\overline{\xi}_1, \overline{\xi}_2) \exp\left\{-\int_{\overline{\xi}_1}^{\xi_1} \tilde{C}_{22}(\sigma, \overline{\xi}_2)d\sigma\right\} \text{ on } \xi_2 = \overline{\xi}_2. \quad (4.3.88b)$$

For the special case where the λ_i and C_{ij} are constants and $F_1 = F_2 = 0$, equations(4.3.88) reduce to

$$\mu(\overline{\xi}_1, \xi_2) = \mu_0 \exp(-\tilde{C}_{11}\xi_2) \text{ on } \xi_1 = \overline{\xi}_1, \quad (4.3.89a)$$

$$\kappa(\xi_1, \overline{\xi}_2) = \kappa_0 \exp(-\tilde{C}_{22}\xi_1) \text{ on } \xi_2 = \overline{\xi}_2, \quad (4.3.89b)$$

where μ_0 and κ_0 are the constants

$$\mu_0 = \mu(\overline{\xi}_1, \overline{\xi}_2) \exp(\tilde{C}_{11}\overline{\xi}_2), \quad \kappa_0 = \kappa(\overline{\xi}_1, \overline{\xi}_2) \exp(\tilde{C}_{22}\overline{\xi}_1).$$

Since $\xi_1 = x - \lambda_1 t$ and $\xi_2 = x - \lambda_2 t$, we have $\xi_2 - \xi_1 = (\lambda_1 - \lambda_2)t$. Thus, along the characteristic $\xi_1 = \overline{\xi}_1 =$ constant, $\xi_2 = \overline{\xi}_1 + (\lambda_1 - \lambda_2)t$, and along the characteristic $\xi_2 = \overline{\xi}_2 =$ constant, $\xi_1 = \overline{\xi}_2 + (\lambda_2 - \lambda_1)t$. Using these results and the definitions of the $\tilde{C}_{ij}$ given in (4.3.30) we can write (4.3.89) in the form

$$\mu(\overline{\xi}_1, \xi_2) = \overline{\mu} \exp(-C_{11}t) \text{ on } \xi_1 = \overline{\xi}_1 = \text{constant}, \quad (4.3.90a)$$

$$\kappa(\xi_1, \overline{\xi}_2) = \overline{\kappa} \exp(-C_{22}t) \text{ on } \xi_2 = \overline{\xi}_2 = \text{constant}, \quad (4.3.90b)$$

where $\overline{\mu}$ and $\overline{\kappa}$ are the constants

$$\overline{\mu} = \mu_0 \exp\left(-\frac{C_{11}\overline{\xi}_1}{\lambda_1 - \lambda_2}\right), \quad \overline{\kappa} = \kappa_0 \exp\left(-\frac{C_{22}\overline{\xi}_2}{\lambda_2 - \lambda_1}\right).$$

Thus, the conditions

$$C_{11} \geq 0, \quad C_{22} \geq 0 \quad (4.3.90c)$$

are necessary for bounded solutions as $t \to \infty$.

(ii) *Solution in a domain bounded by a spacelike arc and timelike arc*

Consider now the solution in the domain $\mathcal{D}_1 + \mathcal{D}_4$ of Figure 4.15. This domain is bounded by the timelike arc $\mathcal{T}_0$ and that portion of the spacelike arc $\mathcal{S}_0$ to the right of the origin. We need to specify both U_1 and U_2 on $\mathcal{S}_0$, and the solution in $\mathcal{D}_4$ then follows according to the construction discussed in the previous subsection. In particular, U_1 and U_2 can be derived everywhere in $\mathcal{D}_4$ including the bounding characteristic, which is the positive ξ_1-axis.

To see what sort of condition is appropriate on $\mathcal{T}_0$, consider the three infinitesimally close points $\alpha = (\Delta\xi_1, \Delta\xi_2)$ on $\mathcal{T}_0$, $\beta = (\Delta\xi_0, 0)$ on the ξ_1-axis, and the origin $O = (0, 0)$. We know U_1 and U_2 at β. Therefore, (4.3.85a) gives $U_1(\alpha)$ in the finite difference form

$$U_1(\alpha) = U_1(\beta) + \Delta\xi_2[-\tilde{C}_{11}(\beta)U_1(\beta) - \tilde{C}_{12}(\beta)U_2(\beta) + \tilde{F}_1(\beta)]. \quad (4.3.91)$$

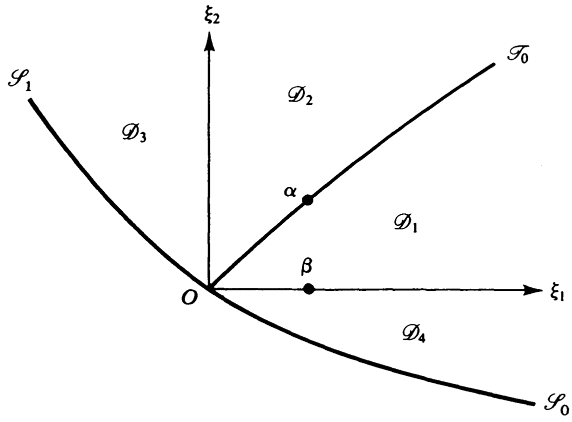

FIGURE 4.15. Solution in $\mathcal{D}_1 + \mathcal{D}_4$

We conclude that we *cannot specify* $U_1(\alpha)$ arbitrarily, as this would be inconsistent with (4.3.91). In terms of the original variables u_1, u_2 of (4.3.10), we note that we cannot prescribe $V_{11}u_1 + V_{12}u_2$ arbitrarily (cf. (4.3.26d)). The most general linear boundary condition at α has the form

$$U_2 + cU_1 = d, \tag{4.3.92}$$

where c and d are arbitrary finite constants. Thus, given $U_1(\alpha)$ from (4.3.91) we compute $U_2(\alpha) = d - cU_1(\alpha)$ from (4.3.92). Let us temporarily exclude the case $c = 0$ and allow c and d to be prescribed functions along $\mathcal{T}_0$. It is easily seen that a unique solution can be constructed in $\mathcal{D}_1$ using the given Cauchy data in $\mathcal{S}_0$ and the boundary data (4.3.92). The solution at all interior points in $\mathcal{D}_1$ is exactly analogous to that for the Cauchy problem. The calculation of U_1 and U_2 on $\mathcal{T}_0$ is analogous to the case just discussed for the point α.

Now, if $c = 0$, i.e., we prescribe U_2 on $\mathcal{T}_0$, the solution procedure is unaffected as long as U_{20}^*, the limiting value of U_2 as $\xi_2 \to 0^+$ along $\mathcal{T}_0$, is the same as $\overline{U}_{20}$, the limiting value of U_2 as $\xi_1 \to 0^+$ for the Cauchy data. If, however, $U_{20}^* \neq \overline{U}_{20}$, we cannot use the value of U_2 at β predicted by the solution in $\mathcal{D}_4$ in (4.3.91). In fact, U_2 will be *discontinuous* across the ξ_1-axis, and we need to compute $U_2(\xi_1, 0^+)$

by solving the ordinary differential equation

$$\frac{\partial U_2}{\partial \xi_1} + \sum_{k=1}^{2} C_{2k}(\xi_1, 0)U_k = \tilde{F}_2(\xi_1, 0) \tag{4.3.93a}$$

that results from (4.3.84b) subject to the initial condition $U_2(\xi_1, 0) = U_{20}^*$. In particular, we now have

$$U_2(\beta) = U_{20}^* + \Delta\xi_1[-\tilde{C}_{21}(0,0)U_1(0,0) - \tilde{C}_{22}(0,0)U_{20}^* + \tilde{F}_2(0,0)]. \tag{4.3.93b}$$

Note, incidentally, that U_1 is continuous accross the ξ_1-axis, and $U_1(\xi_1, 0)$ is given by the solution in $\mathcal{D}_4$. Once $U_2(\xi_1, 0^+)$ has been defined, the solution in $\mathcal{D}_1$ will proceed as before.

In the characteristic boundary-value problem, we wish to solve (4.3.84) in $\mathcal{D}_1 + \mathcal{D}_2$, and we proceed as above for given boundary values of U_1 on the ξ_1-axis and for U_2 on the ξ_2-axis.

(iii) *Discontinuities in the dependent variable itself for a second-order hyperbolic equation*

In Section 4.2.3 we discussed the propagation of discontinuities in the first derivatives for a second-order hyperbolic equation. We deferred the discussion of how discontinuities in the dependent variable itself propagate to this section in order to study the problem in terms of a pair of first-order equations. As we have seen in Section 4.3.5, a given second-order hyperbolic equation has many possible decompositions into a pair of first-order equations. The question then arises whether solutions involving discontinuities are correctly described regardless of the choice of decomposition. We verify next that this is the case for a simple example.

Consider the following initial-value problem on $-\infty < x < \infty$ for the wave equation

$$u_{tt} - u_{xx} = 0, \tag{4.3.94a}$$

$$u(x,0) = \begin{cases} 2 & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 1, \end{cases} = 2[H(x+1) - H(x-1)], \tag{4.3.94b}$$

$$u_t(x,0) = 0. \tag{4.3.94c}$$

According to (3.4.18), the solution is given by

$$u(x,t) = H(x+t+1) - H(x+t-1) + H(x-t+1) - H(x-t-1), \tag{4.3.93}$$

where H denotes the Heaviside function.

Let us first solve this problem using the decomposition (4.3.79). The problem transforms to the system

$$\frac{\partial U_1}{\partial t} + \frac{\partial U_1}{\partial x} = U_2; \quad \frac{\partial U_2}{\partial t} - \frac{\partial U_2}{\partial x} = 0, \tag{4.3.96a}$$

$$U_1(x,0) = 0, \quad U_2(x,0) = u_x(x,0) = 2[\delta(x+1) - \delta(x-1)], \tag{4.3.96b}$$

where δ denotes the Dirac delta function.

Now introduce the characteristic independent variables $\xi_1 = x - t$ and $\xi_2 = x + t$ and denote $\tilde{U}_i(\xi_1, \xi_2) = U_i\left(\frac{\xi_1+\xi_2}{2}, \frac{\xi_2-\xi_1}{2}\right)$, $i = 1, 2$. The system (4.3.96) transforms to

$$\frac{\partial \tilde{U}_1}{\partial \xi_2} = \frac{1}{2}\tilde{U}_2, \quad \frac{\partial \tilde{U}_2}{\partial \xi_1} = 0, \tag{4.3.97a}$$

$$\tilde{U}_1(\xi_1, \xi_1) = 2[H(\xi_1 + 1) - H(\xi_1 - 1)], \tag{4.3.97b}$$

$$\tilde{U}_2(\xi_2, \xi_2) = 2[\delta(\xi_2 + 1) - \delta(\xi_2 - 1)]. \tag{4.3.97c}$$

Solving the second equation in (4.3.97a) for $\tilde{U}_2$ and imposing the initial condition gives

$$\tilde{U}_2(\xi_1, \xi_2) = 2[\delta(\xi_2 + 1) - \delta(\xi_2 - 1)]. \tag{4.3.98a}$$

Using this expression in the right-hand side of the $\tilde{U}_1$ equation and integrating gives

$$\tilde{U}_1 = H(\xi_2 + 1) - H(\xi_2 - 1) + f(\xi_1),$$

where f is an arbitrary function. The initial condition (4.3.97b) determines f, and we have

$$\tilde{U}_1 = H(\xi_2 + 1) - H(\xi_2 - 1) + H(\xi_1 + 1) - H(\xi_1 - 1). \tag{4.3.98b}$$

When this result is expressed in terms of the u, x, t variables, we see that it is identical to (4.3.95).

If we use the alternative decomposition (4.3.82) with $k = 0$ and introduce the characteristic independent variables, (4.3.94) becomes

$$\frac{\partial \tilde{U}_1}{\partial \xi_2} = 0, \quad \frac{\partial \tilde{U}_2}{\partial \xi_1} = 0, \tag{4.3.99a}$$

$$\tilde{U}_1(\xi_1, \xi_1) = -u_x(x, 0) = -2[\delta(\xi + 1) - \delta(\xi_1 - 1)], \tag{4.3.99b}$$

$$\tilde{U}_2(\xi_2, \xi_2) = u_x(x, 0) = 2[\delta(\xi_2 + 1) - \delta(\xi_2 - 1)]. \tag{4.3.99c}$$

Now the solution for $\tilde{U}_1$ and $\tilde{U}_2$ is just the initial value. Therefore,

$$u_t = \frac{1}{2}(\tilde{U}_1 + \tilde{U}_2) = -\delta(\xi_1 + 1) + \delta(\xi_1 - 1) + \delta(\xi_2 + 1) - \delta(\xi_2 - 1),$$

$$u_x = \frac{1}{2}(-\tilde{U}_1 + \tilde{U}_2) = \delta(\xi_1 + 1) - \delta(\xi_1 - 1) + \delta(\xi_2 + 1) - \delta(\xi_2 - 1).$$

These are precisely the t and x derivatives of the expression for u given in (4.3.95).

We have verified that for this simple example, a solution with initial discontinuities is correctly described regardless of the decomposition into a pair of first-order equations.

Problems

4.3.1 Consider the linearized equations for shallow-water flow with axial symmetry (see (3.9.73) and drop the subscripts)

$$h_t + u_r + \frac{u}{r} = 0, \quad u_t + h_r = 0. \tag{4.3.100}$$

Regard this system as a special case of (4.3.10) with $x \to r$. Thus,

$$\mathbf{u} = (h, u), \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1/r \\ 0 & 0 \end{pmatrix}. \tag{4.3.101}$$

a. Use the transformation (4.3.22) to show that (4.3.25) becomes

$$\frac{\partial U_1}{\partial t} + \frac{\partial U_1}{\partial r} + \frac{1}{2r}(U_1 - U_2) = 0, \quad \frac{\partial U_2}{\partial t} - \frac{\partial U_2}{\partial r} + \frac{1}{2r}(U_1 - U_2) = 0, \tag{4.3.101}$$

where

$$U_1 = -\frac{1}{2}(h + u), \quad U_2 = -\frac{1}{2}(h - u). \tag{4.3.102}$$

b. Introduce the characteristic independent variables $\xi_1 = r - t$ and $\xi_2 = r + t$ to show that (4.3.29) is

$$\frac{\partial U_1}{\partial \xi_2} + \frac{1}{2(\xi_1 + \xi_2)}(U_1 - U_2) = 0, \tag{4.3.103a}$$

$$\frac{\partial U_2}{\partial \xi_1} - \frac{1}{2(\xi_1 + \xi_2)}(U_1 - U_2) = 0 \tag{4.3.103b}$$

in this case. This result is not helpful for computing the analytic solution we derived in Problem 3.9.6. However, it is a useful starting point for a numerical solution.

c. Now consider the mathematical model (cf. (4.3.100))

$$\frac{\partial u_1}{\partial t} + \frac{\partial u_2}{\partial x} + b(x)u_2 = 0, \quad \frac{\partial u_2}{\partial t} + \frac{\partial u_1}{\partial x} + b(x)u_1 = 0, \tag{4.3.104a}$$

with initial conditions

$$u_1(x, 0) = f(x), \quad u_2(x, 0) = g(x). \tag{4.3.104b}$$

Here $b(x)$, $f(x)$, and $g(x)$ are prescribed arbitrarily. Derive the equations corresponding to (4.3.103) for U_1 and U_2 in this case and solve these to obtain

$$U_1(x, t) = F(x - t)\exp\left(-\int_{x-t}^{x} b(s)ds\right), \tag{4.3.105a}$$

$$U_2(x, t) = G(x + t)\exp\left(-\int_{x+t}^{x} b(s)ds\right), \tag{4.3.105b}$$

where

$$F(x) = -\frac{1}{2}[f(x) + g(x)], \quad G(x) = -\frac{1}{2}[f(x) - g(x)]. \tag{4.3.105c}$$

4.3.2 In Problem 3.2.1 we derived the shallow-water equations for flow over a variable bottom. Specializing these results to the one-dimensional case and the problem of an isolated bump of unit length moving uniformly with dimensionless speed (Foude number) F = constant > 0 to the left, we have (see Figure 4.16a)

$$\bar{h}_t + (\bar{u}\bar{h})_{\bar{x}} = 0, \quad \bar{u}_{\bar{t}} + \bar{u}\,\bar{u}_{\bar{x}} + (\bar{h} + \epsilon\bar{b}_{\bar{x}}) = 0, \tag{4.3.106}$$

where the bump height is $\epsilon\bar{b}$ with

$$\bar{b}(\bar{x}, \bar{t}) \equiv B(\bar{x} + F\bar{t}). \tag{4.3.107}$$

For an isolated bump, the function B vanishes when $|\bar{x} + F\bar{t}| > \frac{1}{2}$.

a. Introduce the Galilean transformation (see Figure 4.16b)

$$x = \bar{x} + F\bar{t}, \quad t = \bar{t}, \tag{4.3.108a}$$

$$\bar{h}(\bar{x}, \bar{t}) = h(x, t), \quad \bar{u}(\bar{x}, \bar{t}) = u(x, t) - F, \tag{4.3.108b}$$

to a coordinate system attached to and moving with the bump. Thus, $\bar{b}(\bar{x}, \bar{t}) = B(x)$. Show that (4.3.106) transforms to

$$h_t + (uh)_x = 0, \quad u_t + uu_x + (h + \epsilon B)_x = 0. \tag{4.3.109}$$

b. Consider the initial-value problem for (4.3.106), where the bump is at rest in quiescent water for $t < 0$ and is started impulsively at $t = 0$.

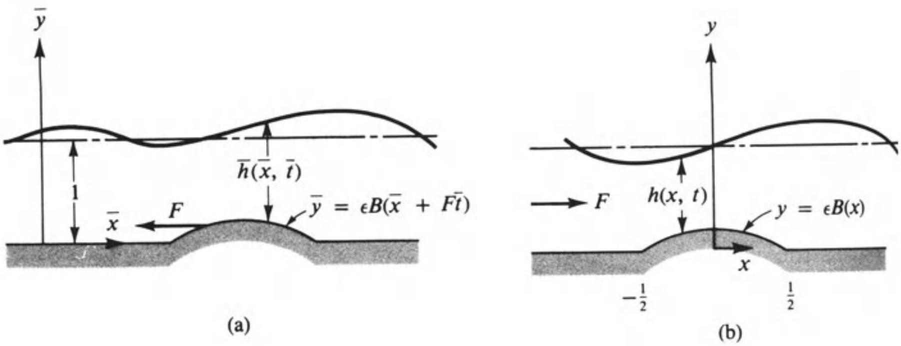

FIGURE 4.16. Bump on bottom in fixed and moving coordinates

We thus have

$$h(\overline{x}, 0) = 1 - \epsilon\overline{b}(\overline{x}, 0), \quad \overline{u}(\overline{x}, 0) = 0. \tag{4.3.110}$$

The corresponding initial conditions for (4.3.109) are

$$h(x, 0) = 1 - \epsilon B(x), \quad u(x, 0) = F. \tag{4.3.111}$$

Assume that h and u have the following expansions for small ϵ

$$h(x, t; \epsilon, F) = 1 + \epsilon h_1(x, t; F) + O(\epsilon^2), \tag{4.3.112a}$$

$$u(x, t; \epsilon, F) = F + \epsilon u_1(x, t; F) + O(\epsilon^2), \tag{4.3.112b}$$

and show that h_1 and u_1 satisfy

$$\frac{\partial h_1}{\partial t} + F\frac{\partial h_1}{\partial x} + \frac{\partial u_1}{\partial x} = 0, \quad \frac{\partial u_1}{\partial x} + F\frac{\partial u_1}{\partial x} + \frac{\partial h_1}{\partial x} = B'(x), \tag{4.3.113a}$$

with the initial conditions

$$h_1(x, 0) = -B(x), \quad u_1(x, 0) = 0. \tag{4.3.113b}$$

c. Show that the diagonal characteristic form (4.3.25) for this example is

$$\frac{\partial U_1}{\partial t} + (F+1)\frac{\partial U_1}{\partial x} = \frac{1}{2}B'(x), \quad \frac{\partial U_2}{\partial t} + (F-1)\frac{\partial U_2}{\partial x} = -\frac{1}{2}B'(x), \tag{4.3.114a}$$

with initial conditions

$$U_1(x, 0) = U_2(x, 0) = \frac{1}{2}B(x). \tag{4.3.114b}$$

d. Solve (4.3.114) then transform back to the h_1, u_1 variables to obtain the solution in the form

$$h_1(x, t) = \frac{F}{F^2 - 1}B(x) - \frac{F}{2(F + 1)}B[x - (F + 1)t] - \frac{F}{2(F - 1)}B[x - (F - 1)t], \tag{4.3.115a}$$

$$u_1(x, t) = -\frac{F}{F^2 - 1}B(x) - \frac{F}{2(F + 1)}B[x - (F + 1)t] + \frac{F}{2(F - 1)}B[x - (F - 1)t]. \tag{4.3.115b}$$

Notice that the solution is *singular* if $F = 1$. This singularity is similar to the one encountered for $M = 1$ in (4.2.70). A discussion of the correct expansion for $F \approx 1$ can be found in Section 6.2 of [26].

For $F \neq 1$, the solution for h_1 and u_1 consists of three components: a stationary disturbance over the bump, plus a right-going wave given by the $B[x - (F + 1)t]$ term and a slower wave (that is left-going if $F < 1$) given by the $B[x - (F - 1)t]$ term.

4.3.3a Work out the details of the results in Section 4.3.4i up to the derivation of (4.3.46).

b. Consider the initial-value problem for (4.3.36) where

$$g_1(x) = 0, \quad g_2(x) = \sin x \tag{4.3.117}$$

in (4.3.37). Show that in region (1) of Figure 4.12, u_1 and u_2 are given by

$$u_1(x,t) = \frac{1}{2}[\sin(x-t) - \sin(x+t)], \tag{4.3.118a}$$

$$u_2(x,t) = \frac{1}{2}[\sin(x-t) + \sin(x+t)]. \tag{4.3.118b}$$

Thus, $u_1 = u_2 = -\frac{1}{2}\sin 2t$ on the ray $x = -t$ ($\xi_2 = -1$).

c. For the case (4.3.118), evaluate $\Gamma(\xi_1)$ and show that (4.3.53) gives $U_2(0) = 0$. Investigate where discontinuities in u_1 and u_2 occur.

4.3.4 Consider the system (4.3.25) with constant λ_i and C_{ij} and prescribed $F_i(x,t)$.

a. Change dependent variables from U_i to V_i according to

$$U_i(x,t) = \phi(x,t)V_i(x,t), \tag{4.3.119a}$$

where

$$\phi(x,t) \equiv \exp\left[\frac{(C_{22} - C_{11})x + (C_{11}\lambda_2 - C_{22}\lambda_1)t}{\lambda_1 - \lambda_2}\right], \tag{4.3.119b}$$

to remove the diagonal terms in the $\{C_{ij}\}$ matrix and transform (4.3.25) to the system

$$\frac{\partial V_1}{\partial t} + \lambda_1\frac{\partial V_1}{\partial x} + C_{12}V_2 = G_1(x,t), \tag{4.3.120a}$$

$$\frac{\partial V_2}{\partial t} + \lambda_2\frac{\partial V_2}{\partial x} + C_{21}V_1 = G_2(x,t), \tag{4.3.120b}$$

where

$$G_i(x,t) \equiv \frac{1}{\phi(x,t)}F_i(x,t), \quad i = 1, 2. \tag{4.3.120c}$$

Thus, given initial conditions

$$U_i(x,0) = U_i^*(x), \quad i = 1, 2, \tag{4.3.121}$$

transform to

$$V_i(x,0) = \frac{1}{\phi(x,0)}U_i^*(x) \equiv V_i^*(x), \quad i = 1, 2. \tag{4.3.122}$$

b. Now introduce the new independent variables

$$\bar{x} = x - \frac{(\lambda_1 + \lambda_2)}{2}t, \quad \bar{t} = \frac{(\lambda_1 - \lambda_2)}{2}t, \tag{4.3.123}$$

and denote

$$\overline{V}_i(\overline{x}, \overline{t}) \equiv V_i\left(\overline{x} + \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2}\overline{t},\ \frac{2}{\lambda_1 - \lambda_2}\overline{t}\right) \tag{4.3.124}$$

to derive the following wave equation for $\overline{V}_1$ or $\overline{V}_2$,

$$\frac{\partial^2 \overline{V}_i}{\partial \overline{t}^2} - \frac{\partial \overline{V}_i}{\partial \overline{x}^2} + c\overline{V}_i = H_i, \quad i = 1, 2, \tag{4.3.125}$$

where

$$c = -\frac{4C_{12}C_{21}}{(\lambda_1 - \lambda_2)^2}, \tag{4.3.126a}$$

$$H_1 = \frac{2}{(\lambda_1 - \lambda_2)}\left[\frac{\partial G_1}{\partial \overline{t}} - \frac{\partial G_1}{\partial \overline{x}}\right] - \frac{4C_{12}}{(\lambda_1 - \lambda_2)^2}G_2, \tag{4.3.126b}$$

$$H_2 = \frac{2}{(\lambda_1 - \lambda_2)}\left[\frac{\partial G_2}{\partial \overline{t}} + \frac{\partial G_2}{\partial \overline{x}}\right] - \frac{4C_{21}}{(\lambda_1 - \lambda_2)^2}G_1. \tag{4.3.126c}$$

Show that the initial conditions that are implied by (4.3.122) are

$$\overline{V}_i(\overline{x}, 0) = V_i^*(\overline{x}), \quad i = 1, 2, \tag{4.3.127a}$$

$$\frac{\partial \overline{V}_1}{\partial \overline{t}}(\overline{x}, 0) = -\frac{dV_1^*}{dx}(\overline{x}) - \frac{2C_{12}}{\lambda_1 - \lambda_2}V_2^*(\overline{x}) + \frac{2}{\lambda_1 - \lambda_2}G_1(\overline{x}, 0) \equiv W_1^*(\overline{x}), \tag{4.3.127b}$$

$$\frac{\partial \overline{V}_2}{\partial \overline{t}}(\overline{x}, 0) = \frac{dV_2^*}{dx}(\overline{x}) - \frac{2C_{21}}{\lambda_1 - \lambda_2}V_1^*(\overline{x}) + \frac{2}{\lambda_1 - \lambda_2}G_2(\overline{x}, 0) \equiv W_2^*(\overline{x}). \tag{4.3.127c}$$

The solution of (4.3.125) subject to the initial conditions (4.3.127) is given in (4.2.62) for the case $c > 0$. For $c < 0$, the solution is formally the same, except that J_0 is everywhere replaced by I_0. Once the $\overline{V}(\overline{x}, \overline{t})$ are defined, the solution for the $U_i(x, t)$ follows from the transformation relations (4.3.123) and (4.3.119).

4.3.5 Consider the wave equation (4.3.76) with $k = 0$ on $-\infty < x < \infty$ subject to the initial conditions

$$u(x, 0) = f(x), \quad u_t(x, 0) = h(x). \tag{4.3.128}$$

The D'Alembert solution was calculated in Chapter 3 in the form

$$u(x, t) = \frac{1}{2}[f(x + t) + f(x - t)] + \frac{1}{2}\int_{x-t}^{x+t} h(s)ds + \frac{1}{2}\int_0^t d\tau \int_{x-(t-\tau)}^{x+(t-\tau)} p(s, \tau)ds. \tag{4.3.129}$$

a. Use the decomposition (4.3.79) with $k = 0$ to show that the initial-value problem reduces to the following pair of equations for $U_1(\xi_1, \xi_2)$, $U_2(\xi_1, \xi_2)$,

$$\frac{\partial U_1}{\partial \xi_2} = \frac{1}{2} U_2, \quad \frac{\partial U_2}{\partial \xi_1} = \frac{1}{2} p\left(\frac{\xi_2 - \xi_1}{2}, \frac{\xi_2 + \xi_1}{2}\right), \quad (4.3.130a)$$

where $\xi_1 = t - x$, $\xi_2 = t + x$, and that the conditions (4.3.128) imply

$$U_1(\xi_1, -\xi_1) = f(-\xi_1), \quad U_2(-\xi_2, \xi_2) = h(\xi_2) + f'(\xi_2). \quad (4.3.130b)$$

b. Solve (4.3.130) and show that your result agrees with (4.3.129).

4.3.6 Consider the system (4.3.25) with constant λ_i, C_{ij} and $F_i = 0$.

a. Show that the U_i obey the second-order equation

$$\left(\frac{\partial}{\partial t} + \lambda_1 \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \lambda_2 \frac{\partial}{\partial x}\right) U_i + \left((C_{11} + C_{22})\frac{\partial}{\partial t}\right.$$

$$\left. + (C_{11}\lambda_2 + C_{22}\lambda_1)\frac{\partial}{\partial x}\right) U_i + (C_{11}C_{22} - C_{12}C_{21})U_i = 0, \; i = 1, 2. \quad (4.3.131)$$

Use the approach discussed in Section 4.2.4 to study the stability of (4.3.131). In particular, show that solutions are bounded as $t \to \infty$ for *all* disturbance wave numbers k if

$$C_{11} \geq 0, \quad C_{22} \geq 0, \quad C_{11}C_{22} - C_{12}C_{21} \geq 0. \quad (4.3.132)$$

Verify that for the linear system (4.3.74) that we derived for channel flow these conditions hold if $0 \leq F \leq 2$.

4.3.7 Consider the linear system of n first-order equations defined in vector form by

$$\overline{A}\frac{\partial \mathbf{u}}{\partial t} + \overline{B}\frac{\partial \mathbf{u}}{\partial x} + \overline{C}\mathbf{u} = \mathbf{f}, \quad (4.3.132)$$

where $\mathbf{u} = (u_1, \ldots, u_n)$, $\mathbf{f} = (f_1(x,t), f_2(x,t), \ldots, f_n(x,t))$, and $\overline{A}$, $\overline{B}$, $\overline{C}$ are $n \times n$ matrices with components that are given functions of x, t. Show that the condition under which $\phi(x,t) = \xi_0 =$ constant is a characteristic curve is that the determinant of the matrix $\overline{D} \equiv \phi_x \overline{B} + \phi_t \overline{A}$ vanishes. Equation (4.3.15b) is the special case of this for $n = 2$ and under the assumption that $\overline{A}^{-1}$ exists.

4.3.8 In Section 3.9.4i we derived an expression for the pressure perturbation due to a bursting balloon. In particular, we found the jump in pressure along the characteristic $r - t = 1$ to be $1/2r$ and the jump in pressure along the characteristic $r - t = -1$ to be $-1/2r$. These results are valid as long as $r > 1$. Reconsider this problem as the solution of

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r} - \frac{\partial^2 \phi}{\partial t^2} = 0, \quad 0 \leq r, \quad 0 \leq t, \quad (4.3.133a)$$

with initial condition

$$\phi(r, 0) = 0, \tag{4.3.133b}$$

$$\phi_t(r, 0) = \begin{cases} -\frac{1}{\gamma}, & \text{if } r < 1, \\ 0, & \text{if } r > 1. \end{cases} \tag{4.3.133c}$$

Since we have invoked spherical symmetry and reduced the governing equation to the one-dimensional form (4.3.133a) with r restricted to $0 \leq r$, we must specify a boundary condition at the origin. Because of the absence of a source at the origin, we must have [see (3.9.68)]

$$\lim_{r \to 0} r^2 \phi_r(r, t) = 0, \quad t \geq 0. \tag{4.3.134}$$

In light of our discussion of Section 4.3.6, derive the expression for the propagation of the initial discontinuity at $r = 1$ along the outgoing characteristic $r - t = 1$ and compare your result with the value derived in Chapter 3. Next, consider the incoming characteristic $r + t = 1$, which reflects from the t-axis and becomes the outgoing characteristic $r - t = -1$. How does the initial discontinuity propagate inward and then reflect? Again, compare your result with the expression derived in Chapter 3.

5

The Scalar Quasilinear First-Order Equation

The first four chapters have been almost exclusively concerned with linear partial differential equations. In this chapter we study the quasilinear partial differential equation of first order for the scalar dependent variable. Some aspects of systems of quasilinear equations are also discussed here and more fully in Chapter 7. The scalar nonlinear equation is covered in Chapter 6.

5.1 Conservation Laws in Two Independent Variables

5.1.1 Systems of Conservation Laws

In previous chapters we have derived a number of integral conservation laws governing physical systems. For example, heat conservation in a one-dimensional conductor obeys (1.1.6), mass and momentum conservation in shallow-water flow obey (3.2.6) and (3.2.10), whereas in channel flow, these equations generalize to (4.3.61), and so on. In the absence of dissipation, a general system of n integral conservation laws in one spatial dimension has the form

$$\begin{aligned}\frac{d}{dt}\int_{x_1}^{x_2} &\Psi_i(\mathbf{u}(x,t),x,t)dx \\ &= \Phi_i(\mathbf{u}(x_1,t),x_1,t) - \Phi_i(\mathbf{u}(x_2,t),x_2,t) \\ &+ \int_{x_1}^{x_2} \lambda_i(\mathbf{u}(x,t),x,t)dx, \quad i = 1,\ldots,n. \end{aligned} \tag{5.1.1}$$

Here x_1 and x_2 are arbitrary fixed values of x $(x_1 < x_2)$, the notation $\mathbf{u}$ indicates the n components $u_1(x,t), u_2(x,t), \ldots, u_n(x,t)$, where the u_i are the dependent variables, the Ψ_i are the conserved quantities, the Φ_i are the fluxes, and the λ_i are the source densities. For example, in (4.3.61) $u_1 = h, u_2 = u, \Psi_1 = u_1, \Psi_2 = u_1u_2$, $\Phi_1 = u_1u_2, \Phi_2 = u_1u_2^2 + u_1^2/2, \lambda_1 = 0, \lambda_2 = u_1 - (1 + 2\sigma u_1)u_2^2/F^2$. Note that in the general form (5.1.1), we have allowed the Ψ_i, Φ_i, and λ_i also to depend on x and t. However, the system (5.1.1) does not include dissipative terms as in (3.3.5b) and (3.3.5c).

We have seen that if $\mathbf{u}$, $\mathbf{u}_x$, and $\mathbf{u}_t$ are continuous, the system (5.1.1) implies the vector system in divergence form

$$\mathbf{p}_t + \mathbf{q}_x = \boldsymbol{\lambda}, \tag{5.1.2}$$

where $\mathbf{p}$ and $\mathbf{q}$ are the n-vectors with components

$$p_i(x,t) \equiv \Psi_i(\mathbf{u}(x,t), x, t), \tag{5.1.3a}$$

$$q_i(x,t) \equiv \Phi_i(\mathbf{u}(x,t), x, t). \tag{5.1.3b}$$

A solution of (5.1.1) where $\mathbf{u}$, $\mathbf{u}_x$, and $\mathbf{u}_t$ are all continuous in some domain of the xt-plane will henceforth be referred to as a *strict solution.*

When the partial derivatives $\mathbf{p}_t$ and $\mathbf{q}_x$ are evaluated, (5.1.2) gives

$$P\mathbf{u}_t + Q\mathbf{u}_x = \mathbf{g}, \tag{5.1.4}$$

where P and Q are the Jacobian matrices

$$P \equiv \frac{\partial(\Psi_1, \ldots, \Psi_n)}{\partial(u_1, \ldots, u_n)}, \quad Q \equiv \frac{\partial(\Phi_1, \ldots, \Phi_n)}{\partial(u_1, \ldots, u_n)}, \tag{5.1.5}$$

and the components of $\mathbf{g}$ are

$$g_i(\mathbf{u}, x, t) \equiv \lambda_i(\mathbf{u}(x,t), x, t) - \frac{\partial \Psi_i}{\partial t}(\mathbf{u}(x,t), x, t) - \frac{\partial \Phi_i}{\partial x}(\mathbf{u}(x,t), x, t). \tag{5.1.6}$$

If P^{-1} exists, (5.1.4) simplifies further to the system (cf. (4.3.1))

$$\mathbf{u}_t + A\mathbf{u}_x = \mathbf{r}, \tag{5.1.7}$$

where A and $\mathbf{r}$ are

$$A(\mathbf{u}, x, t) = P^{-1}Q, \quad \mathbf{r}(\mathbf{u}, x, t) = P^{-1}\mathbf{g}. \tag{5.1.8}$$

The component form of (5.1.7) is a system of n quasilinear first-order partial differential equations

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} A_{ij}(\mathbf{u}, x, t)\frac{\partial u_j}{\partial x} = r_i(\mathbf{u}, x, t), \quad i = 1, \ldots, n, \tag{5.1.9}$$

which we shall study in Chapter 7. Again, returning to the example of (4.3.61) for channel flow, (5.1.9) corresponds to the system (4.3.63a) and (4.3.65), where A and $\mathbf{r}$ are defined in (4.3.66).

In general, an accurate description of a physical process requires the simultaneous application of all the component equations in (5.1.9). In some applications, an empirical formula may be used to express one of the dependent variables in terms of the others and thus reduce the order of (5.1.9). For the example of channel flow, we can argue that for small disturbances the balance between frictional and gravitational forces is approximately maintained even though u and h are no longer constant. This gives the empirical relation $u = Fh^{1/2}$ obtained by setting the right-hand side of (4.3.65) equal to zero. When this empirical relation linking

u to h is used in (4.3.63a), we end up with the following scalar equation for h

$$h_t + \frac{3F}{2} h^{1/2} h_x = 0.$$

In effect, the assumption $u = Fh^{1/2}$ means that the mass flux is no longer a function of u and h but is simply given by $\Phi = Fh^{3/2}$. In the next subsection we study a scalar conservation law with an empirically defined flux.

5.1.2 The Scalar Conservation Law; Traffic Flow

Let us specialize (5.1.1) to the scalar case ($n = 1$) and omit the subscript. Let $\Psi_1 = \rho$, a density, and assume an empirical relation for the flux that involves only ρ, x, and t:

$$q(x,t) = \Phi(\rho(x,t), x, t). \tag{5.1.10}$$

The general scalar integral conservation law is then

$$\begin{aligned} \frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)dx =& \Phi(\rho(x_1,t), x_1, t) - \Phi(\rho(x_2,t), x_2, t) \\ &+ \int_{x_1}^{x_2} \lambda(\rho(x,t), x, t)dx. \end{aligned} \tag{5.1.11}$$

For a given flux Φ we can define a "flow speed" $v(x,t)$ by

$$v(x,t) = \frac{q}{\rho}. \tag{5.1.12a}$$

A second quantity with units of speed is

$$c(\rho, x, t) = \frac{\partial \Phi}{\partial \rho}, \tag{5.1.12b}$$

and we shall see later on that c defines a local *signal speed* for the flow.

The scalar divergence relation corresponding to (5.1.2) if ρ, ρ_x, and ρ_t are continuous is

$$\rho_t + q_x = \lambda, \tag{5.1.13a}$$

and (5.1.9) becomes

$$\rho_t + c(\rho, x, t)\rho_x = \lambda(\rho, x, t) - \Phi_x(\rho, x, t). \tag{5.1.13b}$$

Note that even if the original conservation law has no source term, $\lambda = 0$, the x-dependence in Φ introduces a right-hand side in (5.1.13b).

As an illustration, consider traffic on a one-lane road with no on and off ramps. Let X denote the dimensional value of the distance along the road, T denote the dimensional time, and ρ denote the traffic density—that is, the number of cars per unit distance of road. In this application, (5.1.10),

$$q(X,T) = \Phi(\rho(X,T), X, T), \tag{5.1.14a}$$

is a reasonable choice for the flux if the dependence of Φ on ρ has the qualitative behavior shown in Figure 5.1.

This graph depicts Φ at a given point X on the road and at a given time T. It shows the obvious fact that if there are no cars ($\rho = 0$), then $\Phi = 0$. Conversely, there is a maximum density $\rho_{\max}$ of cars for which $\Phi = 0$ also, because the cars are stacked bumper to bumper and cannot move. At some intermediate density ρ_0, the flux has a maximum value $\Phi_{\max}$. The flux also depends in general on X and T to reflect driving conditions at each location along the road and time of day. One could derive these empirical relationships by making a large number of observations covering all traffic conditions at different locations and times.

One could improve the model in (5.1.14a) by allowing q to depend also on ρ_X, to reflect the fact that drivers tend to slow down when moving into a region of increasing traffic density and vice versa. Thus, a plausible refinement would rephrase (5.1.14a) as

$$q(X, T) = \Phi(\rho(X, T), X, T) - k\rho_X(X, T), \tag{5.1.14b}$$

where k is a positive constant. The simplest expression (5.1.14b) that retains all the essential features has a quadratic dependence of Φ on ρ—that is,

$$q(X, T) = R\rho(\rho_{\max} - \rho) - k\rho_X, \tag{5.1.15}$$

with R, $\rho_{\max}$, and k constant.

In this case (5.1.13a) gives

$$\rho_T + [R\rho(\rho_{\max} - \rho) - k\rho_X]_X = 0 \tag{5.1.16}$$

for a strict solution.

Let us introduce the dimensionless variables

$$u \equiv \frac{\rho_{\max}/2 - \rho}{\rho_{\max}/2}, \quad x \equiv \frac{X}{L_0}, \quad t \equiv \frac{T}{L_0/R\rho_{\max}}, \tag{5.1.17}$$

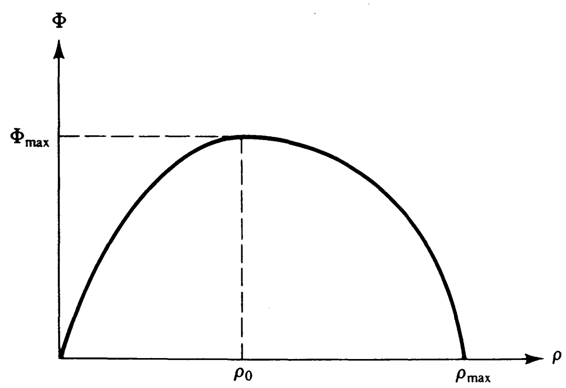

FIGURE 5.1. Flux as a function of traffic density

where L_0 is a characteristic length scale associated with the initial condition for (5.1.16). We then obtain the divergence relation

$$u_t + \left[\frac{1}{2}(u^2 - 1) - \epsilon u_x\right]_x = 0, \tag{5.1.18a}$$

or

$$u_t + uu_x = \epsilon u_{xx}, \quad \epsilon \equiv \frac{k}{R\rho_{\max} L_0}, \tag{5.1.18b}$$

where ϵ is a dimensionless parameter that is the ratio of the *diffusion length* $k/R\rho_{\max}$ to L_0. Equation (5.1.18b) is Burgers' equation, discussed in Chapter 1. Of course, in the present interpretation, solutions of (5.1.18) are valid only for $0 \le \rho \le \rho_{\max}$, that is, for $|u| \le 1$.

A number of other physical problems may be modeled by (5.1.11); the reader is referred to Chapter 3 of [42] for a detailed discussion and references to original sources.

5.2 Strict Solutions in Two Independent Variables

In this section we first study the geometric properties of strict solutions of the quasilinear first-order partial differential equation in two independent variables as a guide in deriving their analytic form. We then point out how the requirement that u, u_x, and u_t be continuous may break down in physically realistic situations. This motivates the idea of a *weak solution* discussed in Section 5.3. These weak solutions are based on the integral conservation law formulation and admit discontinuities.

5.2.1 Geometrical Aspects of Solutions

The general quasilinear equation for u as a function of the two independent variables x, y is

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u), \tag{5.2.1}$$

where a, b, and c are prescribed functions of x, y, u. We also assume that in some domain of interest, a, b, and c are continuous and have continuous first derivatives and that a and b do not vanish simultaneously in this domain along a given solution. The problem is quasilinear because a, b, and c depend on u but do not involve u_x or u_y. The fact that u is involved in these coefficients means that we must take into consideration the solution $u(x, y)$ in any geometric interpretation of (5.2.1).

Therefore, let us examine (5.2.1) in the three-dimensional Cartesian space x, y, u. Let $P = (x_0, y_0, u_0)$ be a given point on a solution $u = \phi(x, y)$ of (5.2.1); we write this solution in the form

$$\psi(x, y, u) \equiv \phi(x, y) - u = 0. \tag{5.2.2}$$

A normal vector $\mathbf{n}$ to the solution surface $\psi = 0$ has components $(u_x, u_y, -1)$. Neither the magnitude of this normal nor the fact that the normal points in the negative u direction is relevant; these are consequences of an arbitrary constant multiplier that does not alter (5.2.2). At the point P, the coefficients a, b, and c each equal a number, which we denote by $a_0 \equiv a(x_0, y_0, u_0)$, $b_0 \equiv b(x_0, y_0, u_0)$, and $c_0 \equiv c(x_0, y_0, u_0)$. We note that at P, (5.2.1) is just a statement of the vanishing of the dot product between the normal vector $\mathbf{n}_0 \equiv (u_x(x_0, y_0), u_y(x_0, y_0), -1)$ and the constant vector $\boldsymbol{\sigma}_0 \equiv (a_0, b_0, c_0)$. Since (5.2.1) is a *linear* algebraic relation linking u_x and u_y at P, it follows that the normals to all the possible solution surfaces through P must lie *in a plane perpendicular to* $\boldsymbol{\sigma}_0$. In other words, $\boldsymbol{\sigma}_0$ *is a tangent vector* to all the possible solution surfaces through P. One may also interpret $\boldsymbol{\sigma}_0$ as the intersection of the one-parameter family of possible solution surfaces of (5.2.1) at P, as sketched in Figure 5.2a. Again, the magnitude and sense of $\boldsymbol{\sigma}_0$ are irrelevant.

The curve generated in x, y, u space by following the local $\boldsymbol{\sigma}$ direction from a given initial point is called a *characteristic curve*; its projection on the $u = 0$ plane is called a *characteristic ground curve*.

Let $\mathcal{S}$ be a reference surface on which the characteristic direction at every point is not tangent to $\mathcal{S}$. Therefore, the characteristic curves that pass through $\mathcal{S}$ can be labeled using two parameters (corresponding, for example, to the two coordinates needed to specify each point on $\mathcal{S}$). This two-parameter family of curves fills the x, y, u space, at least in some neighborhood of $\mathcal{S}$. In order to isolate a specific solution surface $\psi = 0$ generated by a one-parameter subfamily of characteristic curves, let us pick some curve C_0 on $\mathcal{S}$ and consider only those characteristics that pass through C_0. These characteristics generate a surface, on every point of which (5.2.1) is satisfied. This solution surface may be geometrically visualized by regarding it as an infinitely dense set of characteristic curves one layer thick.

We see that in order to specify a solution surface, we must require that it contain a prescribed *initial curve* C_0. The local construction of the solution surface near C_0 may be visualized as follows: We introduce the parameter τ, which varies along C_0, and the parameter s (this may be an arc length), which varies along each characteristic curve emerging from C_0. By extending the characteristic curves a short distance Δs, we generate a thin strip Δs wide to one side of C_0 because none of the characteristic directions are tangent to C_0 (see Figure 5.2b). The new boundary of this strip is the cuve C_1.

We now repeat this process at C_1 and extend the solution over a new thin strip to C_2, and so on. This construction generates a solution surface as long as the successive curves $C_1, C_2, \ldots$ do not become characteristic over a finite arc.

5.2.2 Characteristic Curves; the Solution Surface

We have seen that at each point $P = (x_0, y_0, u_0)$, there is a unique characteristic vector $\boldsymbol{\sigma}_0 = (a_0, b_0, c_0)$ that is tangent to the characteristic curve passing through P. If s is a parameter that varies monotonically along a characteristic curve, the

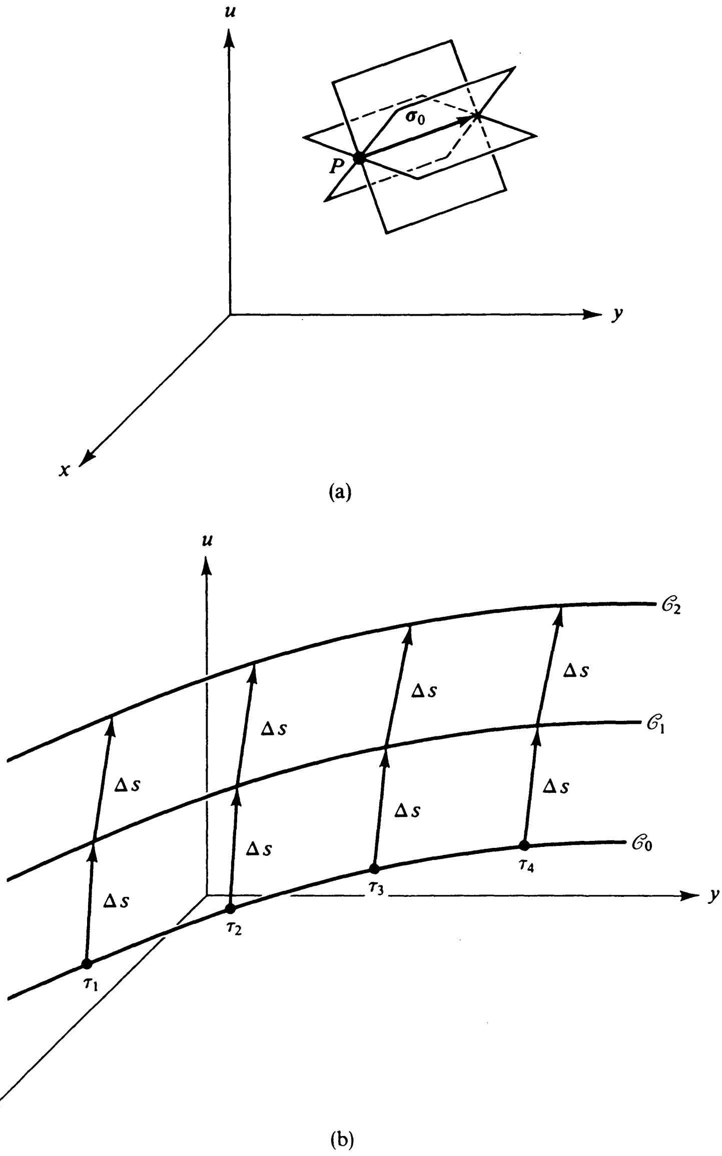

FIGURE 5.2. a. Possible infinitesimal solution surface at P; b. Extending the solution from the curve $\mathcal{C}_0$

coordinates $Q = (x_0 + \Delta x, y_0 + \Delta y, u_0 + \Delta u)$ of a point located at a small distance from P in the characteristic direction of increasing s are obtained by setting $\Delta x = a_0 \Delta s$, $\Delta y = b_0 \Delta s$, $\Delta u = c_0 \Delta s$. Thus, in the limit as $\Delta s \to 0$, the characteristic curves that are so generated obey the system of differential equations

$$\frac{dx}{ds} = a(x, y, u), \quad \frac{dy}{ds} = b(x, y, u), \quad \frac{du}{ds} = c(x, y, u). \tag{5.2.3}$$

Since the system (5.2.3) is autonomous, its solution can be expressed in terms of $s - s_0$, where s_0 is a constant, and two other arbitrary constants c_1 and c_2. We may set $s_0 = 0$ with no loss of generality because $s - s_0$ may be regarded as a new variable along each characteristic. Thus, the solution of (5.2.3) has the form

$$x = \overline{X}(s, c_1, c_2), \tag{5.2.4a}$$
$$y = \overline{Y}(s, c_1, c_2), \tag{5.2.4b}$$
$$u = \overline{U}(s, c_1, c_2). \tag{5.2.4c}$$

We can choose c_1 and c_2 such that (5.2.4) passes through a given point x_0, y_0, u_0. The form of (5.2.4) also confirms the claim made earlier that the characteristics are a two-parameter family of curves; each family member is identified by the two constants c_1, c_2.

We can isolate any desired one-parameter subfamily of (5.2.4) by regarding c_1 and c_2 as functions of a single parameter τ. This gives

$$x = \overline{X}(s, c_1(\tau), c_2(\tau)) \equiv X(s, \tau), \tag{5.2.5a}$$
$$y = \overline{Y}(s, c_1(\tau), c_2(\tau)) \equiv Y(s, \tau), \tag{5.2.5b}$$
$$u = \overline{U}(s, c_1(\tau), c_2(\tau)) \equiv U(s, \tau). \tag{5.2.5c}$$

The particular one-parameter subfamily that passes through the initial curve C_0 is obtained as follows. We specify C_0 in the parametric form

$$x = x_0(\tau), \quad y = y_0(\tau), \quad u = u_0(\tau), \tag{5.2.6}$$

where x_0, y_0, and u_0 are continuously differentiable functions of τ, and we assume that the ground curve $x_0(\tau)$, $y_0(\tau)$ does not intersect with itself. Then we require the functions $X(s, \tau)$, $Y(s, \tau)$, and $U(s, \tau)$ in (5.2.5) to satisfy the initial conditions

$$X(0, \tau) = x_0(\tau), \quad Y(0, \tau) = y_0(\tau), \quad U(0, \tau) = u_0(\tau). \tag{5.2.7}$$

Thus, we have set $s = 0$ on the initial curve. In practice, we shall derive the solution of (5.2.3) directly in the form (5.2.5), then apply the initial conditions (5.2.7) to determine the unknown functions of τ that arise.

To exhibit the solution surface $u = \phi(x, y)$ that passes through C_0, we solve (5.2.5a) and (5.2.5b) for s and τ in terms of x and y. This is always possible as long as the Jacobian

$$\Delta(s, \tau) \equiv X_s Y_\tau - Y_s X_\tau \tag{5.2.8}$$

does not vanish. We know that if the characteristics are nowhere tangent to C_0, then the directions of increasing s and τ along C_0 are not collinear. Therefore, at least in some neighborhood of $s = 0$ we have $\Delta \neq 0$, and we express the solutions of (5.2.5a) and (5.2.5b) for s and τ in the form

$$s = S(x, y), \quad \tau = T(x, y). \tag{5.2.9}$$

Substituting these into (5.2.5c) defines the solution surface in the form

$$u = U(S(x, y),\ T(x, y)) \equiv \phi(x, y). \tag{5.2.10}$$

The result (5.2.10) is available whenever $\Delta \neq 0$, and we can prove the following theorem (for instance, see Chapter II, Section 1.1, of [13]). Every integral surface is generated (in the sense just discussed) by a one-parameter family of characteristic curves. Conversely, every one-parameter family of characteristic curves generates an integral surface.

If $\Delta(0, \tau) = 0$, we have two possibilities: Either the initial curve is a characteristic curve, in which case an infinite number of solutions pass through $\mathcal{C}_0$, or $\mathcal{C}_0$ is not characteristic, and it is not possible to calculate a continuously differentiable solution. We now illustrate these features by studying different initial conditions for the inviscid Burgers' equation ($\epsilon = 0$) with a constant source term [see (5.1.18b) with $t \to y$]

$$uu_x + u_y = 1. \tag{5.2.11}$$

This equation is also discussed in [13]. It is ideal for purposes of illustration, as one can readily construct explicit solutions that exhibit the various features of quasilinear equations.

The characteristic differential equations for (5.2.11) are

$$\frac{dx}{ds} = u, \quad \frac{dy}{ds} = 1, \quad \frac{du}{ds} = 1. \tag{5.2.12}$$

Solving these subject to an unspecified initial curve $\mathcal{C}_0$: $x_0(\tau)$, $y_0(\tau)$, $u_0(\tau)$, we obtain

$$x = \frac{s^2}{2} + su_0(\tau) + x_0(\tau) \equiv X(s, \tau), \tag{5.2.13a}$$

$$y = s + y_0(\tau) \equiv Y(s, \tau), \tag{5.2.13b}$$

$$u = s + u_0(\tau) \equiv U(s, \tau). \tag{5.2.13c}$$

Note that at $s = 0$ we have $x = x_0$, $y = y_0$, and $u = u_0$, as required by (5.2.7).

The Jacobian of the transformation (5.2.13a)–(5.2.13b) is

$$\Delta(s, \tau) \equiv [s + u_0(\tau)]y_0'(\tau) - [su_0'(\tau) + x_0'(\tau)], \tag{5.2.14}$$

and this may vanish at $s = 0$ or along some other curve, depending on the choice of initial data.

(i) $\Delta(0, \tau) \neq 0$

To illustrate a case where a unique strict solution exists near $\mathcal{C}_0$, let

$$x_0(\tau) = \tau, \quad y_0(\tau) = \tau, \quad u_0(\tau) = 2. \tag{5.2.15a}$$

Another way of expressing the initial condition is to say that

$$u(x, x) = 2. \tag{5.2.15b}$$

We then calculate $\Delta(s, \tau) = s + 1$ from (5.2.14), and this does not vanish at $s = 0$. We therefore expect a strict solution to exist for $s > -1$. The difficulty at $s = -1$ will become clear when we derive the details of the solution.

The one-parameter family of characteristic curves (5.2.13) for this case is given by

$$x = \frac{s^2}{2} + 2s + \tau, \tag{5.2.16a}$$

$$y = s + \tau, \tag{5.2.16b}$$

$$u = s + 2. \tag{5.2.16c}$$

Consider first the characteristic ground curves defined by (5.2.16a)–(5.2.16b). Eliminating s gives the one-parameter family of parabolas defined by

$$F(x, y, \tau) \equiv x + 2 - \tau - \frac{1}{2}(y + 2 - \tau)^2 = 0, \tag{5.2.17}$$

which are sketched in Figure 5.3. To generate the family, we just translate the parabola $x = y^2/2$ corresponding to $\tau = 2$ in the direction $x - y =$ constant > 0. It is easy to show that these parabolas have an envelope along the straight line $y = x + \frac{1}{2}$, which, in fact, corresponds to $\Delta(s, \tau) = 0$, that is, $s = -1$.

Recall that if a one-parameter family of curves defined in the implicit form $F(x, y, \tau) = 0$ has an envelope, then this envelope is defined by eliminating τ from the two expressions $F(x, y, \tau) = 0$ and $F_\tau(x, y, \tau) = 0$. For (5.2.17) we compute $F_\tau = 1 + y - \tau$, and setting this equal to zero gives $\tau = 1 + y$. When we use this expression for τ in (5.2.17), we obtain $x - y + \frac{1}{2} = 0$, the envelope. We can also calculate this result by setting $\Delta(s, \tau) = 0$. In our case, $\Delta(s, \tau) = s + 1$. Thus, $s = -1$ is where the transformation $(s, \tau) \leftrightarrow (x, y)$ breaks down. Now setting $s = -1$ in (5.2.16a)—(5.2.16b) gives the parametric representation $x = -\frac{3}{2} + \tau$, $y = -1 + \tau$ or $x - y + \frac{1}{2} = 0$ again.

To see what happens at $s = -1$, we eliminate τ from (5.2.16a)–(5.2.16b). This gives the quadratic expression $s^2 + 2s + 2(y - x) = 0$ for s. The root that corresponds to $y = x$ at $s = 0$ is $s = -1 + \sqrt{1 + 2(x - y)}$. Therefore, the solution is given by

$$u = 1 + \sqrt{1 + 2(x - y)}. \tag{5.2.18}$$

Thus, u is a constant along the lines $x - y =$ constant. The radical in (5.2.18) vanishes along the line $y = x + \frac{1}{2}$, that is, $s = -1$, and the solution does not exist to the left of this line. Moreover, u_x and u_y become infinite along this line, which represents an edge of regression. In fact, we might be tempted to claim the existence of a second branch for u corresponding to the negative sign in front of the radical. This branch gives the mirror image of the surface $u(x, y)$ relative to the plane $u = 1$. This claim is, however, entirely inconsistent; it would result

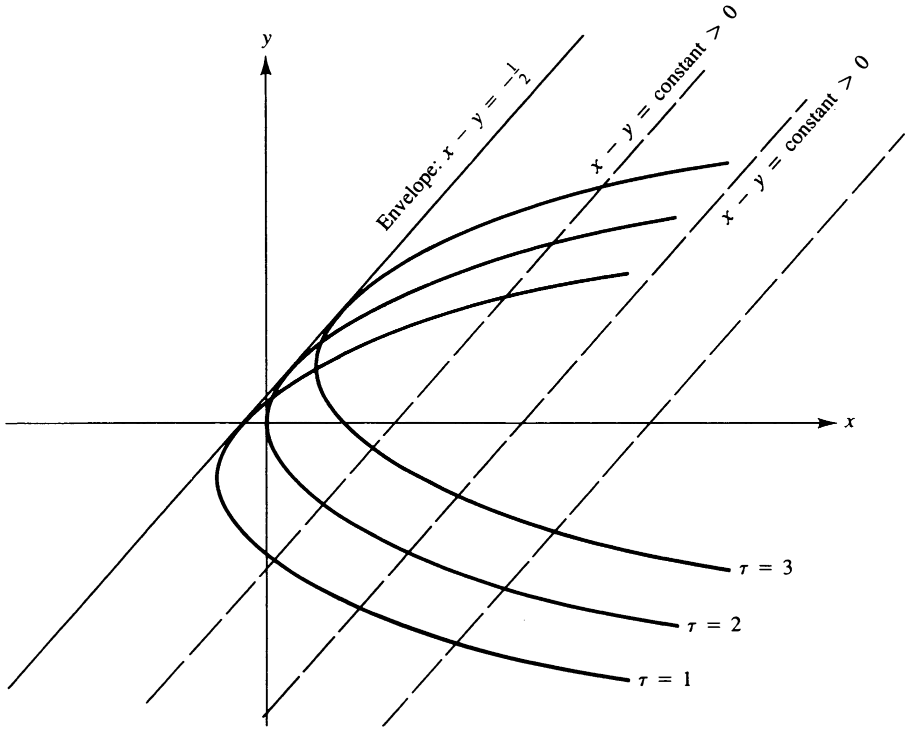

FIGURE 5.3. Characteristic ground curves and their envelope

in a two-valued solution with the lower branch not satisfying the correct initial condition.

Notice that the curve defined by $\Delta(s, \tau) = 0$, $u = 1$, that is, the straight line $y = x + \frac{1}{2}$, $u = 1$, *is not a characteristic curve*; its projection on the $u = 0$ plane is the envelope of characteristic ground curves.

(ii) $\Delta(0, \tau) = 0$, *C_0 is not characteristic*

The difficulty encountered at $s = -1$ in the previous example occurs on the initial curve for the following choice of C_0:

$$x_0 = \tau^2, \quad y_0 = 2\tau, \quad u_0 = \tau. \tag{5.2.19}$$

It is easily seen that $\Delta(0, \tau) = 0$. In order for the initial curve to be characteristic, it must satisfy $dx_0/d\tau = u_0(\tau)$, $dy_0/d\tau = 1$, and $du_0/d\tau = 1$, which it does not. We consider the characteristic curves

$$x = \frac{s^2}{2} + s\tau + \tau^2, \tag{5.2.20a}$$

$$y = s + 2\tau, \tag{5.2.20b}$$

$$u = s + \tau, \tag{5.2.20c}$$

and eliminate s from (5.2.20a)–(5.2.20b) to obtain the one-parameter family of parabolas defined by

$$F(x, y, \tau) \equiv x - \frac{\tau^2}{2} - \frac{1}{2}(y - \tau)^2 = 0. \tag{5.2.21}$$

Again, each member of this family is obtained by translating the parabola $x - y^2/2 = 0$, as indicated in (5.2.21). The family (5.2.21) has the envelope $x - y^2/4 = 0$ (obtained by eliminating τ from (5.2.21) and from the expression for $F_\tau = 0$). Thus, the projection of the initial curve on the xy-plane is now the envelope of characteristic ground curves.

If we calculate a formal expression for $u(x, y)$ from (5.2.20), we obtain $u = y/2 \pm (x - y^2/4)^{1/2}$. Therefore, the initial curve is an edge of regression from which the two branches of the radical define two surfaces over the domain $x > y^2/4$. Again, *this is not a solution* in the strict sense, as u_x and u_y are infinite on the initial curve and u is two-valued for $x > y^2/4$. For the linear equation (5.2.1) with coefficients a, b, c that do not depend on u, the case $\Delta = 0$ on a noncharacteristic curve C means that the characteristic equations (5.2.3) define a vertical surface through C (see Problem 5.2.5).

(iii) $\Delta(0, \tau) = 0$, *C_0 is a characteristic*

Now we consider the initial curve

$$x_0 = \tau^2/2, \quad y_0 = \tau, \quad u_0 = \tau,$$

and obtain $\Delta(0, \tau) = 0$. We also verify that $dx_0/d\tau = \tau = u_0$, $dy_0/d\tau = 1$, and $du_0/d\tau = 1$. Therefore, the initial curve is a characteristic. In fact, the one-parameter family (5.2.13) now has the degenerate form

$$x = \frac{(s + \tau)^2}{2}, \quad y = s + \tau, \quad u = s + \tau, \tag{5.2.22}$$

in terms of the *single* parameter $\sigma \equiv s + \tau$, which implies that no unique surface $u(x, y)$ is possible. For example, any algebraic relation $w(u - y) + u^2/2 - x = 0$ that can be solved for u defines a solution surface $u(x, y)$ implicitly. Here w is an arbitrary function of a single variable with $w(0) = 0$.

(iv) *Steepening of a wave*

In Section 5.1.2 we showed that (5.1.18b) describes the evolution of traffic density for an idealized model. Consider the limiting case $\epsilon = 0$ (drivers do not react to density gradients), for which we have

$$u_t + uu_x = 0, \tag{5.2.23a}$$

and let the initial condition be

$$u(x,0) = \begin{cases} \cos 2\pi x & \text{if } 0 \le x \le 1, \\ 1 & \text{if } x \le 0 \text{ or } x \ge 1. \end{cases} \tag{5.2.23b}$$

Note that $u(x,0)$ and $u_x(x,0)$ are both continuous on $-\infty < x < \infty$. According to (5.1.17), $u = 1$ corresponds to zero density, whereas $u = -1$ has density equal to the maximum value at which there is zero flux. Therefore, the initial condition (5.2.23b) represents an isolated initial distribution of traffic over the unit interval with a peak density ($u = -1$) at $x = \frac{1}{2}$ and no traffic initially outside this interval.

We parameterize (5.2.23b) to read

$$x_0 = \tau, \tag{5.2.24a}$$

$$t_0 = 0, \tag{5.2.24b}$$

$$u_0 = \begin{cases} \cos 2\pi\tau & \text{if } 0 \le \tau \le 1, \\ 1 & \text{if } \tau \le 0 \text{ or } \tau \ge 1, \end{cases} \tag{5.2.24c}$$

and obtain the characteristic curves

$$x = u_0(\tau)s + \tau, \; t = s, \; u = u_0(\tau). \tag{5.2.25}$$

The solution for u can be expressed in implicit form as

$$u = u_0(x - ut), \tag{5.2.26}$$

where u_0 is the function defined by (5.2.24c). The characteristic ground curves are the one-parameter family of straight lines $x = u_0(\tau)t + \tau$, on which u remains a constant equal to its initial value $u_0(\tau)$. The slope dx/dt of each characteristic ground curve equals u_0, and this varies over the unit interval. The resulting pattern is shown in Figure 5.4, where an envelope is clearly defined for sufficiently large t.

We can calculate the envelope of characteristic ground curves either by setting $\Delta(s,\tau) = 0$ or by eliminating τ between $F(x,t,\tau) \equiv u_0(\tau)t + \tau - x = 0$ and $F_\tau = 0$. Let us take the second approach: Setting $F_\tau = 0$ gives $\tau = (1/2\pi)\sin^{-1}(1/2\pi t)$. Therefore, the envelope first forms at $t = 1/2\pi$. Setting $\tau = (1/2\pi)\sin^{-1}(1/2\pi t)$ in $F = 0$ gives two branches defined for $t \ge 1/2\pi$. One branch has

$$x = x_R(t) \equiv \left(t^2 - \frac{1}{4\pi^2}\right)^{1/2} + \frac{1}{2\pi}\sin^{-1}\left(\frac{1}{2\pi t}\right), \quad \frac{1}{4} \le x < \infty, \tag{5.2.27a}$$

with $0 < \sin^{-1}(1/2\pi t) \le \pi/2$, and defines the envelope of the characteristic ground curves emerging from $0 \le x \le \frac{1}{4}$, $t = 0$. The second branch is given by

$$x = x_L(t) \equiv -\left(t^2 - \frac{1}{4\pi^2}\right)^{1/2} + \frac{1}{2\pi}\sin^{-1}\left(\frac{1}{2\pi t}\right), \quad -\infty < x \le \frac{1}{4}, \tag{5.2.27b}$$

with $\pi/2 \le \sin^{-1}(1/2\pi t) < \pi$, and this branch defines the envelope of the characteristic ground curves emerging from $\frac{1}{4} \le x \le \frac{1}{2}$, $t = 0$. The second

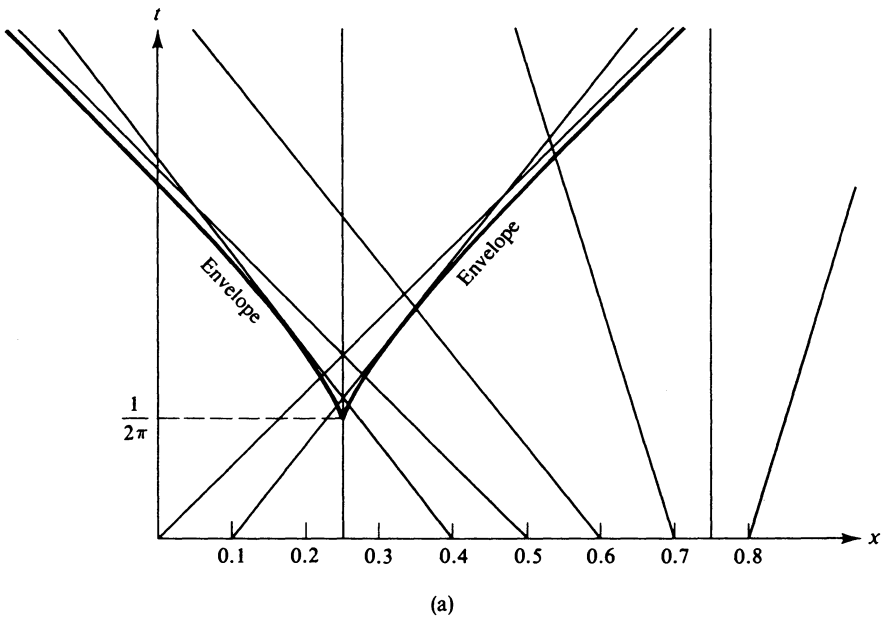

FIGURE 5.4. Characteristic ground curves for (5.2.23)

branch is just the reflection of the first branch with respect to the line $x = \frac{1}{4}$. We note that the two branches form a cusp at $x = \frac{1}{4}$, $t = 1/\pi$; that is, $(dx_R/dt) = (dx_L/dt) = 0$ there. The first branch approaches the characteristic ground curve $\tau = 0$, that is, $x = t$ as $t \to \infty$, whereas the second branch approaches the characteristic ground curve $\tau = \frac{1}{2}$, that is, $x = \frac{1}{2} - t$ as $t \to \infty$. The expression for $u(x, t)$ that results from (5.2.25) is single-valued in the domain of the xt-plane below the two envelope curves, whereas above these curves u is three-valued.

The "solution" (5.2.26) for u as a function of x is sketched in Figure 5.5 for the times $t = 0, 0.1, 1/2\pi$, and 0.4. The points A, B, C, D, and E on the initial profile locate the values of u at $x = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, and 1. These same phases are located by primes for $t = 0.1$, double primes for $t = 1/2\pi$, and triple primes for $t = 0.4$. Since the characteristic ground curves that originate over the interval $0 \leq x \leq \frac{1}{2}$ converge, the initial profile ABC over this subinterval steepens, and at $t = 1/2\pi$, u_x has an infinite slope at B, that is, $x = \frac{1}{4}$. For values of $t > 1/2\pi$, the "solution" is triple-valued in the subinterval $x_L < x < x_R$ spanned by the envelope and is therefore undefined. In the next section, we shall see that a *weak solution*, where a discontinuity in u is allowed, can be constructed for all $t \geq 1/2\pi$. Such a solution will have a stationary discontinuity along $x = \frac{1}{4}$, $t \geq 1/2\pi$ preventing the crossing of the characteristics originating from $x > \frac{1}{4}$ with those originating

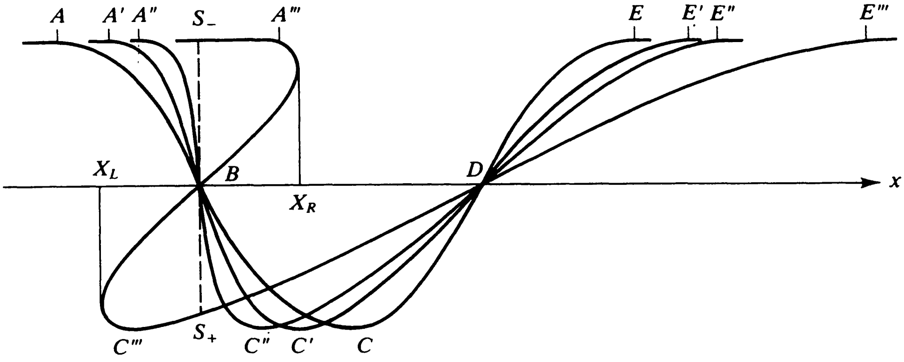

FIGURE 5.5. u as a function of x for various times

from $x < \frac{1}{4}$. Thus, the weak solution will consist of the profile to the left of S_- (if $u > 0$) and to the right of S_+ (if $u < 0$) in Figure 5.5.

The steepening of the profile ABC and the flattening of the profile CDE with time is physically consistent with the traffic behavior we have postulated. In particular, whenever the density is less than $\rho_{\max}$, there is a flow of vehicles to the right. This causes the cars behind the point of maximum initial density to tend to pile up, so the point at which $\rho = \rho_{\max}$ moves to the left. Conversely, the cars initially located in the subinterval $\frac{1}{2} \leq x \leq 1$ gradually spread out over a larger and larger interval, so the initial profile flattens out. We shall comment further on the behavior of the solution for $t > 1/2\pi$ after we have discussed shocks in the next section.

The preceding phenomena are not restricted to (5.1.23a) and carry over to the general quasilinear equation (5.1.13) resulting from the conservation law (5.1.11). The signal speed c given by (5.2.12b) is the characteristic speed at which the initial data are locally propagated. Also, since for $\lambda = 0$ $d\rho/ds = -\Phi_x$ along a given characteristic, we see that the initial value of ρ remains constant if $\Phi_x = 0$; it increases if $\Phi_x < 0$ and decreases if $\Phi_x > 0$. Now, if $\partial c/\partial \rho > 0$, characteristics locally tend to bunch up (hence waves steepen) if $\rho_x < 0$; they spread out (and waves flatten) if $\rho_x > 0$. The opposite behavior occurs if $\partial c/\partial \rho < 0$.

Problems

5.2.1 Consider the two-parameter family of ellipses

$$\frac{1}{4}(x - x_0)^2 + (y - y_0)^2 = 1, \qquad (5.2.28)$$

which may be written in the form

$$x = x_0 + 2\cos\tau, \quad y = y_0 + \sin\tau. \tag{5.2.29}$$

Show that in order for these ellipses to form an envelope on the unit circle $x^2 + y^2 = 1$, we must have

$$x_0 = \pm\frac{1}{\sqrt{1 + 4\tan^2\tau}} - 2\cos\tau, \tag{5.2.30a}$$

$$y_0 = \pm\frac{2\tan\tau}{\sqrt{1 + 4\tan^2\tau}} - \sin\tau. \tag{5.2.30b}$$

Describe these one-parameter subfamilies geometrically.

5.2.2 Calculate the solution of

$$u_x + u_y = u^2 \tag{5.2.31}$$

passing through the curve $u = x$ on $y = -x$ and show that this solution becomes infinite along the hyperbola $x^2 - y^2 = 4$. What is the significance of this hyperbola?

5.2.3 Show that for any given one-parameter family of smooth curves

$$x = X(s,\tau), \quad y = Y(s,\tau), \quad u = U(s,\tau) \tag{5.2.32}$$

for which (5.2.8) does not vanish in some region, we may associate a *linear* first-order partial differential equation

$$a(x,y)u_x + b(x,y)u_y = c(x,y), \tag{5.2.33}$$

such that the function $u(x, y)$ obtained from (5.2.32) solves (5.2.33). Thus, given the solution $u(x, y)$ of a *quasilinear* equation, we can always interpret this as the solution of another linear equation.

Specialize your results to the example (5.2.18) and show that one can interpret this as the solution of

$$\left(1 + \sqrt{1 + 2(x - y)}\right)u_x + u_y = 1. \tag{5.2.34}$$

5.2.4 Solve (5.2.11) for the following initial conditions:

$$u(x,0) = x, \tag{5.2.35}$$

$$u(x,0) = x^2. \tag{5.2.36}$$

In each case, discuss where the solution breaks down and the nature of the singularity there.

5.2.5 Consider the special case of (5.2.1)

$$\tilde{a}(x,y)u_x + \tilde{b}(x,y)u_y = c(x,y,u), \tag{5.2.37}$$

with given continuously differentiable coefficients $\tilde{a}$, $\tilde{b}$, c, where $\tilde{a}$ and $\tilde{b}$ do not depend on u. Assume also that $\tilde{a}$ and $\tilde{b}$ do not vanish simultaneously in the domain of interest.

a. Show that the characteristic ground curves are independent of the initial data and depend only on $\tilde{a}$ and $\tilde{b}$. Therefore, they define a one-parameter family of nonintersecting curves with no singular points in the domain of interest.

b. Specialize your results in part (a) to the case $\tilde{a} = -y$, $\tilde{b} = x$, $c = -u + 1$, and assume that the projection of the initial curve on the xy-plane is the positive x-axis. What is the largest domain in the xy-plane over which a solution of (5.2.37) can be found? Calculate this solution for the initial curve $u(x, 0) = \sin x$ for $x > 0$.

c. Now assume that the solution of the characteristic equations (5.2.3) for (5.2.37) have been found in the form (5.2.5) and that these functions satisfy (5.2.7) for a certain initial curve C_0 : $x_0(\tau)$, $y_0(\tau)$, $u_0(\tau)$ for which $\Delta(0, \tau) = 0$. Assume also that $Y_s U_\tau - U_s Y_\tau \neq 0$ on C_0 so that one can solve the system (5.2.5) for x as a function of y and u in the form

$$x = f(y, u). \tag{5.2.38}$$

We wish to prove that f is actually independent of u—that is, that u is a *vertical* surface through C_0 in this case. Show first that

$$\frac{\partial \Delta(s, \tau)}{\partial s} = [\tilde{a}_x(X, Y) + \tilde{b}_y(X, Y)]\Delta(s, \tau). \tag{5.2.39}$$

Therefore, $\Delta(s, \tau) \equiv 0$ if $\Delta(0, \tau) = 0$. Next, show that

$$\Delta(s, \tau) = (U_s Y_\tau - Y_s U_\tau)\frac{\partial f}{\partial u}(Y, U) \tag{5.2.40}$$

when (5.2.38) is used. Therefore, $\partial f/\partial u = 0$; that is, f depends only on y.

d. Given a particular example of the situation described in part (c).

5.2.6 Consider (5.2.23a) with the smooth initial condition

$$u(x, 0) = \begin{cases} 1 & \text{if } x \leq 0, \\ \dfrac{1}{1 + x} & \text{if } x \geq 0. \end{cases} \tag{5.2.41}$$

a. Calculate the characteristic ground curves and show that these have the envelope

$$t = \frac{1}{4}(x + 1)^2, \quad 1 \leq x < \infty. \tag{5.2.42}$$

b. Show that a strict solution exists *outside* the wedge-shaped domain bounded by (5.2.42) and the line $t = x$, $1 \leq x < \infty$, and this solution is given by

$$u(x, t) = \begin{cases} 1 \text{ if } t \geq x, 0 \leq x \leq 1, \\ 1 \text{ if } t > \frac{1}{4}(x + 1)^2, 1 \leq x < \infty, \\ \dfrac{2}{x + 1 + \sqrt{(x + 1)^2 - 4t}} \text{ if } t < x, 0 \leq x < \infty. \end{cases} \tag{5.2.43}$$

5.2.7 Calculate the solution of

$$tu_x + xu_t = -u, \tag{5.2.44a}$$
$$u(0, t) = 1, \ t > 0,$$

for all $t > 0$ and $0 \leq x < t$.

5.2.8 Consider the linear advection equation

$$u_t + \frac{1}{1 + \epsilon \cos x} u_x = 0, \tag{5.2.45}$$

where ϵ is a constant $0 < \epsilon < 1$.

a. Solve the signaling problem

$$u(x, 0) = 0, \ \ u(0, t) = f(t), \ \ 0 < t < \infty, \tag{5.2.46}$$

explicitly in $0 \leq x < \infty, 0 \leq t < \infty$, in the form

$$u(x, t) = f(t - x - \epsilon \sin x). \tag{5.2.47}$$

b. Solve the initial-value problem

$$u(x, 0) = g(x), \ \ -\infty < x < \infty, \tag{5.2.48}$$

implicitly in $-\infty < x < \infty, 0 \leq t < \infty$, in the form

$$u = g(\xi), \ \ x - t + \epsilon \sin x - \xi - \epsilon \sin \xi = 0. \tag{5.2.49}$$

c. Show that for $0 < \epsilon \ll 1$, the solution in part (b) has the expansion

$$u(x, t) = g(x - t) + \epsilon g'(x - t)[\sin x - \sin(x - t)] + O(\epsilon^2). \tag{5.2.50}$$

5.3 Weak Solutions; Shocks, Fans, and Interfaces

In many physical applications, strict solutions, as postulated in the derivations in the previous sections, are not possible; discontinuities may arise either in the initial data or because the coefficients of the governing equations are discontinuous as, for example, at the interface between two different media. We have also seen that for quasilinear problems, even if the differential equation and initial data are smooth, solutions may steepen and "break," and we should look for a description involving discontinuities when this occurs.

It is evident from the derivations of all the integral conservation laws we have studied so far in this book that these remain valid if the dependent variable or its partial derivatives with respect to x and t are discontinuous at some point in the interval $x_1 \leq x \leq x_2$. For example, the scalar conservation law (5.1.11) still holds if ρ is discontinuous. In this section we shall study how such discontinuities propagate in a solution framework that does not require u, u_x, and u_t to be continuous everywhere.

5.3.1 Shock Speed for a System of Integral Conservation Laws

Here and in the next three subsections we broaden our scope to the system of integral conservation laws (5.1.1) because the derivations are not altered significantly.

$$\frac{d}{dt}\int_{x_1}^{x_2} \Psi_i(\mathbf{u}(x,t),x,t)dx = \Phi_i(\mathbf{u}(x_1,t),x_1,t)$$
$$-\Phi_i(\mathbf{u}(x_2,t),x_2,t) + \int_{x_1}^{x_2} \lambda_i(\mathbf{u}(x,t),x,t)dx,\ i = 1,\ldots,n. \quad (5.3.1a)$$

Henceforth, we shall also omit stating in every case that $i = 1, \ldots, n$ in our results. We assume that the Ψ_i, Φ_i, and λ_i are well-behaved functions of their arguments. For the time being, we regard these functions as continuous and as having continuous first partial derivatives with respect to the u_i, x, t for all $t \geq 0$ and all $x : x_1 \leq x \leq x_2$. This assumption will be relaxed when we consider the behavior of solutions at an interface.

If the u_i are smooth functions of x and t, that is, they are continuous and have continuous first partial derivatives with respect to x and t, then (5.3.1a) implies that they obey the system of *divergence relations* (5.1.2)

$$\mathbf{p}_t + \mathbf{q}_x = \boldsymbol{\lambda}. \quad (5.3.1b)$$

We have argued that the system of integral conservation laws (5.3.1a) remains valid—that is, physically consistent—even if the u_i, or $\partial u_i/\partial t$, $\partial u_i/\partial x$ have discontinuities. In fact, it is this system of integral conservation laws that gives the basic problem description; the divergence relations (5.3.1b) as well as the system of partial differential equations (5.1.9) that results from simplifying these divergence relations are valid only for strict solutions. The implications of this point are considered in detail in Section 5.3.3. Here we wish to explore what restrictions, if any, are implied by (5.3.1a) on possible discontinuities in the solution.

To be more specific, assume that a curve Γ_s, defined by $x = \xi(t)$ in the xt-plane, divides the domain of interest $D : x_1 \leq x \leq x_2, 0 \leq t \leq T$ into two subdomains, $D_1 : x_1 \leq x < \xi(t)$ and $D_2 : \xi(t) < x \leq x_2$, as shown in Figure 5.6. Also assume that the u_i, $\partial u_i/\partial x$, and $\partial u_i/\partial t$ are continuous in D_1 and D_2. Thus, Γ_s is a locus of possible discontinuities in the u_i, $\partial u_i/\partial x$, or $\partial u_i/\partial t$, and we shall also refer to Γ_s as a *shock* for the case where the u_i are discontinuous. The question is, what information concerning $\xi(t)$ can we draw from (5.3.1a)?

To this end we approximate (5.3.1a) in the limit as x_1 and x_2 are taken sufficiently close to ξ. We split the interval of integration for the term on the left-hand side of (5.3.1a) into the two subintervals (x_1, ξ) and (ξ, x_2). If we then regard $(x_1 - \xi)$ and $(\xi - x_1)$ as being small, we obtain

$$\int_{x_1}^{x_2} \Psi_i(\mathbf{u},x,t)dx = \int_{x_1}^{\xi(t)} \Psi_i(\mathbf{u},x,t)dx + \int_{\xi(t)}^{x_2} \Psi_i(\mathbf{u},x,t)dx$$
$$= \Psi_i^-(\xi,t)(\xi(t) - x_1) + \Psi_i^+(\xi,t)(x_2 - \xi(t))$$

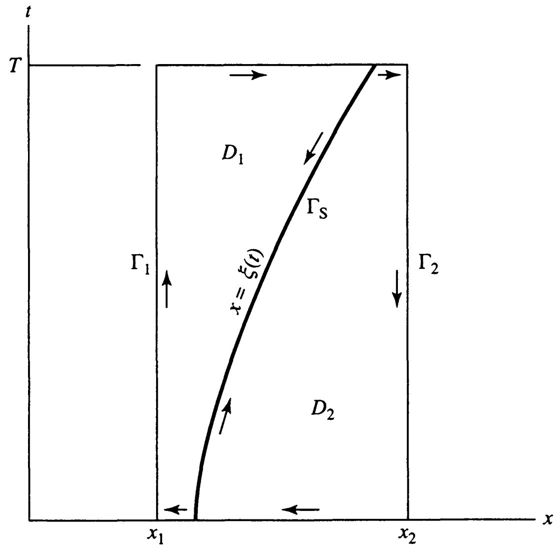

FIGURE 5.6. Domain D containing a shock Γ_s

$$+ O((\xi - x_1)^2) + O((x_2 - \xi)^2), \tag{5.3.2}$$

where we have used the notation

$$\Psi_i^-(\xi, t) \equiv \Psi_i(\mathbf{u}(\xi^-(t), t), \xi^-(t), t), \tag{5.3.3a}$$

$$\Psi_i^+(\xi, t) \equiv \Psi_i(\mathbf{u}(\xi^+(t), t), \xi^+(t), t). \tag{5.3.3b}$$

Differentiating (5.3.2) with respect to t gives

$$\frac{d}{dt}\int_{x_1}^{x_2} \Psi_i(\mathbf{u}, x, t)dx = -\dot{\xi}(t)[\Psi_i] + O(\xi - x_1) + O(x_2 - \xi), \tag{5.3.4}$$

and the [ ] notation means

$$[\Psi_i] \equiv \Psi_i^+(\xi, t) - \Psi_i^-(\xi, t). \tag{5.3.5}$$

The right-hand side of (5.3.1a) contributes only $-[\Phi_j]$ if terms of order $(\xi - x_1)$ and $(x_2 - \xi)$ are neglected. Therefore, in the limit as $x_1 \uparrow \xi$ and $x_2 \downarrow \xi$, we have

$$\dot{\xi}(t)[\Psi_i] = [\Phi_i]. \tag{5.3.6}$$

This condition defines the shock speed $\dot{\xi}$ in terms of the values of the solution on either side. Illustrative examples are discussed later.

For the special case where the Ψ_i and Φ_i are linear in the u_i, we express these in the form

$$\Psi_i = \sum_{j=1}^{n} \overline{A}_{ij}(x, t)u_j, \quad \Phi_i = \sum_{j=1}^{n} \overline{B}_{ij}(x, t)u_j, \tag{5.3.7}$$

where the elements of the matrices $\{\overline{A}_{ij}\}$ and $\{\overline{B}_{ij}\}$ are continuous and have continuous partial derivatives with respect to x and t. Equation (5.3.6) implies that

$$\sum_{j=1}^{n}\{\dot{\xi}(t)\overline{A}_{ij}(x,t) - \overline{B}_{ij}(x,t)\}[u_j] = 0. \tag{5.3.8}$$

In order that (5.3.8) hold, either (1) solutions are continuous, that is, $[u_j] = 0$ for each $j = 1, \ldots, n$, or (2) if the vector with components $[u_1], [u_2], \ldots, [u_n]$ is nonzero, the coefficient matrix in (5.3.8) has a zero determinant. This latter case gives precisely the condition that $x = \xi(t)$ is a characteristic curve for the linear system

$$\sum_{j=1}^{n}\left\{\overline{A}_{ij}\frac{\partial u_j}{\partial t} + \overline{B}_{ij}\frac{\partial u_j}{\partial x}\right\} + \ldots = 0, \tag{5.3.9}$$

associated with (5.3.1b) for this case (see Problem 4.3.7).

The preceding discussion confirms our observation, based on the numerical solution of (5.3.9) by the method of characteristics in Chapter 4, that discontinuities propagate only along characteristics for the linear problem. Specializing further to the scalar linear equation

$$a(x,t)u_t + b(x,t)u_x = c(x,t)u, \tag{5.3.10}$$

we see that for $[u] \neq 0$, (5.3.8) simply defines the characteristic ground curves as loci of discontinuity in u. Moreover, in the linear case, such discontinuities occur only if the initial data are prescribed to be discontinuous; smooth initial data cannot lead to discontinuous solutions as in the quasilinear problem.

5.3.2 Formal Definition of a Weak Solution

We now present a formal definition of a *weak solution* of the system of divergence relations (5.3.1b).

To begin with, consider only a strict solution of (5.3.1b) in some domain D of the xt-plane with boundary Γ. Let $\zeta(x,t)$ be any function of x and t that vanishes on Γ. Now, since (5.3.1b) holds *everywhere* in D, we have

$$\iint_D \zeta\left(\frac{\partial \Psi_i}{\partial t} + \frac{\partial \Phi_i}{\partial x} - \lambda_i\right) dx\, dt = 0. \tag{5.3.11a}$$

We can write (5.3.11a) as

$$\iint_D \{(\zeta\Psi_i)_t + (\zeta\Phi_i)_x\} dx\, dt = \iint_D (\zeta_t\Psi_i + \zeta_x\Phi_i + \zeta\lambda_i) dx\, dt, \tag{5.3.11b}$$

and since ζ, Ψ_i, and Φ_i are continuous and have continuous x and t derivatives, the left-hand side of (5.3.11b) can be expressed as a contour integral over Γ, using the two-dimensional Gauss theorem (see (2.3.14)). This contour integral vanishes identically because the integrand has ζ as a factor and $\zeta = 0$ on Γ. Thus, we conclude that for a strict solution of (5.3.1b), the following integral [the right-hand side of (5.3.11b)] must vanish for all smooth functions ζ such that $\zeta = 0$ on Γ:

$$\iint_D (\zeta_t \Psi_i + \zeta_x \Phi_i + \zeta \lambda_i) dx\, dt = 0. \tag{5.3.12}$$

Notice that the integrand in (5.3.12) does not involve any derivatives of Ψ_i or Φ_i; the only derivatives that occur are for ζ. We take advantage of this feature to define a *weak solution* of (5.3.1b) as one that satisfies (5.3.12) in D for any smooth ζ that vanishes on Γ. This definition makes sense even if the u_i, $\partial u_i/\partial t$, and $\partial u_i/\partial x$ have finite discontinuities in D because the integral (5.3.12) remains well-defined. If the u_i, $\partial u_i/\partial t$, and $\partial u_i/\partial x$ are continuous in D, the statement that (5.3.12) holds immediately implies (5.3.1b). Therefore, we have produced a more general definition of a solution, which does not require the smoothness of the u_i. Let us now see whether this definition results in a shock speed formula that is consistent with (5.3.6).

As in Section 5.3.1, we consider a domain D that is divided into two parts D_1 and D_2 by a shock curve $\Gamma_s : x = \xi(t)$. Assume that in each of the domains D_1 and D_2 the solutions of (5.3.1b) are smooth, and denote the boundary of D_1 by $\Gamma + \Gamma_s$ and the boundary of D_2 by $\Gamma_2 + \Gamma_s$, as indicated in Figure 5.6.

Since the solution for the u_i is smooth in each subdomain D_1 and D_2, we have [according to (5.3.11b) and the two-dimensional Gauss theorem]

$$\iint_{D_1} (\zeta_t \Psi_i + \zeta_x \Phi_i + \zeta \lambda_i) dx\, dt = \iint_{D_1} \{(\zeta \Psi_i)_t + (\zeta \Phi_i)_x\} dx\, dt$$
$$= \int_{\Gamma_s} \{\zeta \Psi_i dx - \zeta \phi_i dt\}, \tag{5.3.13a}$$

$$\iint_{D_2} (\zeta_t \Psi_i + \zeta_x \Phi_i + \zeta \lambda_i) dx\, dt = \iint_{D_2} \{(\zeta \Psi_i)_t + (\zeta \Phi_i)_x\} dx\, dt$$
$$= -\int_{\Gamma_s} \{\zeta \Psi_i dx - \zeta \Phi_i dt\}, \tag{5.3.13b}$$

where now the integrals over Γ_s, on which $\zeta \neq 0$, remain. Note that in the right-hand side of (5.3.13b) we have inserted a minus sign in front of the integral over Γ_s. Therefore, the contour integral in (5.3.13b) (before it is multiplied by -1) is evaluated in the same direction along Γ_s as the contour integral in (5.3.13a). Adding (5.3.13a) and (5.3.13b) and using the definition of a weak solution to set equal to zero the sum of the integrals over D_1 and D_2 that results on the left-hand side, we obtain

$$0 = \int_{\Gamma_s} \zeta \{[\Psi_i] dx - [\Phi_i] dt\} = 0, \tag{5.3.14}$$

where [ ] again denotes the difference in values on either side of the shock curve. Since (5.3.14) is true for arbitrary ζ, the integrand must vanish, and we obtain the shock relation (5.3.6) derived earlier.

Up until now, we have assumed that the Ψ_i, Φ_i, and λ_i are smooth functions of their arguments everywhere in D; discontinuities in these functions arose only because the u_i are discontinuous on the shock curve $x = \xi(t)$. This is not true if $x = \xi(t)$ is an *interface* separating the domains D_1 and D_2, where the Ψ_i, Φ_i, and λ_i may have a different functional dependence on the u_i, x, or t. However, our results remain valid as long as the Ψ_i, Φ_i, and λ_i are smooth functions of their arguments in the individual domains D_1 and D_2. This allows us to use (5.3.6) when $x = \xi(t)$ is such an interface, as is illustrated in the next section.

5.3.3 The Correct Shock and Interface Conditions

We have seen in Sections 5.3.1 and 5.3.2 that the physically relevant integral conservation law (5.3.1a) is the starting point for the definition of the shock condition (5.3.6). In this section we shall illustrate the fact that the system of partial differential equations we obtain from (5.3.1b) for strict solutions may be associated with *different divergence relations*. Therefore, without knowledge of the underlying physical principles, if we were given only a system of partial differential equations governing strict solutions, we would be unable to deduce from these the correct conservation laws or shock conditions.

(i) *A scalar problem*

Consider the scalar problem [see (5.1.18)] for traffic flow with $\epsilon = 0$. The integral conservation law is

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)dx = \frac{1}{2}\{u^2(x_1,t) - u^2(x_2,t)\}, \tag{5.3.15a}$$

and this implies the divergence relation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{5.3.15b}$$

for strict solutions. This simplifies to

$$u_t + uu_x = 0. \tag{5.3.15c}$$

Equations (5.3.15a)—(5.3.15c) give three levels of description: The most general, valid for discontinuous solutions, is (5.3.15a). Equation (5.3.15b) is a direct consequence of (5.3.15a) for strict solutions. Equation (5.3.15c) is, in this case, a trivial consequence of (5.3.15c). The shock condition (5.3.6) for (5.3.15a) is

$$\dot{\xi}(t)[u] = \left[\frac{u^2}{2}\right],$$

or

$$\dot{\xi}(u^+ - u^-) = \left\{ \frac{(u^+)^2}{2} - \frac{(u^-)^2}{2} \right\}.$$

Factoring the right-hand side and simplifying gives

$$\dot{\xi} = \frac{1}{2}(u^+ + u^-). \tag{5.3.16}$$

We shall illustrate in later sections how this result can be used to derive a weak solution.

Now, suppose we don't have (5.3.15a) and start from (5.3.15c). The transformation of dependent variable

$$u(x, t) = e^{v(x,t)} \tag{5.3.17}$$

implies that if u is a *strict solution of* (5.3.15c), *then* $v(x, t)$ *is a strict solution of*

$$v_t + e^v v_x = 0 \tag{5.3.18}$$

for u and v related by (5.3.17). This is as far as we can go. If we were to claim that (5.3.18) is a consequence of the divergence relation

$$v_t + (e^v)_x = 0, \tag{5.3.19}$$

we would obtain the shock condition

$$\dot{\xi}[v] = [e^v],$$

or

$$\dot{\xi} = \frac{u^+ - u^-}{\log u^+ - \log u^-}, \tag{5.3.20}$$

which does not agree with (5.3.16). In fact, we could also set $u = v^2$ to obtain $v_t + v^2 v_x = 0$ or $v_t + (v^3/3)_x = 0$ and incorrectly claim

$$\dot{\xi} = \frac{1}{3}\frac{[v^3]}{[v]} = \frac{1}{3}\frac{(u^+)^{3/2} - (u^-)^{3/2}}{(u^+)^{1/2} - (u^-)^{1/2}}, \tag{5.3.21}$$

and so on. We cannot tell what is the correct shock condition if the only information available is (5.3.15c). We need the basic integral conservation law (5.3.15a) in order to select the correct shock condition.

(ii) *Shallow-water waves, the bore conditions*

To illustrate ideas further, consider the integral conservation laws for shallow-water flow [see (3.2.6)] and (3.2.10)]. Mass conservation requires

$$\frac{d}{dt}\int_{x_1}^{x_2} h(x, t)dx = u(x, t)h(x, t)\Bigg|_{x=x_2}^{x=x_1}, \tag{5.3.22a}$$

and momentum conservation requires

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)h(x,t)dx = \{u^2(x,t)h(x,t) + \frac{1}{2}h^2(x,t)\}\Big|_{x=x_2}^{x=x_1}. \quad (5.3.22b)$$

The divergence relations associated with (5.3.22) are

$$h_t + (uh)_x = 0, \quad (5.3.23a)$$

$$(uh)_t + \left(u^2h + \frac{h^2}{2}\right)_x = 0. \quad (5.3.23b)$$

Therefore, the correct shock conditions are

$$\dot{\xi}[h] = [uh], \quad \dot{\xi}[uh] = \left[u^2h + \frac{h^2}{2}\right]. \quad (5.3.24)$$

In this context a shock is called a *bore*.

Observe that for strict solutions, equations (5.3.22) are equivalent to (see (3.2.12))

$$h_t + uh_x + hu_x = 0, \quad (5.3.25a)$$

$$u_t + uu_x + h_x = 0. \quad (5.3.25b)$$

Now, if we were to interpret (5.3.25b) as being the result of the divergence relation

$$u_t + \left(\frac{u^2}{2} + h\right)_x = 0, \quad (5.3.26)$$

we would obtain the physically inconsistent second shock condition

$$\dot{\xi}[u] = \left[\frac{u^2}{2} + h\right]$$

instead of the one in (5.3.24).

A second observation regarding (5.3.25) is in order. We pointed out earlier [see the remarks following (3.2.18)] that for strict solutions, the integral conservation law (5.3.22b) of momentum and the integral conservation law of energy, given by (3.2.17), both lead to (5.3.25b). If we adopt conservation of momentum as the basic law governing discontinuous solutions, then energy *will not be conserved across a bore*. In fact, we shall show in Section 5.3.4 that the shock relations (5.3.24) admit two types of discontinuities characterized by the relative water levels on either side of the discontinuity; these are bores that propagate into regions of either *lower* or *higher* water than found behind the bore. We shall show that in the former case—that is, if the water behind the bore is higher than the water in front—the total energy in some interval $x_1 \leq x \leq x_2$ containing the bore *will decrease with time*. This behavior is selected as being physically realistic because in a dissipative model, the turbulence generated in the bore would tend to decrease the energy. The case of a bore propagating into a region of higher water will be excluded because it implies the physically unrealistic result that the energy increases in the interval $x_1 \leq x \leq x_2$.

(iii) *Inviscid non-heat-conducting gas; the Rankine–Hugoniot conditions; the interface conditions*

A somewhat more involved problem occurs in gas dynamics, and we refer to the dimensional integral conservation laws of mass, momentum, and energy [see (3.3.5) with $\mu = 0, \lambda = 0$]:

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)dx = \rho(x,t)u(x,t)\Bigg|_{x=x_2}^{x=x_1}, \tag{5.3.27a}$$

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)u(x,t)dx = \{\rho(x,t)u^2(x,t) + p(x,t)\}\Bigg|_{x=x_2}^{x=x_1}, \tag{5.3.27b}$$

$$\begin{aligned}\frac{d}{dt}\int_{x_1}^{x_2} &\left\{\frac{1}{2}\rho(x,t)u^2(x,t) + \frac{p(x,t)}{\gamma-1}\right\}dx \\ &= \left\{u(x,t)\left(\frac{1}{2}\rho(x,t)u^2(x,t) + \frac{\gamma}{\gamma-1}p(x,t)\right)\right\}\Bigg|_{x=x_2}^{x=x_1}.\end{aligned} \tag{5.3.27c}$$

In (5.3.27c) we have used the equation of state $p = \rho R\theta$ (see (3.3.3)) to express the temperature in terms of the pressure p and density ρ. We have also used the definitions $R \equiv C_p - C_v$ and $\gamma \equiv C_p/C_v$.

Now, we have three integral conservation laws for the three dependent variables ρ, u, p. For strict solutions, equations (5.3.27) imply the divergence relations

$$\rho_t + (u\rho)_x = 0, \tag{5.3.28a}$$

$$(\rho u)_t + (\rho u^2 + p)_x = 0, \tag{5.3.28b}$$

$$\left\{\rho\frac{u^2}{2} + \frac{p}{\gamma-1}\right\}_t + \left\{u\left(\rho\frac{u^2}{2} + \frac{\gamma}{\gamma-1}p\right)\right\}_x = 0. \tag{5.3.28c}$$

The shock conditions implied by (5.3.27) are

$$\dot{\xi}[\rho] = [u\rho], \tag{5.3.29a}$$

$$\dot{\xi}[\rho u] = [\rho u^2 + p], \tag{5.3.29b}$$

$$\dot{\xi}\left[\rho\frac{u^2}{2} + \frac{p}{\gamma-1}\right] = \left[u\left(\rho\frac{u^2}{2} + \frac{\gamma}{\gamma-1}p\right)\right]. \tag{5.3.29c}$$

An alternative way of writing (5.3.29) is

$$[\rho v] = 0, \tag{5.3.30a}$$

$$[\rho u^2 + p] = 0, \tag{5.3.30b}$$

$$\left[\frac{v^2}{2} + \frac{\gamma}{\gamma-1}\frac{p}{\rho}\right] = 0, \tag{5.3.30c}$$

where $v(t)$ is the flow speed relative to the shock; that is,

$$v^+(t) \equiv u(\xi^+(t), t) - \dot{\xi}(t), \tag{5.3.31a}$$

$$v^-(t) \equiv u(\xi^-(t), t) - \dot{\xi}(t). \tag{5.3.31b}$$

Equations (5.3.30) are the *Rankine–Hugoniot* relations.

For strict solutions, (5.3.28) simplify to yield

$$\rho_t + (\rho u)_x = 0, \tag{5.3.32a}$$

$$u_t + uu_x + \frac{p_x}{\rho} = 0, \tag{5.3.32b}$$

$$\left(\frac{p}{\rho^\gamma}\right)_t + u\left(\frac{p}{\rho^\gamma}\right)_x = 0. \tag{5.3.32c}$$

Equation (5.3.32c) states that (p/ρ^γ) remains constant along particle paths—that is, on curves $dx/dt = u$. Now, since the entropy is a function of (p/ρ^γ), this implies that the entropy remains constant along particle paths, a result that is physically consistent only if the particle path does not cross a shock.

We can establish how the entropy behaves across a shock from the vantage of the more accurate flow description provided by the conservation equations in which μ and λ are retained [see (3.3.5)]. In this description, *all flows for a gas having constant ambient properties are smooth*; an initial discontinuity immediately evolves into a thin region (shock layer) across which the flow changes rapidly. One can then show that the entropy must *increase* downstream of such a shock layer; for example, see Section 6.15 of [42]. We shall explore this question in more detail for Burgers' equation in Section 5.3.6.

(iv) *Interface between two different gases*

Suppose now that we consider two different gases on either side of a diaphragm that is suddenly removed. In order to exhibit a distinct interface $\xi(t)$, we make an assumption that is physically somewhat unrealistic for gases; namely, that the two gases do not mix across the interface. What are the consequences of the conditions (5.3.29) in this context?

If ξ is such an interface, then $u(\xi(t), t)$ must be continuous for all t, and (5.3.29a) gives $\dot{\xi}[\rho] = u(\xi, t)[\rho]$, that is, the intuitively obvious result that the interface moves with the speed $\dot{\xi} = u(\xi, t)$ of the gas on either side. Since $u(\xi, t)$ is continuous and equals $\dot{\xi}$, (5.3.29b) reduces to $[p] = 0$; that is, the interface cannot sustain a pressure difference. These results, when used in (5.3.29c), give an identity. In summary, for two inviscid non-heat-conducting gases at an interface, we must require the speed to be continuous and equal to the interface speed, and we must require the pressure to be continuous. The density (and hence the temperature) and the gas constant γ may be discontinuous.

Consider now the more accurate description where the two gases on either side of the interface are regarded as viscous and heat-conducting. The integral conservation laws of mass, momentum, and energy are given by (3.3.5), and these lead to the following interface conditions:

$$\dot{\xi}[\rho] = [\rho u], \tag{5.3.33a}$$

$$\dot{\xi}[\rho u] = \left[\rho u^2 + p - \frac{4}{3}\mu u_x\right], \tag{5.3.33b}$$

$$\dot{\xi}\left[\rho\left(\frac{u^2}{2} + C_v\theta\right)\right] = \left[\rho u\left(\frac{u^2}{2} + C_v\theta\right) + pu - \frac{4}{3}\mu u u_x - \lambda\theta_x\right]. \tag{5.3.33c}$$

The pressure p, temperature θ, and density ρ are related according to the equation of state (3.3.3). The constants μ, λ, C_p, and C_v are all, in general, different for the two gases.

Again, we must have u continuous at ξ, and (5.3.33a) gives the result

$$\dot{\xi} = u(\xi^+, t) = u(\xi^-, t), \tag{5.3.34a}$$

that the interface moves with the speed of the gas on either side. When this result is used in (5.3.33b), we conclude that

$$[p - \frac{4}{3}\mu u_x] = 0, \tag{5.3.34b}$$

which gives a balance between the pressure and the viscous stress across the interface. Equations (5.3.34a) and (5.3.34b) simplify (5.3.33c) to the physically obvious result

$$[\lambda\theta_x] = 0. \tag{5.3.34c}$$

That is, the heat flux is continuous at the interface.

To calculate a flow with an interface, we must solve the governing equations on either side; these are (3.3.33), (3.3.6a), (3.3.7a), and (3.3.7b). This calculation provides a solution with certain unknown constants, which are determined when the interface conditions (5.3.34) are imposed. Problem (3.3.2) illustrates the ideas for a small disturbance theory for which $|u| \ll 1$; hence the interface is stationary in the first approximation.

5.3.4 Constant-Speed Shocks; Nonuniqueness of Weak Solutions

Solutions for which the dependent variables remain constant on either side of a shock are interesting and simple special cases that provide much insight into the behavior of more complicated situations. We see immediately from (5.3.6) that if the u_i on either side of the shock are constant and if the Ψ_i, Φ_i do not depend on x or t, then the $[\Psi_i]$, $[\Phi_i]$ and hence $\dot{\xi}$ are also all constant. Such special solutions occur either if the initial data are appropriately chosen constants on either side of a point on the x-axis, or if they exhibit an appropriate symmetry.

(i) *The scalar problem; shocks and fans*

Both shocks and fans occur for the scalar example (5.3.15), which we consider next.

First, assume that the initial condition is the piecewise constant function

$$u(x,0) = \begin{cases} u_1 = \text{ constant} & \text{if } x < x_0, \\ u_2 = \text{ constant } < u_1 & \text{if } x > x_0. \end{cases} \tag{5.3.35}$$

Thus, the characteristics emerging from the $x < x_0$ portion of the x-axis all have the same speed u_1, which is *greater* than the speed u_2 of the characteristics emerging from $x > x_0$. These two families of characteristics immediately intersect, and the solution is not defined in the triangular region $x_0 + u_2 t < x < x_0 + u_1 t$, $t > 0$. This is shown in Figure 5.7a, where the arrows indicate the direction of increasing t. For $x > x_0 + u_1 t$, we have $u = u_2 =$ constant, whereas for $x < x_0 + u_2 t$, $u = u_1 =$ constant.

It is clear that a shock must be introduced at the point $x = x_0$, $t = 0$, and the initial speed of this shock follows immediately from the shock condition (5.3.16). We have $\dot{\xi}(0^+) = (u_1 + u_2)/2$. But for this problem, the shock propagates into a region where the values of u on either side of the shock remain constant, and we conclude that $\dot{\xi}(t) =$ constant $= (u_1 + u_2)/2$; that is, $\xi(t) = x_0 + (u_1 + u_2)t/2$. Thus, the shock has a speed equal to the average of the speeds of the characteristics on either side, and the characteristics converge (as t increases) toward the shock. Once the shock is inserted, we exclude the characteristics with speed u_1 to the right of the shock, and those with speed u_2 to the left. In the limit of a weak discontinuity—that is, $u_1 \approx u_2$—we see that $\dot{\xi} \to u$, the characteristic speed.

We also observe that for this initial-value problem, it is not possible to have more than one shock emerging from $(x_0, 0)$. For instance, let us assume the situation depicted in Figure 5.7b, where the two shocks with constant speeds, $\dot{\xi}_1 = (u_1 + u_0)/2$ and $\dot{\xi}_2 = (u_2 + u_0)/2$, bound the triangular domain $\xi_1 < x < \xi_2$, in which $u = u_0 =$ constant. In order to have $\dot{\xi}_2 > \dot{\xi}_1$, we must have $u_2 > u_1$, but this contradicts the original premise that $u_2 < u_1$. It is easily seen that no choice of u_0 leads to a consistent picture. So we conclude that the weak solution

$$u(x,t) = \begin{cases} u_1 = \text{ constant} & \text{if } x < x_0 + (u_1 + u_2)t/2, \\ u_2 = \text{ constant } < u_1 & \text{if } x > x_0 + (u_1 + u_2)t/2, \end{cases} \tag{5.3.36}$$

is unique if $u_2 < u_1$.

Now suppose that we reverse the inequality relating u_1 and u_2 and consider the initial-value problem

$$u(x,0) = \begin{cases} u_1 = \text{ constant} & \text{if } x < x_0, \\ u_2 = \text{ constant } > u_1 & \text{if } x > x_0. \end{cases} \tag{5.3.37}$$

The picture in the xt-plane is shown in Figure 5.8a, where no characteristics enter the triangular region

$$T: \quad x_0 + u_1 t < x < x_0 + u_2 t, \quad t > 0.$$

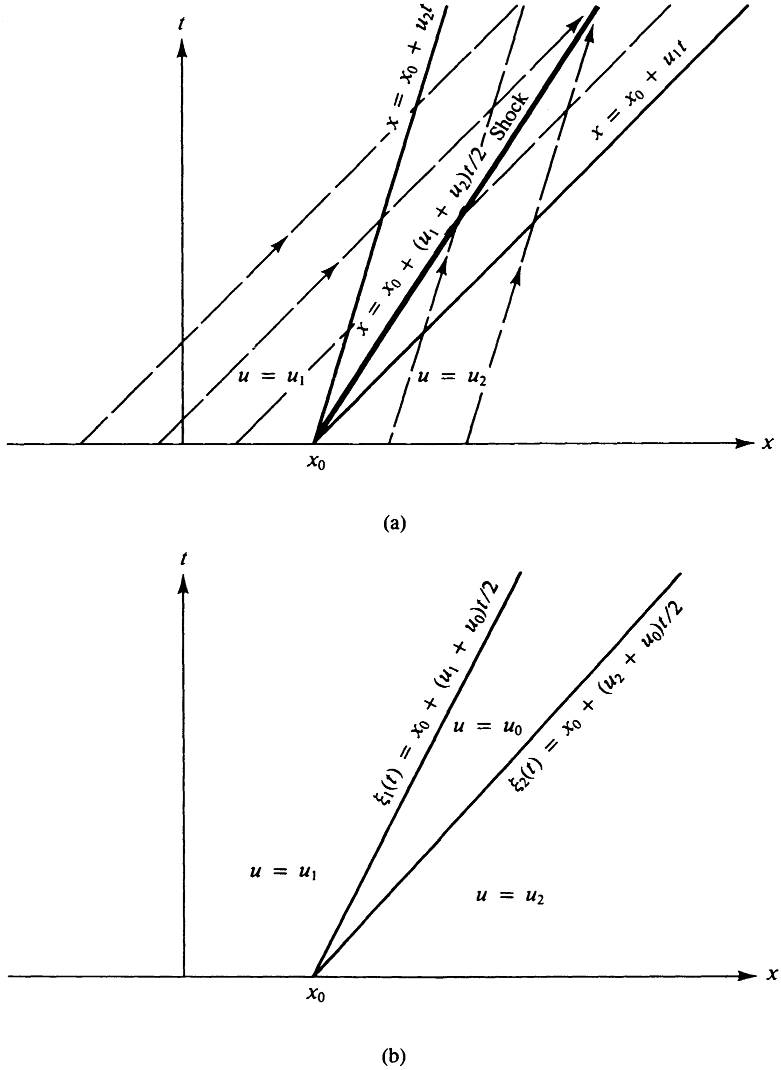

FIGURE 5.7. Piecewise constant initial data for a shock

In this case, there are *infinitely* many possible weak solutions that satisfy the integral conservation law (5.3.15a) and jump condition (5.3.16). We can insert N shocks in T, where $N = 1, 2, \ldots$, without violating (5.3.16). For example, with $N = 1$, we have the situation sketched in Figure 5.8b, where

$$u(x,t) = \begin{cases} u_1 = \text{ constant} & \text{if } x < x_0 + (u_1 + u_2)t/2, \\ u_2 = \text{ constant } > u_1 & \text{if } x > x_0 + (u_1 + u_2)t/2. \end{cases} \tag{5.3.38}$$

This is formally identical to (5.3.26), except that now $u_2 > u_1$. As a result, the characteristics diverge from the shock as t increases. In Figure 5.8c we show the case of $N = 2$. We have the two shocks with speeds $\dot{\xi}_1 = (u_1 + u_0)/2$ and $\dot{\xi}_2 = (u_2 + u_0)/2$ dictated by the *arbitrary* choice of u_0 in the interval $u_1 < u_0 < u_2$. The solution is

$$u(x,t) = \begin{cases} u_1 = \text{ constant} & \text{if } x < x_0 + (u_1 + u_0)t/2, \\ u_0 = \text{ constant} & \text{if } x_0 + (u_1 + u_0)t/2 < x < x_0 + (u_2 + u_0)t/2, \\ u_2 = \text{ constant} & \text{if } x > x_0 + (u_2 + u_0)t/2, \end{cases} \tag{5.3.39}$$

which is perfectly consistent with (5.3.16). We can continue this process for any N.

We conclude that for this case the integral conservation law *does not specify a unique weak solution*; we must invoke some further information to decide what is an *admissible weak solution.*

One point of view is to regard the initial condition (5.3.37) as the limiting case as $\delta \to 0$ of the *smooth initial condition*

$$u(x,0) = \begin{cases} u_1 = \text{ constant} & \text{if } x \le x_0 - \delta, \\ f(x) & \text{if } x_0 - \delta \le x \le x_0 + \delta, \\ u_2 = \text{ constant} & \text{if } x \ge x_0 + \delta, \end{cases} \tag{5.3.40}$$

for some monotone increasing function $f(x)$ with $f(x_0 - \delta) = u_1$, $f'(x_0 - \delta) = 0$, $f(x_0 + \delta) = u_2$, and $f'(x_0 + \delta) = 0$. The characteristics now fan smoothly out of the interval $x_0 - \delta \le x \le x_0 + \delta$, as shown in Figure 5.9a. The solution is strict everywhere and has the parametric form

$$x = \tau + f(\tau)t, \ u = f(\tau), \tag{5.3.41}$$

for $x_0 - \delta \le \tau \le x_0 + \delta$. For $x \le x_0 - \delta + u_1 t$, the solution is $u(x,t) = u_1$, and for $x \ge x_0 + \delta + u_2 t$, the solution is $u(x,t) = u_2$. In the limit as $\delta \to 0$, we obtain the *centered fan* shown in Figure 5.9b, where now $u(x,t) = (x - x_0)/t$ in T. Therefore, we have obtained a weak solution for the initial-value problem (5.3.37) in the form

$$u(x,t) = \begin{cases} u_1 = \text{ constant} & \text{if } x \le x_0 + u_1 t, \\ (x - x_0)/t & \text{if } x_0 + u_1 t \le x \le x_0 + u_2 t, \\ u_2 = \text{ constant} & \text{if } x \ge x_0 + u_2 t. \end{cases} \tag{5.3.42}$$

Note that u is continuous along the rays $t > 0$, $x = x_0 + u_1 t$, and $x = x_0 + u_2 t$, but u_x and u_t are not continuous on these rays.

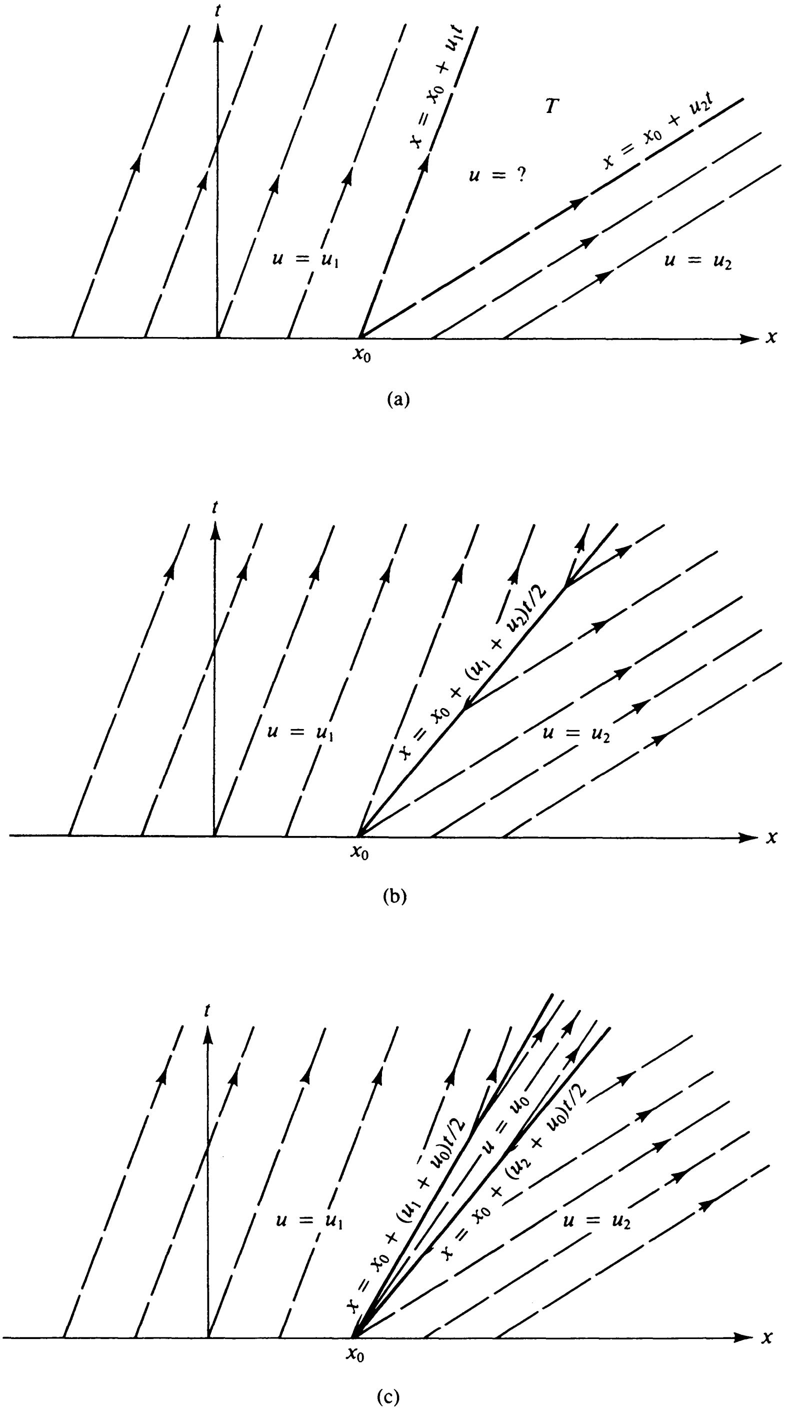

FIGURE 5.8. Piecewise constant initial data for a fan: nonuniqueness of weak solution

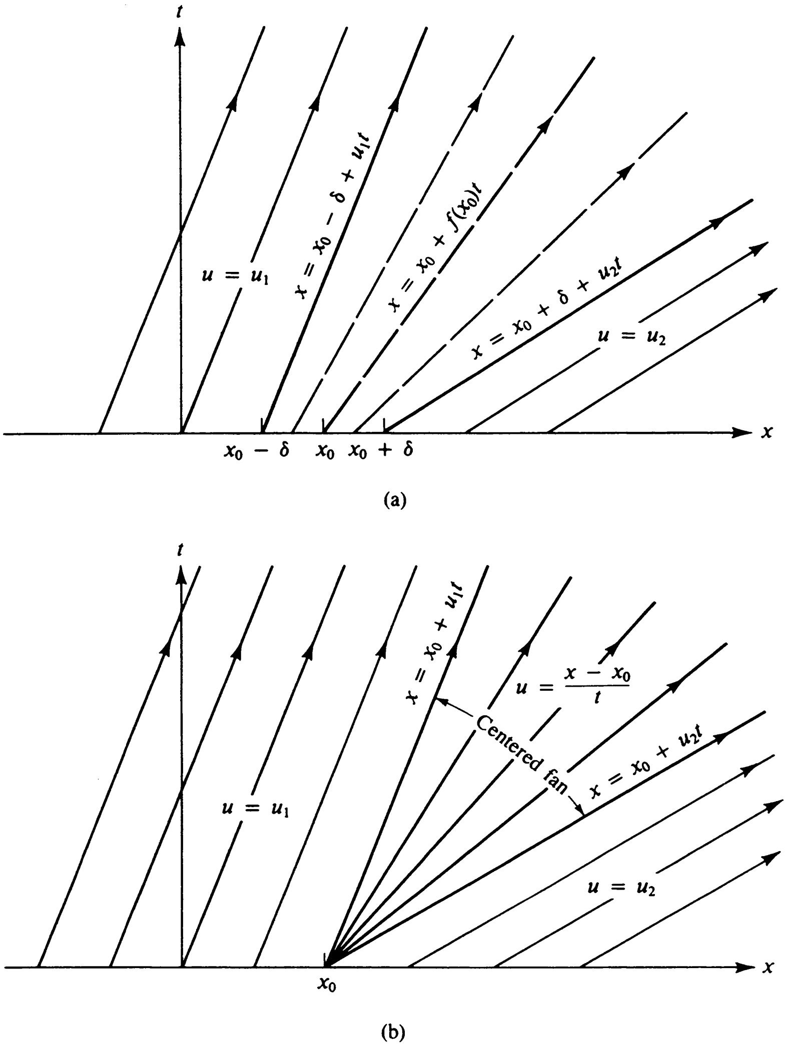

FIGURE 5.9. Centered fan as the limit of smooth initial data

The solution (5.3.42) may be regarded as the limiting case as $N \to \infty$ of the sequence of solutions (5.3.38)–(5.3.39) with N finite discontinuities. Thus, each member of the one-parameter family of rays emerging from x_0 may be regarded as the locus of an infinitesimally weak discontinuity.

Since (5.3.15c) has straight characteristics on which u is constant, we conclude that a centered fan is appropriate at $x = x_0, t = 0$ whenever we have discontinuous initial data such that $u(x_0^+, 0) > u(x_0^-, 0)$; the initial data need not be piecewise constant. It is interesting to note that a centered fan is also a similarity solution of (5.3.15c) in T with boundary conditions $u = u_1$ on $x = x_0 + u_1 t$ and $u = u_2$ on $x = x_0 + u_2 t$.

An alternative view of the solution of (5.3.15c), subject to any initial condition, is to regard it as the limiting case as $\epsilon \to 0$ of the corresponding initial-value problem for Burgers' equation (5.3.18b). This quasilinear second-order equation is parabolic and has smooth solutions for $t > 0$ even if the initial data are discontinuous. In Section 5.3.6 we shall study the exact solution for the two initial conditions (5.3.35) and (5.3.37) and show that as $\epsilon \to 0$, these solutions indeed tend to the weak solutions (5.3.36) and (5.3.42), respectively, that we have derived.

A concise condition, which excludes solutions such as (5.3.38) and (5.3.39), is to require

$$u^+ \leq \dot{\xi} \leq u^- \tag{5.3.43}$$

for admissible weak solutions. Equation (5.3.43) is referred to as an *entropy condition* because in the context of gas dynamics, the corresponding condition excludes discontinuities across which the entropy does not increase. One can prove (see [34]) that the condition (5.3.43) is sufficient to isolate a unique weak solution of (5.3.15) in all cases. For the more general scalar divergence relation

$$u_t + \{\phi(u)\}_x = 0, \tag{5.3.44}$$

the corresponding entropy condition is

$$\phi'(u^+) \leq \dot{\xi} \leq \phi'(u^-). \tag{5.3.45}$$

Finally, we note that shocks are not associated only with discontinuous initial data; the problem discussed in Section 5.2.2iv gives an example of smooth initial data for which the solution breaks down at some subsequent time. In this example, we derive a weak solution for $t \geq 1/2\pi$ by inserting a stationary shock at $x = \frac{1}{4}$, $t \geq 1/2\pi$. Because of the symmetry of the initial data relative to the point $x = \frac{1}{4}$, the values of u on either side of the line $x = \frac{1}{4}$, $t \geq 1/2\pi$ are equal in magnitude but differ in sign. Therefore, the shock condition (5.3.16) remains valid for all $t \geq 1/2\pi$ for this stationary shock. The weak solution for $t \geq 1/2\pi$ is the waveform to the left of S^- and to the right of S^+, with a stationary discontinuity at the point B, as shown in Figure 5.5.

(ii) *The uniformly propagating bore*

Because u = constant and h = constant solve the shallow-water equations (5.3.25), it is natural to ask whether there exist weak solutions of these equations that correspond to a bore propagating with uniform speed. This would require u and h to have different constant values ahead of and behind the bore. Using an appropriate coordinate frame (Galilean transformation) and dimensionless variables, we can, with no loss of generality, regard the water ahead of the bore to be at rest and of unit depth, as shown in Figure 5.10.

Such a bore is an idealization of the steady flow produced by an incoming high tide into an estuary. The coordinate system moves with the speed of the estuary

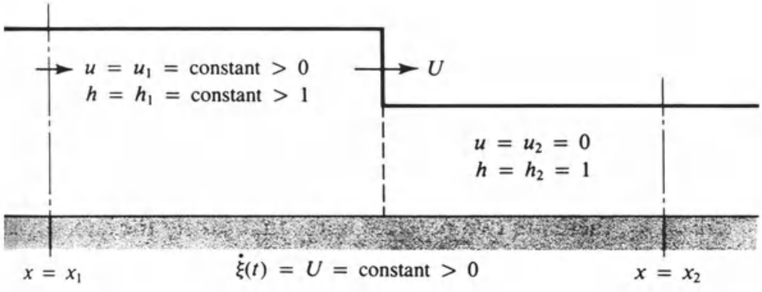

FIGURE 5.10. Constant-speed bore propagating into water at rest

outflow. In the laboratory, we could generate this flow by impulsively setting a wavemaker into motion with constant speed u_1 relative to water at rest (see Figure 3.7). In the first case, the height h_1 of the incoming tide is prescribed, whereas in the laboratory model, the speed u_1 corresponds to the wavemaker speed and is prescribed.

Inserting the values $u^+ = 0$, $h^+ = 1$, $u^- = u_1$, and $h^- = h_1$ into the bore conditions (5.3.24) and denoting the bore speed by $U =$ constant gives

$$U = \frac{u_1 h_1}{h_1 - 1}, \quad U = \frac{u_1^2 h_1 + h_1^2/2 - \frac{1}{2}}{u_1 h_1}. \tag{5.3.46}$$

Thus, we have two equations for the two unknowns (U, h_1) if u_1 is specified or the unknowns (U, u_1) if h_1 is specified. If the unknowns are U and h_1, one obtains a cubic for h_1 after eliminating U from the two equations (5.3.46) (see Problem 5.3.8). Here, we consider the case where (U, u_1) are the unknowns, and we can solve (5.3.46) for these quantities as explicit functions of h_1 in the form

$$u_1 = \pm(h_1 - 1)\left(\frac{h_1 + 1}{2h_1}\right)^{1/2}, \tag{5.3.47a}$$

$$U = \pm\left[\frac{h_1(h_1 + 1)}{2}\right]^{1/2}. \tag{5.3.47b}$$

Given h_1, there are two possible bores that satisfy the shock relations (5.3.46). The upper sign corresponds to bores propagating to the right, and the speed of the water behind these bores is positive or negative depending on the sign of $h_1 - 1$. The reverse is true for the lower sign. Of course, we must keep in mind that we are viewing the problem from a coordinate frame with respect to which the water ahead of the bore is at rest.

To decide which of the two possible solutions in (5.3.47) is physically realistic, let us calculate $\dot{E}(t)$, the time rate of change of energy less the net influx of energy and work done in some fixed interval $x_1 \le x \le x_2$ containing a uniformly propagating bore. If no energy is added or dissipated in this interval, $\dot{E}(t) = 0$.

Referring to the expression (3.2.17), we have

$$2\dot{E}(t) = \frac{d}{dt}\int_{x_1}^{x_2}(u^2h + h^2)dx = (u^3h + 2uh^2)\Big|_{x=x_1}^{x=x_2}. \tag{5.3.48a}$$

Using (5.3.4) with $\dot{\xi} = U =$ constant, we obtain

$$2\dot{E}(t) = -U[u^2h + h^2] + [u^3h + 2uh^2], \tag{5.3.48b}$$

or

$$2\dot{E}(t) = -U(1 - u_1^2h_1 - h_1^2) - u_1^3h_1 - 2u_1h_1^2. \tag{5.3.48c}$$

If we now insert the solution (5.3.47) for u_1 and U in (5.3.48c) and simplify the result, we find, after some algebra, that

$$2\dot{E} = \mp\frac{(h_1 - 1)^3}{2}\left(\frac{h_1 + 1}{2h_1}\right)^{1/2}, \tag{5.3.49}$$

where the upper and lower signs in (5.3.49) correspond to the upper and lower signs in (5.3.47), respectively.

For a physically realistic bore that dissipates energy, we must have $\dot{E} < 0$; therefore, we must pick the upper sign if $h_1 > 1$ and the lower sign if $h_1 < 1$. Figure 5.10 corresponds to the case $h_1 > 1$. For the second alternative, $h_1 < 1$, $u_1 > 0$, and $U < 0$. It is easily seen that a Galilean transformation that results in $u_1 = 0$ and a reversing of the flow direction merely reproduces Figure 5.10. Therefore, for flows to the right into quiescent water, Figure 5.10 is the only physically realistic bore. It has $h_1 > 1$ and $U > u_1 > 0$. In general, *an admissible bore must propagate into water of lower height.* The mathematically allowable solution shown in Figure 5.11 for the choice $h_1 > 1$ and the lower signs in (5.3.47) results in a physically unrealistic flow for which $\dot{E} > 0$.

It is also useful to contrast the behavior of solutions resulting from piecewise constant initial data for the scalar problem $u_t + uu_x = 0$ and the present two-component model. In the scalar case, *any* piecewise constant initial condition of

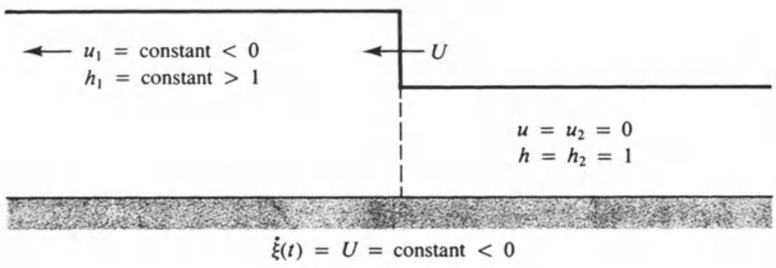

FIGURE 5.11. Physically inconsistent bore

the form (5.3.35) results in a uniformly propagating shock as long as $u_1 > u_2$. In contrast, for the problem governed by (5.3.25), the initial condition

$$h(x,0) = \begin{cases} 1 & \text{if } x > x_0, \\ h_1 > 1 & \text{if } x < x_0, \end{cases} \quad u(x,0) = \begin{cases} 0 & \text{if } x > x_0, \\ u_1 > 0 & \text{if } x < x_0, \end{cases}$$

gives a uniformly propagating shock only if u_1 and h_1 are related by (5.3.47a); for other choices of u_1 and h_1, the solution will also involve a centered fan; this is discussed in Chapter 7.

We also confirm that for weak bores, equations (5.3.47) reduce to the results calculated in Chapter 3 for the linearized theory. If the water level and speed behind the bore do not differ much from those of the quiescent state in front, we have $h_1 = 1 + \epsilon\tilde{h}_1$ and $u_1 = \epsilon\tilde{u}_1$, where ϵ is a small parameter and $\tilde{h}_1$ and $\tilde{u}_1$ are $O(1)$ constants. Substituting this expression for h_1 into (5.3.47b) shows that the bore speed to $O(1)$ is just the unit characteristic speed. Equation (5.3.47a) gives $\tilde{u}_1 = \pm\tilde{h}_1$. For the linear theory, both solutions are possible, and in fact these uniform solutions were essentially calculated in Chapter 3 (see Figure 3.19).

(iii) *The uniformly propagating shock in gas dynamics*

Here, the system of conservation laws (5.3.27) governs the evolution of the three variables ρ, u, p, and we wish to study the problem of a shock propagating into a quiescent gas (density $\rho_2 \equiv \rho_0 =$ constant, $u_2 = 0$, $p_2 \equiv p_0 =$ constant). It is convenient to adopt the dimensionless variables used in (3.3.9)–(3.3.10), where pressures are normalized using p_0, densities using ρ_0, and speeds using the ambient speed of sound $a_0 \equiv \sqrt{\gamma p_0/\rho_0}$. We then normalize the time by T_0, some characteristic time, and normalize distances by $a_0 T_0$. The dimensionless form of (5.3.28) is then

$$\rho_t + (u\rho)_x = 0, \quad \text{(mass)} \tag{5.3.50a}$$

$$(\rho u)_t + \left(\rho u^2 + \frac{p}{\gamma}\right)_x = 0, \quad \text{(momentum)} \tag{5.3.50b}$$

$$\left(\frac{\rho u^2}{2} + \frac{p}{\gamma(\gamma-1)}\right)_t + \left(\frac{\rho u^3}{2} + \frac{pu}{\gamma-1}\right)_x = 0, \quad \text{(energy)} \tag{5.3.50c}$$

where, for simplicity, we use the same notation as in (5.3.28) for the dimensionless variables. The extra factor $1/\gamma$ multiplying the dimensionless p in (5.3.50b)–(5.3.50c) is a direct result of our choice of a_0 rather than $\sqrt{p_0/\rho_0}$ for a velocity scale.

Let us again focus our attention on the problem of a piston that is pushed impulsively with constant speed v into a gas at rest. The picture is analogous to the one discussed in Problem 5.3.8 for a wavemaker being pushed into quiescent water, except that we now have three unknowns instead of two. These are the density and pressure behind the shock and the shock speed. Again, we assume that the gas speed behind the shock equals the piston speed v and is known. Thus, we have $\rho_2 = 1, u_2 = 0, p_2 = 1$, and $u_1 = v =$ prescribed, and we wish to calculate

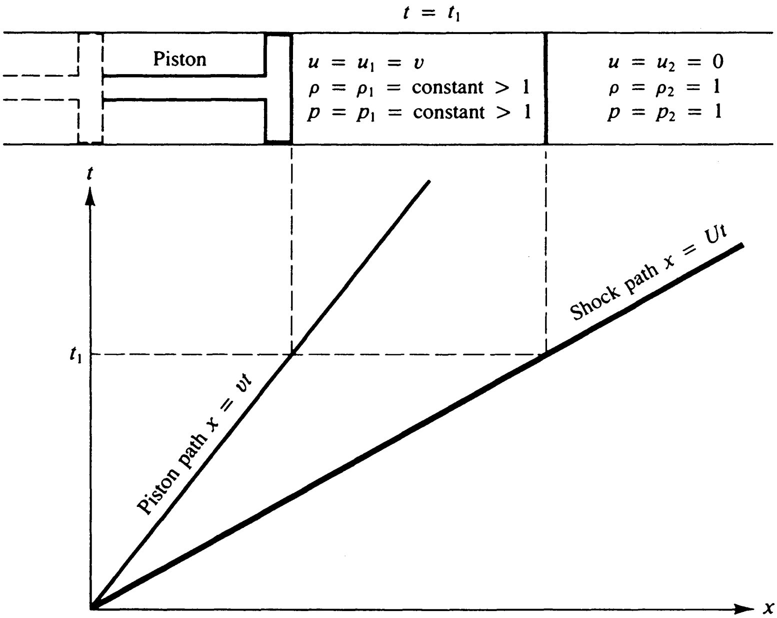

FIGURE 5.12. Piston pushed impulsively into gas at rest

ρ_1, p_1, and the shock speed U (see Figure 5.12). The more general problem of a shock propagating into a gas moving with constant speed u_2 can be obtained by replacing $u_1 \to u_1 - u_2, U \to U - u_2$ in our results.

If we set $\rho^+ = 1, \rho^- = \rho_1, u^+ = 0, u^- = v = u_1, p^+ = 1, p^- = p_1$, and $\dot{\xi} = U$ in the shock conditions associated with (5.3.50), we obtain

$$U(\rho_1 - 1) = u_1\rho_1, \tag{5.3.51a}$$

$$U\rho_1 u_1 = \rho_1 u_1^2 + \frac{1}{\gamma}(p_1 - 1), \tag{5.3.51b}$$

$$U\left(\frac{\rho_1 u_1^2}{2} + \frac{p_1}{\gamma(\gamma - 1)} - \frac{1}{\gamma(\gamma - 1)}\right) = \frac{\rho_1 u_1^3}{2} + \frac{p_1 u_1}{\gamma - 1}. \tag{5.3.51c}$$

We can solve (5.3.51a) for ρ_1 to obtain

$$\rho_1 = \frac{U}{U - u_1}. \tag{5.3.52a}$$

Solving (5.3.51b) for p_1 and using (5.3.52a) gives

$$p_1 = 1 + \gamma u_1 U. \tag{5.3.52b}$$

Finally, using (5.3.52a) and (5.3.52b) to eliminate ρ_1 and p_1 from (5.3.51c) gives

$$U^2 - \frac{(\gamma+1)}{2} U u_1 - 1 = 0. \qquad (5.3.52c)$$

Since u_1 is known, we can solve this quadratic for U and obtain the two roots

$$U^{(1)} \equiv \frac{\gamma+1}{4} u_1 + \left[\left(\frac{\gamma+1}{4} u_1 \right)^2 + 1 \right]^{1/2}, \qquad (5.3.53a)$$

$$U^{(2)} \equiv \frac{\gamma+1}{4} u_1 - \left[\left(\frac{\gamma+1}{4} u_1 \right)^2 + 1 \right]^{1/2}. \qquad (5.3.53b)$$

Each of these roots, when substituted into (5.3.52a) and (5.3.52b), gives a solution for ρ_1 and a solution for p_1. Table 5.1 lists the intervals over which U, ρ_1, and p_1 range as u_1 varies over all possible values.

It is easily seen that if we change the sign of u_1 in case 4, we simply recover case 1, and if we change the sign of u_1 in case 3, we recover case 2. Therefore, cases 3 and 4 are equivalent to cases 2 and 1, respectively, for flow to the left. Our task is now to decide which of cases 1 and 2 corresponds to a physically consistent shock. The answer hinges on how the entropy behaves across the shock in each case.

We can show (see Problem 5.3.10) by direct computation that in case 1 the entropy behind the shock is higher than its upstream value. The reverse is true for case 2, where the entropy decreases downstream of the shock. As it is physically

TABLE 5.1. Range of Values for U, ρ_1, and p_1

	u_1	U	ρ_1	p_1
1	$0 < u_1 < \infty$	$1 < U^{(1)} < \infty$	$1 < \rho_1 < \dfrac{\gamma+1}{\gamma-1}$	$1 < p_1 < \infty$
2	$0 < u_1 < \infty$	$-1 < U^{(2)} < 0$	$1 > \rho_1 > 0$	$1 > p_1 > -\dfrac{\gamma-1}{\gamma+1}$
3	$-\infty < u_1 < 0$	$0 < U^{(1)} < 1$	$0 < \rho_1 < 1$	$-\dfrac{\gamma-1}{\gamma+1} < p_1 < 1$
4	$-\infty < u_1 < 0$	$-\infty < U^{(2)} < -1$	$\dfrac{\gamma+1}{\gamma-1} > \rho_1 > 1$	$\infty > p_1 > 1$

inconsistent to have the entropy decrease, we discard case 2. Thus, shocks behind which the density and pressure drop are ruled out. Actually, case 3 corresponds to an impulsive piston motion with constant speed $u_1 < 0$ *out of* a gas at rest. We shall discuss the solution of this problem in Chapter 7 when we study simple waves.

Therefore, for a shock propagating to the right into a gas at rest, there is only one physically consistent solution given by case 1. The values of U, ρ_1, and p_1 are given by

$$U = \frac{\gamma+1}{4}u_1 + \left[\left(\frac{\gamma+1}{4}u_1\right)^2 + 1\right]^{1/2}$$

$$\to \begin{cases} 1 & \text{as } u_1 \to 0, \\ (\gamma+1)u_1/2 & \text{as } u_1 \to \infty, \end{cases} \tag{5.3.54a}$$

$$\rho_1 = \frac{4 + 4u_1\left[((\gamma+1)u_1/4)^2 + 1\right]^{1/2} + (\gamma+1)u_1^2}{4 + 2(\gamma-1)u_1^2}$$

$$\to \begin{cases} 1 & \text{as } u_1 \to 0, \\ (\gamma+1)/(\gamma-1) & \text{as } u_1 \to \infty, \end{cases} \tag{5.3.54b}$$

$$p_1 = 1 + \frac{\gamma(\gamma+1)}{4}u_1^2 + \gamma u_1\left[\left(\frac{\gamma+1}{4}u_1\right)^2 + 1\right]^{1/2}$$

$$\to \begin{cases} 1 & \text{as } u_1 \to 0, \\ \gamma(\gamma+1)u_1^2/2 & \text{as } u_1 \to \infty. \end{cases} \tag{5.3.54c}$$

These formulas generalize to the case of a shock propagating into a gas that is moving with constant speed u_2 by replacing u_1 everywhere with $u_1 - u_2$ and replacing U with $U - u_2$. It is interesting to note that for an infinitely strong shock, that is, $u_1 \to \infty$, the density ratio across the shock tends to a finite value, $\rho_1 \to (\gamma+1)/(\gamma-1)$, whereas the pressure ratio tends to infinity: $p_1 \to [\gamma(\gamma+1)/2]u_1^2 \to \infty$ as $u_1 \to \infty$.

The properties of weak shocks—that is, $u_1 \ll 1$—are also interesting and will be important in later discussions. We expand each of (5.3.54) in powers of u_1 and retain terms up to $O(u_1^3)$ to find, after some algebra, that

$$U = 1 + \frac{\gamma+1}{4}u_1 + \frac{(\gamma+1)^2}{32}u_1^2 + O(u_1^4), \tag{5.3.55a}$$

$$\rho = 1 + u_1 + \frac{3-\gamma}{4}u_1^2 + \frac{\gamma^2 - 14\gamma + 17}{32}u_1^3 + O(u_1^4), \tag{5.3.55b}$$

$$p = 1 + \gamma u_1 + \frac{\gamma(\gamma+1)}{4}u_1^2 + \frac{\gamma(\gamma+1)^2}{32}u_1^3 + O(u_1^4). \tag{5.3.55c}$$

Note that U tends to the characteristic speed $U \to 1$ as $u_1 \to 0$ and that $\rho - 1$ and $p - 1$ are both of order u_1. However, the change in entropy across a weak shock is extremely small, of order u_1^3 to be precise. To show this, we introduce the dimensionless entropy change s across the shock by (see Problem 5.3.10)

$$s \equiv \frac{S_1 - S_2}{C_v} = \log \frac{p_1}{\rho_1^\gamma}, \tag{5.3.56}$$

and compute the following value for s using (5.3.55b)–(5.3.55c):

$$s = \frac{\gamma(\gamma - 1)}{12} u_1^3 + O(u_1^4). \tag{5.3.57}$$

For example, consider a moderate shock where $u_1 = 0.3$. We obtain $p - 1 = 0.50235$ and $\rho - 1 = 0.33749$, but $s = 0.00275$, a very small number indeed. This fact provides a significant simplification in the analysis of problems with weak shocks. In particular, in any perturbation problem where a weak shock propagates into a domain of constant entropy, we can use the isentropic flow equations for computing the flow up to second order in the disturbances. This question is discussed in more detail in Chapter 7.

Finally, let us reexamine the analogy between shallow-water flow and the flow of a compressible gas. As pointed out in Section 3.3.4, there is an *exact* analogy between *smooth flows* for these two problems if $\gamma = 2$ and if we identify u in both cases and h (or $\sqrt{h}$) with ρ (or a). This analogy *does not carry over exactly* to discontinuous solutions because of the fact that p/ρ^γ does not remain constant across a shock. To see this, compare (5.3.50a)–(5.3.50b), in which $\gamma = 2$, $\rho = h$, with (5.3.23a)–(5.3.23b). These equations correspond if in addition, we set $p = h^2$; that is, $p = \rho^\gamma$ with $\gamma = 2$. But although $p/\rho^\gamma = 1$ in front of the shock, we have just shown that this quantity increases across the shock. Therefore, the analogy between the two discontinuous flows is only qualitative.

It is easy to exhibit the numerical extent of this discrepancy between the shock and bore speeds and the flow speeds behind these discontinuities for various values

TABLE 5.2. Values of u_1 and U as a Function of h

h_1	u_1	U
1	0	1
2	0.866	1.732
3	1.633	2.449
4	2.372	3.162

TABLE 5.3. Values of U and ρ as a Function of u

u_1	U	ρ_1
0	1	1
0.866	1.842	1.887
1.633	2.806	2.393
2.372	3.820	2.639

of ρ (or h). In Table 5.2 we give the values of u_1 and U (behind the bore) predicted by (5.3.47) (with the upper signs) for four values of h_1.

We now use the values of u_1 obtained from Table 5.2 in (5.3.54a)–(5.3.54b) with $\gamma = 2$ to compute U and ρ_1. The results are shown in Table 5.3.

The discrepancies between the bore and shock speeds on the one hand and h_1 and ρ_1 on the other increase as u_1 increases, as expected. In fact, $\rho_1 \to 3$ while $h_1 \to \infty$ as $u_1 \to \infty$. Thus, even for moderate shocks, the hydraulic analogy gives only a qualitative description. Moreover, $\gamma = 7/5$ for a diatomic gas such as air, and the requirement $\gamma = 2$ introduces further discrepancies in this case.

5.3.5 An Example of Shock Fitting for the Scalar Problem

In all the examples discussed in Section 5.3.4, the dependent variables are constant on either side of the shock. This is why the shock moves with constant speed. In this section we give an example that illustrates the basic idea of how to *fit* a curved shock into the xt-plane in order to prevent characteristics having varying values of u from crossing.

Consider the initial-value problem

$$u(x, 0) = \begin{cases} 1 & \text{if } |x| > 1, \\ -1 + |x| & \text{if } |x| < 1, \end{cases} \tag{5.3.58}$$

for the scalar problem (5.3.15). If we interpret u as the dimensionless traffic density defined in (5.1.17), (5.3.58) corresponds to a discrete, piecewise-linear initial density of cars over the interval $-1 \leq x \leq 1$ with a maximum ($u = -1$) at $x = 0$. Outside this interval there are no cars ($u = 1$).

We parameterize the initial curve ($t = 0$) to be

$$x_0(\tau) = \tau, \tag{5.3.59a}$$

$$t_0(\tau) = 0, \tag{5.3.59b}$$

$$u_0(\tau) = \begin{cases} 1 & \text{if } |\tau| > 1, \\ -1 + |\tau| & \text{if } |\tau| < 1. \end{cases} \tag{5.3.59c}$$

The characteristic equations

$$\frac{dt}{ds} = 1, \quad \frac{dx}{ds} = u, \quad \frac{du}{ds} = 0, \tag{5.3.60}$$

are now solved subject to (5.3.59), and we obtain

$$x = u_0(\tau)t + \tau, \quad u = u_0(\tau). \tag{5.3.61}$$

Let us examine the patterns of characteristic ground curves and associated values of u that emerge from all portions of the x-axis according to (5.3.61). Starting with $x < -1$, we have

$$u = 1 \tag{5.3.62a}$$

on the family of straight lines (see Figure 5.13).

$$x = t + \tau, \quad -\infty < \tau < -1. \tag{5.3.62b}$$

The next segment of the x-axis has

$$u = -(1 + \tau), \tag{5.3.63a}$$

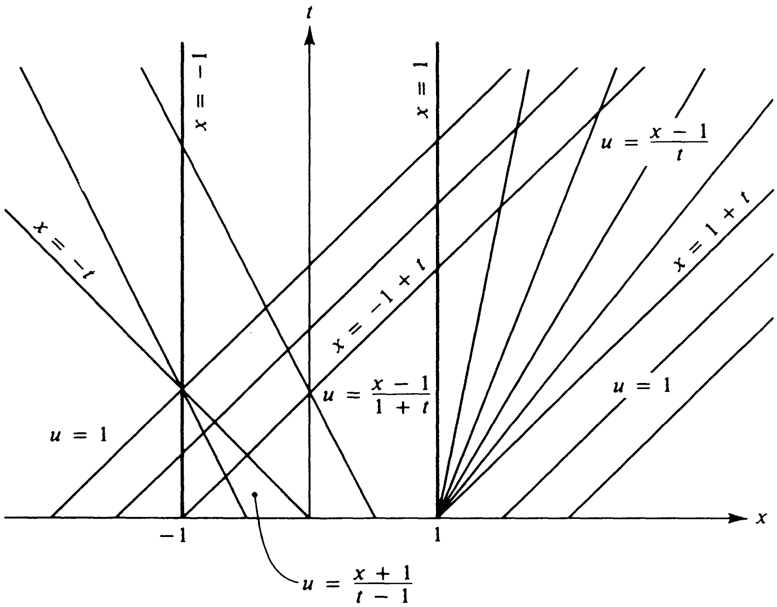

FIGURE 5.13. Characteristic ground curves for (5.3.58)

$$x = -(1 + \tau)t + \tau, \tag{5.3.63b}$$

for $-1 < \tau < 0$; that is,

$$u = \frac{x + 1}{t - 1}, \tag{5.3.64}$$

when we solve (5.3.63b) for τ and use the result in (5.3.63a).

Members of the family (5.3.62b) intersect with members of (5.3.63b) for $t > 0$ starting at $x = -1$. Thus, in the domain covered by both families, the solution is ambiguous. We prevent the crossing of characteristics by inserting a shock that starts at the point $A : t = 0, x = -1$ (see Figure 5.14).

The shock speed obeys (5.3.16) with $u^- = 1$ and $u^+ = (x + 1)/(t - 1)$; that is,

$$\frac{d\xi}{dt} = \frac{1}{2}\left(1 + \frac{\xi + 1}{t - 1}\right). \tag{5.3.65a}$$

Simplifying (5.3.65a), we obtain

$$\frac{d\xi}{dt} + \frac{1}{2(1 - t)}\xi = -\frac{t}{2(1 - t)}. \tag{5.3.65b}$$

This linear equation, subject to the initial condition $\xi(0) = -1$, can be easily solved to give

$$\xi = -2 + t + \sqrt{1 - t}, \quad 0 \leq t \leq \frac{3}{4}. \tag{5.3.66}$$

As noted, the shock (5.3.66) is appropriate only as long as it continues to have $u_1 = 1$ to the left and $u_2 = (x + 1)/(t - 1)$ to the right. The solution for u ceases to be given by (5.3.64) at the point B, where the shock crosses the characteristic $x = -t$ emerging from the origin. Setting $\xi = -t$ in (5.3.66) gives $t = \frac{3}{4}$. Therefore, the shock (5.3.66) is valid up to the point $B : x = -\frac{3}{4}, t = \frac{3}{4}$. This point occurs before the singular point $x = -1, t = 1$ shown in Figure 5.13, where all the lines (5.3.63b) meet, so that this singular behavior is not present in the weak solution, as seen in Figure 5.14.

To continue the shock beyond B, we must use the solution u_2 associated with the characteristics that emerge from $0 < x < 1, t = 0$. This solution is given by

$$u = \tau - 1, \quad x = (\tau - 1)t + \tau, \tag{5.3.67}$$

for $0 < \tau < 1$, or

$$u(x, t) = \frac{x - 1}{1 + t}. \tag{5.3.68}$$

Therefore, the shock speed formula beyond B is

$$\frac{d\xi}{dt} = \frac{1}{2}\left(1 + \frac{\xi - 1}{1 + t}\right). \tag{5.3.69}$$

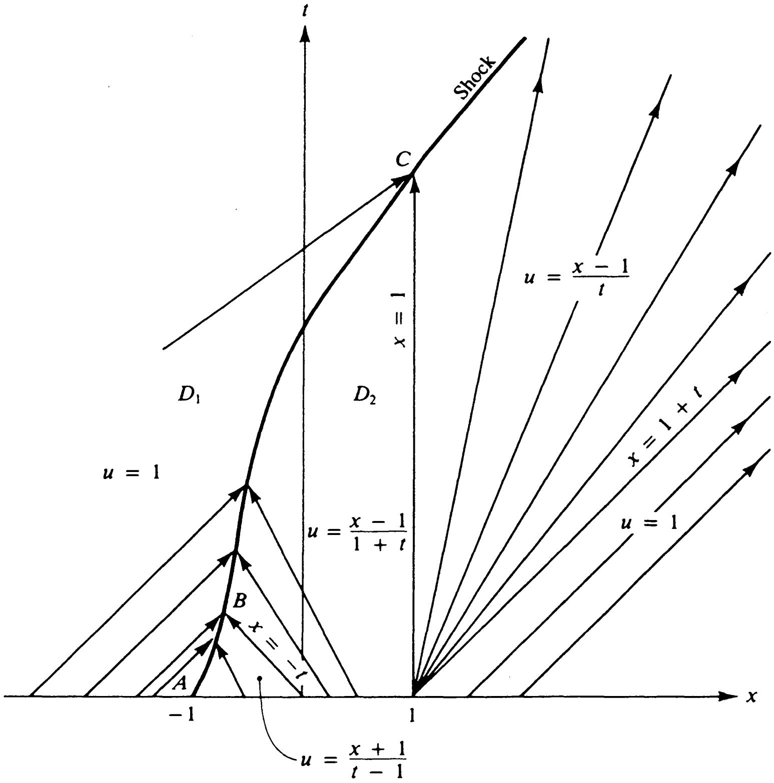

FIGURE 5.14. Weak solution for (5.3.58)

This is again a linear equation, which we solve subject to the initial condition $\xi(\frac{3}{4}) = -\frac{3}{4}$ and obtain

$$\xi = 2 + t - \sqrt{7(1+t)}, \quad \frac{3}{4} \leq t \leq 6. \tag{5.3.70}$$

Equation (5.3.70) is valid as long as (5.3.68) is true—that is, up to the characteristic $x = 1$. The intersection of (5.3.70) with $x = 1$ occurs at $C : x = 1, t = 6$.

The continuation of the solution past $x = 1$ requires that we insert a centered fan at $x = 1, t = 0$, since $u(1^+, 0) > u(1^-, 0)$ [see (5.3.37)]. This is the solution

$$u(x,t) = \frac{x-1}{t}, \quad 1 \leq x \leq 1+t, \quad t > 0. \tag{5.3.71}$$

Therefore, the third segment of the shock satisfies

$$\frac{d\xi}{dt} = \frac{1}{2}\left(1 + \frac{\xi - 1}{t}\right), \quad \xi(6) = 1. \tag{5.3.72}$$

This has the solution

$$\xi = 1 + t - \sqrt{6t}, \quad 6 \leq t < \infty, \tag{5.3.73}$$

and it is easily seen that the curve defined by (5.3.73) remains to the left of the characteristic $x = 1 + t$. In fact, the distance between the shock and $x = 1 + t$ is $\sqrt{6t}$.

The three segments of the shock curve join smoothly (that is, have continuous slopes) at B and C, and the resulting curve divides the xt-plane into two regions: (1) $D_1 : x < \xi(t)$, where the solution is $u = 1$, and (2) $D_2 : x > \xi(t)$, where the solution is defined by (5.3.64), (5.3.68), (5.3.71) and $u = 1$ for $x \geq 1 + t$. The solution in D_2 is smooth everywhere except along the three rays $x = -t$, $x = 1$, and $x = 1 + t$, where u_x and u_t are discontinuous. This completes the weak solution of the initial-value problem (5.3.58). Additional examples are given in Problems 5.3.1–5.3.7.

5.3.6 Exact Solution of Burgers' Equation; Shock Layer, Corner Layer

In Chapter 1 we showed that the initial-value problem for Burgers' equation (5.3.18b) could be solved exactly for initial data on $-\infty < x < \infty$. In this section we use these exact solutions for the two special initial-value problems (5.3.35) and (5.3.37) in order to study the limiting behavior as we let $\epsilon \to 0$.

We consider the equation

$$\overline{u}_t + \overline{u}\,\overline{u}_{\overline{x}} = \epsilon \overline{u}_{\overline{x}\overline{x}}, \quad -\infty < \overline{x} < \infty, \tag{5.3.74}$$

for $\overline{u}(\overline{x}, \overline{t}; \epsilon)$; with no loss of generality, we adopt the simpler initial condition

$$\overline{u}(\overline{x}, 0; \epsilon) = \begin{cases} 1 & \text{if } \overline{x} < 0, \\ -1 & \text{if } \overline{x} > 0, \end{cases} \tag{5.3.75}$$

instead of (5.3.35). It is easily seen that the transformation

$$\overline{x} = \frac{x - x_0 - (u_1 + u_2)t/2}{2/(u_1 - u_2)}, \tag{5.3.76a}$$

$$\overline{t} = \frac{t}{4/(u_1 - u_2)^2}, \tag{5.3.76b}$$

$$\overline{u} = \frac{2u - (u_1 + u_2)}{u_1 - u_2}, \tag{5.3.76c}$$

reduces (5.1.18b) to (5.3.74) (that is, it leaves Burgers' equation invariant) and takes the initial condition (5.3.55) to (5.3.75). Similarly, the transformation

$$\overline{x} = \frac{x - x_0 - (u_1 + u_2)t/2}{2/(u_2 - u_1)}, \tag{5.3.77a}$$

$$\overline{t} = \frac{t}{4/(u_2 - u_1)^2}, \tag{5.3.77b}$$

$$\bar{u} = \frac{2u - (u_1 + u_2)}{u_2 - u_1}, \tag{5.3.77c}$$

also leaves Burgers' equation invariant, and transforms the initial condition (5.3.74) to

$$\bar{u}(\bar{x}, 0; \epsilon) = \begin{cases} -1 & \text{if } \bar{x} < 0, \\ +1 & \text{if } \bar{x} > 0. \end{cases} \tag{5.3.78}$$

Thus, we need study only the initial conditions corresponding to a stationary shock and a symmetric fan on $-\bar{t} \leq \bar{x} \leq \bar{t}$ for the reduced problem where $\epsilon = 0$.

(i) *The shock layer*

We start with (5.3.74) and (5.3.75) and omit the overbars for simplicity. As shown in Problem (1.7.1) (see (1.7.26)), the solution is given explicitly in the form

$$u(x, t; \epsilon) = \frac{e^{-x/\epsilon} \operatorname{erfc}\left(\dfrac{x - t}{2\sqrt{\epsilon t}}\right) - \operatorname{erfc}\left(-\dfrac{x + t}{2\sqrt{\epsilon t}}\right)}{e^{-x/\epsilon} \operatorname{erfc}\left(\dfrac{x - t}{2\sqrt{\epsilon t}}\right) + \operatorname{erfc}\left(-\dfrac{x + t}{2\sqrt{\epsilon t}}\right)}. \tag{5.3.79}$$

The limiting behavior of (5.3.79) as $\epsilon \to 0$ depends on the values of x and t. In particular, we note that for $t > 0$, the arguments of the complementary error functions are positive or negative depending on whether $(x - t)$ and $-(x + t)$ are positive or negative, respectively. Moreover, if $(x - t)/2t^{1/2} \neq 0$ and $-(x + t)/2t^{1/2} \neq 0$ and if these expressions are held fixed as $\epsilon \to 0$, the arguments of the error functions tend to $\pm\infty$ depending on the signs of $(x - t)$ and $-(x + t)$. Therefore, we shall need the asymptotic expansion for $\operatorname{erfc}(y)$ for real y as $y \to \pm\infty$. Integration by parts of the defining integral for $\operatorname{erfc}(y)$ written in the form

$$\operatorname{erfc}(y) = \frac{1}{\sqrt{\pi}} \int_{y^2}^{\infty} e^{-s} s^{-1/2} ds$$

gives

$$\operatorname{erfc}(y) = \frac{e^{-y^2}}{\pi^{1/2} y}[1 + O(y^{-2})], \quad \text{as } y \to \infty. \tag{5.3.80a}$$

To calculate the behavior as $y \to -\infty$, we write $\operatorname{erfc}(y) = 1 - \operatorname{erf}(y)$ and use the fact that $\operatorname{erf}(y)$ is odd to find $\operatorname{erfc}(y) = 1 + \operatorname{erf}(-y) = 2 - \operatorname{erfc}(-y)$. Therefore,

$$\operatorname{erfc}(y) = 2 + \frac{e^{-y^2}}{\pi^{1/2} y}[1 + O(y^{-2})], \quad \text{as } y \to -\infty. \tag{5.3.80b}$$

We subdivide the xt-plane into the three domains:

$$\begin{aligned} &(1) \quad t > 0, \; x - t > 0, \; x + t > 0, \\ &(2) \quad t > 0, \; x - t < 0, \; x + t > 0, \\ &(3) \quad t > 0, \; x - t < 0, \; x + t < 0. \end{aligned} \tag{5.3.81}$$

We use (5.3.80b) to evaluate the leading contribution of the two terms that make up (5.3.79) in (1), (2), and (3) as well as the boundaries between (1) and (2) and between (2) and (3). We then use these expressions to compute the leading term for u as $\epsilon \to 0$. The results are summarized in Table 5.4.

The result,

$$u \to -\tanh \frac{x}{2\epsilon}, \tag{5.3.82}$$

that we have in region (2) is significant only along the $x = 0$ axis, since in the limit $\epsilon \to 0$ with $x > 0$ (5.3.82) gives $u \to -1$, as in (1), and with $x < 0$, we have $u \to 1$, as in (3). In fact, we see that the behavior of the hyperbolic tangent is relevant only in a *thin layer of* $O(\epsilon)$ around $x = 0$. This statement can be formalized as follows: We introduce a rescaled variable $x^* = x/\epsilon$ and define

$$u^*(x^*, t; \epsilon) \equiv u(\epsilon x^*, t; \epsilon). \tag{5.3.83}$$

We then obtain

$$\lim_{\epsilon \to 0} u(\epsilon x^*, t; \epsilon) = -\tanh \frac{x^*}{2}, \tag{5.3.84}$$

if $t > 0$ and x^* is fixed and not equal to zero. Equation (5.3.84) defines the *shock structure* for Burgers' equation and can be derived independently of the exact solution by applying the limit process in (5.3.84) to the differential equation

TABLE 5.4. Asymptotic Values of the Terms in (5.3.79)

Region	$e^{-x/\epsilon} \operatorname{erfc}\left(\frac{x-t}{2\sqrt{\epsilon t}}\right)$	$\operatorname{erfc}\left(-\frac{x+t}{2\sqrt{\epsilon t}}\right)$	u
(1)	$\frac{2(\epsilon t)^{1/2}}{\pi^{1/2}(x-t)} \exp\left[-\frac{(x+t)^2}{4\epsilon t}\right]$	2	-1
$x = t > 0$	$\exp\left(-\frac{x}{\epsilon}\right)$	2	-1
(2)	$2\exp\left(-\frac{x}{\epsilon}\right)$	2	$-\tanh \frac{x}{2\epsilon}$
$x = -t < 0$	$2\exp\left(\frac{t}{\epsilon}\right)$	1	1
(3)	$2\exp\left(-\frac{x}{\epsilon}\right)$	$\frac{2(\epsilon t)^{1/2}}{\pi^{1/2}(x+t)} \exp\left[-\frac{(x+t)^2}{4\epsilon t}\right]$	1

(5.3.74). This idea will be fully explored in Section 8.2 when we study matched asymptotic expansions.

For the time being, we note that the exact solution does indeed tend to the weak solution (5.3.36) with a shock obeying (5.3.16). We see that the term ϵu_{xx} is important only to $O(1)$ in a thin layer of thickness $O(\epsilon)$ centered at the shock location; this term serves to smooth out the discontinuity in the weak solution of the $\epsilon = 0$ problem.

(ii) *The centered fan; corner layer*

According to (1.7.28), the solution for the initial condition (5.3.78) is given by

$$u(x, t; \epsilon) = \frac{-\operatorname{erfc}\left(\dfrac{x+t}{2\sqrt{\epsilon t}}\right) + e^{-x/\epsilon}\operatorname{erfc}\left(\dfrac{t-x}{2\sqrt{\epsilon t}}\right)}{\operatorname{erfc}\left(\dfrac{x+t}{2\sqrt{\epsilon t}}\right) + e^{-x/\epsilon}\operatorname{erfc}\left(\dfrac{t-x}{2\sqrt{\epsilon t}}\right)}. \tag{5.3.85}$$

If we examine the leading contribution as $\epsilon \to 0$ for each of the two terms occurring in (5.3.85) in the three regions listed in (5.3.81), we obtain the expressions in Table 5.5.

We see that in the limit $\epsilon \to 0$, the exact solution indeed tends to the weak solution (5.3.42). As pointed out earlier, this weak solution has discontinuous derivatives u_x and u_t along the rays $x = \pm t$. For example, the weak solution for u as a function of x at a fixed time $t = t_0 > 0$ is the piecewise linear profile shown in Figure 5.15.

It is interesting to show that in the neighborhood of the rays $x = \pm t$, where the weak solution has a *corner*, the asymptotic behavior of the exact solution consists of a *corner layer*, which smooths out this corner. It suffices to consider the corner at

TABLE 5.5. Asymptotic Values of the Terms in (5.3.85)

Region	$\operatorname{erfc}\left(\frac{x+t}{2\sqrt{\epsilon t}}\right)$	$e^{-x/\epsilon}\operatorname{erfc}\left(\frac{t-x}{2\sqrt{\epsilon t}}\right)$	u
(1)	$\frac{2(\epsilon t)^{1/2}}{\pi^{1/2}(x+t)}\exp\left[-\frac{(x+t)^2}{4\epsilon t}\right]$	$2e^{-x/\epsilon}$	1
(2)	$\frac{2(\epsilon t)^{1/2}}{\pi^{1/2}(x+t)}\exp\left[-\frac{(x+t)^2}{4\epsilon t}\right]$	$\frac{2(\epsilon t)^{1/2}}{\pi^{1/2}(x-t)}\exp\left[-\frac{(x+t)^2}{4\epsilon t}\right]$	$\frac{x}{t}$
(3)	2	$\frac{2(\epsilon t)^{1/2}}{\pi^{1/2}(x-t)}\exp\left[-\frac{(x+t)^2}{4\epsilon t}\right]$	-1

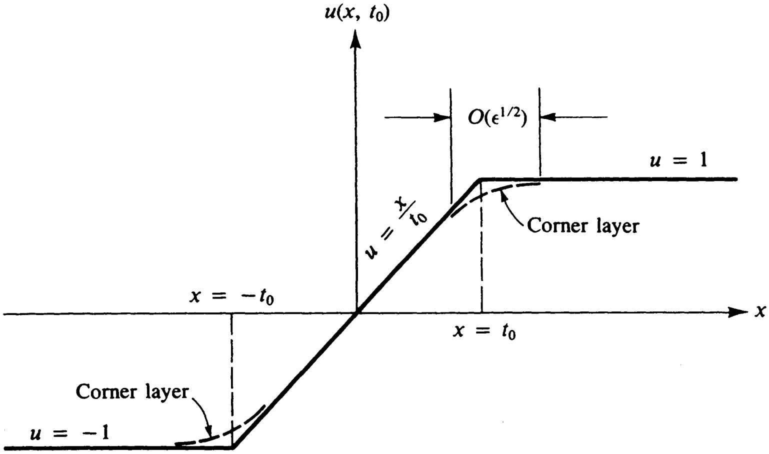

FIGURE 5.15. Corner layers that smooth out discontinuities in u_x and u_t of the weak solution

$x = t$, since the behavior at the other corner follows by symmetry. We note that the arguments of the error functions in (5.3.85) involve $(x - t)/\epsilon^{1/2}$ and $(x + t)/\epsilon^{1/2}$. This suggests that near $x = t$, we must hold the variables $x_c \equiv (x - t)/\epsilon^{1/2}$ and $t_\epsilon \equiv t$ fixed as $\epsilon \to 0$. Thus, we have

$$\operatorname{erfc}\left(\frac{x+t}{2\sqrt{\epsilon t}}\right) = \operatorname{erfc}\left[\left(\frac{t_c}{\epsilon}\right)^{1/2}\left(1 + \frac{x_c}{2t_c^{1/2}}\right)\right],$$

or

$$\operatorname{erfc}\left(\frac{x+t}{2\sqrt{\epsilon t}}\right) = \left(\frac{\epsilon}{\pi t_c}\right)^{1/2} \exp\left(-\frac{t_c}{\epsilon} - \frac{x_c}{\epsilon^{1/2}} - \frac{x_c^2}{4t_c}\right)[1 + O(\epsilon^{1/2})], \tag{5.3.86a}$$

when we use (5.3.80) and regard $(t_c/\epsilon)^{1/2} \to \infty$ and $(1 + x_c/2t_c^{1/2}) = O(1)$. We also obtain

$$e^{-x/\epsilon}\operatorname{erfc}\left(\frac{t-x}{2\sqrt{\epsilon t}}\right) = \exp\left(\frac{t_c}{\epsilon} - \frac{x_c}{\epsilon^{1/2}}\right)\operatorname{erfc}\left(-\frac{x_c}{2t_c^{1/2}}\right). \tag{5.3.86b}$$

Therefore, the dominant behavior for u near this corner is given by

$$u = \frac{-\left(\dfrac{\epsilon}{\pi t_c}\right)^{1/2}\exp\left(-\dfrac{x_c^2}{4t_c}\right) + \operatorname{erfc}\left(-\dfrac{x_c}{2t_c^{1/2}}\right)}{\left(\dfrac{\epsilon}{\pi t_c}\right)^{1/2}\exp\left(-\dfrac{x_c^2}{4t_c}\right) + \operatorname{erfc}\left(-\dfrac{x_c}{2t_c^{1/2}}\right)} + O(\epsilon), \tag{5.3.87}$$

when we substitute (5.3.86) into (5.3.85) and cancel the common factor $\exp(t_c/\epsilon - x_c/\epsilon^{1/2})$. Expanding (5.3.85) further for $\epsilon^{1/2}$ small gives the *corner layer approximation*

$$u = 1 - 2\left(\frac{\epsilon}{\pi t_c}\right)^{1/2} \frac{\exp\left(-\dfrac{x_c^2}{4t_c}\right)}{\operatorname{erfc}\left(-\dfrac{x_c}{2t_c^{1/2}}\right)} + O(\epsilon). \tag{5.3.88}$$

It is easily seen that as $x_c \to -\infty$, the corner layer limit (5.3.88) gives $u \to x/t$ and that $u \to 1$ as $x_c \to \infty$. Thus, (5.3.88) smoothly joins the two linear profiles within a layer of thickness $O(\epsilon^{1/2})$ centered at $x = t$. This is the dotted curve in Figure 5.15. In Chapter 8 we shall see that (5.3.88) may also be regarded as the limiting solution of Burgers' equation as $\epsilon \to 0$ with $u_c = (u - 1)/\epsilon^{1/2}$, x_c and t_c fixed.

In the preceding discussion we have confined our attention to piecewise constant initial values. One can also show, with considerably more effort and the use of asymptotic expansions, that the exact solution of Burgers' equation for more general initial data does, in fact, tend to the admissible weak solution in the limit $\epsilon \to 0$.

Therefore, for the initial-value problem on $-\infty < x < \infty$, use of the correct integral conservation law (5.3.15a) combined with the appropriate entropy condition (5.3.43) provides a very useful approximation everywhere except in $O(\epsilon)$ layers centered at shocks and $O(\epsilon^{1/2})$ layers centered at the two boundary characteristics of centered fans.

The solution of Burgers' equation with boundary conditions leads to $O(\epsilon)$ boundary layers and $O(\epsilon^{1/2})$ transition layers in addition to the shock and corner layers that we found here. A detailed discussion of the various possible approximations is given in Section 3.1.3 of [26]. See also Section 8.2.5iii. In general, it is not possible to satisfy a prescribed boundary condition at $x = 0$, say, for the problem with $\epsilon = 0$. Some examples are illustrated in Problems 5.3.6–5.3.7.

Problems

5.3.1 Consider the initial-value problem

$$u(x, 0) = \begin{cases} f(x) & \text{if } x < 0, \\ 0 & \text{if } x \geq 0 \end{cases} \tag{5.3.89}$$

for (5.3.15c).

a. What conditions must be imposed on the function $f(x)$ in order that the solution be strict in the half-plane $-\infty < x < \infty$, $0 \leq t < \infty$?

b. Now consider the inverse problem where $f(x)$ is unknown. Instead, we are told that a shock, defined by the prescribed function

$$x = \xi(t), \quad 0 \le t < \infty, \tag{5.3.90}$$

separates the half-plane into two domains, in each of which the solution of (5.3.15a) is strict. The shock curve (5.3.90) satisfies (5.3.16) with $\xi(0) = 0$. Show that $\xi(t)$ cannot be prescribed arbitrarily. Derive the conditions that must be imposed on $\xi(t)$ in order to ensure that (5.3.90) is a consistent shock.

c. For functions $\xi(t)$ satisfying the conditions you derived in part (b), calculate $f(x)$ in terms of $\xi(t)$.

d. Show that the special case

$$\xi(t) = (1+t)^{1/2} - 1 \tag{5.3.91}$$

is a consistent shock according to part (b) and leads to

$$f(x) = 1 + x \tag{5.3.92}$$

when you use your result in part (c).

5.3.2 Calculate the weak solution of (5.3.15a) for each of the following initial conditions:

$$u(x,0) = \begin{cases} 1 & \text{if } |x| > 1, \\ 0 & \text{if } |x| < 1. \end{cases} \tag{5.3.93}$$

$$u(x,0) = \begin{cases} 0 & \text{if } |x| > 1, \\ 1 - x & \text{if } 0 < x < 1, \\ -1 - x & \text{if } -1 < x < 0. \end{cases} \tag{5.3.94}$$

$$u(x,0) = \begin{cases} 1 & \text{if } x < -1, \\ 0 & \text{if } -1 < x < 1, \\ -1 & \text{if } 1 < x. \end{cases} \tag{5.3.95}$$

$$u(x,0) = \begin{cases} 1 & \text{if } |x| \ge 1, \\ x^2 & \text{if } |x| \le 1. \end{cases} \tag{5.3.96}$$

5.3.3 Consider the integral conservation law

$$\frac{d}{dt}\int_{x_1}^{x_2} (1 + \epsilon \cos x) u(x,t)\,dx + \frac{1}{2}u^2(x_2,t) - \frac{1}{2}u^2(x_1,t) = 0, \tag{5.3.97}$$

where $x_1 < x_2$ are two arbitrary fixed constants and ϵ is a positive constant < 1.

a. What is the partial differential equation associated with (5.3.97) for strict solutions?

b. What is the shock condition for (5.3.97)?

c. Calculate the weak solution corresponding to the initial condition

$$u(x,0) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } 0 < x \end{cases} \tag{5.3.98}$$

for $t > 0$ and $-\infty < x < \infty$. Derive u explicitly everywhere and give a parameteric representation for the boundaries of the region in the xt-plane where u is not constant.

d. What happens if we replace (5.3.98) by

$$u(x,0) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } 0 < x < 1, \\ -1 & \text{if } 1 < x? \end{cases} \tag{5.3.99}$$

Derive but do not solve the differential equation for the variable shock.

5.3.4 Consider the integral conservation law with a source term

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)dx + \frac{1}{2}\{u^2(x_2,t) - u^2(x_1,t)\} = \int_{x_1}^{x_2} \lambda(x)dx, \tag{5.3.100}$$

where

$$\lambda(x) = \begin{cases} 1 & \text{if } |x| > \frac{1}{2}, \\ \alpha = \text{constant} & \text{if } |x| < \frac{1}{2}. \end{cases}$$

Calculate the weak solution for the initial-value problem

$$u(x,0) = C = \text{constant} \tag{5.3.101}$$

for all ranges of values of the constants α and C. In particular, show that the interface condition at $x = \pm\frac{1}{2}$ is that u is continuous there. Use this condition to connect solutions across the two interfaces, and fit shocks and fans where appropriate.

5.3.5 Consider the integral conservation law

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)dx + \frac{x_2^2}{2}u^2(x_2,t) - \frac{x_1^2}{2}u^2(x_1,t) = 0. \tag{5.3.102}$$

Show that for the initial-value problem

$$u(x,0) = \begin{cases} 1/x & \text{if } 1 \le x \le 2, \\ 0 & \text{if } 2 < x \le 3, \end{cases} \tag{5.3.103}$$

the weak solution is given by

$$u = \begin{cases} 1/x & \text{if } e^t \le x < 2e^{t/2} \text{ and } 0 \le t \le 2\log(3/2), \\ 1/x & \text{if } e^t \le x \le 3 \text{ and } 2\log(3/2) < t \le \log 3, \\ 0 & \text{if } 2e^{t/2} < x \le 3 \text{ and } 0 \le t < 2\log(3/2). \end{cases} \tag{5.3.104}$$

5.3.6 Consider the initial-value/signaling problem

$$u(x,0) = A = \text{constant}, \quad u(0,t) = B = \text{constant} \tag{5.3.105}$$

for (5.3.15a) in the quarter-plane $x \geq 0, t \geq 0$. Show that a weak solution that satisfies both conditions is possible only if

$$0 < |A| < B. \tag{5.3.106}$$

5.3.7 Calculate the weak solution of (5.3.15) for each of the following initial-value/signaling problems:

$$u(x,0) = \begin{cases} 0 & \text{if } 0 < x \leq 1, \\ x-1 & \text{if } 1 \leq x \leq 2, \\ 1 & \text{if } 2 \leq x < \infty, \end{cases} \quad u(0,t) = 2 \text{ if } t > 0. \tag{5.3.107}$$

$$u(x,0) = \begin{cases} -1 & \text{if } 0 < x < 1, \\ 1 & \text{if } 1 < x < \infty, \end{cases} \quad u(0,t) = \begin{cases} 2 & \text{if } 0 < t < 1, \\ 1 & \text{if } 1 < t < \infty. \end{cases} \tag{5.3.108}$$

5.3.8 A wavemaker at the origin is impulsively pushed with constant speed v into quiescent water of unit height over $0 \leq x < \infty$. A bore with speed U = constant > 0 starts propagating to the right into the water at rest. The speed of the water behind the bore is u_1, and the height is h_1. Clearly, we must have $u_1 = v$ in order to satisfy the boundary condition at the wavemaker. Therefore, u_1 is known and U and h_1 are unknown.

a. Show that eliminating U from (5.3.46) gives the cubic equation

$$F_i(h_1) \equiv h_1^3 - h_1^2 - (1 + 2v^2)h_1 + 1 = 0 \tag{5.3.109}$$

for h_1. Show that for any given $v > 0$, this equation has one negative root (which we discard) and two positive roots $h_1^{(1)} < 1$ and $h_1^{(2)} > 1$. Using the first equation in (5.3.46), we see that the root $h_1^{(1)}$ results in a negative bore speed. Therefore, the appropriate solution is $h_1^{(2)}$. Specialize your results to the numerical example $v = \sqrt{3}/2$ for which $h_1^{(2)} = 2$ and $U = \sqrt{3}$.

b. Now assume that there is a vertical wall at some sufficiently large distance $x = x_0$, so the incoming bore will reflect from this wall and propagate back to the left. Equivalently, we may regard the flow for $0 \leq x \leq x_0$ as resulting from the given wavemaker and an image wavemaker starting at $t = 0$ from $x = 2x_0$ and moving impulsively with speed $-v$. Show that the height h_3 of the water behind the reflected bore is given by the larger of the two positive roots of the cubic

$$F_r(h_3) \equiv (h_3 - 1)[h_3^2 + (1 - h_1)h_3 - h_1^3] = 0, \tag{5.3.110a}$$

that is,

$$h_3 = \frac{h_1 - 1 + \sqrt{(h_1 - 1)^2 + 4h_1^3}}{2}, \tag{5.3.110b}$$

and that this value of h_3 is larger than h_1.

5.3.9 Consider shallow-water flow over a bottom surface $y = \epsilon b(x)$, where

$$\epsilon b(x) = \begin{cases} 0 & \text{if } x < x_0, \\ c = \text{constant} & \text{if } x > x_0. \end{cases} \quad (5.3.111)$$

Here x_0 and c are both positive with $0 < c < 1$. At $t = 0$, a rightgoing uniformly propagating bore is located at $x = 0$. The height of the water (measured from the bottom) behind this bore is h_4, a given constant greater than 1. The height of the water in front of the bore is $h_0 = 1$ if $0 < x < x_0$, and $h_1 = 1 - \epsilon b(x)$ if $x_0 < x$. The water in front of the bore is at rest. See Figure 5.16a. Using (5.3.47) we have

$$u_4 = (h_4 - 1)\sqrt{\frac{h_4 + 1}{2h_4}}, \quad V_1 = \sqrt{\frac{h_4(h_4 + 1)}{2}}. \quad (5.3.112)$$

a. Look for a solution with four constant-speed bores as sketched in Figure 5.16b. We have the incident bore V_1, the transmitted bore V_2, the reflected bore V_3, and a stationary bore over the step. Use the divergence relations (see (3.2.51) and (3.2.54) with $w = 0$, $\frac{\partial}{\partial t} = 0$) to show that the following relations must hold across the stationary bore ($V = 0$)

$$u_2h_2 - u_3h_3 = 0, \; u_2^2h_2 + \frac{h_2^2}{2} - u_3^3h_3 - \frac{h_3^2}{2} + c = 0. \quad (5.3.113)$$

b. Show that for the transmitted bore, we must have

$$V_2 = \frac{u_2h_2}{h_2 - (1 - c)} = \frac{u_2^2h_2 + h_2^2/2 - (1 - c)^2/2}{u_2h_2}, \quad (5.3.114)$$

and for the reflected bore, we must have

$$V_3 = \frac{u_3h_3 - u_4h_4}{h_3 - h_4} = \frac{u_3^2h_3 + h_3^2/2 - u_4^2h_4 - h_4^2/2}{u_3h_3 - u_4h_4}. \quad (5.3.115)$$

Equations, (5.3.113)–(5.3.115) give six relations linking the six unknowns V_1, V_3, u_2, h_2, u_3, and h_3.

For $x \neq x_0$, eliminate u_2 from the two equations (5.3.113) to obtain

$$\left(\frac{u_3^2}{h_2} - \frac{1}{2}\right)h_3^2 - u_3^2h_3 + \frac{h_2^2}{2} + c = 0. \quad (5.3.116a)$$

Similarly, eliminate V_2 from the two equations (5.3.114), and V_3 from the two equation (5.3.115), and use $u_2h_2 = u_3h_3$ to obtain

$$u_2^2h_2^2 = (h_2 - 1 + c)[u_3^2h_3^2/h_2 + h_2^2/2 - (1 - c)^2/2], \quad (5.3.116b)$$

$$(u_3h_3 - u_4h_4)^2 = (h_3 - h_4)[u_3^2h_3 + h_3^2/2 - h_4^2/2]. \quad (5.3.116c)$$

c. Solve the system (5.3.116) numerically for the case $c = \frac{1}{2}$, $h_4 = \frac{3}{2}$ (which according to (5.3.112) implies $u_4 = 0.4564$ and $V_1 = 1.3693$)

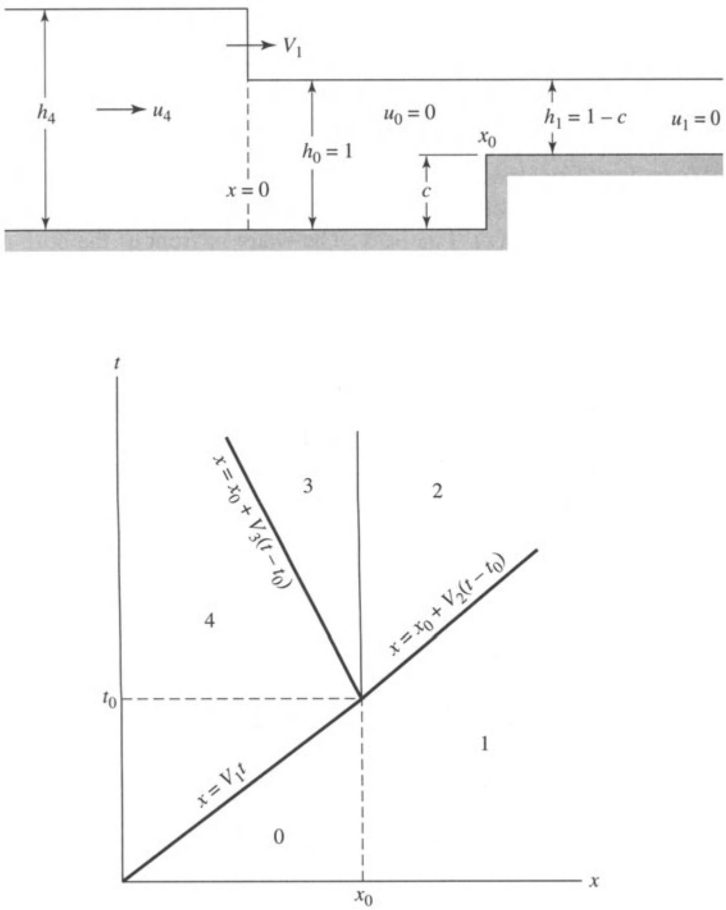

FIGURE 5.16. Incoming bore over a bottom step

to calculate

$$u_3 = 0.3974, \; h_3 = 1.4878, \; h_2 = 0.9877, \qquad (5.3.117)$$
$$V_2 = 1.2122, \; V_3 = -0.5043, \; u_2 = 0.5986.$$

d. Now assume that h_4 is only slightly higher than $h_0 = 1$, and set $h_4 = 1 + \epsilon\tilde{h}$, where $0 < \epsilon \ll 1$ and $\tilde{h}$ is an $O(1)$ constant. show that the linear theory predicts

$$\begin{aligned} V_1 &= V_3 = 1, \\ u_4 &= u_3 = u_2 = \epsilon\tilde{h}, \\ h_4 &= h_3 = 1 + \epsilon\tilde{h},\ h_2 = 1 + \epsilon(\tilde{h} - c), \\ u_0 &= 0, h_0 = 1, u_1 = 0, h_1 = 1 - c. \end{aligned} \tag{5.3.118}$$

Thus, there is no reflected or stationary discontinuity in the linear theory. Verify that your results in part (c) reduce to the above values for ϵ small.

5.3.10 Consider the uniformly propagating shocks corresponding to cases (1) and (2) in Table 5.1. The dimensionless change in the entropy S across a shock is given by (see Section 6.4 of [42])

$$\frac{S_1 - S_2}{C_v} \equiv s = \log \frac{p_1}{\rho_1^\gamma}, \tag{5.3.119}$$

where the subscript 1 denotes values behind the shock and 2 denotes values ahead. Show that s is positive for case (1) and negative for case (2).

5.3.11 At time $t = 0$ a bore traveling with speed $V_1 > 0$ is at $x = 0$, and a second bore traveling with speed $V_2 < 0$ is at $x = 1$. The water in the interval $0 < x < 1$ is at rest and has unit height. The water to the left of the bore V_1 has a constant height $h_1 > 1$ and constant speed $u_1 > 0$, whereas the water to the right of the bore V_2 has constant height $h_2 > 1$ and constant speed $u_2 < 0$. Let us specify these bores by fixing h_1 and h_2 and let us assume that $h_1 > h_2$. Thus, according to (5.3.47), we have

$$u_1 = (h_1 - 1)\left(\frac{h_1 + 1}{2h_1}\right)^{1/2}, \quad V_1 = \left[\frac{h_1(h_1 + 1)}{2}\right]^{1/2}, \tag{5.3.120}$$

and

$$u_2 = -(h_2 - 1)\left(\frac{h_2 + 1)}{2h_2}\right)^{1/2}, \quad V_2 = -\left[\frac{h_2(h_2 + 1)}{2}\right]^{1/2}. \tag{5.3.121}$$

We want to show that for $t > 1/(V_1 - V_2) \equiv t_c$, when the two bores have interacted, the solution still consists of two bores having speeds $\overline{V}_1 > 0$, $\overline{V}_2 < 0$ bounding the interval $V_1 t_c + \overline{V}_2(t - t_c) < x < V_1 t_c + \overline{V}_1(t - t_c)$, in which the speed is a constant u_3 and the height is a constant h_3. The water to the right of the $\overline{V}_1$ bore has $u = u_2$, $h = h_2$, whereas to the left of the $\overline{V}_2$ bore, $u = u_1$, $h = h_1$. See Figure 5.17.

a. Verify that the four bore conditions governing $u_3, h_3, \overline{V}_1, \overline{V}_2$ in terms of the known quantities u_1, h_1, u_2, h_2 are

$$\overline{V}_1 = \frac{u_2 h_2 - u_3 h_3}{h_2 - h_3}, \tag{5.3.122a}$$

$$\overline{V}_1 = \frac{u_2^2 h_2 - u_3^2 h_3 + h_2^2/2 - h_3^2/2}{u_2 h_2 - u_3 h_3}, \tag{5.3.122b}$$

$$\overline{V}_2 = \frac{u_1 h_1 - u_3 h_3}{h_1 - h_3}, \tag{5.3.123a}$$

$$\overline{V}_2 = \frac{u_1^2 h_1 - u_3^2 h_3 + h_1^2/2 - h_3^2/2}{u_1 h_1 - u_3 h_3}. \tag{5.3.123b}$$

Equate the two expressions for $\overline{V}_1$ and $\overline{V}_2$ and show that u_3 and h_3 are the solutions of the two equations

$$u_3 = u_2 + (h_3 - h_2)\left(\frac{h_3 + h_2}{2h_2 h_3}\right)^{1/2}, \tag{5.3.124a}$$

$$u_3 = u_1 - (h_3 - h_1)\left(\frac{h_3 + h_1}{2h_1 h_3}\right)^{1/2}. \tag{5.3.124b}$$

Once u_3 and h_3 have been calculated from (5.3.124), we obtain $\overline{V}_1$ from either (5.3.122a) or (5.3.122b), and $\overline{V}_2$ from either (5.3.123a) or (5.3.123b).

b. Consider the special case $h_1 = 1.5, h_2 = 1.25$, for which (5.3.120) and (5.3.121) give $u_1 = 0.4564$, $V_1 = 1.3693$, $u_2 = -0.2372$, and $V_2 = -1.1859$. Show that (5.3.124a)–(5.3.124b) give $u_3 = 0.2188$, $h_3 = 1.8041$, and (5.3.122)–(5.3.123) give $\overline{V}_1 = 1.2474$, $\overline{V}_2 = -0.9531$. What numerical scheme would you use in general to calculate u_3, h_3 from (5.3.124)?

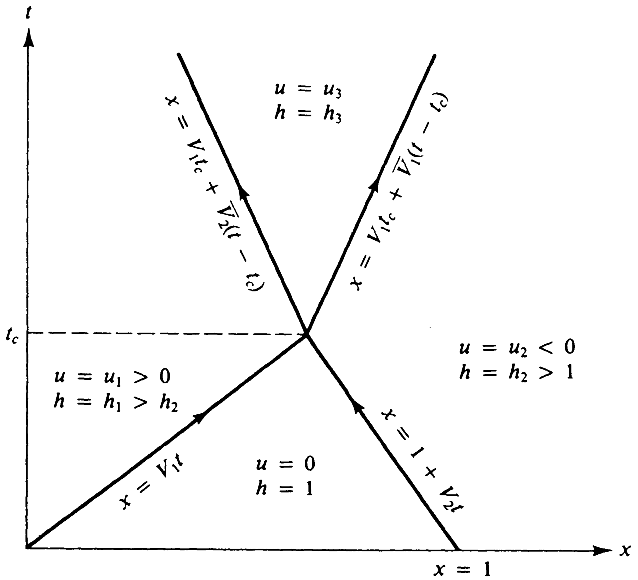

FIGURE 5.17. Intersecting constant-speed bores

c. Compare your results with those predicted for the linear theory. See (3.4.35)–(3.4.36) and Figure 3.19.

5.3.12 Consider a uniform shock propagating to the right into a quiescent gas. Use the dimensionless variables in Section 5.3.4iii so that $u = 0$, $\rho = 1$, $p = 1$ ahead of the shock, and denote the pressure behind the shock by p_i. When this shock reflects from a wall, it moves to the left into a gas with pressure p_i. Denote the pressure behind the reflected shock by p_r.

Show that

$$\frac{p_r - p_i}{p_i} = \frac{p_i - 1}{1 + (\gamma - 1)(p_i - 1)/2\gamma}. \tag{5.3.125}$$

5.4 The Scalar Quasilinear Equation in n Independent Variables

The general quasilinear equation in n independent variables $x_1, \ldots, x_n$ has the form [see (5.2.1)]

$$\sum_{j=1}^{n} a_j(\mathbf{x}, u) \frac{\partial u}{\partial x_j} = a(\mathbf{x}, u), \tag{5.4.1}$$

where the a_i and a are given functions of $\mathbf{x} = x_1, \ldots, x_n$ and u. It is assumed that in some solution domain these functions are continuous and have continuous first partial derivatives and that the a_i do not vanish simultaneously.

5.4.1 The Initial-Value Problem

The geometrical ideas developed in Section 5.2.1 generalize in a straightforward way to the present $(n + 1)$-dimensional problem. In particular, (5.4.1) implies that through each point $(\mathbf{x}, u)$ we have a characteristic curve defined by the solution of the system

$$\frac{dx_i}{ds} = a_i(\mathbf{x}, u), \quad i = 1, \ldots, n, \tag{5.2.1a}$$

$$\frac{du}{ds} = a(\mathbf{x}, u). \tag{5.4.2b}$$

Henceforth, we omit the explicit reminder that $i = 1, \ldots, n$. The characteristic curves can be expressed in the parametric form

$$x_i = \overline{X}_i(s - s_0, c_1, \ldots, c_n), \tag{5.4.3a}$$

$$u = \overline{U}(s - s_0, c_1, \ldots, c_n), \tag{5.4.3b}$$

involving the n arbitrary constants $c_1, \ldots, c_n$ and the additive constant s_0. Thus, they define an n-parameter family of curves that fills some portion of the space of $\mathbf{x}, u$.

The solution of an initial-value problem for (5.4.1) consists in finding the n-dimensional manifold $u = \phi(x_1, \ldots, x_n)$ that satisfies (5.4.1) and passes through a given smooth $(n-1)$-dimensional manifold $\mathcal{C}_0$. We may prescribe $\mathcal{C}_0$ in parametric form as follows:

$$x_i = x_i^{(0)}(\tau_1, \ldots, \tau_{n-1}), \quad u = u^{(0)}(\tau_1, \ldots, \tau_{n-1}) \tag{5.4.4}$$

in terms of the $(n-1)$ parameters $\tau_1, \ldots, \tau_{n-1}$. Here, the functions $x_i^{(0)}$, $u^{(0)}$ are continuous and have continuous first partial derivatives with respect to $\tau_1, \ldots,$ τ_{n-1}.

As in the case $n = 2$, we may set $s_0 \equiv 0$ with no loss of generality. Then regarding the c_i as functions of $\tau_1, \ldots, \tau_{n-1}$, we can generate an $(n-1)$-parameter subfamily of (5.4.3) in the form

$$\begin{aligned} x_i &= \overline{X}_i(s, c_1, (\tau_1, \ldots, \tau_{n-1}), \ldots, c_n(\tau_1, \ldots, \tau_{n-1})) \\ &\equiv X_i(s, \tau_1, \ldots, \tau_{n-1}), \end{aligned} \tag{5.4.5a}$$

$$\begin{aligned} u &= \overline{U}(s, c_1(\tau_1, \ldots, \tau_{n-1}), \ldots, c_n(\tau_1, \ldots, \tau_{n-1})) \\ &\equiv U(s, \tau_1, \ldots, \tau_{n-1}). \end{aligned} \tag{5.4.5b}$$

For fixed $\tau_1, \ldots, \tau_{n-1}$, the functions X_i and U of s also define a characteristic curve in the $(n+1)$-dimensional space of $\mathbf{x}$, u.

We now specify the family (5.4.5) by requiring it to pass through the initial manifold $\mathcal{C}_0$; that is, we set

$$X_i(0, \tau_1, \ldots, \tau_{n-1}) = x_i^{(0)}(\tau_1, \ldots, \tau_{n-1}), \tag{5.4.6a}$$

$$U(0, \tau_1, \ldots, \tau_{n-1}) = u^{(0)}(\tau_1, \ldots, \tau_{n-1}). \tag{5.4.6b}$$

For a given manifold $\mathcal{C}_0$, the conditions (5.4.6) fix the n function c_i of $\tau_1, \ldots, \tau_{n-1}$ and define an $(n-1)$-parameter family of characteristic curves that pass through $\mathcal{C}_0$ in the form

$$x_i = X_i(s, \tau_1, \ldots, \tau_{n-1}), \tag{5.4.7a}$$

$$u = U(s, \tau_1, \ldots, \tau_{n-1}). \tag{5.4.7b}$$

In practice, we shall solve the system (5.4.2) directly in the form (5.4.7).

The solution manifold $u = \phi(x_1, \ldots, x_n)$ is obtained by first solving the system (5.4.7a) for s and $\tau_1, \ldots, \tau_{n-1}$ as functions of the x_i and then substituting these expressions into (5.4.7b). We can invert (5.4.7a) as long as the Jacobian

$$\Delta(s, \tau_1, \ldots, \tau_{n-1}) \equiv \frac{\partial(X_1, \ldots, X_n)}{\partial(s, \tau_1, \ldots, \tau_{n-1})} \tag{5.4.8}$$

does not vanish.

5.4.2 The Characteristic Manifold; Existence and Uniqueness of Solutions

In preparation for dealing with the case $\Delta = 0$, we introduce the idea of a characteristic manifold.

$\mathcal{C}$ is a characteristic $(n-1)$-dimensonal manifold in the $(n+1)$-dimensional space of $\mathbf{x}$, u if at every point $(\mathbf{x}, u)$ on $\mathcal{C}$, the characteristic vector

$$\boldsymbol{\sigma} \equiv (a_1, \ldots, a_n, a) \tag{5.4.9}$$

is tangent to $\mathcal{C}$. In order to obtain an analytic description of a characteristic manifold based on this geometric statement, we define the manifold $\mathcal{C}$ by the $(n+1)$ functions

$$x_i = \tilde{X}_i(\tau_1, \ldots, \tau_{n-1}), \tag{5.4.10a}$$

$$u = \tilde{U}(\tau_1, \ldots, \tau_{n-1}). \tag{5.4.10b}$$

Now, the $(n-1)$ vectors

$$\mathbf{T}_m = \left(\frac{\partial \tilde{X}_1}{\partial \tau_m}, \ldots, \frac{\partial \tilde{X}_n}{\partial \tau_m}, \frac{\partial \tilde{U}}{\partial \tau_m}\right), \quad m = 1, \ldots, n-1, \tag{5.4.11}$$

are linearly independent tangent vectors to $\mathcal{C}$ (see (2.3.48)). The characteristic vector $\boldsymbol{\sigma}$ is tangent to $\mathcal{C}$ at some point if $\boldsymbol{\sigma}$ can be expressed as a linear combination of the $\mathbf{T}_m$ at that point—that is, if there exist $n-1$ constants $\lambda_1, \ldots, \lambda_{n-1}$ such that

$$\boldsymbol{\sigma} = \sum_{m=1}^{n-1} \lambda_m \mathbf{T}_m. \tag{5.4.12}$$

The characteristic vector $\boldsymbol{\sigma}$ is *everywhere* tangent to $\mathcal{C}$ if we can find $n-1$ *functions* $\lambda_1, \ldots, \lambda_{n-1}$ of $\tau_1, \ldots, \tau_{n-1}$ such that (5.4.12) holds everywhere on $\mathcal{C}$. In component form, (5.4.12) implies that the $n+1$ conditions

$$a_i = \sum_{m=1}^{n-1} \lambda_m \frac{\partial \tilde{X}_i}{\partial \tau_m}, \tag{5.4.13a}$$

$$a = \sum_{m=1}^{n-1} \lambda_m \frac{\partial \tilde{U}}{\partial \tau_m} \tag{5.4.13b}$$

must hold everywhere on $\mathcal{C}$ for $n-1$ functions $\lambda_1, \ldots, \lambda_{n-1}$ in order that $\mathcal{C}$ be a characteristic manifold.

It is easy to prove that if $\Delta = 0$, the n conditions (5.4.13a) are automatically satisfied. To see this, note that Δ is the determinant of the $n \times n$ matrix

$$\Delta = \det \begin{pmatrix} \dfrac{\partial X_1}{\partial s} & \cdots & \dfrac{\partial X_n}{\partial s} \\ \dfrac{\partial X_1}{\partial \tau_1} & \cdots & \dfrac{\partial X_n}{\partial \tau_1} \\ \vdots & & \vdots \\ \dfrac{\partial X_1}{\partial \tau_{n-1}} & \cdots & \dfrac{\partial X_n}{\partial \tau_{n-1}} \end{pmatrix} = \det \begin{pmatrix} a_1 & \cdots & a_n \\ \dfrac{\partial X_1}{\partial \tau_1} & \cdots & \dfrac{\partial X_n}{\partial \tau_1} \\ \vdots & & \vdots \\ \dfrac{\partial X_1}{\partial \tau_{n-1}} & \cdots & \dfrac{\partial X_n}{\partial \tau_{n-1}} \end{pmatrix}.$$

Therefore, $\Delta = 0$ implies that the first row vector $(a_1, \ldots, a_n)$ is linearly dependent on the $n-1$ row vectors $(\partial X_1/\partial \tau_1, \ldots, \partial X_n/\partial \tau_1), \ldots, (\partial X_1/\partial \tau_{n-1}, \ldots,$

$\partial X_n/\partial \tau_{n-1}$); that is, there exist $n-1$ functions $\lambda_1, \ldots, \lambda_{n-1}$ of $\tau_1, \ldots, \tau_{n-1}$ such that

$$(a_1, \ldots, a_n) = \lambda_1 \left(\frac{\partial X_1}{\partial \tau_1}, \ldots, \frac{\partial X_n}{\partial \tau_1} \right) + \ldots + \lambda_{n-1} \left(\frac{\partial X_1}{\partial \tau_{n-1}}, \ldots, \frac{\partial X_n}{\partial \tau_{n-1}} \right). \tag{5.4.14}$$

Identifying components on each side of (5.4.14) gives (5.4.13a). The converse is also true; that is, if there exist $n-1$ functions $\lambda_1, \ldots, \lambda_{n-1}$ of $\tau_1, \ldots, \tau_{n-1}$ such that (5.4.13a) holds everywhere on some manifold $\mathcal{C}$, then $\Delta = 0$ on $\mathcal{C}$.

Thus, a necessary condition for $\mathcal{C}$ to be a characteristic manifold is $\Delta = 0$. But this is not sufficient; one must also be able to show that (5.4.13b) holds for the functions $\lambda_1, \ldots, \lambda_{n-1}$ used to satisfy (5.4.13a). Some examples will be worked out later on to illustrate these ideas.

One can also prove the following theorem relating a characteristic manifold to a family of characteristic curves (see Section 2, Chapter 2 of [13]). Every characteristic manifold is generated by an $(n-2)$-parameter family of characteristic curves. Conversely, every $(n-2)$-parameter family of characteristic curves generates a characteristic manifold. We shall also illustrate this result for a specific example later on. Note, incidentally, that for $n = 2$, a characteristic manifold is just a characteristic curve; it is only for $n \geq 3$ that the characteristic manifold has a dimension higher than one and differs from a characteristic curve.

The theorem that concerns the existence and uniqueness of the solution of a given initial-value problem is now stated without proof (see [13]): The solution of (5.4.1), subject to the initial condition (5.4.4), exists and is unique in some neighborhood of $\mathcal{C}_0$ if $\Delta(0, \tau_1, \ldots, \tau_{n-1}) \neq 0$. In the event $\Delta = 0$ everywhere on $\mathcal{C}_0$, nonunique solutions exist only if $\mathcal{C}_0$ is a characteristic manifold; if $\Delta = 0$ but $\mathcal{C}_0$ is not a characteristic manifold, one cannot derive a solution of (5.4.1) passing through $\mathcal{C}_0$.

5.4.3 A Linear Example

We study the linear problem

$$x_2 \frac{\partial u}{\partial x_1} - x_1 \frac{\partial u}{\partial x_2} + \frac{\partial u}{\partial x_3} = 1, \tag{5.4.15}$$

and consider first the initial-value problem $u = 0$ on the conical surface $x_3 = x_1^2 + x_2^2$. A parametric form for $\mathcal{C}_0$ is

$$x_1^{(0)} = \tau_1, \ x_2^{(0)} = \tau_2, \ x_3^{(0)} = \tau_1^2 + \tau_2^2, \ u^{(0)} = 0. \tag{5.4.16}$$

The characteristic equations (5.4.2) specialize to

$$\frac{dx_1}{ds} = x_2, \ \frac{dx_2}{ds} = -x_1, \ \frac{dx_3}{ds} = 1, \ \frac{du}{ds} = 1. \tag{5.4.17}$$

The general solution of this system is easy to compute because the first two equations do not involve x_3 and u, and the last two are trivially solved. We obtain [see

(5.4.7)] the following two-parameter family of characteristic curves:

$$x_1 = x_2^{(0)}(\tau_1, \tau_2) \sin s + x_1^{(0)}(\tau_1, \tau_2) \cos s \equiv X_1(s, \tau_1, \tau_2), \tag{5.4.18a}$$

$$x_2 = -x_1^{(0)}(\tau_1, \tau_2) \sin s + x_2^{(0)}(\tau_1, \tau_2) \cos s \equiv X_2(s, \tau_1, \tau_2), \tag{5.4.18b}$$

$$x_3 = s + x_3^{(0)}(\tau_1, \tau_2) \equiv X_3(s, \tau_1, \tau_2), \tag{5.4.18c}$$

$$u = s + u^{(0)}(\tau_1, \tau_2) \equiv U(s, \tau_1, \tau_2). \tag{5.4.18d}$$

The particular two-parameter family that passes through the initial manifold (5.4.16) is

$$x_1 = \tau_2 \sin s + \tau_1 \cos s, \tag{5.4.19a}$$

$$x_2 = \tau_2 \cos s - \tau_1 \sin s, \tag{5.4.19b}$$

$$x_3 = s + \tau_1^2 + \tau_2^2, \tag{5.4.19c}$$

$$u = s. \tag{5.4.19d}$$

Using (5.4.18) in the definition of Δ gives

$$\begin{aligned}\Delta(s, \tau_1, \tau_2) = {} & x_1^{(0)} \left(\frac{\partial x_3^{(0)}}{\partial \tau_2} \frac{\partial x_1^{(0)}}{\partial \tau_1} - \frac{\partial x_3^{(0)}}{\partial \tau_1} \frac{\partial x_1^{(0)}}{\partial \tau_2} \right) \\ & + x_2^{(0)} \left(\frac{\partial x_3^{(0)}}{\partial \tau_2} \frac{\partial x_2^{(0)}}{\partial \tau_1} - \frac{\partial x_3^{(0)}}{\partial \tau_1} \frac{\partial x_2^{(0)}}{\partial \tau_2} \right) \\ & + \left(\frac{\partial x_1^{(0)}}{\partial \tau_1} \frac{\partial x_2^{(0)}}{\partial \tau_2} - \frac{\partial x_1^{(0)}}{\partial \tau_2} \frac{\partial x_2^{(0)}}{\partial \tau_1} \right),\end{aligned} \tag{5.4.20}$$

and for the special case (5.4.16), we have

$$\Delta(s, \tau_1, \tau_2) = \Delta(0, \tau_1, \tau_2) = 1. \tag{5.4.21}$$

Therefore, we expect a unique solution manifold to result from (5.4.19). It is easily verified that this manifold is

$$u = x_3 - (x_1^2 + x_2^2). \tag{5.4.22}$$

Let us now demonstrate that the two-dimensional manifold generated by an arbitrary one-parameter family of characteristic curves is a characteristic manifold. One way to define a general one-parameter family of characteristic curves is to set $\tau_2 = r(\tau_1)$ (for an arbitrary function r) in (5.4.18). To generate a two-dimensional manifold from the resulting one-parameter family of curves, we replace $s \to \tau_2$ and $\tau_1 \to \tau_1$ and regard the new τ_1, τ_2 as the two variables on the manifold. Then $x_2^{(0)}, x_1^{(0)}, x_3^{(0)}$, and $u^{(0)}$ may be regarded as arbitrary functions f, g, h, and k of τ_1, respectively, and we obtain a two-dimensional manifold in the form

$$x_1 = f(\tau_1) \sin \tau_2 + g(\tau_1) \cos \tau_2 \equiv \tilde{X}_1(\tau_1, \tau_2), \tag{5.4.23a}$$

$$x_2 = -g(\tau_1) \sin \tau_2 + f(\tau_1) \cos \tau_2 \equiv \tilde{X}_2(\tau_1, \tau_2), \tag{5.4.23b}$$

$$x_3 = \tau_2 + h(\tau_1) \equiv \tilde{X}_3(\tau_1, \tau_2), \tag{5.4.23c}$$

$$u = \tau_2 + k(\tau_1) \equiv \tilde{U}(\tau_1, \tau_2). \tag{5.4.23d}$$

In order to prove that the manifold (5.4.23) is characteristic, we must find λ_1 and λ_2 such that the four equations (5.4.13) are satisfied. For our case, these are

$$-g(\tau_1)\sin\tau_2 + f(\tau_1)\cos\tau_2 = \lambda_1[f'(\tau_1)\sin\tau_2 + g'(\tau_1)\cos\tau_2] + \lambda_2[f(\tau_1)\cos\tau_2 - g(\tau_1)\sin\tau_2], \qquad (5.4.24a)$$

$$-f(\tau_1)\sin\tau_2 - g(\tau_1)\cos\tau_2 = \lambda_1[-g'(\tau_1)\sin\tau_2 + f'(\tau_1)\cos\tau_2] + \lambda_2[-f(\tau_1)\sin\tau_2 - g(\tau_1)\sin\tau_2], \qquad (5.4.24b)$$

$$1 = \lambda_1 h'(\tau_1) + \lambda_2, \qquad (5.4.24c)$$

$$1 = \lambda_1 k'(\tau_1) + \lambda_2. \qquad (5.4.24d)$$

Solving (5.4.24a)–(5.4.24b) for λ_1 and λ_2 gives $\lambda_1 = 0$ and $\lambda_2 = 1$, and these values indeed also satisfy (5.4.24c)–(5.4.24d). Therefore, the two-dimensional manifold (5.4.23) is a characteristic manifold. The converse is also true—any characteristic manifold can be generated by a one-parameter family of characteristic curves.

For the particular initial-value problem (5.4.16), we see that we may interpret the solution (5.4.22) to be generated either by the two-parameter family of characteristic curves (5.4.19) or the following one-parameter (ξ) family of characteristic manifolds:

$$x_1 = \xi\sin\tau_2 + \tau_1\cos\tau_2 \equiv \tilde{X}_1(\tau_1, \tau_2; \xi), \qquad (5.4.25a)$$

$$x_2 = \xi\cos\tau_2 - \tau_1\sin\tau_2 \equiv \tilde{X}_2(\tau_1, \tau_2; \xi), \qquad (5.4.25b)$$

$$x_3 = \tau_2 + \tau_1^2 + \xi^2 \equiv \tilde{X}_3(\tau_1, \tau_2; \xi), \qquad (5.4.25c)$$

$$u = \tau_2 \equiv \tilde{U}(\tau_1, \tau_2; \xi). \qquad (5.4.25d)$$

We see that for any fixed ξ, (5.4.25) defines a characteristic manifold, and we have already derived the solution (5.4.22) from just such a set of equations (albeit before we had labeled $\xi \to \tau_2, \tau_1 \to \tau_1, \tau_2 \to s$).

Consider now the initial manifold C_0 with

$$x_1^{(0)} = \tau_1\cos\tau_2, \quad x_2^{(0)} = -\tau_1\sin\tau_2, \quad x_3^{(0)} = \tau_2 + \tau_1^2, \quad u^{(0)} = \tau_2. \qquad (5.4.26)$$

It is easily seen that (5.4.20) gives $\Delta(0, \tau_1, \tau_2) = 0$ in this case. The family of characteristic curves (5.4.18) that results for this choice is the degenerate one,

$$x_1 = \tau_1\cos(s + \tau_2), \quad x_2 = -\tau_1\sin(s + \tau_2),$$
$$x_3 = (s + \tau_2) + \tau_1^2, \quad u = (s + \tau_2), \qquad (5.4.27)$$

in which s and τ_2 occur only in the combination $(s + \tau_2)$. This implies that (5.4.27) actually defines just the characteristic manifold C_0 (instead of a one-parameter family of characteristic manifolds or a two-parameter family of characteristic curves). To verify this statement, note that (5.4.13) gives the four conditions

$$-\tau_1\sin(s + \tau_2) = \lambda_1\cos(s + \tau_2) - \lambda_2\tau_1\sin(s + \tau_2),$$
$$\tau_1\cos(s + \tau_2) = -\lambda_1\sin(s + \tau_2) - \lambda_2\tau_1\cos(s + \tau_2),$$
$$1 = 2\lambda_1\tau_1 + \lambda_2, \quad 1 = \lambda_2,$$

which are satisfied with $\lambda_1 = 0, \lambda_2 = 1$. In this case, the solution manifold is not unique. We can exhibit this nonuniqueness by noting that the implicit formula

$$u = x_3 - (x_1^2 + x_2^2) + F\left(u + \tan^{-1}\frac{x_2}{x_1}\right) \tag{5.4.28}$$

defines a solution u for any function F as long as $F(0) = 0$ and (5.4.28) can be solved for u.

Finally, if we define $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}$ as in (5.4.26) but choose $u^{(0)} \neq \tau_2$, we cannot solve the equations that correspond to (5.4.27). In this case, $\Delta = 0$, but $\mathcal{C}_0$ is not a characteristic manifold.

5.4.4 A Quasilinear Example

Consider the generalizaton of (5.3.15c) to three independent variables—that is,

$$\frac{\partial u}{\partial x_1} + u\left(\frac{\partial u}{\partial x_2} + \frac{\partial u}{\partial x_3}\right) = 0. \tag{5.4.29}$$

The characteristics satisfy the equations

$$\frac{dx_1}{ds} = 1, \quad \frac{dx_2}{ds} = u, \quad \frac{dx_3}{ds} = u, \quad \frac{du}{ds} = 0, \tag{5.4.30}$$

which can be solved in the form

$$x_1 = s + x_1^{(0)}(\tau_1, \tau_2), \tag{5.4.31a}$$

$$x_2 = u^{(0)}(\tau_1, \tau_2)s + x_2^{(0)}(\tau_1, \tau_2), \tag{5.4.31b}$$

$$x_3 = u^{(0)}(\tau_1, \tau_2)s + x_3^{(0)}(\tau_1, \tau_2), \tag{5.4.31c}$$

$$u = u^{(0)}(\tau_1, \tau_2). \tag{5.4.31d}$$

Let us restrict attention to solutions that pass through the initial manifold,

$$x_1^{(0)} = 0; \quad x_2^{(0)} = \tau_1; \quad x_3^{(0)} = \tau_2; \quad u^{(0)} = \sin\tau_1 \sin\tau_2, \tag{5.4.32}$$

for which we compute

$$x_1 = s, \tag{5.4.33a}$$

$$x_2 = s\sin\tau_1\sin\tau_2 + \tau_1, \tag{5.4.33b}$$

$$x_3 = s\sin\tau_1\sin\tau_2 + \tau_2, \tag{5.4.33c}$$

$$u = \sin\tau_1\sin\tau_2, \tag{5.4.33d}$$

and

$$\Delta(s, \tau_1, \tau_2) = 1 + s\sin(\tau_1 + \tau_2). \tag{5.4.34}$$

Thus, $\Delta(0, \tau_1, \tau_2) = 1$, and a unique solution exists near the initial manifold. Since we cannot solve for τ_1 and τ_2 in terms of x_1, x_2, and x_3 in closed form, we write the solution in the implicit form

$$u = \sin(x_2 - x_1 u)\sin(x_3 - x_1 u). \tag{5.4.35}$$

This solution first breaks down when $x_1 = 1$, and it cannot be extended to $x_1 > 1$ for values of x_1, x_2, x_3 that lie on the surface

$$\tau_1(x_1, x_2, x_3) + \tau_2(x_1, x_2, x_3) = -\sin^{-1}(1/x_1), \tag{5.4.36}$$

where $\tau_1(x_1, x_2, x_3)$ and $\tau_2(x_1, x_2, x_3)$ are the solutions of (5.4.33b) and (5.4.33c) in which $s = x_1$.

Although some aspects of the theory of weak solutions, as discussed in Section 5.3, can be extended to higher dimensions, we shall not present these results here. Certainly, the geometry of shock manifolds for dimensions greater than 1 becomes more complicated. But the difficulties are not confined just to questions of geometry. For example, it is no longer possible to derive an exact solution of the two-dimensional Burgers' equation

$$\frac{\partial u}{\partial x_1} + u\left(\frac{\partial u}{\partial x_2} + \frac{\partial u}{\partial x_3}\right) = \epsilon\left(\frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}\right). \tag{5.4.37}$$

Therefore, it is more difficult to establish what is an admissible weak solution of the integral conservation law that led to (5.4.29).

Problem

5.4.1 Consider the equation

$$uu_x + u_y + yu_z = 1. \tag{5.4.38}$$

a. Calculate the two-parameter family of characteristic curves.
b. Solve (5.4.38) for the initial-value problem $u = 0$ on $y = x^2 + z^2$.
c. Give an example of a noncharacteristic initial manifold on which $\Delta(0, \tau_1, \tau_2)$, as defined by (5.4.8), vanishes.

6

Nonlinear First-Order Equations

Nonlinear first-order partial differential equations arise in geometrical optics, in the description of dynamical systems by Hamilton–Jacobi theory, and in other applications. In this chapter we begin with a discussion of the underlying physical principles and then study the mathematical theory that provides a unifying description of a number of different problems.

6.1 Geometrical Optics: A Nonlinear Equation

In this section we shall derive a nonlinear equation that is the basic mathematical model in geometrical optics, dynamics, and variational calculus. These links are established in later sections; here our discussion is based on the problem in optics. The results that we shall derive are also valid in acoustics or any process involving the propagation of a disturbance in an isotropic medium with a given space-dependent signal speed. Here, discussion proceeds from physical principles, and the results are shown in Section 6.3 to be consequences of the general theory for the nonlinear first-order equations.

6.1.1 Huygens' Construction; the Eikonal Equation

In geometrical optics, we study the propagation boundary of an optical disturbance (wave front) without regard to such factors as the intensity, frequency, or phase of the light wave. In fact, we only distinguish between domains through which a disturbance has passed and undisturbed ones, and keep track of the boundary separating these two domains at any given time t. Moreover, we assume that a disturbance at some time $t = t_0$ at the point $P_0 = (x_0, y_0, z_0)$ propagates locally in an isotropic manner with speed $c_0 \equiv c(x_0, y_0, z_0)$; that is, at time $t_0 + \Delta t$, the disturbance that originated at t_0 and P_0 has spread along a *spherical* surface of radius $c_0 \Delta t$ and center at P_0. Every point on the disturbance surface or wave front is also regarded as a continuous emitter of disturbances, consequently advancing the wave front into the medium.

J. Kevorkian, *Partial Differential Equations*

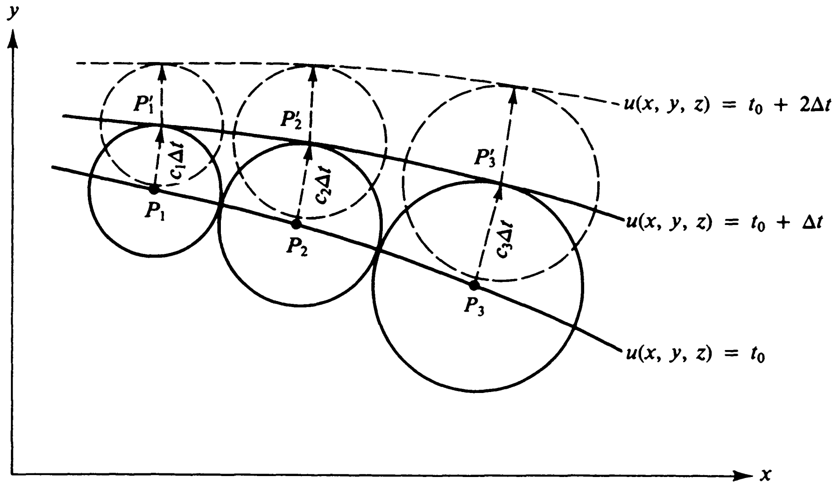

FIGURE 6.1. Disturbance emitted from two consecutive wave fronts

Suppose that we have a medium with a given signal speed $c(x, y, z)$ at every point and assume that at time $t = t_0$, the wave front is defined by the surface

$$u(x, y, z) = t_0 = \text{constant}, \tag{6.1.1}$$

as shown in cross section in Figure 6.1. To fix ideas, let the region toward increasing y be the undisturbed zone at time $t = t_0$.

Now, each point on the surface defined by (6.1.1) emits a disturbance with a propagation speed c, which depends on location. In Figure 6.1, we illustrate the situation where c increases with x. Consider a sequence of points $P_1, P_2, P_3, \ldots$ lying on the surface (6.1.1). At time $t = t_0 + \Delta t$, the disturbances emitted from P_1, P_2, P_3 will be located along the spheres centered at P_1, P_2, P_3 and having radii equal to $c_1\Delta t$, $c_2\Delta t$, $c_3\Delta t, \ldots$. Therefore, the wave front at time $t = t_0 + \Delta t$ will be the *envelope to all these spheres*. This geometrical construction, which is attributed to C. Huygens, can be translated into an analytical description of the surface u once we recognize that *light rays are orthogonal to wave fronts*. In fact, the light rays emanating from a point P are a one-parameter family of radial vectors centered at P, and the particular rays that connect P_1 to P_1', P_2 to $P_2', \ldots$ are each orthogonal to the new front $P_1', P_2', P_3', \ldots$ at time $t = t_0 + \Delta t$. Let us denote the infinitesimal displacement vector along a light ray by $d\boldsymbol{\sigma}$. In Cartesian form, $d\boldsymbol{\sigma} \equiv dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors in the x, y, and z directions, respectively. It then follows from the definition of the gradient of a scalar function (see (2.3.57)) that

$$|\operatorname{grad} u| \cdot |d\boldsymbol{\sigma}| = dt, \tag{6.1.2}$$

because $d\sigma$ is parallel to grad u. But along a light ray we have

$$dt = \frac{|d\sigma|}{c}. \tag{6.1.3}$$

Therefore, eliminating dt from (6.1.2) and (6.1.3) gives

$$|\operatorname{grad} u|^2 = \frac{1}{c^2}, \tag{6.1.4a}$$

or

$$u_x^2 + u_y^2 + u_z^2 = \frac{1}{c^2(x, y, z)}, \tag{6.1.4b}$$

in terms of the Cartesian coordinates x, y, z. Equation (6.1.4) is called the *eikonal* equation. It has a number of other interpretations besides the one just discussed. For example, see (6.2.59), (6.2.60), (6.2.72), and (6.2.118). In Problem 4.2.3 we showed that (6.1.4b) governs characteristic manifolds of the wave equation. Thus, Huygens' construction reconfirms our original interpretation of characteristics as wave fronts along which discontinuities propagate.

Consider the special case of (6.1.4b) where $c = c_0 =$ constant and $\partial/\partial z \equiv 0$, that is, disturbances do not vary in the z-direction, and introduce a point disturbance initially, say at $t = 0$, $x = y = 0$. Actually, this corresponds to a line of disturbances along the z-axis. Since c_0 is constant, this point disturbance in the xy-plane must propagate along the front $x^2 + y^2 = c_0^2 t^2$, which is a circle of radius $c_0 t$ centered at the origin. In xyu-space, this front is the surface of the right circular cone

$$u = \frac{\sqrt{x^2 + y^2}}{c_0}, \tag{6.1.5}$$

which is easily seen to be a solution of (6.1.4b) for this special case. We shall rederive this result in Section 6.3.4i from the general theory (see also Section 6.3.4iii and Problems 6.3.1–6.3.2 for examples with variable c).

6.1.2 *The Equation for Light Rays*

To simplify the derivation, we consider the two-dimensional case and denote $u_x = p$ and $u_y = q$. We shall show that along a light ray, the following system of first-order equations is satisfied:

$$\frac{dx}{d\sigma} = cp, \tag{6.1.6a}$$

$$\frac{dy}{d\sigma} = cq, \tag{6.1.6b}$$

$$\frac{du}{d\sigma} = \frac{1}{c}, \tag{6.1.6c}$$

$$\frac{dp}{d\sigma} = -\frac{c_x}{c^2}, \tag{6.1.6d}$$

$$\frac{dq}{d\sigma} = -\frac{c_y}{c^2}, \tag{6.1.6e}$$

where $d\sigma \equiv (dx^2 + dy^2)^{1/2}$ is the infinitesimal arc length along a light ray.

To prove (6.1.6a)–(6.1.6b), note that the vector with components $(dx/d\sigma, dy/d\sigma)$ is a unit tangent along a light ray. The unit normal to a wave front, defined by the surface $u(x, y) = \text{constant}$, is by definition the vector with components (cp, cq). Therefore, (6.1.6a)–(6.1.6b) just states the already observed fact that light rays are normal to wave fronts. Equation (6.1.6c) is simply a restatement of (6.1.3), and we show next that (6.1.6d)–(6.1.6e) define the curvature of light rays.

To simplify this derivation further, let the wave front at $t = 0$ be tangent to the x-axis at some point $A = (x_0, 0)$, as shown in Figure 6.2.

The sketch shows that $c(A) \equiv c(x_0, 0)$ is smaller than $c(B) \equiv c(x_0 + \Delta x, 0)$ because after a time Δu, the light ray AC emerging from A has traveled a shorter distance than the ray BD.

The vectors **AB**, **AC**, and **BD** have the following components:

$$\mathbf{AB} = (\Delta x, 0), \quad \mathbf{AC} = (0, c(A)\Delta u),$$
$$\mathbf{BD} = (0, c(B)\Delta u) = (0, c(A)\Delta u + c_x(A)\Delta x \Delta u + O(\Delta x^2 \Delta u)).$$

Therefore, the infinitesimal turning angle $d\alpha$ of the wave front is

$$d\alpha \equiv \lim_{\Delta x \to 0} \frac{|\mathbf{BD} - \mathbf{AC}|}{|\mathbf{AB}|} = c_x(A)du.$$

In general, for an arbitrary initial wave front orientation, we would have

$$\frac{d\alpha}{du} = \operatorname{grad} c \cdot \boldsymbol{\tau}, \tag{6.1.7}$$

where $\boldsymbol{\tau}$ is the unit tangent to the wave front—that is,

$$\boldsymbol{\tau} \equiv (cq, -cp), \tag{6.1.8a}$$

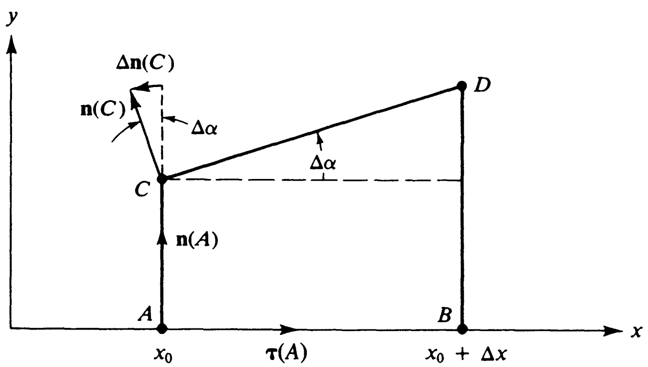

FIGURE 6.2. Curvature of a light ray

since the unit normal is

$$\mathbf{n} \equiv (cp, cq). \tag{6.1.8b}$$

Also, we note from Figure 6.2 that $\mathbf{n}(C) = -(\sin \Delta\alpha)\boldsymbol{\tau}(A) + (\cos \Delta\alpha)\mathbf{n}(A) = -\Delta\alpha\boldsymbol{\tau}(A) + \mathbf{n}(A) + O(\Delta\alpha^2)$. Thererfore, $\Delta\mathbf{n}(C) \equiv \mathbf{n}(C) - \mathbf{n}(A) = -\Delta\alpha\boldsymbol{\tau}(A) + O(\Delta\alpha^2)$, and in the limit $\Delta\alpha \to 0$, the infinitesimal change $d\mathbf{n}$ in the unit normal is given by

$$d\mathbf{n} = -d\alpha\boldsymbol{\tau},$$

or

$$\frac{d\mathbf{n}}{d\sigma} = -\frac{d\alpha}{d\sigma}\boldsymbol{\tau}. \tag{6.1.9}$$

Now

$$\frac{d\alpha}{d\sigma} = \frac{d\alpha}{du}\frac{du}{d\sigma} = \frac{1}{c}\,\text{grad}\, c \cdot \boldsymbol{\tau} = qc_x - pc_y,$$

where we have used (6.1.6c) for $du/d\sigma$.

Therefore, the first component of (6.1.9) is

$$\frac{d}{d\sigma}(cp) = -cq(qc_x - pc_y), \tag{6.1.10a}$$

and the second component is

$$\frac{d}{d\sigma}(cq) = cp(qc_x - pc_y). \tag{6.1.10b}$$

We develop the left-hand side of (6.1.10a) to obtain

$$c\frac{dp}{d\sigma} + p\left(c_x\frac{dx}{d\sigma} + c_y\frac{dy}{d\sigma}\right) = -cc_xq^2 + cc_ypq. \tag{6.1.11}$$

Using (6.1.6a)–(6.1.6b) for $(dx/d\sigma)$ and $(dy/d\sigma)$ in the preceding, canceling cc_ypq from both sides of (6.1.11), and then dividing by c gives (6.1.6d). Similarly, (6.1.6e) follows from (6.1.10b). In Section 6.3.2 we shall show that the system (6.1.6) defines certain *characteristic strips* associated with the eikonal equation $p^2 + q^2 = 1/c^2$.

In the three-dimensional problem, with $x = x_1$, $y = x_2$, $z = x_3$, $(\partial u/\partial x_i) = p_i$, we have the system of seven first-order equations

$$\frac{dx_i}{d\sigma} = cp_i, \quad i = 1, 2, 3, \tag{6.1.12a}$$

$$\frac{du}{d\sigma} = \frac{1}{c}, \tag{6.1.12b}$$

$$\frac{dp_i}{d\sigma} = -\frac{1}{c^2}\frac{\partial c}{\partial x_i}, \quad i = 1, 2, 3. \tag{6.1.12c}$$

An alternative description of the light rays in the two-dimensional problem is to eliminate u (which merely specifies the time along a ray) and σ (which specifies the length of a ray) and to derive the equation governing the ray trajectories in the xy-plane. Let us express y as a function of x along a ray and let $' \equiv d/dx$. We can combine (6.1.6a)–(6.1.6b) to write $y'^2 = q^2/p^2$, or

$$y'^2 = \frac{1 - c^2 p^2}{c^2 p^2}, \tag{6.1.13}$$

when we use the eikonal equation $p^2 + q^2 = 1/c^2$. Solving (6.1.13) for p^2 gives

$$p^2 = \frac{1}{c^2(1 + y'^2)}. \tag{6.1.14}$$

Dividing (6.1.6d) by (6.1.6a) gives $p' = -c_x/c^3 p$, from which it follows that

$$(p^2)' = -2\frac{c_x}{c^3} = \left(\frac{1}{c^2}\right)_x. \tag{6.1.15}$$

We now differentiate (6.1.14) with respect to x and use (6.1.15) for $(p^2)'$ to obtain (after some algebra)

$$cy'' + (c_y - y'c_x)(1 + y'^2) = 0. \tag{6.1.16}$$

This second-order quasilinear equation defines a ray trajectory in a given medium with specified $c(x, y)$ once we prescribe $y(x_0)$ and $y'(x_0)$. In particular, note that if $c =$ constant, (6.1.16) reduces to $y'' = 0$, or $y(x)$ is a straight line, as expected.

A more fundamental interpretation of (6.1.16) is that if a light ray passes through two fixed points (x_0, y_0) and (x_1, y_1), the path that it takes (as defined by a solution of (6.1.16), subject to the two boundary conditions $y(x_0) = y_0$, $y(x_1) = y_1$), is a path of minimum time. This is *Fermat's principle*, discussed in the next section.

6.2 Applications Leading to the Hamilton–Jacobi Equation

In Section 6.1 we studied the basic problem for geometrical optics by looking for surfaces of $t =$ constant along which disturbances propagate, to derive the eikonal equation. The orthogonal trajectories to these surfaces define the light rays. In this section we will show that these light rays may be regarded as certain paths of minimum elapsed time between two fixed points.

This multiplicity of interpretations can be found in a number of applications governed by the Hamilton–Jacobi equation. This equation is a general version of the eikonal equation, and in this section we study how it arises in the calculus of variations, geometry, and dynamics.

6.2.1 The Variation of a Functional

Let the n continuously differentiable functions $q_1(s), q_2(s), \ldots, q_n(s)$ be the components of a vector $\mathbf{q}(s)$. We shall borrow the terminology used in dynamics (which, as we shall show later one, provides one interpretation of our results) and refer to the $q_1, \ldots, q_n$ as *coordinates*. In this case s will be the time, but for the present purposes s is an unspecified independent variable. Let L, the ***Lagrangian***, be a given function of the $2n + 1$ variables $s, q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n$, where a dot denotes a derivative with respect to s. We shall use the abbreviated notation $L(s, q_i, \dot{q}_i)$, where the subscript i indicates that all n components of a vector quantity may occur. We say that a *motion* is given if the q_i are prescribed functions of s. Thus, along a given motion, L is a scalar function of s.

Consider now the following functional:

$$J \equiv \int_{s_I}^{s_F} L(s, q_i, \dot{q}_i)ds. \tag{6.2.1}$$

For a given motion, and if the integral exists over the interval $s_I \leq s \leq s_F$, J is a number. This number depends on the functional form of L, the given motion, and the values of s_I and s_F. Suppose that we now vary the motion and the endpoints slightly and evaluate J; that is, we compute

$$J^* \equiv \int_{s_I^*}^{s_F^*} L(s, q_i^*, \dot{q}_i^*)ds, \tag{6.2.2}$$

where

$$\begin{aligned} s_I^* &\equiv s_I + \delta s_I, \quad s_F^* \equiv s_F + \delta s_F, \\ q_j^*(s) &\equiv q_j(s) + \delta q_j(s), \quad \dot{q}_j^*(s) \equiv \dot{q}_j(s) + \delta \dot{q}_j(s), \quad j = 1, \ldots, n. \end{aligned} \tag{6.2.3}$$

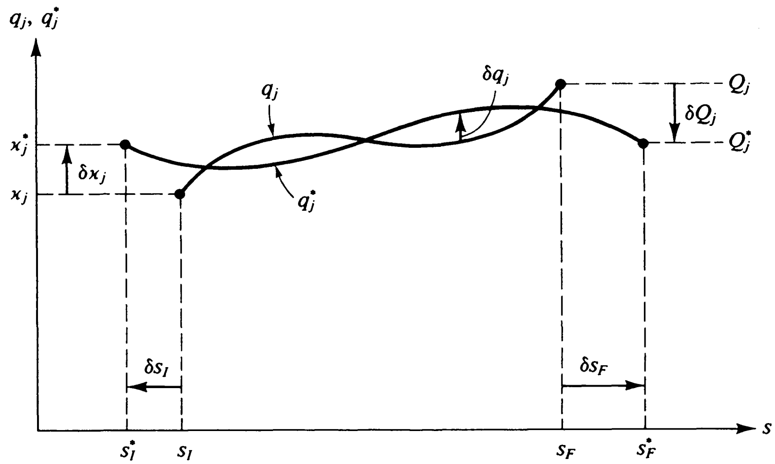

FIGURE 6.3. A given motion q_j and its variation q_j^*

Henceforth, for simplicitly, we shall omit pointing out that $j = 1, \ldots, n$. The δ-notation is somewhat awkward and is adopted only for the sake of tradition; it indicates that the quantity in question is a small perturbation of the associated variable. Note carefully that $\delta q_j(s)$ does not indicate that $q_j(s)$ is multiplied by δ. Rather, $\delta q_j(s)$ is a completely independent function of s that merely introduces a small perturbation to $q_j(s)$. Also, $\delta \dot{q}_j(s) \equiv (d/ds)[\delta q_j(s)]$. We shall refer to δq_j as the variation of q_j; a typical curve for $q_j(s)$ and $q_j^*(s)$ is sketched in Figure 6.3.

As seen in this figure, we have used the notation

$$\kappa_j \equiv q_j(s_I), \quad Q_j \equiv q_j(s_F), \quad \kappa_j^* \equiv q_j^*(s_I^*), \quad Q_j^* \equiv q_j^*(s_F^*), \tag{6.2.4}$$

to indicate values of q_j and q_j^* at the initial and final points. It then follows that

$$\kappa_j^* = \kappa_j + \delta\kappa_j = q_j^*(s_I + \delta s_I) = q_j(s_I + \delta s_I) + \delta q_j(s_I + \delta s_I). \tag{6.2.5}$$

Strictly speaking, q_j is not defined outside the interval $s_I \leq s \leq s_F$, and $q_j(s_I + \delta s_I)$ may be ambiguous if $\delta s_I < 0$. However, regardless of the sign of δs_I, we extend the definition of q_j by linear extrapolation and set

$$q_j(s_I + \delta s_I) = q_j(s_I) + \dot{q}_j(s_I)\delta s_I = \kappa_j + \dot{q}_j(s_I)\delta s_I. \tag{6.2.6a}$$

We also have

$$\delta q_j(s_I + \delta s_I) = \delta q_j(s_I), \tag{6.2.6b}$$

and we have neglected quadratic terms in small quantities in (6.2.6). If we use (6.2.6) in the right-hand side of (6.2.5) and equate perturbation quantities, we obtain

$$\delta\kappa_j = \delta q_j(s_I) + \dot{q}_j(s_I)\delta s_I.$$

The notation

$$\dot{\kappa}_j \equiv \dot{q}_j(s_I), \quad \dot{Q}_j \equiv \dot{q}_j(s_F), \tag{6.2.7}$$

then leads to the formula

$$\delta\kappa_j = \delta q_j(s_I) + \dot{\kappa}_j \delta s_I, \tag{6.2.8a}$$

and the corresponding expression

$$\delta Q_j = \delta q_j(s_F) + \dot{Q}_j \delta s_F \tag{6.2.8b}$$

for the endpoint.

We are now ready to compute $\delta J \equiv J^* - J$. First, we split the interval of integration for J^* in (6.2.2) and write this as

$$J^* = \int_{s_I}^{s_F} L(s, q_i^*, \dot{q}_i^*)ds - \int_{s_I}^{s_I+\delta s_I} L(s, q_i^*, \dot{q}_i^*)ds + \int_{s_F}^{s_F+\delta s_F} L(s, q_i^*, \dot{q}_i^*)ds. \tag{6.2.9}$$

If we ignore quadratic terms in perturbation quantities, the three integrals in (6.2.9) can be approximated as follows:

$$\int_{s_I}^{s_F} L(s, q_i^*, \dot{q}_i^*)ds = \int_{s_I}^{s_F} \left\{ L(s, q_i, \dot{q}_i) + \sum_{j=1}^{n} \frac{\partial L}{\partial q_j}\delta q_j + \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j \right\} ds,$$

$$\int_{s_I}^{s_I+\delta s_I} L(s, q_i^*, \dot{q}_i^*)ds = L(s_I, \kappa_i, \dot{\kappa}_i)\delta s_I,$$

$$\int_{s_F}^{s_F+\delta s_F} L(s, q_i^*, \dot{q}_i^*)ds = L(s_F, Q_i, \dot{Q}_i)\delta s_F.$$

Therefore,

$$\delta J = L(s_F, Q_i, \dot{Q}_i)\delta s_F - L(s_I, \kappa_i, \dot{\kappa}_i)\delta s_I + \int_{s_I}^{s_F} \sum_{j=1}^{n} \left\{ \frac{\partial L}{\partial q_j}\delta q_j + \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j \right\} ds. \tag{6.2.10}$$

Let us define

$$\frac{\partial L}{\partial \dot{q}_j} \equiv p_j, \tag{6.2.11}$$

and refer to the p_j as the *momenta*, using the terminology of dynamics.

Along a given motion, the p_j are known functions of s, and we denote the endpoint values of the p_j by

$$p_j(s_I) \equiv \phi_j, \quad p_j(s_F) \equiv P_j.$$

Now, if we integrate the second term in the integrand in (6.2.10) by parts and collect the coefficients of δs_I and δs_F, we find on using (6.2.8) that

$$\delta J = \left[L(s_F, Q_i, \dot{Q}_i) - \sum_{j=1}^{n} P_j \dot{Q}_j \right] \delta s_F - \left[L(s_I, \kappa_i, \dot{\kappa}_i) - \sum_{j=1}^{n} \phi_j \dot{\kappa}_j \right] \delta s_I$$
$$+ \sum_{j=1}^{n} (P_j \delta Q_j - \phi_j \delta \kappa_j) + \int_{s_I}^{s_F} \sum_{j=1}^{n} \left\{ \frac{\partial L}{\partial q_j} - \frac{d}{ds}\left[\frac{\partial L}{\partial \dot{q}_j} \right] \right\} \delta q_j ds. \tag{6.2.12}$$

This defines the variation of the functional J in (6.2.1) for arbitrary variations of the endpoints, the values of the q_i at the endpoints, and the values of the functions $q_i(s)$ in the interval $s_I \leq s \leq s_F$.

6.2.2 A Variational Principle; the Euler–Lagrange Equations; Examples

Suppose that we require

$$\delta J = 0 \tag{6.2.13a}$$

subject to

$$\delta\kappa_i = \delta Q_i = 0 \tag{6.2.13b}$$

and

$$\delta s_I = \delta s_F = 0. \tag{6.2.13c}$$

This means that we allow the δq_i to be arbitrary (small) functions of s in the fixed interval $s_I \leq s \leq s_F$ with *fixed* values of the q_i at the endpoints, and we look for that set of q_i for which the variation of J is zero. It follows from (6.2.12) that we must have

$$\int_{s_I}^{s_F} \sum_{j=1}^{n} \left[\frac{\partial L}{\partial q_j} - \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{q}_j} \right) \right] \delta q_j ds = 0.$$

This integral will vanish for arbitrary δq_j only if the coefficient of each δq_j in the integrand vanishes—that is,

$$\frac{\partial L}{\partial q_j} - \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{q}_j} \right) = 0. \tag{6.2.14}$$

This system of n second-order equations, attributed to Euler and Lagrange, gives a *necessary condition* for the solution of the variational problem (6.2.13).

We see from (6.2.14) that whenever a particular q_j is absent from the Lagrangian, the associated momentum is a constant along the motion. In dynamics, $L \equiv T - V$, where T is the kinetic energy and V is the potential energy; the variational principle (6.2.13) is called *Hamilton's principle*, and (6.2.14) are called Lagrange's equations governing the evolution of the coordinates q_i. For example, see Chapater 2 of [20].

(i) *Fermat's principle*

Consider two fixed points $P = (x_0, y_0)$ and $Q = (x_1, y_1)$ in a two-dimensional optical medium with speed of light $c(x, y)$. The time elapsed, u, for light to travel from P to Q along a given ray $y(x)$ is given by the line integral

$$u = \int_P^Q \frac{d\sigma}{c(x, y)}, \tag{6.2.15}$$

where $d\sigma$ is the infinitesimal arc length along this ray. Thus, $d\sigma = \sqrt{1 + y'^2(x)}dx$, and (6.2.15) becomes

$$u = \int_{x_0}^{x_1} \frac{\sqrt{1 + y'^2}}{c(x, y)} dx. \tag{6.2.16}$$

This is a special case of (6.2.1) with $n = 1$, $J = u$, $s = x$, $q = y$, and $L(x, y, y') = \sqrt{1 + y'^2}/c(x, y)$. Fermat's principle is just the variational principle (6.2.13), and states that the path $y(x)$ that a light ray follows between any two fixed points along the ray is one that minimizes the elapsed time. To write down the Euler–Lagrange equation, we compute

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = \frac{d}{dx}\left(\frac{y'}{c(x,y)\sqrt{1+y'^2}}\right)$$
$$= \frac{-c_x y' - c_y y'^2}{c^2\sqrt{1+y'^2}} + \frac{y''}{c\sqrt{1+y'^2}} - \frac{y'^2 y''}{c(1+y'^2)^{3/2}}, \quad (6.2.17)$$
$$\frac{\partial L}{\partial y} = -\frac{c_y\sqrt{1+y'^2}}{c^2}. \quad (6.2.18)$$

Substituting (6.2.17)–(6.2.18) into (6.2.14) and simplifying gives the equation for light rays (6.1.16) that we derived earlier using Huygens' construction. This formula becomes awkward if rays have a vertical tangent. A general result valid for arbitrary ray paths and in three dimensions can be derived for a medium with speed of light $c(x_1, x_2, x_3)$ using Fermat's principle or the system (6.1.12). The time elapsed for light to travel along a ray, defined parametrically in the form $x_1(s)$, $x_2(s)$, $x_3(s)$, is given by

$$J = \int_{s_I}^{s_F} \frac{\left(\sum_{j=1}^{3} \dot{x}_j^2\right)^{1/2}}{c(x_1, x_2, x_3)}\, ds, \quad (6.2.19)$$

where s is any parameter that varies monotonically along the ray, such as the arc length or the time. Fermat's principle leads to the system (6.2.14) with $L = (\sum_{j=1}^{3} \dot{x}_j^2)^{1/2}/c$. As expected, the Euler–Lagrange equations associated with (6.2.19) correspond to the equations for light rays that we obtain from (6.1.12) (see Problem 6.2.1).

(ii) *Geodesics*

A related more general problem that arises in geometry has

$$L = \left[\sum_{j=1}^{n}\sum_{k=1}^{n} g_{jk}(q_i)\dot{q}_j\dot{q}_k\right]^{1/2}. \quad (6.2.20)$$

Here $L\,ds$ is the infinitesimal displacement in the n-dimensional space spanned by the curvilinear coordinates q_i, and the $g_{jk} = g_{kj}$ define the fundamental metric tensor (see (2.3.49)). To fix ideas, let q_1, q_2 denote the spherical polar coordinates on $\mathcal{S}$, the unit sphere centered at origin. We express the Cartesian x, y, z coordinates in terms of q_1, q_2 by

$$x = \sin q_1 \cos q_2, \quad y = \sin q_1 \sin q_2, \quad z = \cos q_1. \quad (6.2.21)$$

Thus, q_1 is the colatitude and q_2 is the longitude on $\mathcal{S}$ measured from the x-axis. Now, the infinitesimal displacement between two neighboring points on $\mathcal{S}$ is

$$|d\mathbf{x}| = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{dq_1^2 + \sin^2 q_1 dq_2^2}. \quad (6.2.22)$$

Therefore, for this example $n = 2$, $g_{11} = 1$, $g_{12} = g_{21} = 0$, and $g_{22} = \sin^2 q_1$. If we regard s as an arbitrary parameter along a curve on S, then J is the distance between s_I and s_F. The solution of (6.2.14) for a given Lagrangian (6.2.20) is called a *geodesic* on the corresponding surface, and we refer to J as the *geodetic distance*. Thus, for the special case (6.2.22), the geodesics are great circles. In general, the solution of (6.2.14) associated with the functional (6.2.1) is called an *extremal*, and we next derive an alternative representation of the equations governing extremals.

6.2.3 Hamiltonian Form of the Variational Problem

(i) *Hamilton's differential equations from the Euler–Lagrange equations; Legendre Transformation*

We seek an alternative representation of the equations (6.2.14) as a system of $2n$ first-order equations. We accomplish this by eliminating the $\dot{q}_i$ in favor of a new set of n variables. One choice is to introduce a *Legendre transformation* defined by the *Hamiltonian*

$$H = \sum_{j=1}^{n} p_j \dot{q}_j - L, \tag{6.2.23a}$$

where the p_j are the momenta defined in (6.2.11),

$$p_j \equiv \frac{\partial L}{\partial \dot{q}_j}. \tag{6.2.23b}$$

The transformation (6.2.23) is implemented as follows. We first solve the n equations (6.2.23b) for the $\dot{q}_i$ in terms of the q_i and p_i in the form

$$\dot{q}_j = f_j(s, q_i, p_i), \tag{6.2.24}$$

and then use this result in (6.2.23a) to express the Hamiltonian as a function of s, q_i, p_i. Note that a necessary and sufficient condition for the solvability of the system (6.2.23b) in the form (6.2.24) is that the Jacobian for the system (6.2.23b) be nonzero; that is,

$$\det\left\{\frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_k}\right\} \neq 0. \tag{6.2.25}$$

Note, in particular, that if any one of the $\dot{q}_j$ is absent from L, then a Legendre transformation does not exist. See also the discussion following (6.2.33) for another example where this determinant vanishes.

If (6.2.25) is satisfied, a Legendre transformation exists, and we can express the Hamiltonian in the form $H(s, q_i, p_i)$. Now, if we calculate the differential of H, we have

$$dH = \frac{\partial H}{\partial s} ds + \sum_{j=1}^{n}\left(\frac{\partial H}{\partial q_j} dq_j + \frac{\partial H}{\partial p_j} dp_j\right). \tag{6.2.26a}$$

But according to (6.2.23a), we must also have

$$dH = \sum_{j=1}^{n}\left(p_j d\dot{q}_j + \dot{q}_j dp_j - \frac{\partial L}{\partial q_j}dq_j - \frac{\partial L}{\partial \dot{q}_i}d\dot{q}_j\right) - \frac{\partial L}{\partial s}ds,$$

or

$$dH = -\frac{\partial L}{\partial s}ds + \sum_{j=1}^{n}\left(-\frac{\partial L}{\partial q_j}dq_j + \dot{q}_j dp_j\right), \tag{6.2.26b}$$

when we use (6.2.23b). Therefore, equating the coefficients of ds, dq_j, and dp_j in the two expressions for dH gives

$$\frac{\partial H}{\partial s} = -\frac{\partial L}{\partial s}, \tag{6.2.27a}$$

$$\frac{\partial H}{\partial q_j} = -\frac{\partial L}{\partial q_j}, \tag{6.2.27b}$$

$$\frac{\partial H}{\partial p_j} = \dot{q}_j. \tag{6.2.27c}$$

The formulas (6.2.23) and (6.2.27) define a Legendre transformation from $L(s, q_i, \dot{q}_i)$ and the $(q_i, \dot{q}_i)$ variables to $H(s, q_i, p_i)$ and the (q_i, p_i) variables. Repeating the Legendre transformation—that is, eliminating the p_i in favor of new variables, say u_i—we arrive back at the L, q_i, $\dot{q}_i$ set. Therefore, a Legendre transformation is its own inverse.

Now, if the q_i satisfy the Euler–Lagrange equations (6.2.14), then (6.2.23b) and (6.2.27b) lead to the system of n equations

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}. \tag{6.2.28a}$$

Equations (6.2.27c) give n equations for the $\dot{q}_i$,

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \tag{6.2.28b}$$

and (6.2.27a) relates $(\partial H/\partial s)$ to $(\partial L/\partial s)$. Equations (6.2.28) are *Hamilton's differential equations* associated with the Hamiltonian $H(s, q_i, p_i)$.

Again, we note that if a particular coordinate is absent from H, the corresponding momentum is a constant according to (6.2.28a). This is consistent with the observation made earlier based on the Euler–Lagrange equations: If q_i is absent from L, then it is also absent from H according to (6.2.23a). Similarly, if a particular p_k is absent from H, (6.2.28b) implies that q_k is constant. Another important property of the Hamiltonian function is that along a solution of (6.2.28),

$$\dot{H} = \frac{\partial H}{\partial s}. \tag{6.2.29}$$

To show this result, we compute the general expression for the derivative with respect to s of $H(s, q_i(s), p_i(s))$:

$$\dot{H} = \frac{\partial H}{\partial s} + \sum_{j=1}^{n} \left(\frac{\partial H}{\partial q_j} \dot{q}_j + \frac{\partial H}{\partial p_j} \dot{p}_j \right). \tag{6.2.30}$$

Along a solution of (6.2.28), the terms under the summation sign in (6.2.30) cancel identically, and we obtain (6.2.29). An important special case has L, and hence H, independent of s; that is, $\partial H / \partial s = 0$ [see (6.2.27a)]. In this case $H(q_i, p_i) =$ constant is an integral of the system (6.2.28).

(ii) *Hamilton's differential equation from a variational principle*

We shall now show that the Hamiltonian system of differential equations (6.2.28) also results directly from a variational principle. We introduce the functional

$$I = \int_{s_I}^{s_F} \left\{ \sum_{j=1}^{n} p_j \dot{q}_j - H(s, q_i, p_i) \right\} ds, \tag{6.2.31}$$

which equals J if the Legendre transformation (6.2.23) exists. Consider the variational principle

$$\delta I = 0, \tag{6.2.32a}$$

subject to

$$\delta \kappa_j = \delta Q_j = 0 \tag{6.2.32b}$$

and

$$\delta s_I = \delta s_F = 0. \tag{6.2.32c}$$

In view of (6.2.32b) and (6.2.32c), δI is simply

$$\delta I = \int_{s_I}^{s_F} \left\{ \sum_{j=1}^{n} p_j^* \dot{q}_j^* - H(s, q_i^*, p_i^*) \right\} ds - \int_{s_I}^{s_F} \left\{ \sum_{j=1}^{n} p_j \dot{q}_j - H(s, q_i, p_i) \right\} ds, \tag{6.2.33}$$

where the $q_j^*, \dot{q}_j^*$ are defined in (6.2.3) and $p_j^* \equiv p_j + \delta p_j$. If we ignore quadratic terms in perturbation quantities, (6.2.33) reduces to

$$\delta I = \int_{s_I}^{s_F} \left\{ \sum_{j=1}^{n} p_j \delta \dot{q}_j + \dot{q}_j \delta p_j - \frac{\partial H}{\partial q_j} \delta q_j - \frac{\partial H}{\partial p_j} \delta p_j \right\} ds.$$

Integrating the first term by parts and then using (6.2.32b) gives

$$\delta I = \int_{s_I}^{s_F} \left\{ -\sum_{j=1}^{n} \left(\dot{p}_j + \frac{\partial H}{\partial q_j} \right) \delta q_j + \sum_{j=1}^{n} \left(\dot{q}_j - \frac{\partial H}{\partial p_j} \right) \delta p_j \right\} ds. \tag{6.2.34}$$

Therefore, the variational principle (6.2.32) gives the Hamiltonian system (6.2.28). In dynamics (6.2.32) is called *Hamilton's extended principle.*

A Legendre transformation need not exist. For example, consider the case where the Lagrangian is a homogeneous function of degree one in the $\dot{q}_i$; that is, L has the property $L(s, q_i, \alpha\dot{q}_i) = \alpha L(s, q_i, \dot{q}_i)$ for any $\alpha > 0$. The Lagrangians in (6.2.19)–(6.2.20) have this property. According to Euler's theorem for homogeneous functions, we have the identity

$$\sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j = L. \tag{6.2.35}$$

It then follows from the definition (6.2.23) for H that $H \equiv 0$, so a Legendre transformation is not possible.

(iii) *A problem in dynamics*

Consider a dynamical system having two degrees of freedom characterized by the coordinates q_1, q_2, which evolve as functions of the time t. For the time being, we describe this system somewhat generally by assuming that its Lagrangian has the form

$$L = T - V, \tag{6.2.36a}$$

where T is the kinetic energy, with the form

$$T \equiv \frac{1}{2}[\phi_1(q_1) + \phi_2(q_2)][\dot{q}_1^2 + \dot{q}_2^2], \tag{6.2.36b}$$

and V is the potential energy, with the form

$$V(q_1, q_2) \equiv -\frac{V_1(q_1) + V_2(q_2)}{\phi_1(q_1) + \phi_2(q_2)} \tag{6.2.36c}$$

for prescribed functions $\phi_1(q_1)$, $\phi_2(q_2)$, $V_1(q_1)$, and $V_2(q_2)$. A Lagrangian in the form (6.2.36) is said to be of *Liouville type*. Since L does not depend on t explicitly, H is constant along a solution, so it is useful to calculate H first. We have

$$p_j \equiv \frac{\partial L}{\partial \dot{q}_j} = (\phi_1 + \phi_2)\dot{q}_j. \tag{6.2.37}$$

Therefore, in this case

$$H \equiv \sum_{j=1}^{2} p_j\dot{q}_j - L = T + V = E = \text{total energy} = \text{constant}. \tag{6.2.38}$$

To derive the Lagrange equations, we compute

$$\frac{\partial L}{\partial q_j} = \frac{1}{2}\frac{d\phi_j}{dq_j}(\dot{q}_1^2 + \dot{q}_2^2) + \frac{1}{\phi_1 + \phi_2}\frac{dV_j}{dq_j} - \frac{V_1 + V_2}{(\phi_1 + \phi_2)^2}\frac{d\phi_j}{dq_j},$$

and in view of (6.2.38), this is just

$$\frac{\partial L}{\partial q_j} = \frac{E}{\phi_1 + \phi_2}\frac{d\phi_j}{dq_j} + \frac{1}{\phi_1 + \phi_2}\frac{dV_j}{dq_j}. \tag{6.2.39}$$

Using (6.2.37) and (6.2.39) in Lagrange's equations (6.2.14) gives

$$\frac{d}{dt}[(\phi_1 + \phi_2)\dot{q}_j] - \frac{E}{\phi_1 + \phi_2}\frac{d\phi_j}{dq_j} - \frac{1}{\phi_1 + \phi_2}\frac{dV_j}{dq_j} = 0, \quad j = 1, 2. \quad (6.2.40)$$

These equations admit a second integral, as can be seen by multiplying (6.2.40) by $2(\phi_1 + \phi_2)\dot{q}_j$, to obtain

$$\frac{d}{dt}[(\phi_1 + \phi_2)^2\dot{q}_j^2 - 2E\phi_j - 2V_j] = 0, \quad j = 1, 2, \tag{6.2.41a}$$

or

$$(\phi_1 + \phi_2)^2\dot{q}_j^2 - 2E\phi_j - 2V_j = \delta_j = \text{ constant}, \quad j = 1, 2. \tag{6.2.41b}$$

The two integrals in (6.2.41b) are not independent, as can be seen by adding the two expressions for $j = 1, 2$ to obtain $\delta_1 + \delta_2 = 0$ when the energy integral (6.2.38) is used. The significance of the integrals (6.2.41b) will become clear later on, when we discuss the complete integral in Section 6.4.3.

The Hamiltonian form of the evolution equations are derived from the expression for H in (6.2.38) written as a function of the q_i, p_i; that is,

$$H(q_i, p_i) = \frac{1}{2}\frac{p_1^2 + p_2^2}{\phi_1 + \phi_2} - \frac{V_1 + V_2}{\phi_1 + \phi_2}. \tag{6.2.42}$$

We obtain

$$\dot{q}_j = p_j/(\phi_1 + \phi_2), \tag{6.2.43a}$$

$$\dot{p}_j = \frac{H(q_i, p_i)}{\phi_1 + \phi_2}\frac{d\phi_j}{dq_j} + \frac{1}{\phi_1 + \phi_2}\frac{dV_j}{dq_j}, \quad j = 1, 2, \tag{6.2.43b}$$

which are equivalent to (6.2.40).

A specific example of a Lagrangian of the form (6.2.36) occurs in the *Euler problem* when appropriate q_1, q_2 variables are chosen. Euler's problem consists of the motion of a particle of mass m in the field of two *fixed* Newtonian centers of gravitation having masses M_1 and M_2 and located at $X = \pm D$, $Y = 0$, $Z = 0$. It is clear from symmetry that the particle will remain in the $Z = 0$ plane if Z and dZ/dT both vanish simultaneously, and we shall consider only this planar case. We normalize the X- and Y-coordinates using D, and the time T using $[D^3/\gamma(M_1 + M_2)]^{1/2}$, where γ is the universal gravitational constant (see (2.4.1)). The equations of motion (mass $\times$ acceleration $=$ force) then take the dimensionless form

$$\ddot{x} = -\mu\frac{(x-1)}{r_1^2} - (1-\mu)\frac{(x+1)}{r_2^3}, \tag{6.2.44a}$$

$$\ddot{y} = -\mu\frac{y}{r_1^3} - (1-\mu)\frac{y}{r_2^3}, \tag{6.2.44b}$$

where

$$r_1^2 \equiv (x-1)^2 + y^2, \quad r_2^2 \equiv (x+1)^2 + y^2, \tag{6.2.44c}$$

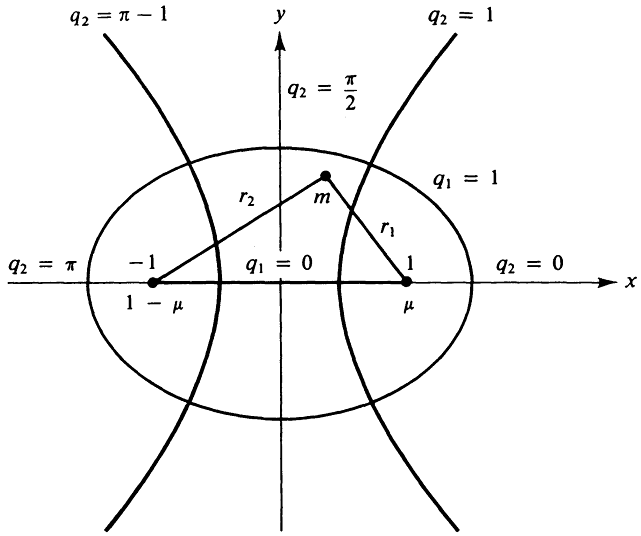

FIGURE 6.4. Elliptic–hyperbolic coordinates

and $\mu \equiv M_1/(M_1 + M_2)$ (see Figure 6.4).

We must note at the outset that having the two gravitational centers M_1 and M_2 fixed in an inertial frame is dynamically inconsistent, so Euler's problem does not model an actual gravitational three-body problem. A more appropriate model that does correspond to a limiting case of the gravitational three-body problem is the so-called restricted three-body problem. Here, M_1 and M_2 move in circular orbits under their mutual gravitation, unaffected by m, which is assumed to be very small. Thus, m moves in the gravitational field that results from the motion of M_1 and M_2 (see Problem 6.2.3).

A limiting case of (6.2.44), which does have some physical significance, corresponds to letting $\mu \to 0$ with $\bar{x}, \bar{y}, \bar{t}$ fixed, where $\bar{x} \equiv (x-1)/\mu^{1/2}, \bar{y} \equiv y/\mu^{1/2}$, $\bar{t} \equiv t/\mu^{1/4}$. In this limit, (6.2.44) tends to

$$\frac{d^2\bar{x}}{d\bar{t}^2} = -\frac{\bar{x}}{\bar{r}^3} - \frac{1}{4}, \tag{6.2.45a}$$

$$\frac{d^2\bar{y}}{d\bar{t}^2} = -\frac{\bar{y}}{\bar{r}^3}, \tag{6.2.45b}$$

where $\bar{r}^2 \equiv \bar{x}^2 + \bar{y}^2$. In this limit, the particle of mass m moves in the field of a Newtonian gravitational center at the origin plus a uniform horizontal field represented by the term $-\frac{1}{4}$ in (6.2.45a). Such a uniform field would result, for example, if we were to account for the light pressure due to the sun on a near-earth satellite. A second interpretation of (6.2.45) is the motion of an electron under the added effect of a uniform external electrostatic field.

In writing (6.2.44), we appealed to Newton's second law of motion. An equivalent starting point would be to construct the Lagrangian for Hamilton's principle. Equations (6.2.44) would then correspond to Lagrange's equations (6.2.14). The advantage of having the Lagrangian in some given coordinate system, such as the Cartesian xy-system, is that we can then transform the equations of motion to any other coordinate system by first transforming the Lagrangian and then evaluating (6.2.14) for the new coordinates. We illustrate this feature next.

The kinetic energy T and potential energy V of the particle in terms of the dimensionless (x, y) variables are

$$T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2), \quad V = -\frac{\mu}{r_1} - \frac{(1-\mu)}{r_2}. \tag{6.2.46}$$

Therefore, the Lagrangian is $L \equiv T - V$, and it is easily seen that Lagrange's equations with the $(x, y, \dot{x}, \dot{y})$ variables are just (6.2.44). It is also clear that V, as given in (6.2.46), is not in the form (6.2.36c), so the integrals (6.2.41b) are not immediately available; the only obvious integral is the Hamiltonian.

Suppose that instead of (x, y), we introduce the curvilinear coordinates (q_1, q_2) defined by (see Figure 6.4)

$$x = \cosh q_1 \cos q_2, \quad y = \sinh q_1 \sin q_2. \tag{6.2.47}$$

Since it follows from (6.2.47) that

$$\frac{x^2}{\cosh^2 q_1} + \frac{y^2}{\sinh^2 q_1} = 1, \quad \frac{x^2}{\cos^2 q_2} - \frac{y^2}{\sin^2 q_2} = 1, \tag{6.2.48}$$

we see that curves of $q_1 =$ constant are confocal ellipses in the xy-plane with foci at $x = \pm 1$, $y = 0$, semimajor axes $\cosh q_1$, and semiminor axes $\sinh q_1$. Similarly, curves of $q_2 =$ constant are confocal hyperbolas orthogonal to the family of ellipses.

If we now compute $(\dot{x}, \dot{y})$ and use these expressions, together with (6.2.47), in (6.2.46), we obtain

$$T(q_1, q_2, \dot{q}_1, \dot{q}_2) = \frac{1}{2}(\cosh^2 q_1 - \cos^2 q_2)(\dot{q}_1^2 + \dot{q}_2^2), \tag{6.2.49a}$$

$$V(q_1, q_2) = -\frac{\cosh q_1 + (2\mu - 1)\cos q_2}{\cosh^2 q_1 - \cos^2 q_2}. \tag{6.2.49b}$$

Therefore, the Lagrangian in terms of the $(q_i, \dot{q}_i)$ variables of (6.2.47) is in the form (6.2.36) with $\phi_1 = \cosh^2 q_1$, $\phi_2 = -\cos^2 q_1$, $V_1 = \cosh q_1$, and $V_2 = (2\mu - 1)\cos q_2$.

The Hamiltonian (6.2.42) is given by

$$\begin{aligned} H(q_1, q_2, p_1, p_2) &= \frac{1}{\cosh^2 q_1 - \cos^2 q_2}\left[\frac{p_1^2 + p_2^2}{2} - \cosh q_1 - (2\mu - 1)\cos q_2\right] \\ &= E = \text{constant}, \end{aligned} \tag{6.2.50}$$

where (p_1, p_2) are related to $(q_1, q_2, \dot{q}_1, \dot{q}_2)$ according to

$$p_1 \equiv \frac{\partial L}{\partial \dot{q}_1} = \frac{\partial T}{\partial \dot{q}_1} = (\cosh^2 q_1 - \cos^2 q_2)\dot{q}_1, \qquad (6.2.51a)$$

$$p_2 \equiv \frac{\partial L}{\partial \dot{q}_2} = \frac{\partial T}{\partial \dot{q}_2} = (\cosh^2 q_1 - \cos^2 q_2)\dot{q}_2. \qquad (6.2.51b)$$

The integrals (6.2.41b) have the form

$$(\cosh^2 q_1 - \cos^2 q_2)^2 \dot{q}_1^2 - 2E\cosh^2 q_1 - 2\cosh q_1 = \delta_1 = \text{constant},$$
$$(\cosh^2 q_1 - \cos^2 q_2)^2 \dot{q}_2^2 + 2E\cos^2 q_1 - 2(2\mu - 1)\cos q_2 = \delta_2 = \text{constant},$$

which can also be written in the form

$$p_1^2 - 2E\cosh^2 q_1 - 2\cosh q_1 = \delta_1 = \text{constant}, \qquad (6.2.52a)$$

$$p_2^2 + 2E\cos^2 q_1 - 2(2\mu - 1)\cos q_2 = \delta_2 = \text{constant}. \qquad (6.2.52b)$$

In Section 6.2.6 we shall give an alternative derivation of these integrals and show that they allow us to reduce to quadrature the solution for the coordinates as functions of time and four constants of integration.

6.2.4 Field of Extremals from a Point

In a sense all our developments so far in this section have been preliminaries in preparation for deriving a partial differential equation that governs certain families of extremals.

(i) *The Hamilton–Jacobi Equation*

Let $s = s_I$, $q_i = \kappa_i$ define a fixed point in the $(n+1)$-dimensonal space of (s, q_i); that is, we specify the $(n+1)$ constants $(s_I, \kappa_1, \ldots, \kappa_n)$. For each choice of endpoint $(s_F, Q_1, \ldots, Q_n)$, if a solution of the two-point boundary-value problem for (6.2.14) exists, we obtain an extremal. Along such an extremal, J, as defined by the integral (6.2.1), can be expressed as a function of the endpoint; that is, $J = J(s_F, Q_i)$.

To see this, note that the extremals emerging from the point (s_I, κ_i) are an n-parameter family of functions $q_i = f_i(s, c_i)$ involving n constants of integration $c_1, \ldots, c_n$; the other n constants of integration have been determined by the requirement $f_i(s_I, c_i) = \kappa_i$. Thus, along any one such extremal, the Lagrangian is $L(s, q_i, \dot{q}_i) = L(s, f_i(s, c_i), (\partial f_i/\partial s)(s, c_i)) \equiv \mathcal{L}(s, c_i)$, a function of s and n constants. Therefore, upon evaluating the integral in (6.2.1) along an extremal, we compute J as a function of s and n constants. But since the c_i are independent, we can invert the expression $q_i = f_i(s, c_i)$ to compute $c_i = g_i(s, q_i)$. When these expressions are substituted into the result for J, one obtains J as a function of s and the q_i. The details for a particular example are worked out later on.

Now consider all possible values of (s_F, Q_i); that is, construct the *field of extremals* through the fixed point (s_I, Q_i). Clearly, J is a scalar function of the

$(n+1)$ variables (s, q_i), and according to (6.2.12), we have

$$\delta J(s, q_i) = \left[L(s, q_i, \dot{q}_i) - \sum_{j=1}^{n} p_j \dot{q}_j \right] \delta s + \sum_{j=1}^{n} p_j \delta q_j. \qquad (6.2.53a)$$

The terms in (6.2.12) multiplied by δs_I and $\delta \kappa_i$ vanish because the point (s_I, κ_i) is fixed. The integral in (6.2.12) also vanishes because we consider only the extremals emerging from (s_I, κ_i).

If a Legendre transformation exists for the given Lagrangian, we use (6.2.23a) to write (6.2.53a) in the form

$$\delta J(s, q_i) = -H(s, q_i, p_i)\delta s + \sum_{j=1}^{n} p_j \delta q_j. \qquad (6.2.53b)$$

This implies

$$\frac{\partial J}{\partial s} = -H(s, q_i, p_i), \quad \frac{\partial J}{\partial q_j} = p_j. \qquad (6.2.54)$$

Combining the two equations (6.2.54) gives the *Hamilton–Jacobi equation*

$$\frac{\partial J}{\partial s} + H\left(s, q_i, \frac{\partial J}{\partial q_i}\right) = 0 \qquad (6.2.55)$$

for $J(s, q_i)$. This is a nonlinear first-order equation for the scalar J over the field of extremals. It is a fundamental equation in a number of applications, and we shall study its significance later on when we derive solutions.

(ii) *The general eikonal equation for a homogeneous Lagrangian of degree one*

Now, suppose that a Legendre transformation does not exist—say, for example, L is a homogeneous function of degree one in the $\dot{q}_i$, as in (6.2.20). In this case, equations (6.2.23) hold with $H \equiv 0$, and the p_i have the form

$$p_j \equiv \frac{\partial L}{\partial \dot{q}_j} = \sum_{\ell=1}^{n} \frac{g_{j\ell}\dot{q}_\ell}{L}. \qquad (6.2.56a)$$

Also, if we divide (6.2.23a) (with $H = 0$) by L, we have

$$1 = \sum_{k=1}^{n} \frac{p_k \dot{q}_k}{L}. \qquad (6.2.56b)$$

Let the matrix $\{b_{jk}\}$ be the inverse of the $\{g_{jk}\}$ matrix. Multiplying (6.2.56a) by b_{kj} and summing over j gives (for each $k = 1, \ldots, n$)

$$\sum_{j=1}^{n} b_{kj} p_j = \sum_{j=1}^{n} \sum_{\ell=1}^{n} \frac{b_{kj} g_{j\ell} \dot{q}_\ell}{L} = \frac{\dot{q}_k}{L}. \qquad (6.2.57)$$

Multiplying this by p_k, summing over k, and using (6.2.56b) gives

$$\sum_{j=1}^{n}\sum_{k=1}^{n} b_{kj} p_j p_k = 1. \tag{6.2.58}$$

Finally, using the second equation in (6.2.54) for the p_i gives

$$\sum_{j=1}^{n}\sum_{k=1}^{n} b_{kj} \frac{\partial J}{\partial q_j}\frac{\partial J}{\partial q_k} = 1. \tag{6.2.59}$$

This is a generalized eikonal equation (see (6.1.4b)). For the case of light rays, J is the time and L is specified by (6.2.19). Therefore, comparing (6.2.19) and (6.2.20) we have $\{g_{jk}\} = \delta_{jk}/c^2$, so $\{b_{jk}\} = c^2\delta_{jk}$, where δ_{jk} is the Kronecker delta. Equation (6.2.59) now reduces to

$$\left(\frac{\partial J}{\partial x_1}\right)^2 + \left(\frac{\partial J}{\partial x_2}\right)^2 + \left(\frac{\partial J}{\partial x_3}\right)^2 = \frac{1}{c^2(x_1, x_2, x_3)}, \tag{6.2.60}$$

which is just (6.1.4b).

(iii) *The analogy between dynamics and geometrical optics*

There is a mathematical analogy between the family of trajectories emerging from a fixed point in certain dynamical systems and the family of light rays emerging from a fixed point in a medium with a given speed of light c.

For simplicity, let us restrict attention to the two-dimensional problem and consider a particle moving in the x_1x_2-plane under the influence of a conservative force field; that is, in appropriate dimensionless variables the equations of motion are

$$\ddot{x}_1 = -\frac{\partial V}{\partial x_1}, \quad \ddot{x}_2 = -\frac{\partial V}{\partial x_2}, \tag{6.2.61}$$

for a given potential energy $V(x_1, x_2)$. It then follows that the Lagrangian $L \equiv T - V$ has the form

$$L(x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - V(x_1, x_2), \tag{6.2.62}$$

and the Hamiltonian is

$$H(x_1, x_2, p_1, p_2) = \frac{1}{2}(p_1^2 + p_2^2) + V(x_1, x_2) = E = \text{constant}. \tag{6.2.63}$$

The Hamilton–Jacobi equation associated with motion originating from some fixed point $t^{(0)}, x_1^{(0)}, x_2^{(0)}$ is

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left(\frac{\partial S}{\partial x_1}\right)^2 + \frac{1}{2}\left(\frac{\partial S}{\partial x_2}\right)^2 + V(x_1, x_2) = 0, \tag{6.2.64}$$

where [see (6.2.55)] we are using $S(t, x_1, x_2)$ instead of J to indicate the integral of the Lagrangian from the fixed point to a variable point (t, x_1, x_2) along solutions of (6.2.61).

Now, if we define

$$S(t, x_1, x_2) \equiv -Et + W(x_1, x_2) \tag{6.2.65}$$

and consider only the family of solutions with the same energy E, it is easily seen that W obeys the two-dimensional eikonal equation [see (6.1.4b) with $\partial/\partial z = 0$]

$$\left(\frac{\partial W}{\partial x_1}\right)^2 + \left(\frac{\partial W}{\partial x_2}\right)^2 = 2[E - V(x_1, x_2)]. \tag{6.2.66}$$

Thus, we identify $W \leftrightarrow u$ and $c^{-2} \leftrightarrow 2[E - V(x_1, x_2)]$, and we note that the equations for the light rays (6.1.6) are then identical to Hamilton's equations corresponding to (6.2.61), that is,

$$\frac{dx_j}{dt} = p_j, \quad j = 1, 2, \tag{6.2.67a}$$

$$\frac{dp_j}{dt} = -\frac{\partial V}{\partial x_j}, \quad j = 1, 2, \tag{6.2.67b}$$

if we identify $d\sigma$ in (6.1.6) with dt/c. This apparent discrepancy in units is due to the fact that u has units of time, whereas W has units of action. One can avoid this discrepancy by either considering the eikonal equation in geometrical optics for the variable $U = uc_0$, where c_0 is some constant reference value of the speed of light, or by using the variable W/E (which has units of time) in (6.2.66).

A simple example that illustrates the preceding is motion starting from the origin under the influence of uniform gravity. We have $\ddot{x}_1 = 0$, $\ddot{x}_2 = -1$. Therefore, $x_1 = ut$ and $x_2 = vt - t^2/2$, where u and v are the arbitrary constant initial values of $\dot{x}_1$ and $\dot{x}_2$ at $t = 0$. The Lagrangian is $L(x_1, x_2, \dot{x}_1, \dot{x}_2) = (\dot{x}_1^2 + \dot{x}_2^2)/2 - x_2$. Therefore, along any motion emerging from the origin, we can express L as a function of the time and the two constants u, v in the form

$$\begin{aligned} L &= \frac{u^2 + (v - t)^2}{2} - \left(vt - \frac{t^2}{2}\right) \\ &= \frac{u^2 + v^2}{2} - 2vt + t^2. \end{aligned} \tag{6.2.68}$$

Integrating this expression with respect to time over the interval $(0, t)$ gives

$$S = \left(\frac{u^2 + v^2}{2}\right) t - vt^2 + \frac{t^3}{3}. \tag{6.2.69}$$

Now, to express S as a function of the endpoint (t, x, y), we use $u = x_1/t$, $v = x_2/t - t/2$ to eliminate u and v, and we obtain

$$S(t, x_1, x_2) = \frac{x_1^2}{2t} + \frac{x_2^2}{2t} - \frac{x_2 t}{2} - \frac{t^3}{24}. \tag{6.2.70}$$

The Hamilton–Jacobi equation is

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left(\frac{\partial S}{\partial x_1}\right)^2 + \frac{1}{2}\left(\frac{\partial S}{\partial x_2}\right)^2 + x_2 = 0, \tag{6.2.71}$$

and it is easily seen that (6.2.70) satisfies (6.2.71). Also, using (6.2.65), we obtain the following eikonal equation for W:

$$\left(\frac{\partial W}{\partial x_1}\right)^2 + \left(\frac{\partial W}{\partial x_2}\right)^2 = 2(E - x_2). \tag{6.2.72}$$

In classical dynamics, the wave fronts W = constant are somewhat artificial mathematically defined surfaces with no particular physical significance. For further details, the reader is referred to [20].

6.2.5 Extremals from a Manifold; Transversality

We can generalize the idea of a field of extremals from a fixed point $s = s_I, q_i = \kappa_i$ to consider extremals originating from the *n-dimensional manifold* defined by the relation

$$\Gamma(s_I, \kappa_i) = 0 \tag{6.2.73}$$

in the $(n + 1)$-dimensional space of s and q_i. We also assume that Γ is a continuously differentiable function of its arguments, so the manifold is smooth. Thus, for example, if $n = 1$, (6.2.73) defines a smooth curve of possible initial points in sq-space. If $n = 2$, (6.2.72) defines a smooth surface in the three-dimensional space of s, q_1, q_2, and in general, (6.2.73) defines an n-dimensional manifold in the $(n + 1)$-dimensional space of s, q_i.

Let B denote the endpoint s_F, Q_i, and consider an extremal for the functional (6.2.1) subject to B being fixed and the initial point A being allowed to lie anywhere on $\Gamma = 0$. Thus, we want $\delta J = 0$ with $\delta s_F = \delta Q_i = 0$. It then follows from (6.2.12) that we must satisfy (6.2.14). In addition, we must have

$$H(s_I, \kappa_i, \Phi_i)\delta s_I - \sum_{j=1}^{n} \phi_j \delta\kappa_j = 0, \tag{6.2.74a}$$

where we have assumed that the Legendre transformation (6.2.23) exists and have used the definition (6.2.23a) for H evaluated on $\Gamma = 0$. But the variations δs_I and $\delta\kappa_i$ are not independent; they must be consistent with the requirement that the initial point s_I, κ_i lie on $\Gamma = 0$. Therefore, we must have

$$\delta\Gamma = 0 = \frac{\partial\Gamma}{\partial s_I}\delta s_I + \sum_{j=1}^{n} \frac{\partial\Gamma}{\partial\kappa_j}\delta\kappa_j. \tag{6.2.74b}$$

Since $H \neq 0$, we can solve for δs_I from (6.2.74a) and substitute this into (6.2.74b). The resulting linear homogeneous expression for the $\delta\kappa_i$ is satisfied for arbitrary variations $\delta\kappa_j$, $j = 1, \ldots, n$, if

$$-\frac{H(s_I, \kappa_i, \phi_i)}{\phi_j} = \frac{\partial\Gamma/\partial s_I}{\partial\Gamma/\partial\kappa_j}. \tag{6.2.75}$$

Equation (6.2.75), called the *transversality condition*, fixes the values of the ϕ_i (or, equivalently, the $\dot{\kappa}_i$) for any point on the manifold $\Gamma = 0$. An extremal emerging from $\Gamma = 0$ that satisfies (6.2.75) is said to be *transverse* to this manifold. If a Legendre transformation does not exist, we can still use the two requirements (6.2.74) to derive the transversality conditions.

To illustrate these ideas, consider the Lagrangian (6.2.20) for the case $n = 2$. Let the function $\Gamma(q_1, q_2) = 0$, independent of s, be given. Thus, $\Gamma = 0$ defines the same curve C in the q_1q_2-plane for all s. We wish to calculate the necessary conditions governing the geodetic distance from the fixed point $B = (q_1^{(0)}, q_2^{(0)})$ to the curve C. The geodesics from B must satisfy the Euler–Lagrange equations, and these follow immediately once we have calculated the expressions

$$\frac{\partial L}{\partial q_1} = \frac{1}{2L}\left[\frac{\partial g_{11}}{\partial q_1}\dot{q}_1^2 + 2\frac{\partial g_{12}}{\partial q_1}\dot{q}_2^2 + \frac{\partial g_{22}}{\partial q_1}\dot{q}_2^2\right], \tag{6.2.76a}$$

$$\frac{\partial L}{\partial q_2} = \frac{1}{2L}\left[\frac{\partial g_{11}}{\partial q_2}\dot{q}_1^2 + 2\frac{\partial g_{12}}{\partial q_2}\dot{q}_1\dot{q}_2 + \frac{\partial g_{22}}{\partial q_2}\dot{q}_2^2\right], \tag{6.2.76b}$$

$$\frac{\partial L}{\partial \dot{q}_1} = \frac{1}{L}[g_{11}\dot{q}_1 + g_{12}\dot{q}_2], \tag{6.2.76c}$$

$$\frac{\partial L}{\partial \dot{q}_2} = \frac{1}{L}[g_{12}\dot{q}_1 + g_{22}\dot{q}_2]. \tag{6.2.76d}$$

As shown earlier, $H = 0$ in this case, and a Legendre transformation does not exist. However, (6.2.74a) with $H = 0$ and (6.2.74b) with $(\partial\Gamma/\partial s_I) = 0$ must still hold on C, and combining these two expressions, we obtain the transversality condition

$$\frac{g_{11}\dot{q}_1 + g_{12}\dot{q}_2}{g_{12}\dot{q}_1 + g_{22}\dot{q}_2} = \frac{\partial\Gamma/\partial\kappa_1}{\partial\Gamma/\partial\kappa_2}. \tag{6.2.77}$$

For the special case of Cartesian coordinates $q_1 = x_1, q_2 = x_2, g_{11} = g_{22} = 1$, $g_{12} = 0$, (6.2.77) reduces to

$$\frac{dx_1}{dx_2} = \frac{\partial\Gamma/\partial\kappa_1}{\partial\Gamma/\partial\kappa_2},$$

that is, the geometrically obvious statement that geodesics are normal to C. For the case of spherical polar coordinates q_1, q_2 on the surface of the unit sphere [see (6.2.22)], the geodesics are the one-parameter family of great circles passing through B. The transversality condition isolates the great circles that intersect a given curve C at right angles (see Problem 6.2.4).

It is clear from the preceding simple examples that the transversality condition introduces a *local* requirement for an allowable extremal emerging from $\Gamma = 0$; there may be a number of points on $\Gamma = 0$ for which the transversality condition is satisfied. In fact, for the case of the unit sphere, we may regard the fixed point B as the north pole, $q_1 = 0$, with no loss of generality, and we see that any simple closed curve C on the sphere will contain at least two points (defining the closest and farthest points from B) where the transversality condition is satisfied. In the degenerate case, where C is the equator, this condition is satisfied at every point on C.

Consider now how we might go about calculating an extremal from the manifold $\Gamma = 0$ to a fixed point B. One approach consists of "forward-shooting" in the following sense. At $s = s_I$ we guess the n initial values κ_i and use (6.2.75) to compute the ϕ_i (or the $\dot{\kappa}_i$). Using these values of κ_i, ϕ_i (or $\dot{Q}_i$) at s_I, we integrate Hamilton's equations (6.2.28) (or the Euler–Lagrange equations (6.2.14)) forward in s up to $s = s_F$ and check whether our solution satisfies the n end conditions $q_i(s_F) = Q_i$. If not, we revise our initial guess and repeat this process until the end conditions are satisfied (if possible). An alternative approach is to guess the n unknown end values P_i (or $\dot{Q}_i$) at B and integrate backward in s to $s = s_I$, where we check whether the n transversality conditions are satisfied.

Of course, in a particular problem, one or the other approach may be preferable depending on the structure of the solution. At any rate, a solution need not exist nor be unique. More details would require digressing into an area that is beyond the scope of this text. Here our main goal is the derivation of the partial differential equation associated with the field of extremals emerging from the manifold $\Gamma = 0$.

We proceed as in the last section except that now the initial point, instead of being fixed, is allowed to range over the manifold $\Gamma = 0$. For every such point, we construct an extremal that is transverse to $\Gamma = 0$. The set of all transverse extremals emerging from $\Gamma = 0$ defines a field (over which J is a scalar function of s and the q_i, with $J = 0$ on $\Gamma = 0$) as long as the extremals *do not lie entirely on* $\Gamma = 0$. We exclude this possiblity and argue, as in the previous section, that $J(s, q_i)$ obeys the Hamilton–Jacobi equation (6.2.55) if the Legendre transformation (6.2.23) exists. We shall discuss the situation where the extremals lie entirely on $\Gamma = 0$ (characteristic initial manifold) and the case where the extremals are all tangent to $\Gamma = 0$ without lying entirely on it (caustic manifold) when we study the solution details of (6.2.55) in Section 6.3.3.

6.2.6 Canonical Transformations

Consider the Hamiltonian system of $2n$ first-order equations (6.2.28) associated with a given $H(s, q_i, p_i)$. We have shown that equations (6.2.23) define the extremals associated with the variational principle (6.2.32).

A *canonical transformation* is a special transformation of the $2n$ variables $\{q_i, p_i\}$ to a new set $\{\overline{q}_i, \overline{p}_i\}$ such that the Hamiltonian form of the governing equations *is preserved.* More precisely, let

$$\overline{q}_j = F_j(s, q_i, p_i), \qquad (6.2.78a)$$

$$\overline{p}_j = G_j(s, q_i, p_i) \qquad (6.2.78b)$$

represent a general transformation of the $2n$ variables $\{q_i, p_i\}$. Note that we do not transform s itself, but instead we allow the relations linking the two sets of coordinates and momenta, $\{q_i, p_i\}$ and $\{\overline{q}_i, \overline{p}_i\}$, to depend on s. The only restriction on the functions F_i and G_i for a general transformation is that the associated Jacobian be nonvanishing for all s in the interval of interest. For such a transformation, the equations governing the $\overline{q}_i, \overline{p}_i$ have the form

$$\dot{\overline{q}}_j = \frac{\partial F_j}{\partial s} + \sum_{k=1}^{n} \left(\frac{\partial F_j}{\partial q_k} \dot{q}_k + \frac{\partial F_j}{\partial p_k} \dot{p}_k \right), \tag{6.2.79a}$$

$$\dot{\overline{p}}_j = \frac{\partial G_j}{\partial s} + \sum_{k=1}^{n} \left(\frac{\partial G_j}{\partial q_k} \dot{q}_k + \frac{\partial G_j}{\partial p_k} \dot{p}_k \right), \tag{6.2.79b}$$

and if the $\{q_i, p_i\}$ satisfy (6.2.28), then the $\{\overline{q}_i, \overline{p}_i\}$ satisfy

$$\dot{\overline{q}}_j = \frac{\partial F_j}{\partial s} + \sum_{k=1}^{n} \left(\frac{\partial F_j}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial F_j}{\partial p_k} \frac{\partial H}{\partial q_k} \right), \tag{6.2.80a}$$

$$\dot{\overline{p}}_j = \frac{\partial G_j}{\partial s} + \sum_{k=1}^{n} \left(\frac{\partial G_j}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial G_j}{\partial p_k} \frac{\partial H}{\partial q_k} \right). \tag{6.2.80b}$$

The right-hand sides of (6.2.80) are known functions of the s, q_i, p_i for a given H and a given transformation (6.2.78). Therefore, inverting (6.2.78) and substituting the result into (6.2.80) gives $2n$ first-order equations of the form

$$\dot{\overline{q}}_j = \Lambda_j(s, \overline{q}_i, \overline{p}_i), \quad \dot{\overline{p}}_j = \Delta_j(s, \overline{q}_i, \overline{p}_i), \tag{6.2.81}$$

for functions Λ_i and Δ_i, which can be calculated in principle.

The transformation (6.2.78) is said to be canonical if there exists a new Hamiltonian $\overline{H}(s, \overline{q}_i, \overline{p}_i)$, depending on s and the new coordinates $\overline{q}_i$ and momenta $\overline{p}_i$, such that (6.2.81) is in Hamiltonian form, that is,

$$\Lambda_j = \frac{\partial \overline{H}}{\partial \overline{p}_j}, \quad \Delta_j = -\frac{\partial \overline{H}}{\partial \overline{q}_j}. \tag{6.2.82}$$

In general, the transformation (6.2.78) is not canonical. To illustrate ideas, consider the Hamiltonian form of the equations describing simple harmonic oscillations. We have $H(q, p) = \frac{1}{2}(p^2 + q^2)$ and $\dot{q} = p$, $\dot{p} = -q$, that is, $\ddot{q} + q = 0$. Now suppose we introduce the transformation $\overline{q} = q^3$, $\overline{p} = p$, which assigns a unique $(\overline{q}, \overline{p})$ to very (q, p) and vice versa according to $q = \overline{q}^{1/3}$, $p = \overline{p}$. The transformmed differential equations are calculated as follows:

$$\dot{\overline{q}} = 3q^2\dot{q} = 3q^2 p = 3\overline{q}^{2/3}\overline{p}, \tag{6.2.83a}$$

$$\dot{\overline{p}} = \dot{p} = -q = -\overline{q}^{1/3}. \tag{6.2.83b}$$

Now, in order for this system to be Hamiltonian, we must be able to find a function $\overline{H}(\overline{q}, \overline{p})$ such that $\partial \overline{H}/\partial \overline{p} = 3\overline{q}^{2/3}\overline{p}$ and $\partial \overline{H}/\partial \overline{q} = \overline{q}^{1/3}$. A necessary condition for the existence of such a function is the consistency condition

$$\frac{\partial}{\partial \overline{q}}(3\overline{q}^{2/3}\overline{p}) = \frac{\partial}{\partial \overline{p}}(\overline{q}^{1/3}), \tag{6.2.84}$$

which is clearly violated. Admittedly, this is a somewhat contrived example, proposed only to show that a transformation of dependent variables need not be canonical. Actually, a large class of transformations for the example in question are indeed canonical. In particular, the reader can verify that the linear transformation $\overline{q} = A_{11}q + A_{12}p, \overline{p} = A_{21}q + A_{22}p$ is canonical for any nonsingular constant matrix $\{A_{ij}\}$.

(i) *Generating function*

The laborious calculations that we have outlined to check whether or not a given transformation is canonical are inefficient and unnecessary, and we next discuss an approach based on the coordinate-invariance of the variational principle that governs the system (6.2.28).

The variational principle (6.2.32) is independent of the variables that we choose to represent a given Hamiltonian system. Let $\{s, q_i, p_i\}$ and $\{s, \overline{q}_i, \overline{p}_i\}$ be two such sets of variables. Then we must have

$$\delta \int_{s_I}^{s_F} \left\{ \sum_{j=1}^{n} p_j \dot{q}_j - H(s, q_i, p_i) \right\} ds = 0, \tag{6.2.85a}$$

subject to $\delta\kappa_i = \delta Q_i = 0$ and $\delta s_I = \delta s_F = 0$. Similarly, we must also have

$$\delta \int_{s_I}^{s_F} \left\{ \sum_{j=1}^{n} \overline{p}_j \dot{\overline{q}}_j - \overline{H}(s, \overline{q}_i, \overline{p}_i) \right\} ds = 0, \tag{6.2.85b}$$

subject to $\delta\overline{\kappa}_i = \delta\overline{q}_i = 0$ and $\delta s_I = \delta s_F = 0$. Since (6.2.85b) is true for any canonical transformation, we conclude by subtracting it from (6.2.85a) that the difference in integrands can at most be the total derivative of an arbitrary function of the two sets of coordinates, momenta, and s. In this case, the integral of the difference will depend only on the endpoints, and since these are fixed, the variation in question will vanish. In summary, if the two sets of coordinates and momenta are related by a canonical transformation, we must have

$$\sum_{j=1}^{n} p_j \dot{q}_j - H(s, q_i, p_i) - \sum_{j=1}^{n} \overline{p}_j \dot{\overline{q}}_j + \overline{H}(s, \overline{q}_i, \overline{p}_i) = \dot{K}, \tag{6.2.86}$$

where K, called the *generating function*, is an arbitrary function of the two sets of coodinates, momenta, and s. But since a canonical transformation must also satisfy the $2n$ conditions (6.2.78), K can depend only on s and $2n$ of the $4n$ variables. In order that K define a transformation, it must involve n of the old variables and n of the new ones, which means it must have one of the following four possible forms:

$$K_1(s, q_i, \overline{q}_i), \quad K_2(s, q_i, \overline{p}_i), \quad K_3(s, \overline{q}_i, p_i), \quad \text{or} \quad K_4(s, p_i, \overline{p}_i).$$

We now show how an arbitrary function $K_1(s, q_i, \overline{q}_i)$ generates a canonical transformation. The total derivative of K_1 is given by

$$\dot{K}_1 = \frac{\partial K_1}{\partial s} + \sum_{j=1}^{n} \left(\frac{\partial K_1}{\partial q_j} \dot{q}_j + \frac{\partial K_1}{\partial \overline{q}_j} \dot{\overline{q}}_j \right), \tag{6.2.87a}$$

and when we use this for the right-hand side of (6.2.86), multiply the result by ds, and collect terms, we obtain

$$\sum_{j=1}^{n}\left(p_j - \frac{\partial K_1}{\partial q_j}\right)dq_j - \sum_{j=1}^{n}\left(\overline{p}_j + \frac{\partial K_1}{\partial \overline{q}_j}\right)d\overline{q}_j + \left(\overline{H} - H - \frac{\partial K_1}{\partial s}\right)ds = 0. \tag{6.2.87b}$$

Since the q_i, $\overline{q}_i$ and s are $(2n + 1)$ independent quantities that can be varied arbitrarily, we conclude that we must have

$$p_j = \frac{\partial K_1}{\partial q_j}, \tag{6.2.88a}$$

$$\overline{p}_j = -\frac{\partial K_1}{\partial \overline{q}_j}, \tag{6.2.88b}$$

$$\overline{H} = H + \frac{\partial K_1}{\partial s}. \tag{6.2.88c}$$

Equations (6.2.88) define a canonical transformation implicitly. To obtain the explicit form (6.2.78), we solve the n equations (6.2.88a) for the $\overline{q}_i$ as functions of s, q_i, p_i; this gives (6.2.87a). We then use the result just computed in the right-hand sides of the n equations (6.2.88b) to obtain (6.2.78b). Finally, the new Hamiltonian $\overline{H}$ is obtained from (6.2.88c), in which H and $\partial K_1/\partial s$ are expressed as functions of the new coordinates, new momenta, and s.

To derive the transformation formulas for the case $K_2(s, q_i, \overline{p}_i)$, we appeal to our knowledge of Legendre transformations. Notice that in going from K_1 to K_2, we are replacing the $\overline{q}_i$ variables in favor of the $\overline{p}_i$. Moreover, the relationship defining the $\overline{p}_i$ is just (6.2.88b). Thus, aside from the minus sign in this equation, we are dealing with precisely the same kind of transformation as we used in (6.2.23). In particular, we define K_2 by [see (6.2.23a)]

$$K_2(s, q_i, \overline{p}_i) = \sum_{j=1}^{n} \overline{p}_j \overline{q}_j + K_1(s, q_i, \overline{q}_i). \tag{6.2.89}$$

Now, if we let K in (6.2.86) be the expression given by (6.2.89) for K_1, we obtain

$$\sum_{j=1}^{n} p_j \dot{q}_j - H(s, q_i, p_i) - \sum_{j=1}^{n} \overline{p}_j \dot{\overline{q}}_j + \overline{H}(s, \overline{q}_i, \overline{p}_i) = \frac{d}{ds}\left[K_2(s, q_i, \overline{p}_i) - \sum_{j=1}^{n} \overline{p}_j \overline{q}_j\right]. \tag{6.2.90a}$$

Upon evaluating the right-hand side and collecting terms, we obtain

$$\sum_{j=1}^{n}\left(p_j - \frac{\partial K_2}{\partial q_j}\right)dq_j + \sum_{j=1}^{n}\left(\overline{q}_j - \frac{\partial K_2}{\partial \overline{p}_j}\right)d\overline{p}_j + \left(\overline{H} - H - \frac{\partial K_2}{\partial s}\right)ds = 0. \tag{6.2.90b}$$

Therefore, the implicit definition of the canonical transformation for a given K_2 is

$$p_j = \frac{\partial K_2}{\partial q_j}, \tag{6.2.91a}$$

$$\overline{q}_j = \frac{\partial K_2}{\partial \overline{p}_j}, \qquad (6.2.91b)$$

$$\overline{H} = H + \frac{\partial K_2}{\partial s}. \qquad (6.2.91c)$$

In this case, we calculate the explicit form (6.2.78b) by solving the system (6.2.91a) for the $\overline{p}_i$ as functions of the q_i, p_i, and s; then we use this result in (6.2.91b) to obtain (6.2.78b).

Similar results can be derived for the generating functions K_3 and K_4, but we do not list these. The interested reader can find these formulas in [20].

(ii) *Hamilton–Jacobi equation*

Let us concentrate on the canonical transformation defined by (6.2.91). We have argued that these equations define, in principle, an explicit canonical transformation of the form (6.2.78). Its inverse would be in the form

$$q_j = \overline{F}_j(s, \overline{q}_i, \overline{p}_i), \quad p_j = \overline{G}_j(s, \overline{q}_i, \overline{p}_i). \qquad (6.2.92)$$

Faced with the task of solving the Hamiltonian system of $2n$ equations (6.2.28) we might ask whether it is possible to transform this system canonically to a simpler form that can be solved more readily. The simplest new Hamiltonian is $\overline{H} \equiv 0$, in which case the $\overline{q}_i$ and $\overline{p}_i$ are $2n$ constants. More importantly, the solution of the original system (6.2.28) is just (6.2.92), which gives the q_i and p_i as functions of s and $2n$ constants of integration!

In essence then, we can solve (6.2.28) if we can find the canonical transformation that renders $\overline{H} \equiv 0$. But this transformation must obey (6.2.91), which gives

$$0 = H\left(s, q_i, \frac{\partial K_2}{\partial q_i}\right) + \frac{\partial K_2}{\partial s}, \qquad (6.2.93)$$

the Hamilton–Jacobi equation for K_2. In Section 6.4 we demonstrate in detail how we can solve the system (6.2.28) if a *complete integral*, which is a solution $K_2(s, q_i, \alpha_i)$ of (6.2.93) involving n independent constants $\alpha_1, \ldots, \alpha_n$, is found.

An important special case has $\partial H/\partial s = 0$, so that $H(q_i, p_i) = \alpha_1 =$ constant. Suppose that we now seek a canonical transformation generated by a function of the type K_2, that is, one that depends on the q_i and $\overline{p}_i$, so that $\overline{H}$ is independent of the $\overline{q}_i$. In this case, the $\overline{p}_i =$ constant $= \gamma_i$. If we denote the generating function for such a canonical transformation by $W(q_i, \overline{p}_i)$, we see from (6.2.91a) and (6.2.91c) that W obeys

$$H\left(q_i, \frac{\partial W}{\partial q_i}\right) = \alpha_1, \qquad (6.2.94)$$

the time-independent Hamilton–Jacobi equation.

A complete integral of (6.2.94) is a solution $W(q_i, \alpha_i)$ involving α_1 and $(n-1)$ additional independent constants $\alpha_2, \ldots, \alpha_n$. Again, we shall show that having such a solution is equivalent to being able to solve the system (6.2.28) associated with H.

At this point, we note that if H is independent of s, then the substitution $K_2 \equiv -\alpha_1 s + W$ reduces (6.2.93) to (6.2.94). We now illustrate ideas with two examples.

(iii) *The linear oscillator*

Consider the linear oscillator with variable frequency $\omega(s)$:

$$\ddot{q} + \omega^2(s)q = 0. \tag{6.2.95}$$

This equation also follows from Hamilton's differential equations for

$$H(s, q, p) \equiv \frac{p^2 + \omega^2(s)q^2}{2}. \tag{6.2.96}$$

We obtain

$$\dot{q} = \frac{\partial H}{\partial p} = p, \quad \dot{p} = -\frac{\partial H}{\partial q} = -\omega^2(s)q, \tag{6.2.97}$$

and eliminating p gives (6.2.95). Note that H is not constant if ω depends on s. In fact,

$$\dot{H} = \frac{\partial H}{\partial s} = \omega\dot{\omega}q^2, \tag{6.2.98}$$

and once $q(s)$ is calculated, this defines $H(s)$ by quadrature along a solution.

Let us now study the properties of the canonical transformation generated by the function

$$K_1(s, \overline{q}) \equiv \frac{\omega(s)}{2} q^2 \cot \overline{q}. \tag{6.2.99}$$

Equations (6.2.88a) and (6.2.88b) give the following two relations linking q and p to $\overline{q}$ and $\overline{p}$:

$$p \equiv \frac{\partial K_1}{\partial q} = \omega q \cot \overline{q}, \tag{6.2.100a}$$

$$\overline{p} \equiv -\frac{\partial K_1}{\partial \overline{q}} = \frac{\omega q^2}{2} \csc^2 \overline{q}. \tag{6.2.100b}$$

If we want to express $\overline{q}$ and $\overline{p}$ as functions of q and p, we first solve (6.2.100a) for $\overline{q}$ to obtain (6.2.78a) for this case:

$$\overline{q} = \cot^{-1} \frac{p}{\omega q}. \tag{6.2.101a}$$

We then use this in (6.2.100b) to derive (6.2.78b):

$$\overline{p} = \frac{p^2 + \omega^2 q^2}{2\omega}. \tag{6.2.101b}$$

The inverse transformation (6.2.92) is given by

$$q = \left(\frac{2\overline{p}}{\omega}\right)^{1/2} \sin \overline{q}, \quad p = (2\omega\overline{p})^{1/2} \cos \overline{q}. \tag{6.2.102}$$

The new Hamiltonian is

$$\begin{aligned}\overline{H} = H + \frac{\partial K_1}{\partial s} &= \frac{1}{2}p^2 + \frac{\omega^2}{2}q^2 + \frac{\dot{\omega}}{2}q^2 \cot \overline{q} \\ &= \frac{1}{2}\left(2\omega\overline{p}\cos^2\overline{q}\right) + \frac{\omega^2}{2}\left(\frac{2\overline{p}}{\omega}\sin^2\overline{q}\right) + \frac{\dot{\omega}}{2}\left(\frac{2\overline{p}}{\omega}\sin^2\overline{q}\right)\cot\overline{q} \\ &= \omega(s)\overline{p} + \frac{\dot{\omega}(s)\overline{p}}{2\omega(s)}\sin 2\overline{q}. \end{aligned} \tag{6.2.103}$$

Therefore, Hamilton's differential equations for the new variables are

$$\dot{\overline{q}} = \omega(s) + \frac{\dot{\omega}(s)}{2\omega(s)}\sin 2\overline{q}, \tag{6.2.104a}$$

$$\dot{\overline{p}} = -\frac{\dot{\omega}(s)\overline{p}}{\omega(s)}\cos 2\overline{q}. \tag{6.2.104b}$$

We note that for ω = constant, $\overline{q} = \omega t + \overline{q}_0$. Thus, $\overline{q}$ is the phase, and $\overline{q}_0$ = constant is the pahse shift. Also, $\overline{p}$ = constant = E/ω, where E is the constant energy. There is no particular advantage associated with the $\overline{p}$, $\overline{q}$ variables if ω depends on s, except if $\dot{\omega}$ is small. In this case, one may construct a perturbation solution having the form

$$\overline{q} = \overline{q}_0 + \int_0^s \omega(\sigma)d\sigma + \ldots, \tag{6.2.105a}$$

$$\overline{p} = \overline{p}_0 + \ldots. \tag{6.2.105b}$$

In fact, a Hamiltonian is said to be in *standard form* if it is independent of the $\overline{q}_i$ to $O(1)$ and is a 2π-periodic function of the $\overline{q}_i$ to higher order. For a large class of problems, one can transform the Hamiltonian to such a standard form as a starting point for a perturbation solution (see Chapter 5 of [26] for more details).

In the preceding discussion, the function K_1 in (6.2.99) was just "pulled out of the hat." Any function K_1 having continuous first partial derivates with respect to its arguments generates a canonical transformation. Suppose that instead of K_1, we were given the explicit transformation (6.2.101) or (6.2.104). How would we test whether such a transformation is canonical without having to find the associated generating function or having to transform the system (6.2.97) to (6.2.104) and show that this latter is derivable from a Hamiltonian? Again, we can appeal to the invariance of the expression $p\,dq - H\,ds$. We have

$$\begin{aligned} p\,dq - H\,ds = {} & (2\omega\overline{p})^{1/2}\cos\overline{q}\left[\left(\frac{2\overline{p}}{\omega}\right)^{1/2}\cos\overline{q}\,d\overline{q} + \frac{1}{(2\omega\overline{p})^{1/2}}\sin\overline{q}\,d\overline{p}\right. \\ & \left. - (2\overline{p})^{1/2}\sin\overline{q}\,\frac{d\omega}{2\omega^{3/2}}\right] - \frac{1}{2}(2\omega\overline{p})\cos^2\overline{q}\,ds - \frac{\omega^2}{2}\frac{2\overline{p}}{\omega}\sin^2\overline{q}\,ds \\ = {} & 2\overline{p}\cos^2\overline{q}\,d\overline{q} + \sin\overline{q}\cos\overline{q}\,d\overline{p} - \overline{p}\sin\overline{q}\cos\overline{q}\,\frac{\dot{\omega}}{\omega}ds - \omega\overline{p}\,ds. \end{aligned} \tag{6.2.106}$$

Anticipating that the right-hand side must be in the form $\overline{p}\,d\overline{q} - \overline{H}\,ds + dK$, we use trigonometric identities to write $2\overline{p}\cos^2\overline{q}\,d\overline{q} = \overline{p}\,d\overline{q} + \overline{p}\cos 2\overline{q}\,d\overline{q}$, and $\sin\overline{q}\cos\overline{q} = \frac{1}{2}\sin 2\overline{q}$. Therefore,

$$p\,dq - H\,ds = \overline{p}\,d\overline{q} - \left(\omega\overline{p} + \frac{\dot{\omega}}{2\omega}\overline{p}\sin 2\overline{q}\right)ds + \left(\overline{p}\cos 2\overline{q}\,d\overline{q} + \frac{1}{2}\sin 2\overline{q}\,d\overline{p}\right). \tag{6.2.107}$$

We identify the second term on the right-hand side with $\overline{H}$ and note that the third term in parentheses is indeed the differential $d[(\overline{p}/2)\sin 2\overline{q}]$. This demonstrates that (6.2.101)–(6.2.104) is a canonical transformation without exhibiting K_1 or calculating the differential equations satisfied by the $\overline{q}$, $\overline{p}$ variables.

Now suppose that we want to find the canonical transformation that results in $\overline{H} = 0$ for new variables $\overline{q}$, $\overline{p}$. The Hamilton–Jacobi equation (6.2.93) for this case is

$$\frac{1}{2}\left(\frac{\partial K_2}{\partial q}\right)^2 + \frac{1}{2}\omega^2(s)q^2 + \frac{\partial K_2}{\partial s} = 0. \tag{6.2.108}$$

Consider the case $\omega =$ constant, where setting $K_2 \equiv -\alpha s + W(q, \alpha)$, with $\alpha =$ constant, gives

$$\frac{1}{2}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{\omega^2}{2}q^2 = \alpha. \tag{6.2.109}$$

We can integrate this equation and obtain

$$W = \int^q (2\alpha - \omega^2\xi^2)^{1/2} d\xi = \frac{\alpha}{\omega}\left[\sin^{-1}\frac{\omega q}{(2\alpha)^{1/2}} + \frac{\omega q}{(2\alpha)^{1/2}}\left(1 - \frac{\omega^2 q^2}{2\alpha}\right)^{1/2}\right], \tag{6.2.110}$$

which gives W as a function of q and α, where α is the energy. In order to define the generating function K_2, we must decide what the constant $\overline{p}$ is. This constant can be any function of α that we wish to prescribe. One obvious choice is to let $\overline{p} = \alpha$, the energy, in which case

$$K_2(s, q, \overline{p}) = -\overline{p}s + W(q, \overline{p}), \tag{6.2.111}$$

where

$$W(q, \overline{p}) = \frac{\overline{p}}{\omega}\left[\sin^{-1}\frac{\omega q}{(2\overline{p})^{1/2}} + \frac{\omega q}{(2\overline{p})^{1/2}}\left(1 - \frac{\omega^2 q^2}{2\overline{p}}\right)^{1/2}\right]. \tag{6.2.112}$$

The canonical transformation generated by K_2 obeys [see (6.2.91)]

$$p = \frac{\partial K_2}{\partial q} = \frac{\partial W}{\partial q} = (2\overline{p} - \omega^2 q^2)^{1/2}, \tag{6.2.113a}$$

$$\overline{q} = \frac{\partial K_2}{\partial \overline{p}} = -s + \frac{\partial W}{\partial \overline{p}} = -s + \frac{1}{\omega}\sin^{-1}\frac{\omega q}{(2\overline{p})^{1/2}}, \tag{6.2.113b}$$

$$\overline{H} = 0 = H + \frac{\partial K_2}{\partial s} = H - \overline{p}. \tag{6.2.113c}$$

The explicit form of this transformation is obtained by solving (6.2.113a)–(6.2.113b) for q and p. We obtain

$$q = \frac{(2\overline{p})^{1/2}}{\omega} \sin \omega(s + \overline{q}), \quad p = (2\overline{p})^{1/2} \cos \omega(s + \overline{q}). \tag{6.2.114}$$

Since $\overline{H} = 0$, it follows that $\overline{p}$ = constant = energy, and $\overline{q}$ = constant = phase shift. Thus, the transformation (6.2.114) is in fact the solution of (6.2.97) with ω = constant.

A second choice is to set $\overline{p} = \alpha/\omega$, and it is easily seen that now

$$K_2(s, q, \overline{p}) \equiv -\omega \overline{p} s + W(q, \overline{p}), \tag{6.2.115}$$

where

$$W(q, \overline{p}) = \overline{p}\left[\sin^{-1}\left(\frac{\omega}{2\overline{p}}\right)^{1/2} q + \left(\frac{\omega}{2\overline{p}}\right)^{1/2}\left(1 - \frac{\omega}{2\overline{p}}q^2\right)^{1/2}\right], \tag{6.2.116}$$

and that W in (6.2.116) generates the same canonical transformation as K_1 in (6.2.99).

Although we have evaluated the integral defining W in (6.2.112) and (6.2.116) explicitly, this calculation is not needed to define the canonical transformation; we need only to evaluate the integral resulting for $\partial W/\partial \overline{p}$.

The variables $\overline{q}$, $\overline{p}$ defined by (6.2.101) are normalized *angle and action* variables, which are important for the asymptotic solution of (6.2.104) for the case where $\dot{\omega}$ is small. In fact, it is in this context and in other perturbation problems (rather than the solution of linear constant-coefficient equations) that canonical transformations play a crucial role (see Chapter 5 of [26]).

If $\dot{\omega}$ is not small, the solution of the Hamilton–Jacobi equation (6.2.108) is no easier and certainly much less direct than the solution of (6.2.95). We pointed out earlier that there exists a connection between the solvability of a given Hamiltonian system of differential equations (6.2.28) on the one hand and the solvability (through the availability of a complete integral) of the associated Hamilton–Jacobi equation (6.2.93) on the other hand. The simple example of the linear oscillator gives a hint of this connection. Euler's problem discussed next provides a less trivial illustration. The detailed discussion of this question will be given in Section 6.4.

(iv) *Euler's problem*

In Section 6.2.3iii we showed by direct calculation that Euler's problem has two independent integrals (see (6.2.52)). It is interesting to see how this property emerges from the solvability of the Hamilton–Jacobi equation in the curvilinear coordinates (6.2.47).

We identify the energy E in (6.2.50) with α_1 and write the Hamiltonian in the form

$$H(q_1, q_2, p_1, p_2) = \frac{1}{2}\frac{p_1^2 + p_2^2}{\cosh^2 q_1 - \cos^2 q_2} - \frac{\cosh q_1 + (2\mu - 1)\cos q_2}{\cosh^2 q_1 - \cos^2 q_2}$$

$$= \alpha_1 = \text{constant.} \tag{6.2.117}$$

Therefore, after multiplying by $2(\cosh^2 q_1 - \cos^2 q_2)$, the time-independent Hamilton–Jacobi equation (6.2.94) for W is

$$\left(\frac{\partial W}{\partial q_1}\right)^2 + \left(\frac{\partial W}{\partial q_2}\right)^2 - 2[\cosh q_1 + (2\mu - 1)\cos q_2] - 2\alpha_1(\cosh^2 q_1 - \cos^2 q_2) = 0. \tag{6.2.118}$$

We have already observed that use of the elliptic–hyperbolic coordinates leads in a rather straightforward way to two independent integrals for the solution. This result is also a consequence of the remarkable simplification of the structure of the solution of (6.2.118) for W in terms of the q_1, q_2 variables. In particular, we see that assuming a solution for W in the *separated form*

$$W = W_1(q_1, \alpha_1, \alpha_2) + W_2(q_2, \alpha_1, \alpha_2) \tag{6.2.119}$$

is consistent with (6.2.118) because upon substitution of (6.2.119) into (6.2.118) and rearrangement of terms, we obtain

$$\left(\frac{\partial W_1}{\partial q_1}\right)^2 - 2\cosh q_1 - 2\alpha_1 \cosh^2 q_1$$
$$= -\left(\frac{\partial W_2}{\partial q_2}\right)^2 + 2(2\mu - 1)\cos q_2 - 2\alpha_1 \cos^2 q_2. \tag{6.2.120}$$

The right-hand side of (6.2.120) depends only on the variable q_2, whereas the left-hand side depends only on q_1. Therefore, each side is equal to a constant, say α_2. We can then calculate W_1 and W_2 by quadrature in the form assumed in (6.2.119). At this point, we may express each of the α_i as any desired function of the new momenta $\overline{p}_i$, which are also constants, since the new Hamiltonian is independent of the $\overline{q}_i$. A simple choice has $\alpha_1 = \overline{p}_1$ and $\alpha_2 = \overline{p}_2$, and we obtain the generating function in the separated form

$$W(q_1, q_2, \overline{p}_1, \overline{p}_2) = W_1(q_1, \overline{p}_1, \overline{p}_2) + W_2(q_2, \overline{p}_1, \overline{p}_2), \tag{6.2.121a}$$

where

$$W_1(q_1, \overline{p}_1, \overline{p}_2) \equiv \int^{q_1} (\overline{p}_2 + 2\cosh\xi + 2\overline{p}_1 \cosh^2 \xi)^{1/2} d\xi, \tag{6.2.121b}$$

$$W_2(q_2, \overline{p}_1, \overline{p}_2) \equiv \int^{q_2} [-\overline{p}_2 + 2(2\mu - 1)\cos\eta - 2\overline{p}_1 \cos^2 \eta]^{1/2} d\eta. \tag{6.2.121c}$$

The canonical transformation generated by W satisfies [see (6.2.91)]

$$p_1 \equiv \frac{\partial W}{\partial q_1} = \frac{\partial W_1}{\partial q_1} = (\overline{p}_2 + 2\cosh q_1 + 2\overline{p}_1 \cosh^2 q_1)^{1/2}, \tag{6.2.122a}$$

$$p_2 \equiv \frac{\partial W}{\partial q_2} = \frac{\partial W_2}{\partial q_2} = [-\overline{p}_2 + 2(2\mu - 1)\cos q_2 - 2\overline{p}_1 \cos^2 q_2]^{1/2}, \tag{6.2.122b}$$

$$\overline{q}_1 \equiv \frac{\partial W}{\partial \overline{p}_1} = \int^{q_1} \frac{\cosh^2 \xi}{(\overline{p}_2 + 2\cosh\xi + 2\overline{p}_1 \cosh^2 \xi)^{1/2}} d\xi$$

$$- \int^{q_2} \frac{\cos^2 \eta}{[-\overline{p}_2 + 2(2\mu - 1)\cos\eta - 2\overline{p}_1 \cos^2\eta]^{1/2}}\, d\eta, \quad (6.2.122c)$$

$$\overline{q}_2 \equiv \frac{\partial W}{\partial \overline{p}_2} = \frac{1}{2} \int^{q_1} \frac{d\xi}{(\overline{p}_2 + 2\cosh\xi + 2\overline{p}_1 \cosh^2\xi)^{1/2}}$$
$$- \frac{1}{2} \int^{q_2} \frac{d\eta}{[-\overline{p}_2 + 2(2\mu - 1)\cos\eta - 2\overline{p}_1 \cos^2\eta]^{1/2}}. \quad (6.2.122d)$$

The new Hamiltonian equals the old one, since W does not involve s explicitly, and we have

$$\overline{H} = \alpha_1 = \overline{p}_1. \quad (6.2.123)$$

Therefore, Hamilton's equations associated with (6.2.123) are

$$\dot{\overline{q}}_1 = \frac{\partial \overline{H}}{\partial \overline{p}_1} = 1, \quad \dot{\overline{q}}_2 = \frac{\partial \overline{H}}{\partial \overline{p}_2} = 0, \quad (6.2.124a)$$

$$\dot{\overline{p}}_1 = -\frac{\partial \overline{H}}{\partial \overline{q}_1} = 0, \quad \dot{\overline{p}}_2 = -\frac{\partial \overline{H}}{\partial \overline{q}_2} = 0. \quad (6.2.124b)$$

These have the solutions

$$\overline{q}_1 = s + \overline{q}_1^{(0)}, \quad \overline{q}_1^{(0)} = \text{constant}, \quad \overline{q}_2 = \overline{q}_2^{(0)} = \text{constant}, \quad (6.2.125a)$$
$$\overline{p}_1 = \text{constant}, \quad \overline{p}_2 = \text{constant}, \quad (6.2.125b)$$

involving the four arbitrary constants $\overline{q}_1^{(0)}, \overline{q}_2^{(0)}, \overline{p}_1, \overline{p}_2$.

Now we show that the solution in terms of the original q_1, q_2, p_1, p_2 variables can be calculated in principle. First note that squaring (6.2.122a)–(6.2.122b), adding, and solving for $\overline{p}_1$ gives (6.2.117)—that is, the energy is conserved. Squaring these two equations and subtracting the result gives

$$\overline{p}_2 = \frac{1}{2}(p_1^2 - p_2^2) - \cosh q_1 + (2\mu - 1)\cos q_2 - \overline{p}_1(\cosh^2 q_1 + \cos^2 q_2), \quad (6.2.126)$$

and we identify the constant $\overline{p}_2$ with $(\delta_1 - \delta_2)/2$ of (6.2.52). Thus, the two independent integrals we derived in Section 6.2.3 arise almost automatically according to this formulation.

To define the explicit canonical transformation for q_1, q_2, p_1, and p_2 as functions of $\overline{q}_1, \overline{q}_2, \overline{p}_1$, and $\overline{p}_2$, we proceed as follows. First we evaluate the integrals in (6.2.122c)–(6.2.122d), which can be expressed in terms of elliptic functions, to give $\overline{q}_1$ and $\overline{q}_2$ in the form

$$\overline{q}_1 = \psi_1(q_1, q_2, \overline{p}_1, \overline{q}_2), \quad (6.2.127a)$$
$$\overline{q}_2 = \psi_2(q_1, q_2, \overline{p}_1, \overline{p}_2). \quad (6.2.127b)$$

Solving these for q_1 and q_2 gives the explicit form (see (6.2.92))

$$q_1 = \overline{F}_1(\overline{q}_1, \overline{q}_2, \overline{p}_1, \overline{p}_2), \quad (6.2.128a)$$
$$q_2 = \overline{F}_2(\overline{q}_1, \overline{q}_2, \overline{p}_1, \overline{p}_2). \quad (6.2.128b)$$

Although this is a straightforward calculation in principle, the details are rather messy and are omitted.

The expressions for p_1 and p_2 now follow in the form

$$\begin{aligned} p_1 &= [\overline{p}_2 + 2\cosh \overline{F}_1(\overline{q}_1, \overline{q}_2, \overline{p}_1, \overline{p}_2) + 2\overline{p}_1 \cosh^2 \overline{F}_1(\overline{q}_1, \overline{q}_2, \overline{p}_1, \overline{p}_2)]^{1/2} \\ &\equiv \overline{G}_1(\overline{q}_1, \overline{q}_2, \overline{p}_1, \overline{q}_2), \qquad (6.2.129a) \\ p_2 &= [-\overline{p}_2 + 2(2\mu - 1)\cos \overline{F}_2(\overline{q}_1, \overline{q}_2, \overline{p}_1, \overline{p}_2) - 2\overline{p}_1 \cos^2 \overline{F}_2(\overline{q}_1, \overline{q}_2, \overline{p}_1, \overline{p}_2)]^{1/2} \\ &\equiv \overline{G}_2(\overline{q}_1, \overline{q}_2, \overline{p}_1, \overline{p}_2), \qquad (6.2.129b) \end{aligned}$$

when we use (6.2.128) in (6.2.122a)–(6.2.122b).

Equations (6.2.128) and (6.2.129) define the explicit canonical transformation generated by (6.2.121). The solution for the q_i and p_i as functions of s and four integration constants now follows immediately when we substitute (6.2.125) into (6.2.128) and (6.2.129).

Problems

6.2.1 Consider light rays in two dimensions, with the speed of light $c(x, y)$ given.

a. Show that the Euler–Lagrange equations (6.2.14) associated with the Lagrangian

$$L(x, y, \dot{x}, \dot{y}) = \sqrt{\dot{x}^2 + \dot{y}^2}/c(x, y) \qquad (6.2.130)$$

reduce to

$$c\dot{y}^2\ddot{x} - c\dot{x}\dot{y}\ddot{y} = -\dot{y}^2(\dot{x}^2 + \dot{y}^2)c_x + \dot{x}\dot{y}(\dot{x}^2 + \dot{y}^2)c_y, \quad (6.2.131a)$$

$$-c\dot{x}\dot{y}\ddot{x} - c\dot{x}^2\ddot{y} = \dot{x}\dot{y}(\dot{x}^2 + \dot{y}^2)c_x - \dot{x}^2(\dot{x}^2 + \dot{y}^2)c_y, \quad (6.2.131b)$$

and that these two equations are not independent in the sense that we cannot solve for $\ddot{x}$ and $\ddot{y}$ as functions of x, y, $\dot{x}$, and $\dot{y}$. This is to be expected, since we must have $dx^2 + dy^2 \equiv d\sigma^2$ along a light ray, where σ is the arc length, and this implies that $\dot{x}$ and $\dot{y}$ are related by $\dot{x}^2 + \dot{y}^2 = \dot{\sigma}^2$, where $\dot{} \equiv d/ds$.

b. Let $s = \sigma$, and show that (6.2.131) reduces to

$$c\frac{d^2x}{d\sigma^2} + c_x\left[1 - \left(\frac{dx}{d\sigma}\right)^2\right] - c_y\frac{dx}{d\sigma}\frac{dy}{d\sigma} = 0, \quad (6.2.132a)$$

$$c\frac{d^2y}{d\sigma^2} + c_y\left[1 - \left(\frac{dy}{d\sigma}\right)^2\right] - c_x\frac{dx}{d\sigma}\frac{dy}{d\sigma} = 0, \quad (6.2.132b)$$

and that these also follow from (6.1.6) when p and q are eliminated.

c. Now let $s = t$, the time, and show that (6.2.131) reduces to

$$c\frac{d^2x}{dt^2} + \left[c^2 - 2\left(\frac{dx}{dt}\right)^2\right]c_x - 2c_y\frac{dx}{dt}\frac{dy}{dt} = 0, \quad (6.2.133a)$$

$$c\frac{d^2y}{dt^2} + \left[c^2 - 2\left(\frac{dy}{dt}\right)^2\right]c_y - 2c_x\frac{dx}{dt}\frac{dy}{dt} = 0, \quad (6.2.133b)$$

which also follows form (6.1.6) when p and q are eliminated and $u = t$ is chosen as the independent variable.

d. Finally, let $s = x$, and write $dy/dx \equiv y' = \dot{y}/\dot{x}$. Therefore, $y'' = \ddot{y}/\dot{x}^2 - \dot{y}\ddot{x}/\dot{x}^3$. Show that (6.2.131a) reduces to (6.1.16) and, using corresponding expressions with x and y interchanged in (6.2.131a), gives a formula of the form (6.1.16) with x and y interchanged.

6.2.2 In a two-dimensional isotropic medium, the light rays emanating from some initial wave front are given by the one-parameter family

$$y - ke^x = 0, \quad (6.2.134)$$

where k is an arbitrary constant. Show that this information specifies the speed of light $c(x, y)$ in the form

$$c(x, y) = \frac{g(y^2 + 2x)}{(1 + y^2)^{1/2}}, \quad (6.2.135)$$

where g is an arbitrary function of its argument. Derive this result in two ways:

a. Use the eikonal equation

$$u_x^2 + u_y^2 = \frac{1}{c^2(x, y)} \quad (6.2.136)$$

and the fact that light rays are the orthogonal trajectories of the $u =$ constant curves. In this case, verify that (6.2.134) is a solution of the differential equation (6.1.16) for the light rays with c given by (6.2.135).

b. Regard (6.1.16) as a quasilinear first-order partial differential equation for $c(x, y)$ and solve it for an unknown initial curve.

c. What additional information is needed in order to specify g?

6.2.3 The *circular restricted three-body problem* is a dynamically consistent generalization of Euler's problem, where a point of mass μ and a point of mass $(1 - \mu)$ describe circular orbits about their common mass center. In the resulting gravitational field, we introduce a particle that does not disturb the motion of the two circling masses (the primaries); this particle merely moves under the influence of the Newtonian gravitational forces exerted by the primaries.

If we choose dimensionless variables such that lengths are normalized by the constant distance between the primaries, and the time is normalized by the reciprocal angular velocity of the circular motion, we obtain the following equations for the special case where the particle moves in the plane of the circular motion:

$$\ddot{\bar{x}} = -\frac{(1 - \mu)(\bar{x} - \bar{\xi}_1)}{\bar{r}_1^3} - \frac{\mu(\bar{x} - \bar{\xi}_2)}{\bar{r}_2^3}, \quad (6.2.137a)$$

$$\ddot{\bar{y}} = -\frac{(1-\mu)(\bar{y}-\bar{\eta}_1)}{\bar{r}_1^3} - \frac{\mu(\bar{y}-\bar{\eta}_2)}{\bar{r}_2^3}. \qquad (6.2.137b)$$

Here, $\bar{x}$, $\bar{y}$ are Cartesian coordinates in the inertial frame with origin at the center of mass of the primaries (see Figure 6.5). Hence, we have

$$\bar{r}_1^2 \equiv (\bar{x}-\bar{\xi}_1)^2 + (\bar{y}-\bar{\eta}_1)^2, \quad \bar{r}_2^2 \equiv (\bar{x}-\bar{\xi}_2)^2 + (\bar{y}-\bar{\eta}_2)^2, \qquad (6.2.138)$$

where

$$\bar{\xi}_1 \equiv -\mu \cos t, \quad \bar{\eta}_1 \equiv -\mu \sin t, \qquad (6.2.139a)$$

$$\bar{\xi}_2 \equiv (1-\mu)\cos t, \quad \bar{\eta}_2 \equiv (1-\mu)\sin t. \qquad (6.2.139b)$$

a. Show that equations (6.2.137) follow from the time-dependent Lagrangian

$$\bar{L}(t, \bar{x}, \bar{y}, \dot{\bar{x}}, \dot{\bar{y}}) = \frac{1}{2}(\dot{\bar{x}}^2 + \dot{\bar{y}}^2) + \frac{(1-\mu)}{\bar{r}_1} + \frac{\mu}{\bar{r}_2}. \qquad (6.2.140)$$

b. Introduce the coordinate system x, y defined by

$$\bar{x} \equiv x\cos t - y\sin t, \quad \bar{y} \equiv x\sin t + y\cos t, \qquad (6.2.141)$$

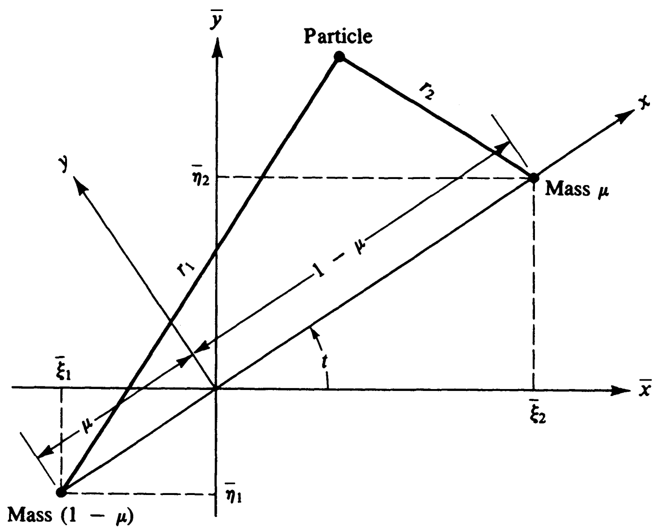

FIGURE 6.5. Inertial and rotating frames

in which the two primaries lie at $x = 1 - \mu$, $y = 0$, and $x = -\mu$, $y = 0$; then show that the Lagrangian that results is time-independent:

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2+\dot{y}^2)+\frac{1}{2}(x^2+y^2)+(x\dot{y}-y\dot{x})+\frac{(1-\mu)}{r_1}+\frac{\mu}{r_2}, \quad (6.2.142)$$

where

$$r_1^2 \equiv (x+\mu)^2 + y^2, \quad r_2^2 \equiv (x-1+\mu)^2 + y^2. \quad (6.2.143)$$

Derive (6.2.142) directly in the rotating x, y frame by introducing appropriate centrifugal and Coriolis forces.

c. Show that the Hamiltonian corresponding to (6.2.142) is

$$H(q_i, p_i) = \frac{1}{2}(p_1^2+p_2^2)+p_1q_2-p_2q_1-\frac{(1-\mu)}{r_1}-\frac{\mu}{r_2}, \quad (6.2.144)$$

where

$$p_1 = \dot{x} - y, \quad p_2 = \dot{y} + x, \quad (6.2.145a)$$

$$q_1 = x, \quad q_2 = y. \quad (6.2.145b)$$

Since (6.2.144) is time-independent, it is a constant of the motion called the *Jacobi integral.* Verify that the Hamilton–Jacobi equation associated with (6.2.144) for W *cannot be solved* by separation of variables.

6.2.4 The Lagrangian for geodesics on the unit sphere is [see (6.2.22)]

$$L(\dot{q}_1^2 + \dot{q}_2^2 \sin^2 q_1)^{1/2}. \quad (6.2.146)$$

a. Derive the Euler–Lagrange equations and show that the equation for q_2 integrates to

$$\dot{q}_2 \sin^2 q_1 = \lambda L, \quad (6.2.147)$$

where λ is an arbitrary constant.

b. Show that the Euler–Lagrange equation for q_1 is identically satisfied by (6.2.147). Therefore, this equation suffices to define the geodesics. Guided by the special cases $\lambda = 0$ and $\lambda = 1$, interpret λ geometrically in the three-dimensional space containing the unit sphere. In particular, show that $\lambda \equiv \cos i$, where i is the inclination angle between the polar axis and the normal to the local geodesic plane. This plane is defined by $\mathbf{r}$, the unit displacement vector from the origin to the geodesic, and $\dot{\mathbf{r}}$, the vector tangent to the geodesic. Thus, since i = constant, the geodesic is given by the intersection with the unit sphere of a plane inclined at the angle i to the equatorial plane.

c. Let $\Gamma(q_1, q_2) = 0$ be a given curve on the surface of the unit sphere. Show that the transversality condition on $\Gamma = 0$ reduces to

$$\frac{\partial\Gamma/\partial q_1}{\partial\Gamma/\partial q_2} = \frac{\dot{q}_1}{\dot{q}_2 \sin^2 q_1}. \quad (6.2.148)$$

Show that (6.2.148) merely states that a transversal geodesic must be normal to $\Gamma = 0$.

6.3 The Nonlinear Equation

The essential difference between the quasilinear and nonlinear problems is that in the first instance the partial differential equation specifies a unique characteristic direction at each point, whereas for the nonlinear case, we have a "cone" of possible characteristic directions. This feature necessitates that we keep track of certain characteristic strips (which are characteristic curves embedded in an infinitesimal surface strip) in order to construct a solution. We begin our discussion with the case of two independent variables for which the geometry is easily visualized in the three-dimensional space of x, y, u.

6.3.1 The Geometry of Solutions

We consider the general nonlinear equation

$$F(x, y, u, p, q) = 0 \tag{6.3.1}$$

for the independent variables x and y and the dependent variable u, and we let $p \equiv u_x$ and $q \equiv u_y$. It will be useful to refer to the special case of the eikonal equation for which F does not depend on u and has the form [see (6.1.4b)]

$$F \equiv p^2 + q^2 - \frac{1}{c^2(x, y)} = 0. \tag{6.3.2}$$

As in Section 5.2.1, let us examine the geometrical constraints imposed by (6.3.1) on possible solution surfaces through a given point $P = (x_0, y_0, u_0)$. A normal vector $\mathbf{n}$ to a possible solution surface at P again has components $\mathbf{n} = (p, q, -1)$, but now the relation between p and q that is dictated by (6.3.1) is nonlinear. In particular, the family of possible normals will, in general, not lie in a plane; instead, this family generates a curved surface centered at P. This surface is actually a right circular cone for the eikonal equation [because the equation linking p and q defines a circle of radius $1/c(x_0, y_0)$], but in general the relation $F(x_0, y_0, u_0, p, q) = 0$ describes a more complicated surface. The tangent vectors to a possible solution surface through P, each of which is perpendicular to the cone of normals, will therefore also lie on a curved surface, called the *Monge cone* (see Figure 6.6).

Again, the Monge cone is only an inverted right circular cone with apex at P for the case (6.3.2). In general, it is a more complicated local surface dictated by the dependence of F on p and q. Notice that for the quasilinear problem, the Monge cone degenerates to a single characteristic ray. To see how we can isolate a solution surface in this space "filled" with Monge cones, let us assume, as we did in Section 5.2.1, that we want to consider only those solutions that pass through a given curve C_0. We see immediately that even for the simple case of (6.3.2) depicted in Figure

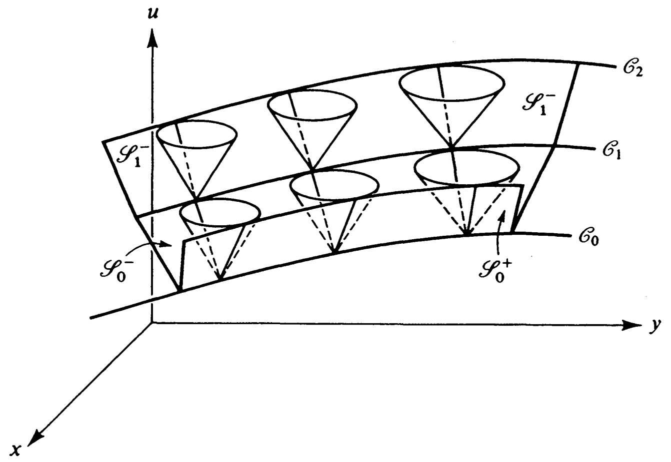

FIGURE 6.6. Monge cones for points on C_0 and C_1

6.6, *specifying the curve C_0 does not isolate a unique solution surface near C_0.* In fact, for this case, there are two possible infinitesimal tangent surfaces $\mathcal{S}_0^+$ and $\mathcal{S}_0^-$, each of which contains C_0 and is tangent to all the Monge cones on C_0. The reason we have two surfaces is that (6.3.2) is quadratic in p and q; in the general case (6.3.1), we may have more possible infinitesimal strips through C_0.

Based on the preceding observation, we conclude that in addition to specifying C_0 we must also specify a *strip condition* that isolates the particular strip we wish to follow. The analytical details that establish this strip condition are given in Section 6.3.3.

Next, we proceed a distance Δs along the generators of each of the Monge cones that are embedded in the strip we have chosen, say $\mathcal{S}_0^-$, as indicated in Figure 6.6. This takes us to the new curve C_1, where we repeat our construction of the Monge cones. But now there is no longer any ambiguity as to which strip we must choose; *only one strip joins smoothly with $\mathcal{S}_0^-$*. In our case this is the strip $\mathcal{S}_1^-$, because if we were to choose the new strip $\mathcal{S}_0^+$ (which is omitted from Figure 6.6 for clarity), the rays would have a finite discontinuity in the first derivative along C_1 (the apex angle of each of the Monge cones on C_1 is finite for a nonlinear equation), and this is inconsistent with a smooth solution surface.

6.3.2 Focal Strips and Characteristic Strips

According to our geometrical description, a solution surface u is everywhere tangent to a Monge cone along one of its generators. This particular generator, together with the associated infinitesimal tangent plane, forms a strip that is used to construct the solution surface.

We shall define the generator of a Monge cone as the intersection of two planes tangent to the cone in the limit as the lines of tangency approach one another.

In Figure 6.7 we show a portion of the Monge cone with apex at the point $P(x_0, y_0, u_0)$. We identify the one-parameter family of generators by the parameter λ and consider two planes A and B tangent to the Monge cone; the plane A is tangent along the generator λ, and the plane B is tangent along the neighboring generator $\lambda + \Delta\lambda$.

The points x, y, u lying on A satisfy

$$u - u_0 = (x - x_0)p(\lambda) + (y - y_0)q(\lambda), \tag{6.3.3a}$$

where p and q are values consistent with (6.3.1) at P. Similarly, points lying on B satisfy

$$\begin{aligned} u - u_0 &= (x - x_0)p(\lambda + \Delta\lambda) + (y - y_0)q(\lambda + \Delta\lambda) \\ &= (x - x_0)p(\lambda) + (y - y_0)q(\lambda) \\ &\quad + \{(x - x_0)p'(\lambda) + (y - y_0)q'(\lambda)\}\Delta\lambda + O((\Delta\lambda)^2). \end{aligned} \tag{6.3.3b}$$

Therefore, points x, y, u that lie on the intersection PQ (see Figure 6.7) satisfy both equations. Subtracting these and taking the limit $\Delta\lambda \to 0$ gives the condition

$$(x - x_0)p'(\lambda) + (y - y_0)q'(\lambda) = 0, \tag{6.3.4}$$

which must hold on the generator of a Monge cone. We must also satisfy (6.3.1) on the generator; that is,

$$F(x_0, y_0, u_0, p(\lambda), q(\lambda)) = 0, \tag{6.3.5a}$$

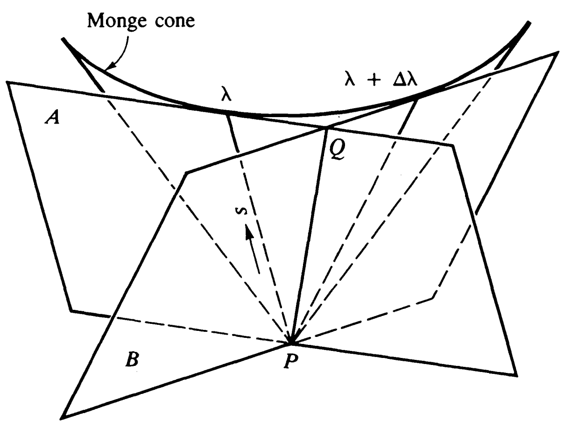

FIGURE 6.7. Planes tangent to generators of a Monge cone

from which it follows by differentiation with respect to λ that

$$F_p p'(\lambda) + F_q q'(\lambda) = 0. \qquad (6.3.5b)$$

Let s be a parameter that varies along a generator. We see that for x, y, u close to P, (6.3.4) and (6.3.5b) give

$$\frac{dx}{ds} = F_p, \qquad (6.3.6a)$$

$$\frac{dy}{ds} = F_q, \qquad (6.3.6b)$$

when we let $x - x_0 = dx$, $y - y_0 = dy$ and identify the coefficients of p' and q' in these two equations. Along a fixed generator, we must also satisfy the limiting form (6.3.3a), that is,

$$\frac{du}{ds} = p\frac{dx}{ds} + q\frac{dy}{ds},$$

which, in view of (6.3.6a)–(6.3.6b), gives

$$\frac{du}{ds} = pF_p + qF_q. \qquad (6.3.6c)$$

The condition (6.3.6c) is called a *strip condition* because [see (6.3.3)] it assigns an infinitesimal plane to the curve $x(s)$, $y(s)$, $u(s)$.

A curve $x(s)$, $y(s)$, $u(s)$, that *for some given* $p(s)$, $q(s)$ satisfies the three equations (6.3.6) is called a *focal curve*. We see that for the quasilinear problem, the system (6.3.6) corresponds to the characteristic system (5.2.3), and this system defines a unique curve passing through a given point P. In contrast, for the nonlinear problem, the four equations (6.3.6), (6.3.1) do not define $x(s)$, $y(s)$, $u(s)$, $p(s)$, and $q(s)$ uniquely; we need one more condition to ensure that the *focal strip* defined by this system is tangent to a solution surface.

To illustrate this point, consider the counterexample that results if we choose an *arbitrary surface* $u = \phi(x, y)$ that is not a consistent solution of (6.3.1). If we ignore the requirement $p = \phi_x, q = \phi_y$, we can satisfy (6.3.1), (6.3.6) as follows. Equation (6.3.1) gives one algebraic relation linking p, q, x, and y in the form

$$F(x, y, \phi(x, y), p, q) = 0. \qquad (6.3.7)$$

Equation (6.3.6c) gives

$$\frac{du}{ds} = pF_p(x, y, \phi(x, y), p, q) + qF_q(x, y, \phi(x, y), p, q). \qquad (6.3.8)$$

But since $u = \phi(x, y)$, we also have $du/ds = \phi_x(dx/ds) + \phi_y(dy/ds)$, and using (6.3.6a)–(6.3.6b) gives

$$\frac{du}{ds} = \phi_x(x, y)F_p(x, y, \phi(x, y), p, q) + \phi(x, y)F_q(x, y, \phi(x, y), p, q). \qquad (6.3.9)$$

Subtracting (6.3.9) from (6.3.8) gives a second algebraic equation linking p, q:

$$[p-\phi_x(x,y)]F_p(x,y,\phi(x,y),p,q)+[q-\phi_y(x,y)]F_q(x,y,\phi(x,y),p,q)=0. \tag{6.3.10}$$

Solving (6.3.7) and (6.3.10) gives $p(x, y)$, $q(x, y)$; using these in (6.3.6a)–(6.3.6b) leads to $x(s)$, $y(s)$. The strip $x(s)$, $y(s)$, $u(s)$, $p(s)$, $q(s)$ that results is *not necessarily tangent* to the surface $u = \phi(x, y)$. In summary, each of the many possible solutions of (6.3.1), (6.3.6) defines a focal strip that is not necessarily tangent to a solution surface. Our next task is to isolate from the preceding family of focal strips the one strip along which $p = \phi_x$, $q = \phi_y$ for a solution surface $u = \phi(x, y)$.

On a given solution surface $u = \phi(x, y)$, (6.3.1) must be satisfied identically; that is,

$$F(x,y,\phi(x,y),\phi_x(x,y),\phi_y(x,y)) \equiv 0. \tag{6.3.11a}$$

Moreover, the partial derivatives of this expression with respect to x and y must also vanish; that is, denoting $\phi_x = p$, $\phi_y = q$, we must have

$$F_x + F_u p + F_p p_x + F_q q_x = 0, \tag{6.3.11b}$$

$$F_y + F_u q + F_p p_y + F_q q_y = 0. \tag{6.3.11c}$$

The solution surface $u = \phi(x, y)$ must contain the focal curves; hence $F_p = dx/ds$, $F_q = dy/ds$. In addition, we must have $p_y = q_x$ for consistency. Therefore, (6.3.11b) may also be written in the form

$$F_x + F_u p + p_x \frac{dx}{ds} + p_y \frac{dy}{ds} = 0,$$

or

$$\frac{dp}{ds} = -(F_x + pF_u). \tag{6.3.12a}$$

Similarly, (6.3.11c) gives

$$\frac{dq}{ds} = -(F_y + qF_u). \tag{6.3.12b}$$

The crux of the derivation of (6.3.12) is the fact that we have identified p and q with ϕ_x and ϕ_y on a given solution surface $u = \phi(x, y)$.

The system (6.3.6) and (6.3.12) of five equations for the five variables x, y, u, p, q defines a four-parameter family of strips; one of the five integration constants is s_0, which appears only in the additive form $(s - s_0)$ in the solution because this system is autonomous, and we may set $s_0 = 0$ with no loss of generality.

We show next that (6.3.1) is an integral of the system (6.3.6), (6.3.12), that is, that $F(x, y, u, p, q)$ is a constant along any solution of this system. To prove this, we differentiate the expression for F with respect to s and obtain

$$\frac{dF}{ds} = F_x \frac{dx}{ds} + F_y \frac{dy}{ds} + F_u \frac{du}{ds} + F_p \frac{dp}{ds} + F_q \frac{dq}{ds}.$$

Along a solution of (6.3.6), (6.3.12), this vanishes identically because upon substituting the expression for $(dx/ds)\dots(dq/ds)$ we obtain

$$\frac{dF}{ds} = F_xF_p + F_yF_q + F_u(pF_p + qF_q) - F_p(F_x + pF_u) - F_q(F_y + qF_u) = 0.$$

Therefore, F = constant, and when we require this constant to be zero [in order to conform with (6.3.1)], we reduce the solutions of (6.3.6), (6.3.12) to a three-parameter family. We shall refer to a solution of (6.3.6), (6.3.12) along which $F = 0$ as a *characteristic strip*. In the next section we show how an integral surface that passes through a prescribed initial strip can be isolated from the three-parameter family of characteristic strips associated with (6.3.1).

In Section 6.1.2 we used physical arguments to derive the system (6.1.6) for the light rays associated with the two-dimensional eikonal equation

$$F \equiv p^2 + q^2 - \frac{1}{c^2(x,y)} = 0. \tag{6.3.13}$$

Using our general theory for (6.3.13), we compute the following special case of (6.3.6), (6.3.12) with $F_u = 0$:

$$\frac{dx}{ds} = F_p = 2p, \tag{6.3.14a}$$

$$\frac{dy}{ds} = F_q = 2q, \tag{6.3.14b}$$

$$\frac{du}{ds} = pF_p + qF_q = 2p^2 + 2q^2 = \frac{2}{c^2(x,y)}, \tag{6.3.14c}$$

$$\frac{dp}{ds} = -F_x = -\frac{2c_x}{c^3}, \tag{6.3.14d}$$

$$\frac{dq}{ds} = -F_y = -\frac{2c_y}{c^3}. \tag{6.3.14e}$$

If we identify $(2/c)ds$ with $d\sigma$, the infinitesimal distance along a light ray, we see that (6.3.14) is identical with (6.1.6). In particular, we note that the light rays are mathematically the projections of the characteristic curves on the xy-plane.

In view of the mathematical analogy between (6.3.13) and (6.2.66), we also conclude that (6.3.14a)–(6.3.14b) correspond to Hamilton's equations (6.2.67a) for the coordinates, and (6.3.14d)–(6.3.14e) correspond to (6.2.67b) for the momenta. The role of the Hamilton–Jacobi equation in dynamics is explored in more detail in Section 6.4.3.

6.3.3 The Initial-Value Problem

We are given a noncharacteristic initial strip $\mathcal{S}_0$ defined parametrically in the form

$$x = x_0(\tau),\quad y = y_0(\tau),\quad u = u_0(\tau),\quad p = p_0(\tau),\quad q = q_0(\tau), \tag{6.3.15}$$

for functions x_0, y_0, u_0, p_0, q_0 that are continuous and have a continuous first derivative.

The five functions in (6.3.15) are not entirely arbitrary. To begin with, we must again exclude the situation where the ground curve $x_0(\tau)$, $y_0(\tau)$ has intersections. More importantly, we must require the strip (6.3.15) to be (1) self-consistent, that is, to satisfy the strip condition

$$\frac{du_0}{d\tau} = p_0(\tau)\frac{dx_0}{d\tau} + q_0(\tau)\frac{dy_0}{d\tau}, \tag{6.3.16a}$$

and (2) to be consistent with (6.3.1), that is,

$$F(x_0(\tau), y_0(\tau), u_0(\tau), p_0(\tau), q_0(\tau)) = 0. \tag{6.3.16b}$$

Equations (6.3.15)–(6.3.16) impose three independent conditions to be satisfied by the three-parameter family of characteristic strips obtained by solving (6.3.6), (6.3.12) subject to $F = 0$. We shall demonstrate next that these three conditions specify a unique solution surface as long as a certain Jacobian does not vanish.

We interpret the construction of the solution surface u in the sense depicted in Figure 6.8, of smoothly joining all the characteristic strips $\mathcal{C}_i$ that emerge from the initial strip $\mathcal{S}_0$. Thus, for any fixed value τ_j of the parameter along $\mathcal{S}_0$, we generate a characteristic strip $\mathcal{C}_j$, which for $s = 0$ coincides with $\mathcal{S}_0$ at τ_j. The converse of this construction, which will be useful in later discussion, is that whenever two integral surfaces join smoothly, this juncture occurs along a characteristic strip.

To implement the construction of a solution surface that passes through $\mathcal{S}_0$, we need to express the solution of (6.3.6), (6.3.12) in the form of a one-parameter family of characteristic strips. Let the general solution of (6.3.6), (6.3.12), subject to (6.3.1), be given in the form (note we have set $s_0 = 0$)

$$x = \overline{X}(s, c_1, c_2, c_3), \tag{6.3.17a}$$

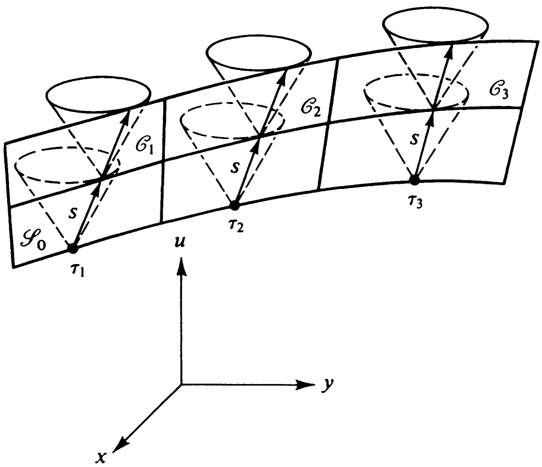

FIGURE 6.8. Solution passing through an initial strip

$$y = \overline{Y}(s, c_1, c_2, c_3), \tag{6.3.17b}$$

$$u = \overline{U}(s, c_1, c_2, c_3), \tag{6.3.17c}$$

$$p = \overline{P}(s, c_1, c_2, c_3), \tag{6.3.17d}$$

$$q = \overline{Q}(s, c_1, c_2, c_3), \tag{6.3.17e}$$

involving the three arbitrary constants c_1, c_2, c_3. A general one-parameter subfamily of (6.3.17) is obtained by regarding c_1, c_2, and c_3 as arbitrary functions of the parameter τ; that is,

$$x = X(s, \tau) \equiv \overline{X}(s, c_1(\tau)c_2(\tau), c_3(\tau)), \tag{6.3.18a}$$

$$y = Y(s, \tau) \equiv \overline{Y}(s, c_1(\tau), c_2(\tau), c_3(\tau)), \tag{6.3.18b}$$

$$u = U(s, \tau) \equiv \overline{U}(s, c_1(\tau), c_2(\tau), c_3(\tau)), \tag{6.3.18c}$$

$$p = P(s, \tau) \equiv \overline{P}(s, c_1(\tau), c_2(\tau), c_3(\tau)), \tag{6.3.18d}$$

$$q = Q(s, \tau) \equiv \overline{Q}(s, c_1(\tau), c_2(\tau), c_3(\tau)). \tag{6.3.18e}$$

The dependence of the functions X, Y, U, P, Q on τ is specified by requiring the one-parameter family (6.3.18) to contain the initial strip (6.3.15); that is,

$$X(0, \tau) = x_0(\tau), \quad Y(0, \tau) = y_0(\tau), \quad U(0, \tau) = u_0(\tau),$$
$$P(0, \tau) = p_0(\tau), \quad Q(0, \tau) = q_0(\tau).$$

Again, in practice, we calculate the solution directly in the form (6.3.18).

If the Jacobian

$$\Delta(s, \tau) \equiv X_s Y_\tau - Y_s X_\tau \tag{6.3.19}$$

does not vanish, we can invert (6.3.18a)–(6.3.18b) to express s and τ as functions of x, y. Using this result in (6.3.18c)–(6.3.18e) gives $u(x, y)$, $p(x, y)$, $q(x, y)$. It is easy to prove that this result satisfies (6.3.1) with $p = u_x$ and $q = u_y$ (for example, see Section 3.2 of [13]).

If $\Delta(0, \tau) \equiv 0$ and $\mathcal{S}_0$ is a characteristic strip, the solution of (6.3.1) is nonunique, as in the quasilinear problem. In the exceptional case, where $\Delta(0, \tau) = 0$ but $\mathcal{S}_0$ is not a characteristic strip, $\mathcal{S}_0$ must be a focal strip because $\Delta(0, \tau) = 0$ implies that (6.3.6a)–(6.3.6b) are satisfied on $\mathcal{S}_0$, and the strip condition (6.3.6c) is always required on $\mathcal{S}_0$ [see (6.3.16a)]. Thus, in this case we are unable to satisfy (6.3.12) on $\mathcal{S}_0$, and the projections of the characteristic curves must therefore have an envelope (a *caustic*) along the projection of the initial curve on the xy-plane. In the context of geometrical optics, a *focal curve* is indeed a curve along which light rays have an envelope (the degenerate case corresponds to a focal point), and this is the origin of the terminology used in describing the curves obeying the general system (6.3.6).

The extension of the above results to n independent variables is straight-forward, and we list the results with no further discussion. The partial differential equation in n variables has the form

$$F(x_i, u, p_i) = 0, \tag{6.3.20}$$

where x_i indicates $x_1, \ldots, x_n$ and $p_j \equiv \partial u/\partial x_j$. The characteristic strips are governed by the system of $(2n+1)$ equations

$$\frac{dx_j}{ds} = \frac{\partial F}{\partial p_j}, \tag{6.3.21a}$$

$$\frac{du}{ds} = \sum_{j=1}^{n} p_j \frac{\partial F}{\partial p_j}, \tag{6.3.21b}$$

$$\frac{dp_j}{ds} = -\left(\frac{\partial F}{\partial x_j} + p_j \frac{\partial F}{\partial u}\right). \tag{6.3.21c}$$

6.3.4 Example Problems for the Eikonal Equation

The simplest initial-value problem for the eikonal equation consists of a *point* disturbance initially. The resulting wave front is called an *integral conoid*, and we calculate this solution next for the case $n = 2$ and $c = c_0 = \text{constant}$.

(i) *Integral conoid*

The initial strip degenerates in the case of the integral conoid to just the Monge cone, which we take at $x = y = u = 0$. Thus, $x_0 = 0$, $y_0 = 0$, $u_0 = 0$. We see that the strip condition (6.3.16a) is then identically satisfied, so the only restriction on $p_0(\tau)$ and $q_0(\tau)$ is (6.3.16b); that is, $p_0^2(\tau) + q_0^2(\tau) = 1/c_0^2 = \text{constant}$. The characteristic strips obey

$$\frac{dx}{ds} = 2p, \quad \frac{dy}{ds} = 2q, \quad \frac{du}{ds} = 2(p^2 + q^2) = \frac{2}{c_0^2}, \quad \frac{dp}{ds} = 0, \quad \frac{dq}{ds} = 0. \tag{6.3.22}$$

Therefore, $p = p_0(\tau)$; $q = q_0(\tau)$; $x = 2p_0(\tau)s$; $y = 2q_0(\tau)s$; $u = 2s/c_0^2$. Squaring the expressions for x and y and adding gives $x^2 + y^2 = 4s^2(p_0^2 + q_0^2) = 4s^2/c_0^2$. Therefore, $u = \sqrt{x^2 + y^2}/c_0$, as is obvious from Huygens' construction [see (6.1.5)].

(ii) *Moving disturbance*

Suppose we have a point disturbance that moves with constant speed v along a straight line in the xy-plane. With no loss of generality, we may assume the motion to occur along the x-axis and to pass through the origin when $u = 0$. The initial values for x_0, y_0, and u_0 are then $x_0 = \tau$, $y_0 = 0$, $u_0 = \tau/v$ because $dx_0/du_0 = (dx_0/d\tau)/(du_0/d\tau) = v$ and $dy_0/du_0 = 0$.

The strip condition (6.3.16a) requires that $1/v = p_0(\tau)$, whereas (6.3.16b) requires $p_0^2(\tau) + q_0^2(\tau) = 1/c_0^2$. This latter condition is expressed more conveniently if we denote $p_0 \equiv (\cos\theta)/c_0$ and $q_0 \equiv (\sin\theta)/c_0$ in terms of a new parameter θ, in which case the strip condition defines the parameter θ as $\theta \equiv \cos^{-1}(c_0/v)$. This has two real values in $(0, 2\pi)$ as long as $v > c_0$. Let us restrict our attention to the value where $0 \leq \cos^{-1}(c_0/v) \leq \pi/2$. The requirement $v > c_0$ is geometrically obvious because the wave fronts have no envelope for finite x, y if $v < c_0$. The requirement $v > c_0$ violates physical law for light, so in this particular example we may wish to regard the physical problem as one in acoustics, where $v > c_0$ corresponds to a supersonic disturbance speed.

The solution of (6.3.22) now takes the form

$$p = \frac{1}{v}, \quad q = \frac{(v^2 - c_0^2)^{1/2}}{c_0 v}, \quad x = \frac{2s}{v} + \tau,$$
$$y = \frac{2(v^2 - c_0^2)^{1/2} s}{c_0 v}, \quad u = \frac{2s}{c_0^2} + \frac{\tau}{v},$$

and we see that

$$\Delta(s, \tau) \equiv -\frac{2(v^2 - c_0^2)^{1/2}}{c_0 v}$$

does not vanish as long as $v > c_0$.

The solution for $u(x, y)$ is the plane

$$u = \frac{1}{v}[x + (M^2 - 1)^{1/2} y], \tag{6.3.23a}$$

where $M \equiv v/c_0$ is the Mach number. The other choice of θ results in a negative q_0, and

$$u = \frac{1}{v}[x - (M^2 - 1)^{1/2} y]. \tag{6.3.23b}$$

(iii) *Variable c*

As a final illustration, consider the case where $c = |x|$ and let the initial front be the straight line $y = ax$, where $a =$ constant > 0. The two unit normals to the front are $\mathbf{n}_1 \equiv (a/\sqrt{1 + a^2}, -1/\sqrt{1 + a^2})$ and $\mathbf{n}_2 \equiv (-a/\sqrt{1 + a^2}, 1/\sqrt{1 + a^2})$. If we choose $\mathbf{n}_1$, our initial strip will be defined parametrically in the form

$$x_0 = \tau, \quad y_0 = a\tau, \quad u_0 = 0,$$
$$p_0 = \frac{a}{|\tau|(1 + a^2)^{1/2}}, \quad q_0 = -\frac{1}{|\tau|(1 + a^2)^{1/2}}.$$

The forms for p_0 and q_0 ensure that the strip condition associated with $\mathbf{n}_1$ and the eikonal equation are initially satisfied. Let us temporarily consider only the characteristics with $\tau > 0$ and omit the absolute value signs.

The characteristic strips obey

$$\frac{dx}{ds} = 2p, \quad \frac{dy}{ds} = 2q, \quad \frac{du}{ds} = \frac{2}{x^2}, \quad \frac{dp}{ds} = -\frac{2}{x^3}, \quad \frac{dq}{ds} = 0. \tag{6.3.24}$$

Solving for q gives $q = q_0 = -1/\tau(1 + a^2)^{1/2}$. Therefore, we obtain p directly from the eikonal equation,

$$p = \pm\left[\frac{1}{x^2} - \frac{1}{\tau^2(1 + a^2)}\right]^{1/2} = \pm\frac{[\tau^2(1 + a^2) - x^2]^{1/2}}{x\tau(1 + a^2)^{1/2}}, \tag{6.3.25}$$

or less directly by integrating the equation $dp/dx = -1/px^3$, which results from dividing dp/ds by dx/ds. Initially, we must use the plus sign in (6.3.25) in front of the radical in conformity with our choice of initial strip, where p is positive. The sign for p along a given ray τ = constant changes whenever $x = \pm\tau(1+a^2)^{1/2}$.

We now express dx/ds in the form

$$\frac{dx}{ds} = \pm 2\frac{[\tau^2(1+a^2) - x^2]^{1/2}}{x\tau(1+a^2)^{1/2}},$$

and integrate this subject to $x = \tau$ at $s = 0$ to obtain

$$s = \mp\frac{1}{2}\{\tau[\tau^2(1+a^2) - x^2]^{1/2} \mp \tau a\}(1+a^2)^{1/2}. \tag{6.3.26}$$

The solution for y is just

$$y = 2q_0 s + a\tau = -\frac{2s}{\tau(1+a^2)^{1/2}} + a\tau,$$

or

$$s = -\frac{\tau}{2}(y - a\tau)(1+a^2)^{1/2}. \tag{6.3.27}$$

It follows from (6.3.26) and (6.3.27) that the upper and lower signs in the preceding equations correspond to $y > 0$ and $y < 0$, respectively. If we now eliminate s from (6.3.26) and (6.3.27), we find that the projections of the characteristics on the xy-plane are the one-parameter family of concentric circles centered at the origin with radius $\tau(1+a^2)^{1/2}$; that is,

$$\tau = \left[\frac{x^2+y^2}{1+a^2}\right]^{1/2} \quad \text{for } \tau > 0. \tag{6.3.28}$$

The remaining equation for u may be expressed in the form

$$\frac{du}{dx} = \frac{du/ds}{dx/ds} = \pm\frac{\tau(1+a^2)^{1/2}}{x[\tau^2(1+a^2) - x^2]^{1/2}}. \tag{6.3.29}$$

Integrating (6.3.29) subject to $u = 0$, $x = \tau$ and using (6.3.28) gives

$$u(x,y) = \mp\log[\sqrt{1+(y/x)^2} \pm y/|x|] + \log[\sqrt{1+a^2} + a]\operatorname{sgn} x. \tag{6.3.30}$$

The solution becomes infinite as $x \to 0$, and this is expected because the disturbance speed $c \to 0$. Therefore, it takes an infinite time for the initial front to approach the y-axis from either side. We note the similarity behavior u = constant on rays (y/x) = constant, and it is easily verified that in this case the eikonal equation can be solved directly using similarity arguments (see Problem 6.3.3).

Problems

6.3.1 In (6.2.72), fix the constant E, so you restrict attention to planar motions of a given energy under the influence of gravity.

a. Find the integral conoid centered at the origin in parametric form, and indicate how we could, in principle, calculate $W(x, y)$.
b. Let x and y be small and show that the approximate expression for the conoid is

$$W(x, y) \approx \frac{r(2E - y/2)}{\sqrt{2E}}, \tag{6.3.31}$$

where $r^2 = x^2 + y^2$. Sketch curves of W = constant in the xy-plane and argue that the conoid is not a cone.
c. Give an optical as well as a dynamical interpretation for your results. In particular, identify the trajectories emerging from the origin in terms of the solutions of the characteristic equations.

6.3.2 The eikonal equation in cylindrical polar coordinates (r, θ) is

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \frac{1}{c^2(r, \theta)}. \tag{6.3.32}$$

a. Without taking advantage of cylindrical symmetry, show that the integral conoid from the origin for the case where $c = (1 + r^2)^{1/2}$ is

$$u(r, \theta) = \log[r + (1 + r^2)^{1/2}]. \tag{6.3.33}$$

b. Rederive (6.3.33) by noting that if c depends only on r, then u does not depend on θ for the integral conoid. Therefore, (6.3.32) reduces to

$$\frac{du}{dr} = \frac{1}{c(r)}, \tag{6.3.34}$$

and integrating this for $c = (1 + r^2)^{1/2}$ gives (6.3.33).

6.3.3 Solve the problem discussed in Section 6.3.4iii using similarity.

6.3.4 The three-dimensional eikonal equation for a medium with constant signal speed (normalized to equal unity) is

$$u_x^2 + u_y^2 + u_z^2 = 1. \tag{6.3.35}$$

a. Solve the initial-value problem $u = k$ = constant on the plane $\alpha x + \beta y + \gamma z = 0$, where α, β, and γ are arbitrary constants.
b. Construct the integral conoid by a process of envelope formation using your solution in part (a). What is the projection of the integral conoid on the manifold u = constant?

6.4 The Complete Integral; Solutions by Envelope Formation

In this section we shall study an alternative approach that bypasses the necessity of integrating the system (6.3.6), (6.3.12) wherever a *complete integral* of (6.3.1) is available.

A complete integral of (6.3.1) is simply a solution involving two arbitrary constants a, b:

$$u = \phi(x, y, a, b). \tag{6.4.1}$$

We ensure that these constants are independent [that is, do not occur in a particular combination $f(a, b)$ in (6.4.1)] by requiring the determinant

$$D \equiv \phi_{xa}\phi_{yb} - \phi_{ya}\phi_{xb} \tag{6.4.2}$$

to be nonzero.

In the special case where F is independent of u (for example, the eikonal equation or the Hamilton–Jacobi equation), one of the constants in the complete integral is additive, and (6.4.1) has the form

$$u = \phi(x, y, a) + b. \tag{6.4.3}$$

This follows immediately from the fact that if $u = \phi(x, y, a)$ solves (6.3.1) for arbitrary a, then $u = \phi(x, y, a) + b$ is also a solution for arbitrary b if u does not occur in F.

6.4.1 Envelope Surfaces Associated with the Complete Integral

Suppose we are given a complete integral of (6.3.1) in the form (6.4.1). We shall show now that we can construct another solution by a process of envelope formation.

Let us specify an arbitrary relation

$$b = w(a), \tag{6.4.4}$$

linking b to a, so that (6.4.1) now reads

$$u = \phi(x, y, a, w(a)). \tag{6.4.5a}$$

If this one-parameter family of surfaces has an envelope, we must be able to solve

$$\phi_a(x, y, a, w(a)) + \phi_b(x, y, a, w(a))w'(a) = 0 \tag{6.4.5b}$$

for $a(x, y)$. Substituting this expression in (6.4.5a) gives

$$u = \Psi(x, y) \equiv \phi(x, y, a(x, y), w(a(x, y))), \tag{6.4.6}$$

which is a surface involving the arbitrary function w. We now show that $u = \Psi(x, y)$, as given by (6.4.6), also solves (6.3.1). We compute

$$\Psi_x = \phi_x + (\phi_a + \phi_b w'(a))a_x, \tag{6.4.7a}$$

$$\Psi_y = \phi_y + (\phi_a + \phi_b w'(a))a_y. \tag{6.4.7b}$$

But $a(x, y)$ was determined from the requirement (6.4.5b). Therefore, the coefficients of a_x and a_y vanish identically in (6.4.7), and we have $\Psi_x = \phi_x$, $\Psi_y = \phi_y$. We know that $F(x, y, \phi, \phi_x, \phi_y) = 0$, since ϕ is a complete integral. We also know that $\phi = \Psi$ and have shown that $\phi_x = \Psi_x, \phi_y = \Psi_y$. Therefore, $\Psi(x, y)$, as defined by (6.4.6), solves (6.3.1).

The *singular integral* of (6.3.1) is the envelope of the two-parameter family of solutions resulting from varying a and b independently in the complete integral; that is, we eliminate a and b from the three expressions

$$u = \phi(x, y, a, b), \quad \phi_a(x, y, a, b) = 0, \quad \phi_b(x, y, a, b) = 0. \tag{6.4.8}$$

If a singular integral exists, we can also derive it directly from (6.3.1) by eliminating p and q from the three relations

$$F(x, y, u, p, q) = 0, \quad F_p(x, y, u, p, q) = 0, \quad F_q(x, y, u, p, q) = 0. \tag{6.4.9}$$

To show that (6.4.8) and (6.4.9) define the same function, note that the function of x, y, a, b defined by substituting the complete integral into (6.3.1) vanishes identically, that is,

$$F(x, y, \phi, \phi_x, \phi_y) \equiv 0. \tag{6.4.10}$$

In particular, the total derivatives dF/da and dF/db of (6.4.10) also vanish, where

$$\frac{dF}{da} = F_u\phi_a + F_p\phi_{xa} + F_q\phi_{ya} = 0, \tag{6.4.11a}$$

$$\frac{dF}{db} = F_u\phi_b + F_p\phi_{xb} + F_q\phi_{yb} = 0. \tag{6.4.11b}$$

On the singular integral, $\phi_a = 0$ and $\phi_b = 0$; the homogeneous system that results from (6.4.11) can hold with $D \neq 0$ only if $F_p = F_q = 0$. Thus, we have shown that the two conditions (6.4.8), (6.4.9) are equivalent.

For purposes of illustration, consider the paraboloid of revolution defined by

$$u = 1 - (x - a)^2 - (y - b)^2 \equiv \phi(x, y, a, b), \tag{6.4.12}$$

where a and b are arbitrary constants. The surface $u = \phi(x, y, a, b)$ is generated by rotating the parabola $u = 1 - x^2$ around the vertical axis $x = a, y = b$ in xyu-space. Working backward, we compute $\phi_x = -2(x - a), \phi_y = -2(y - b)$, from which it is easily seen that (6.4.12) is a complete integral of the nonlinear equation

$$F \equiv 1 - u - \frac{1}{4}(p^2 + q^2) = 0. \tag{6.4.13}$$

It is geometrically obvious that the singular integral of (6.4.13) is the horizontal plane $u = 1$, and this result follows immediately either from the conditions (6.4.8) applied to (6.4.12) or the conditions (6.4.9) applied to (6.4.13).

We can also construct the envelope of the one-parameter family of surfaces

$$u = 1 - (x - a)^2 - [y - w(a)]^2, \tag{6.4.14}$$

which corresponds to requiring the axis of the paraboloid to lie on the curve $y = w(x)$. This envelope is defined by (6.4.14) and

$$2(x - a) + 2[y - w(a)]w'(a) = 0. \tag{6.4.15}$$

For example, if w is the straight line $y = \alpha x$, α = constant, (6.4.15) gives $(x - a) + (y - \alpha a)\alpha = 0$, or $a(x, y) = (x + \alpha y)/(1 + \alpha^2)$, for which (6.4.14) defines the surface

$$u = 1 - \left[x - \frac{x + \alpha y}{1 + \alpha^2}\right]^2 - \left[y - \frac{\alpha(x + \alpha y)}{1 + \alpha^2}\right]^2$$
$$= 1 - \frac{(y - \alpha x)^2}{1 + \alpha^2}. \tag{6.4.16}$$

We verify that this surface is a solution of (6.4.13), since (6.4.16) implies that $1 - u = (y - \alpha x)^2/(1 + \alpha^2)$ and $p = 2\alpha(y - \alpha x)/(1 + \alpha^2)$, $q = -2(y - \alpha x)/(1 + \alpha^2)$. Therefore, $1 - u = (p^2 + q^2)/4$.

6.4.2 Relationship Between Characteristic Strips and the Complete Integral

In the previous section, we demonstrated that the complete integral can be used to generate *certain* solutions by envelope formation. Actually, the complete integral is more far-reaching; it can be used to generate the *three-parameter* family of characteristic strips that gives the general solution of the system (6.3.6), (6.3.12) discussed in Section 6.3.2.

As a first step in demonstrating this property, we recall that if two solution surfaces join smoothly along a curve, this curve, along with the attached strip, is characteristic. Next, suppose that we generate a solution surface from the envelope of the family of complete integrals

$$u = \phi(x, y, a, w_1(a))$$

for a given $w_1(a)$. A second solution surface can be generated from the family

$$u = \phi(x, y, a, w_2(a)).$$

If for some $a = a_0$ we have $w_1(a_0) = w_2(a_0)$ and $w_1'(a_0) = w_2'(a_0)$, then the two envelope surfaces join smoothly along the intersection strip associated with the *three* constants a_0, $b_0 \equiv w(a_0)$, $c_0 \equiv w'(a_0)$. Since w and w' can be chosen arbitrarily, a_0, b_0, c_0 are arbitrary, and we conclude that the strip that smoothly joins any two envelope surfaces of a complete integral is a characteristic strip.

We now prove this result formally for the equation

$$F(x, y, u, p, q) = 0, \tag{6.4.17}$$

for which we assume to have found the complete integral

$$u = \phi(x, y, a, b). \tag{6.4.18}$$

Let us define a three-parameter family of strips, $x(\sigma; a, b, c)$, $y(\sigma; a, b, c)$, $u(\sigma; a, b, c)$, $p(\sigma; a, b, c)$, and $q(\sigma; a, b, c)$, using (6.4.18) and the following five conditions:

$$\phi_a(x, y, a, b) = \alpha\sigma, \quad \alpha = \text{constant}, \tag{6.4.19a}$$

$$\phi_b(x, y, a, b) = \beta\sigma, \quad \beta = \text{constant}, \tag{6.4.19b}$$

$$p = \phi_x(x, y, a, b), \tag{6.4.19c}$$

$$q = \phi_y(x, y, a, b), \tag{6.4.19d}$$

$$0 = \alpha\sigma + \beta w'\sigma \text{ or } \frac{\alpha}{\beta} = -w' \equiv -c. \tag{6.4.19e}$$

To see that (6.4.19) indeed defines a three-parameter family of strips in parametric form [(a, b, c) are constants that identify each member of the family and σ varies along each strip], note that we can solve (6.4.19a)–(6.4.19b) for x and y, since (6.4.2) does not vanish. Moreover, using (6.4.19e) we can express this result in the form $x(\sigma; a, b, c)$, $y(\sigma; a, b, c)$. Substituting these expressions for x and y into (6.4.17) gives $u(\sigma; a, b, c)$, and (6.4.19c)–(6.4.19d) give $p(\sigma; a, b, c)$, $q(\sigma; a, b, c)$.

Having shown that (6.4.19) does indeed define a three-parameter family of strips, our next task is to show that these strips are characteristic. We differentiate (6.4.19a)–(6.4.19b) with respect to σ to obtain

$$\phi_{ax}\frac{dx}{d\sigma} + \phi_{ay}\frac{dy}{d\sigma} = \alpha, \tag{6.4.20a}$$

$$\phi_{bx}\frac{dx}{d\sigma} + \phi_{by}\frac{dy}{d\sigma} = \beta. \tag{6.4.20b}$$

Next, we note that (6.4.18) satisfies (6.4.17) identically for any a and b; therefore,

$$\frac{dF}{da} = F_u\phi_a + F_p\phi_{xa} + F_q\phi_{ya} = 0, \tag{6.4.21a}$$

$$\frac{dF}{db} = F_u\phi_b + F_p\phi_{xb} + F_q\phi_{yb} = 0. \tag{6.4.21b}$$

Solving (6.4.20) for $(dx/d\sigma)$ and $(dy/d\sigma)$ gives

$$\frac{dx}{d\sigma} = \frac{1}{D}(\alpha\phi_{by} - \beta\phi_{ay}), \tag{6.4.22a}$$

$$\frac{dy}{d\sigma} = \frac{1}{D}(\beta\phi_{ax} - \alpha\phi_{bx}), \tag{6.4.22b}$$

where D is the determinant (6.4.2). Similarly, solving (6.4.21) for F_p and F_q gives

$$F_p = -\frac{F_u}{D}(\phi_a\phi_{yb} - \phi_b\phi_{ya}) = -F_u\sigma\frac{dx}{d\sigma}, \tag{6.4.23a}$$

$$F_q = -\frac{F_u}{D}(\phi - b\phi_{xa} - \phi_a\phi_{xb}) = -F_u\sigma\frac{dy}{d\sigma}, \tag{6.4.23b}$$

when (6.4.19a)–(6.4.19b) and (6.4.22) are used.

So far, we have shown that the three-parameter family of strips defined by (6.4.19) satisfies

$$\frac{dx}{d\sigma} = -\frac{1}{\sigma F_u}F_p, \tag{6.4.24a}$$

$$\frac{dy}{d\sigma} = -\frac{1}{\sigma F_u}F_q, \tag{6.4.24b}$$

which, after the change of variable $\sigma \to s$ defined by $(ds/d\sigma) = -1/\sigma F_u$, give (6.3.6a), (6.3.6b). The proof that the remaining three equations, (6.3.6c), (6.3.12a), and (6.3.12b), are also satisfied parallels the steps used in Section 6.3.2 and is not repeated.

We conclude this section by demonstrating that each member of the three-parameter family of characteristic strips associated with the example problem (6.4.13) is simply the branch strip that smoothly joins any two envelope surfaces of (6.4.14). The characteristic strips of (6.4.13) satisfy the system

$$\frac{dx}{ds} = -\frac{p}{2}, \quad \frac{dy}{ds} = -\frac{q}{2}, \quad \frac{du}{ds} = -\frac{1}{2}(p^2 + q^2) = 2(u - 1),$$
$$\frac{dp}{ds} = p, \quad \frac{dq}{ds} = q.$$

We can solve these equations in the form

$$x = \frac{p_0}{2}(1 - e^s) + x_0, \tag{6.4.25a}$$

$$y = \frac{q_0}{2}(1 - e^s) + y_0, \tag{6.4.25b}$$

$$u = 1 + (u_0 - 1)e^{2s}, \tag{6.4.25c}$$

$$p = p_0 e^s, \tag{6.4.25d}$$

$$q = q_0 e^s, \tag{6.4.25e}$$

where x_0, y_0, u_0, p_0, q_0 must satisfy

$$\dot{u}_0 = p_0 \dot{x}_0 + q_0 \dot{y}_0, \tag{6.4.26a}$$

$$1 - u_0 - \frac{1}{4}(p_0^2 + q_0^2) = 0. \tag{6.4.26b}$$

If we let $p_0/2 + x_0 = a$ and $q_0/2 + y_0 = b$, then (6.4.25a)–(6.4.25b) give $(x - a)^2 + (y - b)^2 = (e^{2s}/4)(p_0^2 + q_0^2)$. Using (6.4.26b) to eliminate p_0, q_0 from the right-hand side of the last equation gives

$$(x - a)^2 + (y - b)^2 = e^{2s}(1 - u_0),$$

and using (6.4.25c), we obtain the complete integral

$$(x - a)^2 + (y - b)^2 = 1 - u. \tag{6.4.27}$$

Now, (6.4.25a)–(6.4.25b) also imply that we must have

$$\frac{y - b}{x - a} = \frac{q_0}{p_0} = \text{constant}. \tag{6.4.28}$$

This defines a vertical plane in xyu-space; it intersects the $u = 0$ plane along the straight line with slope $(dy/dx) = q_0/p_0$ defined by (6.4.28). Thus, the characteristic strip is the infinitesimal surface tangent to (6.4.27) along the curve where the vertical plane passing through (6.4.28) and the surface defined by (6.4.27) intersect (see Figure 6.9, which shows the $u \geq 0$ half-space).

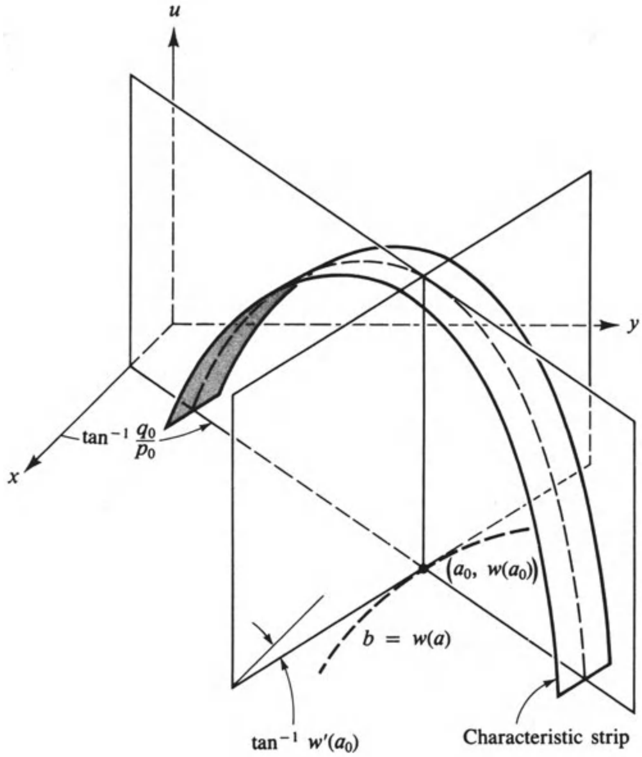

FIGURE 6.9. Characteristic strip

We have argued earlier that by displacing the complete integral an infinitesimal amount in some arbitrary direction associated with the tangent $w'(a)$ to a curve $w(a)$, we generate a characteristic strip. It then follows that the line with slope $w'(a)$ passing through $x = a_0$, $y = b_0 = w(a_0)$ in the xy-plane *must be normal* to the vertical plane (6.4.28). We now demonstrate this for our example, where

$$w'(a) = \frac{db}{da} = \frac{\dot{b}}{\dot{a}} = \frac{\dot{q}_0/2 + \dot{y}_0}{\dot{p}_0/2 + \dot{x}_0}. \tag{6.4.29}$$

But subtracting the derivative of (6.4.26b) with respect to τ from (6.4.26a) gives

$$0 = p_0\left(\dot{x}_0 + \frac{\dot{p}_0}{2}\right) + q_0\left(\dot{y}_0 + \frac{\dot{q}_0}{2}\right).$$

Therefore, we find that

$$w'(a) = -\frac{p_0}{q_0}, \tag{6.4.30}$$

which is the negative of the reciprocal of the slope of the straight line defined by (6.4.28). This proves that the vertical plane tangent to the curve that generates the envelope is indeed normal to the plane in which the characteristic curve lies (see Figure 6.9).

6.4.3 The Complete Integral of the Hamilton–Jacobi Equation

In Section 6.2.4 we showed that the Hamilton–Jacobi equation [see (6.2.55) with $s = t$]

$$\frac{\partial J}{\partial t} + H\left(t, q_i, \frac{\partial J}{\partial q_i}\right) = 0 \tag{6.4.31}$$

for the scalar $J(t, q_i)$ governs the field of extremals from a fixed point $t = t_I$, $q_i = \kappa_i$ for the variational principle (6.2.13). We also showed [see (6.2.93) with $J = K_2$ and $s = t$] that (6.4.31) defines the generating function, K_2, of a canonical transformation to a new Hamiltonian, which vanishes identically. As a result, the new set of coordinates and momenta are constants, and the transformation relations (6.2.92) define the solution of Hamilton's differential equations (6.2.28).

Let us now study the partial differential equation (6.4.31) written in the form (6.3.20). We use the notation

$$u \equiv J, \tag{6.4.32a}$$

$$x_j \equiv q_j, \quad j = 1, \ldots, n, \tag{6.4.32b}$$

$$p_j \equiv \frac{\partial J}{\partial q_j}, \quad j = 1, \ldots, n, \tag{6.4.32c}$$

$$x_{n+1} \equiv t, \tag{6.4.32d}$$

$$p_{n+1} \equiv \frac{\partial J}{\partial t}, \tag{6.4.32e}$$

and observe that the definition (6.4.32c) for the p_j is notationally consistent with the second equation in (6.2.54) or (6.2.91a), and that regarding $t = x_{n+1}$ implies that (6.4.32e) is the extension of (6.4.32c) to $j = n + 1$. In effect, we have the $(n + 1)$-dimensional equation

$$F(x_1, \ldots, x_{n+1}, p_1, \ldots, p_{n+1}) \equiv p_{n+1} + H(x_1, \ldots, x_{n+1}, p_1, \ldots, p_n) = 0, \tag{6.4.33}$$

which does not involve u explicitly.

According to (6.3.21), the characteristics of (6.4.33) obey

$$\frac{dx_j}{ds} = \frac{\partial H}{\partial p_j}, \quad j = 1, \ldots, n, \tag{6.4.34a}$$

$$\frac{dx_{n+1}}{ds} = 1, \tag{6.4.34b}$$

$$\frac{du}{ds} = \sum_{j=1}^{n} p_j \frac{\partial H}{\partial p_j} + p_{n+1}, \tag{6.4.34c}$$

$$\frac{dp_j}{ds} = -\frac{\partial H}{\partial x_j}, \quad j = 1, \ldots, n, \tag{6.4.34d}$$

$$\frac{dp_{n+1}}{ds} = -\frac{\partial H}{\partial x_{n+1}}. \tag{6.4.34e}$$

Thus, (6.4.34a) and (6.4.34d) give Hamilton's differential equations (6.2.28), in which we use (6.4.34b) to set $x_{n+1} = s$. This system of $2n$ equations does not involve u or p_{n+1}. Once the x_j and p_j have been computed, (6.4.34e) gives p_{n+1} by quadrature. In fact, (6.4.34e) is just (6.2.29). We can then compute u by quadrature from (6.4.34c).

We see that the basic system of $2n$ equations (6.4.34a) and (6.4.34d) defining a given dynamical system recurs as the essential part of the characteristic system for (6.4.33). It would therefore appear to be of no particular advantage to recast a given Hamiltonian system of $2n$ ordinary differential equations in terms of the associated Hamilton–Jacobi partial differential equation. This is generally true if a complete integral of the Hamilton–Jacobi equation is not available. However, if a complete integral of (6.4.33) can be derived directly—that is, without relying on a solution of (6.4.34)—we expect [based on the examples that we studied in Sections 6.2.6ii and 6.4.2] to be able to *bypass having to solve* (6.4.34a) and (6.4.34d). The proof of this statement is discussed next.

In direct analogy with the two-dimensional case (6.4.2), since u does not occur explicitly in (6.4.33), a complete integral is the $(n + 1)$-dimensional manifold

$$u = \phi(x_1, \ldots, x_{n+1}, a_1, \ldots, a_n) + a_{n+1}, \tag{6.4.35}$$

which satisfies (6.4.33) identically. Here, $a_1, \ldots, a_n$ are n independent constants—that is,

$$\det\left\{\frac{\partial^2 \phi}{\partial x_j \partial a_k}\right\} \neq 0 \tag{6.4.36}$$

—and a_{n+1} is an arbitrary additive constant. We shall now prove that given a complete integral, the solution of Hamilton's equations (6.4.34a) and (6.4.34d) are defined implicitly by the $2n$ algebraic relations

$$\frac{\partial \phi}{\partial a_j} = b_j = \text{constant}, \tag{6.4.37a}$$

$$\frac{\partial \phi}{\partial x_j} = p_j. \tag{6.4.37b}$$

A solution of the system (6.4.34a) and (6.4.34d) consists of $2n$ functions x_i, p_i of s and $2n$ arbitrary constants. Let us first show that (6.4.37) defines such a set of functions and then show that the result satisfies (6.4.34a) and (6.4.34d). The system (6.4.37a) can be solved for the x_i because (6.4.36) holds. This results in n functions x_i of s and the $2n$ constants a_i, b_i. Substituting this result into the left-hand side of (6.4.37b) directly gives a set of n functions p_i of s and the a_i, b_i. Here we use (6.4.34b) to identify $x_{n+1} = s$.

To prove that the $2n$ functions q_i, p_i of s defined by (6.4.37) satisfy (6.4.34a) and (6.4.34d), we begin by taking the total derivative of (6.4.37a) with respect to s:

$$\frac{\partial^2 \phi}{\partial a_j \partial s} + \sum_{k=1}^{n} \frac{\partial^2 \phi}{\partial a_j \partial x_k} \frac{dx_k}{ds} = 0. \tag{6.4.38}$$

Now, since (6.4.35) is a solution of (6.4.33), we have the identity

$$\frac{\partial \phi}{\partial s} + H\left(s, x_i, \frac{\partial \phi}{\partial x_i}\right) = 0. \tag{6.4.39}$$

Therefore, the partial derivative of this with respect to a_j gives

$$\frac{\partial^2 \phi}{\partial a_j \partial s} + \sum_{k=1}^{n} \frac{\partial H}{\partial p_k} \frac{\partial^2 \phi}{\partial a_j \partial x_k} = 0. \tag{6.4.40}$$

Subtracting (6.4.40) from (6.4.38) gives the identity

$$\sum_{k=1}^{n} \frac{\partial^2 \phi}{\partial a_j \partial x_k} \left(\frac{dx_k}{ds} - \frac{\partial H}{\partial p_k}\right) = 0, \tag{6.4.41}$$

which is a homogeneous system of n algebraic equations for the $z_i \equiv (dx_i/ds - \partial H/\partial p_i)$. Since the determinant (6.4.36) of coefficients does not vanish, we conclude that each of the z_i must vanish, and this gives (6.4.34a).

To prove that (6.4.34d) holds, we take the total derivative of (6.4.37b) with respect to s:

$$\frac{dp_j}{ds} = \frac{\partial^2 \phi}{\partial x_j \partial s} + \sum_{k=1}^{n} \frac{\partial^2 \phi}{\partial x_j \partial x_k} \frac{dx_k}{ds}. \tag{6.4.42}$$

Next, we take the partial derivative of (6.4.39) with respect to x_j:

$$0 = \frac{\partial^2 \phi}{\partial x_j \partial s} + \sum_{k=1}^{n} \frac{\partial H}{\partial p_k} \frac{\partial^2 \phi}{\partial x_k \partial x_j} + \frac{\partial H}{\partial x_j}. \tag{6.4.43}$$

Subtracting (6.4.43) from (6.4.42) and noting (6.4.41) gives (6.4.34d).

This completes the proof that knowing the complete integral leads (after some algebra) to the general solution of the Hamiltonian system of equations (6.4.34a) and (6.4.34d). This result means that the search for a canonical transformation generated by K_2 to a new Hamiltonian that vanishes identically (see (6.2.93)) is exactly equivalent to finding the complete integral for the Hamiltonian in a given set of variables.

It is instructive, although somewhat repetitive, to rederive the solution of Euler's problem [see (6.2.117) and the discussion in Section 6.2.6ii] from the point of view of calculating the complete integral for the Hamilton–Jacobi equation

$$\frac{\partial u}{\partial x_3} + \frac{1}{2(\cosh^2 x_1 - \cos^2 x_2)}[p_1^2 + p_2^2 - 2\cosh x_1 - 2(2\mu - 1)\cos x_2] = 0. \tag{6.4.44}$$

We assume that the complete integral of (6.4.44),

$$u = \phi(x_1, x_2, x_3, a_1, a_2) + a_3, \tag{6.4.45}$$

has the separated form

$$\phi = \phi_1(x_1, a_1, a_2) + \phi_2(x_2, a_1, a_2) + \phi_3(x_3, a_1, a_2). \tag{6.4.46}$$

Substituting (6.4.46) into (6.4.44) gives [see the calculations that lead to (6.2.121)]

$$\phi_1 = \int^{x_1} (a_2 + 2\cosh\xi + 2a_1\cosh^2\xi)^{1/2} d\xi, \tag{6.4.47a}$$

$$\phi_2 = \int^{x_2} [-a_2 + 2(2\mu - 1)\cos\eta - 2a_1\cos^2\eta]^{1/2} d\eta, \tag{6.4.47b}$$

$$\phi_3 = -a_1 x_3, \tag{6.4.47c}$$

and this defines the complete integral.

Equation (6.4.37a) for $j = 1$ and $j = 2$ gives

$$\frac{\partial\phi}{\partial a_1} = -x_3 + \int^{x_1} \frac{\cosh^2\xi}{(a_2 + 2\cosh\xi + 2a_1\cosh^2\xi)^{1/2}} d\xi - \int^{x_2} \frac{\cos^2\eta}{[-a_2 + 2(2\mu - 1)\cos\eta - 2a_1\cos^2\eta]^{1/2}} d\eta = b_1, \tag{6.4.48a}$$

$$\frac{\partial\phi}{\partial a_2} = \frac{1}{2}\int^{x_1} \frac{d\xi}{(a_2 + 2\cosh\xi + 2a_1\cosh^2\xi)^{1/2}} - \frac{1}{2}\int^{x_2} \frac{d\eta}{[-a_2 + 2(2\mu - 1)\cos\eta - 2a_1\cos^2\eta]^{1/2}} = b_2, \tag{6.4.48b}$$

and (6.4.37b) gives

$$\frac{\partial\phi}{\partial x_1} = (a_2 + 2\cosh x_1 + 2a_1\cosh^2 x_1)^{1/2} = p_1, \tag{6.4.49a}$$

$$\frac{\partial\phi}{\partial x_2} = [-a_2 + 2(2\mu - 1)\cos x_2 - 2a_1\cos^2 x_2]^{1/2} = p_2. \tag{6.4.49b}$$

Comparing (6.4.48)–(6.4.49) with (6.2.122)–(6.2.125), we see that we have derived identical results when we identify $p_1 \to p_1$, $p_2 \to p_2$, $x_1 \to q_1$, $x_2 \to q_2$, $x_3 \to s$, $a_1 \to \overline{p}_1$, $a_2 \to \overline{p}_2$, $b_1 \to \overline{q}_1^{(0)}$, and $b_2 \to \overline{q}_2^{(0)}$. Thus, the calculation of the solution via canonical transformation to a zero Hamiltonian is exactly equivalent to the calculation using (6.4.37) for the complete integral.

We have seen that solvability of the Hamilton–Jacobi equation is intimately connected with the integrability of the Hamiltonian system (6.4.34a) and (6.4.34d) of $2n$ differential equations. We have used the idea of separation of variables to solve the Hamilton–Jacobi equation directly in the form of a complete integral. Whether a given Hamiltonian is separable or not [in the sense discussed in Section 6.2.6 and above] is easy to establish by trial substitution. As we have observed, separability is a property of the particular choice of variables. Therefore, the question of whether or not we can compute the complete integral directly also depends on this choice of variables. The answer to the more fundamental question of whether for a given Hamiltonian there exists a set of variables in terms of which the Hamilton–Jacobi equation becomes separable (hence solvable) is not known in general.

Problems

6.4.1 Given a two-parameter family of surfaces

$$u = \Phi(x, y, a, b), \tag{6.4.50}$$

show that there exists a unique partial differential equation of the first order for which (6.4.50) is a complete integral. Derive this partial differential equation. Specialize your results to the case

$$u = abxy + ax^2. \tag{6.4.51}$$

6.4.2 Consider the equation

$$u^2(1 + p^2 - q^2) - 1 = 0. \tag{6.4.52}$$

a. Calculate the complete integral and describe it geometrically. Calculate the singular integral.
b. Construct a solution involving an arbitrary function by envelope formation from the complete integral.
c. Calculate the characteristic strips of (6.4.52), and discuss how they are formed from the complete integral.

6.4.3 Clairaut's equation is defined as

$$F(x, y, u, p, q) \equiv xp + yq + f(p, q) - u = 0, \tag{6.4.53}$$

for a given function f.

a. Use separation of variables to show that the two-parameter family of planes

$$u = ax + by + f(a, b) \tag{6.4.54}$$

is a complete integral of (6.4.53).
b. For the case $f(a, b) \equiv -\frac{1}{2}(a^2 + b^2)$, calculate the singular integral of (6.4.53) and interpret this result geometrically.
c. Calculate the characteristic strips of (6.4.53) and discuss how they are formed from the complete integral.

6.4.4 For a given function $H(x, y, p)$, we want to study the nonlinear equation

$$F \equiv H(x, y, p) + q = 0, \tag{6.4.55}$$

where $p = u_x, q = u_y$.

a. Show that the equations for the characteristic strips for (6.4.55) can be separated into two equations involving x and p as dependent variables and y as independent variable.

b. Let

$$u = \phi(x, y, a) + b \tag{6.4.56}$$

be a complete integral of (6.4.55).

Consider now the two equations

$$\phi_a = \alpha = \text{constant}, \tag{6.4.57a}$$

$$\phi_x = p. \tag{6.4.57b}$$

Assume that $x = X(y, a, \alpha)$ is the solution of (6.4.57a) for x and also assume that if the preceding $X(y, a, \alpha)$ is substituted into (6.4.57b) for x, this equation takes the form $p = P(y, a, \alpha)$. Prove that the two functions $X(y, a, \alpha)$ and $P(y, a, \alpha)$ so calculated define the general solution of the two ordinary differential equations you derived in part (a).

c. In this part, interpret y as the time, and consider the nonlinear oscillator defined by

$$\frac{d^2x}{dy^2} + f'(x) = 0, \tag{6.4.58}$$

with $p = x'$. The energy equation for (6.4.58) is

$$H = \frac{p^2}{2} + f(x) = \text{constant}. \tag{6.4.59}$$

What are the two differential equations for x and p corresponding to those in part (a)? Use (6.4.59) for H in (6.4.55) and derive an expression for the complete integral for this case. Show by explicit calculation that use of the complete integral, as indicated in part (b), leads to a solution of either (6.4.58) or, equivalently, of the two differential equations for x and p.

7

Quasilinear Hyperbolic Systems

Much of Chapter 4 concerned the linear hyperbolic equation of second order with two independent variables, or the system of two first-order equations with two independent variables. Typically, such linear equations govern the perturbation due to a known solution of a quasilinear problem. In Chapter 5, we discussed some aspects of weak solutions for quasilinear equations associated with systems of integral conservation laws. In particular, we derived the shock conditions and used these results to study simple solutions consisting of uniformly propagating shocks.

In this chapter we consider more general initial- and boundary-value problems for quasilinear hyperbolic equations, and we again restrict the discussion to the case of two independent variables. Our approach here parallels that used in Chapter 4 in many respects. The fundamental difference is that for the quasilinear problem, the characteristics depend on the solution and are therefore defined only locally by the governing equations. The same geometrical concepts remain valid locally as long as the characteristics are well-defined; the essential added complication is that the calculations for the characteristic curves are coupled with those for the dependent variables. As in the scalar quasilinear problem, characteristics of a given family may cross, in which event we look for a weak solution containing discontinuities.

7.1 The Quasilinear Second-Order Hyperbolic Equation

The general quasilinear second-order equation in two independent variables has the form (cf. (4.1.1))

$$au_{xx} + 2bu_{xy} + cu_{yy} + d = 0, \tag{7.1.1}$$

where a, b, c, and d are now functions of x, y, u, u_x, and u_y, and we shall consider only the hyperbolic problem, where $\Delta \equiv b^2 - ac > 0$ in some solution domain. Note that since Δ now also depends on u, u_x, and u_y, the type of a given equation is generally not defined by the functional form of the coefficients a, b, c, and d as in the linear case; one must also account for the solution in evaluating Δ. Thus,

Δ may well be positive in some domain for certain Cauchy data and negative for other choices.

The geometerical interpretation of the characteristic curves is locally identical to the situation for the linear problem. More precisely, we again define a characteristic curve $x(s)$, $y(s)$ as one for which knowledge of u, u_x, and u_y does not define the second derivative leading out of the curve (see Section 4.2.1i).

7.1.1 Transformation to Characteristic Independent Variables

Let C be a smooth nonintersecting curve defined in the xy-plane in parametric form by $x = X(s)$, $y = Y(s)$. Denote $u_x \equiv p$, $u_y \equiv q$ and assume that $u = U(s)$, $p = P(s)$, and $q = Q(s)$ are also specified on C in a consistent manner—that is, that we require

$$\dot{U}(s) = P(s)\dot{X}(s) + Q(s)\dot{Y}(s). \tag{7.1.2}$$

We wish to use the preceding Cauchy data together with (7.1.1) to calculate all three second derivatives u_{xx}, u_{xy}, and u_{yy} on C. In addition to (7.1.1), we have at our disposal the two equations that result from differentiating $p = u_x(x, y)$ and $q = u_y(x, y)$ with respect to s on C. Thus, we must have

$$A(s)u_{xx} + 2B(s)u_{xy} + C(s)u_{yy} = -D(s), \tag{7.1.3a}$$

$$\dot{X}(s)u_{xx} + \dot{Y}(s)u_{xy} = \dot{P}(s), \tag{7.1.3b}$$

$$\dot{X}(s)u_{xy} + \dot{Y}(s)u_{yy} = \dot{Q}(s), \tag{7.1.3c}$$

where we use capital letters to denote the coefficients calculated along C. For instance, $A(s) \equiv a(X(s), Y(s), U(s), P(s), Q(s))$, and so on.

If for a given C and given Cauchy data U, P, and Q, the determinant of coefficients in (7.1.3) does not vanish, we can calculate u_{xx}, u_{xy}, and u_{yy} on C. A characteristic curve is defined to be a curve along which Cauchy data do not determine these three unknowns. This occurs only if the determinant of coefficients in (7.1.3) vanishes, that is,

$$\begin{vmatrix} A & 2B & C \\ \dot{X} & \dot{Y} & 0 \\ 0 & \dot{X} & \dot{Y} \end{vmatrix} = A\dot{Y}^2 - 2B\dot{X}\dot{Y} + C\dot{X}^2 = 0. \tag{7.1.4}$$

Let us divide (7.1.4) by $\dot{X}^2$ and denote $\dot{Y}/\dot{X} = dy/dx = y'$. We see that the characteristic slope y' obeys [see (4.1.13b)]

$$Ay'^2 - 2By' + C = 0, \tag{7.1.4}$$

which has real solutions if $B^2 - AC > 0$. In this case, we denote the two characteristic slopes by λ^+ and λ^-, where

$$\lambda^+ = \frac{B + \sqrt{B^2 - AC}}{A}, \tag{7.1.6a}$$

$$\lambda^- = \frac{B - \sqrt{B^2 - AC}}{A}. \tag{7.1.6b}$$

An alternative way of expressing this result is to denote by ξ the parameter s that varies along the characteristic curve with slope λ^-, and let η be the parameter that varies along the characteristic curve with slope λ^+. We may then regard x and y as functions of ξ and η. Using the notation $x = X(\xi, \eta)$, $y = Y(\xi, \eta)$ as before, we have the following pair of partial differential equations governing the characteristic curves:

$$Y_\xi(\xi, \eta) - \lambda^-(\xi, \eta)X_\xi = 0, \tag{7.1.7a}$$

$$Y_\eta(\xi, \eta) - \lambda^+(\xi, \eta)X_\eta = 0. \tag{7.1.7b}$$

Henceforth, for brevity, we shall also refer to the curves on which η = constant as the λ^- characteristics, and the curves on which ξ = constant as the λ^+ characteristics. At this stage λ^+ and λ^- are unknown because they involve u, p, and q in addition to x and y. Thus, (7.1.7) defines the characteristic curves only locally in a small neighborhood of a point where u, p, and q are given.

Now let us reexamine the system (7.1.3) along a particular characteristic strip—that is, an infinitesimal solution surface attached to a characteristic curve. If a solution $u(x, y)$ that contains this characteristic strip exists, the three algebraic relations that result from solving (7.1.3) for u_{xx}, u_{xy}, and u_{yy} must be compatible. Each of these relations is in the form of a numerator determinant divided by the same denominator determinant (7.1.4). In particular, since the denominator determinant vanishes along a characteristic curve, each of the three numerator determinants associated with the solutions for u_{xx}, u_{xy}, and u_{yy} must also vanish in order for a solution to exist. For example, in the solution for u_{xy} from (7.1.3), we have the numerator determinant

$$N \equiv \begin{vmatrix} A & -D & C \\ \dot{X} & \dot{P} & 0 \\ 0 & \dot{Q} & \dot{Y} \end{vmatrix}, \tag{7.1.8a}$$

which must vanish. This gives

$$A\dot{P}\dot{Y} + D\dot{X}\dot{Y} + C\dot{X}\dot{P} = 0. \tag{7.1.8b}$$

Dividing (7.1.8b) by $A\dot{Y}$ gives

$$\dot{P} + \frac{C}{A}\frac{\dot{X}}{\dot{Y}}\dot{Q} + \frac{D}{A}\dot{X} = 0, \tag{7.1.8c}$$

which must hold along either the η = constant characteristic or the ξ = constant characteristic. Along the η = constant characteristic, we let $s = \xi$, hence $d/ds \to \partial/\partial\xi$, and (7.1.8c) gives

$$P_\xi + \frac{C}{A\lambda^-}Q_\xi + \frac{D}{A}X_\xi = 0. \tag{7.1.8d}$$

Now, according to (7.1.6b),

$$\frac{C}{A\lambda^-} = \frac{CA}{A[B-(B^2-AC)^{1/2}]} = \frac{C[B+(B^2-AC)^{1/2}]}{B^2-(B^2-AC)}$$
$$= \frac{B+(B^2-AC)^{1/2}}{A} = \lambda^+.$$

Therefore, (7.1.8d) may also be written as

$$P_\xi + \lambda^+ Q_\xi + \frac{D}{A} X_\xi = 0. \tag{7.1.9a}$$

Similarly, along the ξ = constant characteristic, we have

$$P_\eta + \lambda^- Q_\eta + \frac{D}{A} X_\eta = 0. \tag{7.1.9b}$$

We can also verify that setting the numerator determinant for either u_{xx} or u_{yy} equal to zero gives the result (7.1.9), as expected.

To complete the system of charcteristic equations, we need to use the consistency condition (7.1.2) written either in terms of ξ,

$$U_\xi = PX_\xi + QY_\xi, \tag{7.1.10a}$$

or η,

$$U_\eta = PX_\eta + QY_\eta. \tag{7.1.10b}$$

In summary, we have derived the system of six partial differential equations (7.1.7), (7.1.9), and (7.1.10) for the five variables X, Y, U, P, and Q. It is easily seen that this system is not overdetermined and that in fact, any five of these equations imply the sixth. For example, let us show that (7.1.10b) is a consequence of (7.1.7), (7.1.9), and (7.1.10a). We denote

$$F(\xi, \eta) \equiv U_\eta - PX_\eta - QY_\eta, \tag{7.1.11a}$$

and wish to prove that $F(\xi, \eta) \equiv 0$ if (7.1.7), (7.1.9), and (7.1.10a) hold. We calculate

$$F_\xi = U_{\xi\eta} - P_\xi X_\eta - PX_{\xi\eta} - Q_\xi Y_\eta - QY_{\xi\eta}. \tag{7.1.11b}$$

Taking the partial derivative of (7.1.10a) with respect to η gives

$$0 = U_{\xi\eta} - P_\eta X_\xi - PX_{\xi\eta} - Q_\eta Y_\xi - QY_{\xi\eta}, \tag{7.1.11c}$$

and when this expression is subtracted from (7.1.11b), we obtain

$$F_\xi = P_\eta X_\xi - P_\xi X_\eta + Q_\eta Y_\xi - Q_\xi Y_\eta. \tag{7.1.12}$$

Now we use (7.1.7) to express Y_ξ and Y_η in terms of X_ξ and X_η in (7.1.12) to get

$$F_\xi = P_\eta X_\xi - P_\xi X_\eta + Q_\eta \lambda^- X_\xi - Q_\xi \lambda^+ X_\eta. \tag{7.1.13}$$

Using (7.1.9a) and (7.1.9b) to eliminate $\lambda^+ Q_\xi$ and $\lambda^- Q_\eta$, we obtain $F_\xi = 0$. Therefore, $F = f(\eta)$, a constant along each characteristic curve η = constant.

But since F vanishes on some noncharacteristic initial curve, we conclude that $f \equiv 0$; hence, $F(\xi, \eta) \equiv 0$, which is just (7.1.10b).

For the special case where the coefficients of (7.1.1) do not depend on u, (7.1.7) and (7.1.9) decouple from (7.1.10), and we can solve these first for X, Y, P and Q. Either (7.1.10a) or (7.1.10b) then gives u by quadrature.

7.1.2 The Cauchy Problem; the Numerical Method of Characteristics

Let $\mathcal{C}_0$ be a noncharacteristic initial curve on which u, p, and q are specified consistently with (7.1.2). We discretize this curve be selecting along it a spacing of points that is appropriate to the rate of change of the given data (see Figure 7.1a).

Let us use the same subscript notation as in Section 4.2.2 and indicate different values of ξ and η by the first and second subscripts, respectively. With no loss of generality, we may choose the point $\xi = 0$ and $\eta = 0$ to lie on $\mathcal{C}_0$. Points on $\mathcal{C}_0$ will then be denoted as follows:

$$x = X(\xi_i, \eta_{-i}) \equiv X_{i,-i}, \quad y = Y(\xi_i, \eta_{-i}) \equiv Y_{i,-i},$$

which assumes that the directions of increasing ξ and η on $\mathcal{C}_0$ are both above it, as indicated in Figure 7.1a.

Consider now two adjacent points $(m-1, n)$ and $(m, n-1)$ at which the values of x, y, u, p, and q are known. Assuming that $\lambda^+_{m,n-1} > \lambda^-_{m-1,n}$, the point (m, n) is located by the intersection of the $\eta =$ constant characteristic curve from $(m-1, n)$ with the $\xi =$ constant characteristic from $(m, n-1)$, as shown Figure 7.1a.

Therefore, the x- and y-coordinates of the point (m, n) are approximately defined by the following forward difference formulas associated with (7.1.7):

$$Y_{m,n} - Y_{m-1,n} - \lambda^-_{m-1,n}(X_{m,n} - X_{m-1,n}) = 0, \qquad (7.1.14a)$$

$$Y_{m,n} - Y_{m,n-1} - \lambda^+_{m,n-1}(X_{m,n} - X_{m,n-1}) = 0. \qquad (7.1.14b)$$

Since $\lambda^-_{m-1,n}$ and $\lambda^+_{m,n-1}$ are known (they depend on the known values of x, y, u, p, and q at $(m-1, n)$ and $(m, n-1)$, respectively), we can solve the two linear equations (7.1.14) for $X_{m,n}$ and $Y_{m,n}$ to obtain

$$X_{m,n} = \frac{1}{\lambda^+_{m,n-1} - \lambda^-_{m-1,n}} \Big[Y_{m-1,n} - Y_{m,n-1} + \lambda^+_{m,n-1} X_{m,n-1} - \lambda^-_{m-1,n} X_{m-1,n} \Big], \qquad (7.1.15a)$$

$$Y_{m,n} = \frac{1}{\lambda^+_{m,n-1} - \lambda^-_{m-1,n}} \Big[\lambda^+_{m,n-1} Y_{m-1,n} - \lambda^-_{m-1,n} Y_{m,n-1} \; \lambda_{m,n-1} \lambda^-_{m-1,n} (X_{m,n-1} - X_{m-1,n}) \Big]. \qquad (7.1.15b)$$

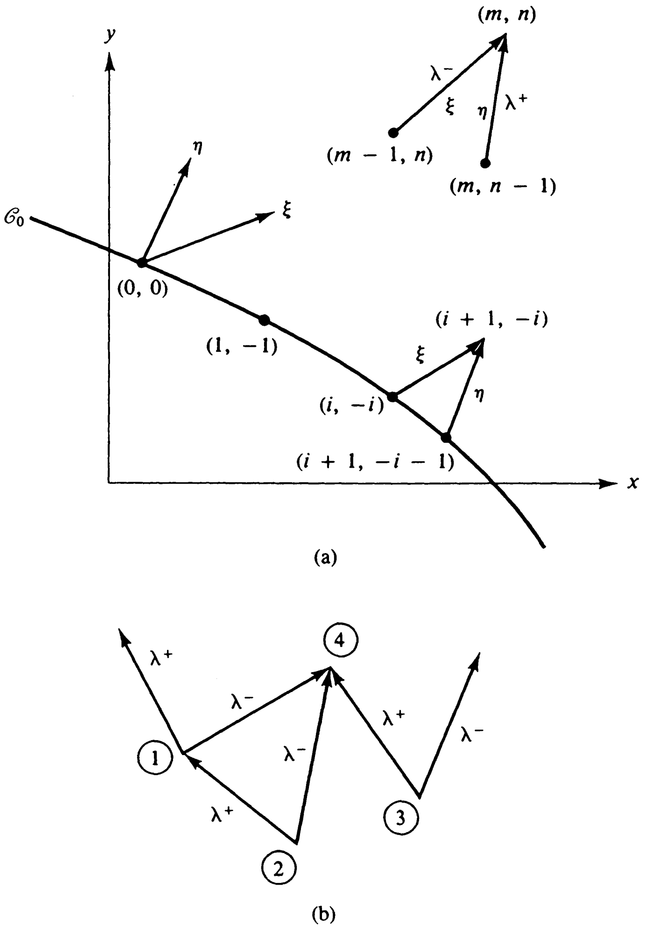

FIGURE 7.1. Network of characteristics

This result breaks down if $\lambda^+_{m,n-1} \approx \lambda^-_{m-1,n}$, but this can occur only if $B^2 - AC = 0$ at some point between $(m, n-1)$ and $(m-1, n)$, a situation that we have ruled out for a solution that remains hyperbolic in some domain.

The finite difference form of (7.1.9) is

$$P_{m,n} - P_{m-1,n} + \lambda^+_{m-1,n}(Q_{m,n} - Q_{m-1,n}) = -\widetilde{D}_{m-1,n}(X_{m,n} - X_{m-1,n}), \tag{7.1.16a}$$

$$P_{m,n} - P_{m,n-1} + \lambda^-_{m,n-1}(Q_{m,n} - Q_{m,n-1}) = -\widetilde{D}_{m,n-1}(X_{m,n} - X_{m,n-1}), \tag{7.1.16b}$$

where $\widetilde{D} = D/A$ and the right-hand sides of (7.1.16) are known. These two linear equations for $P_{m,n}$ and $Q_{m,n}$ can also be solved, and we obtain

$$P_{m,n} = \frac{1}{\lambda^+_{m-1,n} - \lambda^-_{m,n-1}} (\beta_{m,n}\lambda^+_{m-1,n} - \alpha_{m,n}\lambda^-_{m,n-1}), \tag{7.1.17a}$$

$$Q_{m,n} = \frac{1}{\lambda^+_{m-1,n} - \lambda^-_{m,n-1}} (\alpha_{m,n} - \beta_{m,n}), \tag{7.1.17b}$$

where

$$\alpha_{m,n} \equiv P_{m-1,n} + \lambda^+_{m-1,n} Q_{m-1,n} - \frac{\widetilde{D}_{m-1,n}}{\lambda^+_{m,n-1} - \lambda^-_{m-1,n}} \Big[Y_{m-1,n} - Y_{m,n-1} + \lambda^+_{m,n-1}(X_{m,n-1} - X_{m-1,n}) \Big], \tag{7.1.18a}$$

$$\beta_{m,n} \equiv P_{m,n-1} + \lambda^-_{m,n-1} Q_{m,n-1} - \frac{\widetilde{D}_{m,n-1}}{\lambda^+_{m,n-1} - \lambda^-_{m-1,n}} \Big[Y_{m-1,n} - Y_{m,n-1} + \lambda^-_{m-1,n}(X_{m,n-1} - X_{m-1,n}) \Big]. \tag{7.1.18b}$$

Equations (7.1.17) and (7.1.18) define P and Q at (m, n) in terms of known quantities at $(m - 1, n)$ and $(m, n - 1)$. Again, we rule out $\lambda^+_{m-1,n} = \lambda^-_{m,n-1}$ and $\lambda^+_{m,n-1} = \lambda^-_{m-1,n}$ for hyperbolic problems.

To complete the solution at (m, n), we need u there, and this can be calculated from either (7.1.10a),

$$U^{(\xi)}_{m,n} = U_{m-1,n} + P_{m-1,n}(X_{m,n} - X_{m-1,n}) + Q_{m-1,n}(Y_{m,n} - Y_{m-1,n}), \tag{7.1.19a}$$

or (7.1.10b),

$$U^{(\eta)}_{m,n} = U_{m,n-1} + P_{m,n-1}(X_{m,n} - X_{m,n-1}) + Q_{m,n-1}(Y_{m,n} - Y_{m,n-1}). \tag{7.1.19b}$$

All the terms on the right-hand sides of (7.1.19) have been calculated at this stage. As in (4.2.28), we may use the weighted average of the two expressions in (7.1.19) to define $U_{m,n}$.

This procedure defines x, y, u, p, and q uniquely as long as characteristics of the same family from adjacent gridpoints do not intersect. For example, consider the situation depicted in Figure 7.1b, where the values of x, y, u, p, and q are given (from previous calculations) at the three adjacent points (1), (2), and (3). We have the λ^- characteristics from (1) and (2) intersecting at the same point, (4), as the λ^+ characteristic from (3). The values of p, q (and u) at (4) are ambiguous because they depend on whether we use the pair of points (1), (3) or (2), (3) to compute them.

A similar situation was encountered in Chapter 5, where one-parameter families of characteristic curves had intersections beyond a certain envelope curve. We then approached the problem from the vantage of weak solutions and prevented the crossing of characteristics by inserting appropriate shocks. We shall postpone discussion of weak solutions until we have considered a hyperbolic system of two first-order equations in the next section.

Problems

7.1.1 Consider steady two-dimensional compressible flow over a body. The dimensionless equations for the velocity potential ϕ^* and density ρ^* are given by (3.3.51) and (3.3.52) with $\partial/\partial t^* \equiv 0$. Dropping the asterisks, we have

$$\frac{1}{2}(\phi_x^2 + \phi_y^2) + \frac{1}{\gamma - 1}\rho^{\gamma-1} = \frac{1}{\gamma - 1} + \frac{M^2}{2}, \quad (7.1.20a)$$

$$\rho(\phi_{xx} + \phi_{yy}) + \rho_x\phi_x + \rho_y\phi_y = 0. \quad (7.1.20b)$$

a. Introduce the dimensionless speed of sound $a = \rho^{(\gamma-1)/2}$ (cf. (3.3.22)) to show that equations (7.1.20) transform to

$$(a^2 - \phi_x^2)\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + (a^2 - \phi_y^2)\phi_{yy} = 0, \quad (7.1.21a)$$

where

$$a^2 = 1 + \frac{\gamma - 1}{2}M^2 - \frac{\gamma - 1}{2}(\phi_x^2 + \phi_y^2). \quad (7.1.21b)$$

b. Show that (7.1.21a) is hyperbolic for supersonic flow, that is,

$$\phi_x^2 + \phi_y^2 > a^2, \quad (7.1.22)$$

and that the characteristics according to (7.1.6) are given by

$$\lambda^{\pm} = \frac{1}{a^2 - \phi_x^2}\left[-\phi_x\phi_y \pm a(\phi_x^2 + \phi_y^2 - a^2)^{1/2}\right]. \quad (7.1.23)$$

7.1.2 Consider the following signaling problem for the quasilinear wave equation on $0 \le x < \infty, 0 \le y < \infty$,

$$u_{xx} - a^2(u, u_x, u_y)u_{yy} = 0, \quad (7.1.24a)$$

$$u(x, 0) = f(x),\ u_y(x, 0) = g(x), \quad (7.1.24b)$$

$$u(0, y) = h(y), \quad (7.1.24c)$$

where a, f, g, and h are prescribed and $f(0) = h(0)$.

Generalize the discussion in Section 7.1.2 to include the boundary condition (7.1.24c) assuming that $\lambda^+ > 0$ and $\lambda^- < 0$.

7.2 Systems of n First-Order Equations

The discussion in this section parallels that in Section 4.3. As in the case of the quasilinear second-order equation, the essential difference from the corresponding linear problem is that characteristic slopes depend on the solution, and therefore the geometry of characteristics can be derived only locally.

The physical problem is modeled by a system of n first-order equations of the form

$$\frac{\partial u_j}{\partial t} + \sum_{k=1}^{n} A_{jk}\frac{\partial u_k}{\partial x} = f_j, \quad j = 1, \ldots, n, \quad (7.2.1)$$

for the n dependent variables $u_1(x, t)$, $u_2(x, t)$, ..., $u_n(x, t)$. The matrix components A_{jk} and the components f_j depend on x, t as well as u_1, u_2, ..., u_n.

As pointed out in Section 4.3.2, we may regard (7.2.1) as the component form (with respect to some unspecified constant basis $\mathbf{b}_1$, $\mathbf{b}_2$, ..., $\mathbf{b}_n$) of the vector equation

$$\mathbf{u}_t + A\mathbf{u}_x = \mathbf{f}. \tag{7.2.2a}$$

Thus (cf. (4.3.18a))

$$\mathbf{u} = \sum_{k=1}^{n} u_k \mathbf{b}_k, \quad \mathbf{f} = \sum_{k=1}^{n} f_k \mathbf{b}_k, \tag{7.2.2b}$$

and the components of the linear operator A with respect to the $\{\mathbf{b}_i\}$ basis are defined in the usual way (cf. (4.3.18b)),

$$A\mathbf{b}_j = \sum_{k=1}^{n} A_{kj} \mathbf{b}_k. \tag{7.2.2c}$$

The fact that the basis $\{\mathbf{b}_i\}$ does not depend on x, t is reflected by the statements

$$\mathbf{u}_t = \sum_{k=1}^{n} \frac{\partial u_k}{\partial t} \mathbf{b}_k, \quad \mathbf{u}_x = \sum_{k=1}^{n} \frac{\partial u_k}{\partial x} \mathbf{b}_k \tag{7.2.2d}$$

that follow from (7.2.2b) with the derivatives of the $\mathbf{b}_k$ all equal to zero.

7.2.1 Characteristic Curves and the Normal Form

We define a characteristic curve C, as in Section 4.3.2, by the property that along C Cauchy data together with (7.2.1) do not define the outward derivative of each of the u_i. The calculation of the characteristic condition for C is identical to that for the linear problem, since the A_{ij} are known along C once the u_i are specified there. Therefore, the characteristic speeds at a given point x, t where the u_i are known are defined by the n roots of the following nth-degree polynomial in λ:

$$\det\{A_{ij} - \delta_{ij}\lambda\} = 0, \tag{7.2.3}$$

where δ_{ij} is the Kronecker delta.

Equation (7.2.3) also defines the eigenvalues of the $\{A_{ij}\}$ matrix. A system (7.2.1) is called hyperbolic for a given solution in some domain of the xt-plane if the eigenvalues of $\{A_{ij}\}$ are real and distinct in that domain. As for the single second-order equation (7.1.1), hyperbolicity depends on the actual solution being considered because the A_{ij} also involve the $u_i(x, t)$. Henceforth, we shall restict attention to hyperbolic systems.

Let $\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \ldots, \boldsymbol{\ell}_n$ be the left eigenvectors of the $\{A_{ij}\}$ matrix, and denote the corresponding eigenvalues by $\lambda_1, \lambda_2, \ldots, \lambda_n$. Thus, if the $\boldsymbol{\ell}_i$ have components

$$\boldsymbol{\ell}_i = \sum_{j=1}^{n} \ell_{ij}\mathbf{b}_j, \tag{7.2.4}$$

the definition of a left eigenvector gives

$$(\ell_{i1}, \quad \ell_{i2}, \quad \ldots, \quad \ell_{in}) \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1n} \\ A_{21} & A_{22} & \ldots & A_{2n} \\ \vdots & \vdots & & \\ A_{n1} & A_{n2} & \ldots & A_{nn} \end{pmatrix}$$
$$= (\lambda_i\ell_{i1}, \lambda_i\ell_{i2}, \ldots, \lambda_i\ell_{in}), \; i = 1, \ldots, n, \tag{7.2.5a}$$

or

$$\begin{aligned} &\ell_{i1}(A_{11} - \lambda_i) + \ell_{i2}A_{21} + \ldots + \ell_{in}A_{n1} = 0, \\ &\ell_{i1}A_{12} + \ell_{i2}(A_{22} - \lambda_i) + \ldots + \ell_{in}A_{n2} = 0, \\ &\vdots \\ &\ell_{i1}A_{1n} + \ell_{i2}A_{2n} + \ldots + \ell_{in}(A_{nn} - \lambda_i) = 0, \; i = 1, \ldots, n. \end{aligned} \tag{7.2.5b}$$

Let us express (7.2.1) in matrix form and multiply this expression from the left by the row vector $(\ell_{i1}, \ell_{i2}, \ldots, \ell_{in})$. After using (7.2.5a) to simplify the second term on the right-hand side, we obtain the following *normal form*:

$$\sum_{k=1}^{n} \ell_{ik}\left(\frac{\partial u_k}{\partial t} + \lambda_i\frac{\partial u_k}{\partial x}\right) = \sum_{k=1}^{n} \ell_{ik}f_k, \; i = 1, \ldots, n. \tag{7.2.6}$$

Notice that the directional derivative

$$\partial_i \equiv \frac{\partial}{\partial t} + \lambda_i\frac{\partial}{\partial x} \tag{7.2.7}$$

along the ith characteristic is the only one that occurs in the ith equation.

If $n = 2$, we can simplify (7.2.7) further by introducing characteristic coordinates, and this is discussed in Section 7.3. If $n > 2$, we have more characteristic curves passing through a given point than we have coordinates (x, t), so we cannot use the characteristic curves as a coordinate system. We can, however, exploit the fact that the ith equation involves only the derivative along the ith characteristic curve in the following way: For each i we select the pair of coordinate families consisting of the family $\phi_i(x, t) = \xi_i$ = constant and the family $t = \sigma_i$ = constant. Here, the curves $\phi_i(x, t) = \xi_i$ = constant make up the family of characteristic curves having slope $(dx/dt) = \lambda_i$, that is,

$$\frac{dx}{dt} = \lambda_i = -\frac{\partial\phi_i/\partial t}{\partial\phi_i/\partial x}, \tag{7.2.8}$$

and the lines $t = \sigma_i$ = constant are not tangent to any of the characteristic curves as long as λ_i is finite.

If we now regard all the u_k in the ith equation as functions of the two characteristic variables ξ_i and σ_i, we can write (7.2.6) in the following form, which is convenient for solution using forward differences along characteristic directions:

$$\sum_{k=1}^{n} \ell_{ik} \frac{\partial u_k}{\partial \sigma_i} = \sum_{k=1}^{n} \ell_{ik} f_k. \tag{7.2.9}$$

The notation $(\partial u_k/\partial \sigma_i)$ in (7.2.9) is somewhat misleading. We reiterate that it means $(\partial u_k/\partial t)$ in the ith equation holding ξ_i fixed. Of course, instead of t we could have chosen σ_i to be any parameter that increases monotonically along a characteristic. The choice $\sigma_i = t$ gives the simplest form for (7.2.9).

For the quasilinear problem with $n > 2$, this is as far as we can simplify the system (7.2.1); an illustrative example is discussed in Section 7.2.2. In Section 7.3.1 we show that if $n = 2$, we can use the ξ_i as coordinates instead of x, t, and we can then simplify our results further.

It is also interesting to consider the special case where (7.2.1) is semilinear (that is, the A_{ij} do not depend on the u_i, but the f_i may depend on the u_i in an arbitrary way). We introduce the following linear transformation from the original components $u_1, u_2, \ldots, u_n$ to new components $U_1, U_2, \ldots, U_n$ according to

$$U_i = \sum_{k=1}^{n} \ell_{ik} u_k, \quad i = 1, \ldots, n, \tag{7.2.10a}$$

with $\{\ell_{ij}\}^{-1} \equiv \{\mu_{ij}\}$. The inverse transformation is

$$u_i = \sum_{k=1}^{n} \mu_{ik} U_k, \quad i = 1, \ldots, n. \tag{7.2.10b}$$

We compute

$$\frac{\partial U_i}{\partial t} = \sum_{k=1}^{n} \left(\ell_{ik} \frac{\partial u_k}{\partial t} + \frac{\partial \ell_{ik}}{\partial t} u_k \right), \tag{7.2.11a}$$

$$\frac{\partial U_i}{\partial x} = \sum_{k=1}^{n} \left(\ell_{ik} \frac{\partial u_k}{\partial x} + \frac{\partial \ell_{ik}}{\partial x} u_k \right). \tag{7.2.11b}$$

Therefore, after we multiply (7.2.11b) by λ_i, add this result to (7.2.11a), then use (7.2.6), we find

$$\frac{\partial U_i}{\partial t} + \lambda_i \frac{\partial U_i}{\partial x} = \sum_{k=1}^{n} \left(\ell_{ik} f_k + u_k \partial_i (\ell_{ik}) \right), \quad i = 1, \ldots, n.$$

Using (7.2.10b) to express the u_k in terms of the U_k gives the diagonal characteristic form

$$\frac{\partial U_i}{\partial t} + \lambda_i \frac{\partial U_i}{\partial x} + \sum_{k=1}^{n} C_{ik} U_k = F_i, \quad i = 1, \ldots, n, \tag{7.2.12a}$$

where

$$C_{ik} = -\sum_{m=1}^{n}(\partial_i \ell_{im})\mu_{mk}, \quad F_i = \sum_{k=1}^{n} \ell_{ik} f_k. \tag{7.2.12b}$$

It is easy to verify that (4.3.25) is a special case of (7.2.12) for $n = 2$ and the F_i independent of the the U_i. See Problem 7.2.2.

Note that if the A_{ij} depend on the u_i, then so do the ℓ_{ij}. Therefore, the right-hand sides in (7.2.11) will involve additional terms like $(\partial \ell_{ik}/\partial u_k) \cdot (\partial u_k/\partial t)$ and $(\partial \ell_{ik}/\partial u_k) \cdot (\partial u_k/\partial x)$, and this precludes the diagonal form (7.2.12).

We now illustrate the calculations leading to (7.2.6) and (7.2.12) for two examples with $n = 3$. See also Problem 7.2.1, where shallow-water flow in a two-layer model leads to a hyperbolic system with $n = 4$.

7.2.2 Unsteady Nonisentropic Flow

In Chapter 5 we derived the three equations governing unsteady compressible flow where the entropy is different along each particle path (see (5.3.32)). In order to illustrate the role of the f_j in (7.2.1), let us assume a distribution of mass and energy sources whose strength depends on x, t, and the local flow speed. Then the dimensional equations for mass, momentum, and energy conservation become

$$\rho_t + (\rho u)_x = Q(x, t, u), \tag{7.2.13a}$$

$$u_t + uu_x + \frac{1}{\rho} p_x = 0, \tag{7.2.13b}$$

$$(p/\rho^\gamma)_t + u(p/\rho^\gamma)_x = E(x, t, u). \tag{7.2.13c}$$

We will introduce the entropy s instead of the density ρ as a dependent variable. The relation between s, p, and ρ is

$$s \equiv \log\left(\frac{p}{\rho^\gamma}\right) \tag{7.2.14}$$

(see (5.3.119)). Equation (7.2.13c) becomes

$$s_t + us_x = e^{-s}E(x, t, u) \equiv h(x, t, u, s). \tag{7.2.15}$$

The definition of the entropy in (7.2.14) implies that the density ρ and the speed of sound c (see (3.3.11)) are the following functions of p and s:

$$\rho(p, s) = (pe^{-s})^{1/\gamma}, \tag{7.2.16a}$$

$$c(p, s) = \sqrt{\gamma p^{(\gamma-1)/\gamma} e^{s/\gamma}}. \tag{7.2.16b}$$

The flow is defined in terms of the three variables u, p, and s, and we need to transform (7.2.13a). Equation (7.2.14) implies that $\rho^\gamma = pe^{-s}$. Therefore,

$$\rho_t = \frac{e^{-s}}{\gamma\rho^{\gamma-1}}(p_t - ps_t) = \frac{1}{c^2}(p_t - ps_t), \quad \rho_x = \frac{1}{c^2}(p_x - ps_x).$$

Using these expressions in (7.2.13a) gives

$$\frac{1}{c^2}(p_t - ps_t + up_x + ups_x) + \rho u_x = Q,$$

and using (7.2.15), this simplifies to

$$p_t + up_x + \gamma p u_x = c^2 Q + ph \equiv q(x, t, u, p, s). \tag{7.2.17}$$

The system of equations corresponding to (7.2.1) is given by (7.2.13b), (7.2.17), and (7.2.15) for the dependent variables u, p, and s, respectively. In particular, we have

$$\mathbf{u} = u\mathbf{b}_1 + p\mathbf{b}_2 + s\mathbf{b}_3, \tag{7.2.18a}$$

$$A_{ij} = \begin{pmatrix} u & \rho^{-1} & 0 \\ \gamma p & u & 0 \\ 0 & 0 & u \end{pmatrix}, \tag{7.2.18b}$$

$$\mathbf{f} = q\mathbf{b}_2 + h\mathbf{b}_3. \tag{7.2.18c}$$

The eigenvalues of the $\{A_{ij}\}$ matrix are defined by the vanishing of the determinant (7.2.3), which reduces to

$$(u - \lambda)^3 - (u - \lambda)c^2 = 0.$$

That is,

$$\lambda_1 = u + c, \quad \lambda_2 = u - c, \quad \lambda_3 = u, \tag{7.2.19}$$

which are real and distinct.

The component form (7.2.5b) for the left eigenvectors gives the following system of homogeneous linear algebraic equations governing the $\{\ell_{ij}\}$ for each i:

$$(u - \lambda_i)\ell_{i1} + \gamma p \ell_{i2} = 0, \tag{7.2.20a}$$

$$\frac{1}{\rho}\ell_{i1} + (u - \lambda_i)\ell_{i2} = 0, \tag{7.2.20b}$$

$$(u - \lambda_i)\ell_{i3} = 0. \tag{7.2.20c}$$

For $i = 1$, either (7.2.20a) or (7.2.20b) gives $(\ell_{11}/\ell_{12}) = c\rho$, whereas (7.2.20c) gives $\ell_{13} = 0$, because $u - \lambda_1 = -c \neq 0$. We choose

$$\boldsymbol{\ell}_1 = c\rho\mathbf{b}_1 + \mathbf{b}_2. \tag{7.2.21a}$$

Similarly, we obtain $(\ell_{21}/\ell_{22}) = -cp$ for $i = 2$, and $\ell_{31} = \ell_{32} = 0$, $\ell_{33} =$ arbitrary for $i = 3$. We then choose

$$\boldsymbol{\ell}_2 = -c\rho\mathbf{b}_1 + \mathbf{b}_2, \tag{7.2.21b}$$

$$\boldsymbol{\ell}_3 = \mathbf{b}_3. \tag{7.2.21c}$$

Thus,

$$\{\ell_{ij}\} = \begin{pmatrix} c\rho & 1 & 0 \\ c\rho & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.2.22}$$

Note again that we cannot specify all three components of the eigenvectors; we only have two independent conditions linking these three quantities.

Using (7.2.22) in (7.2.6) gives the three equations

$$c\rho(u_t + (u+c)u_x) + (p_t + (u+c)p_x) = q, \tag{7.2.23a}$$

$$c\rho(u_t + (u-c)u_x) - (p_t + (u-c)p_x) = -q, \tag{7.2.23b}$$

$$s_t + us_x = h. \tag{7.2.23c}$$

This system can also be written in the form (7.2.9),

$$c\rho\frac{\partial u}{\partial\sigma_1} + \frac{\partial p}{\partial\sigma_1} = q, \tag{7.2.24a}$$

$$c\rho\frac{\partial u}{\partial\sigma_2} - \frac{\partial p}{\partial\sigma_2} = -q, \tag{7.2.24b}$$

$$\frac{\partial s}{\partial\sigma_3} = h. \tag{7.2.24c}$$

Equations (7.2.24) are a convenient starting point for a numerical solution using forward differencing along characteristics. Consider two adjacent points (1) and (2) in the xt-plane, as sketched in Figure 7.2, and assume that u, p, and s are known there. For instance, (1) and (2) may be two adjacent points on an initial spacelike arc or two adjacent points in the calculation of the previous step. Assume also that $0 < u < c$ at (1) and (2), so that the characteristic directions that emerge have $\lambda_2 < 0, 0 < \lambda_3 < \lambda_1$. We wish to calculate the values of u, p, and s at (3), which is located by the intersection of the λ_1 characteristic from (1) and the λ_2 characteristic from (2).

First we locate (3) using the forward difference approximation of (7.2.8) with $j = 1$ and $j = 2$:

$$x^{(3)} - x^{(1)} = \lambda_1^{(1)}\left(t^{(3)} - t^{(1)}\right), \tag{7.2.25a}$$

$$x^{(3)} - x^{(2)} = \lambda_2^{(2)}\left(t^{(3)} - t^{(2)}\right), \tag{7.2.25b}$$

where we are using superscripts to indicate the values at a given point. The unknowns are $x^{(3)}$ and $t^{(3)}$, as all the other terms are evaluated at (1) or (2), where u, p, and s, as well as x and t, are specified. Solving this linear system gives

$$x^{(3)} = \frac{1}{\lambda_1^{(1)} - \lambda_2^{(2)}}\left[\lambda_1^{(1)}(x^{(2)} - \lambda_2^{(2)}t^{(2)}) - \lambda_2^{(2)}(x^{(1)} - \lambda_1^{(1)}t^{(1)})\right], \tag{7.2.26a}$$

$$t^{(3)} = \frac{1}{\lambda^{(1)} - \lambda_2^{(2)}}\left[x^{(2)} - \lambda_2^{(2)}t^{(2)} - x^{(1)} + \lambda_1^{(1)}t^{(1)}\right]. \tag{7.2.26b}$$

Next, we consider the following difference form of (7.2.24a) and (7.2.24b) to compute u and p at (3):

$$\rho^{(1)}c^{(1)}(u^{(3)} - u^{(1)}) + p^{(3)} - p^{(1)} = q^{(1)}(t^{(3)} - t^{(1)}), \tag{7.2.27a}$$

$$\rho^{(2)}c^{(2)}(u^{(3)} - u^{(2)}) + p^{(3)} + p^{(2)} = -q^{(2)}(t^{(3)} - t^{(2)}), \tag{7.2.27b}$$

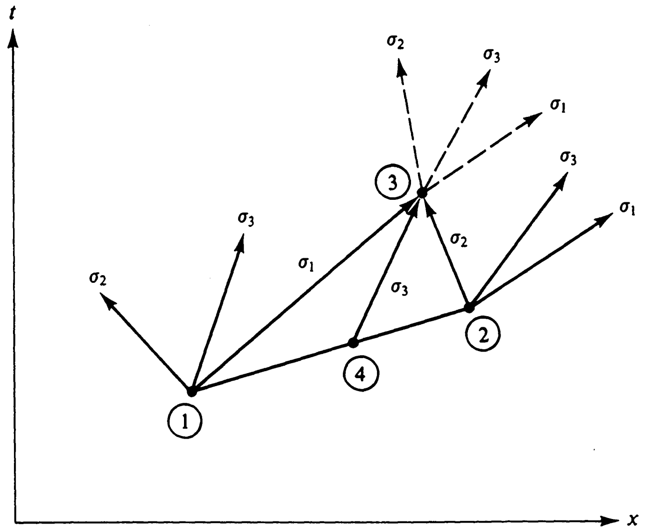

FIGURE 7.2. Characteristics emerging from the spacelike arc (1)-(4)-(2)

where $t^{(3)}$ is given in (7.2.26b). Solving the system (7.2.27) for u and p gives

$$u^{(3)} = \frac{1}{\rho^{(1)}c^{(1)} + \rho^{(2)}c^{(2)}} [\rho^{(1)}u^{(1)}c^{(1)} + p^{(1)} + \rho^{(2)}u^{(2)}c^{(2)} - p^{(2)} + q^{(1)}(t^{(3)} - t^{(1)}) - q^{(2)}(t^{(3)} - t^{(2)})], \quad (7.2.28a)$$

$$p^{(3)} = \frac{1}{\rho^{(1)}c^{(1)} + \rho^{(2)}c^{(2)}} \Big\{\rho^{(1)}c^{(1)}[-\rho^{(2)}u^{(2)}c^{(2)} + p^{(2)} + q^{(2)}(t^{(3)} - t^{(2)})] + \rho^{(2)}c^{(2)}[\rho^{(1)}c^{(1)}u^{(1)} + p^{(1)} + q^{(1)}(t^{(3)} - t^{(1)})]\Big\}. \quad (7.2.27b)$$

To complete the solution at (3), we need to calculate s there. Since s is propagated along the λ_3 characteristic, we need to define starting values at an appropriate intermediate point (4) on the straight line joining the points (1) and (2). One possible approach is to use linear interpolation to determine (4) and the values of u, p, and s there, based on data at (1) and (2).

The straight-line approximation for the λ_3 characteristic joining (4) to (3) follows from (7.2.8),

$$x^{(3)} - x^{(4)} = \lambda_3^{(4)}(t^{(3)} - t^{(4)}), \quad (7.2.29a)$$

but now $\lambda_3^{(4)}$, $x^{(4)}$, and $t^{(4)}$ are not known directly. We use linear interpolation between (1) and (2) to write

$$\frac{\lambda_3^{(4)} - \lambda_3^{(1)}}{\sigma_{14}} = \frac{\lambda_3^{(2)} - \lambda_3^{(1)}}{\sigma_{12}}, \quad (7.2.29b)$$

$$\frac{t^{(4)} - t^{(1)}}{x^{(4)} - x^{(1)}} = \frac{t^{(2)} - t^{(1)}}{x^{(2)} - x^{(1)}}, \tag{7.2.29c}$$

where σ_{ij} denotes

$$\sigma_{ij} \equiv [(x^{(i)} - x^{(j)})^2 + (t^{(i)} - t^{(j)})^2]^{1/2}. \tag{7.2.29d}$$

Equations (7.2.29a)–(7.2.29c) define $x^{(4)}$, $t^{(4)}$, and $\lambda_3^{(4)}$ in principle, but an explicit solution is not practical. An efficient iterative approach is to guess a value of $\lambda_3^{(4)}$ (for example, the average of the values $\lambda_3^{(1)}$ and $\lambda_3^{(2)}$) and to use this initial guess in the two linear equations (7.2.29a) and (7.2.29c) for $x^{(4)}$ and $t^{(4)}$. Solving these gives

$$x^{(4)} = \frac{(x^{(2)} - x^{(1)})(\lambda_3^{(4)} t^{(3)} x^{(3)}) + \lambda_3^{(4)}(x^{(1)}t^{(2)} - x^{(2)}t^{(1)})}{\lambda_3^{(4)}(t^{(2)} - t^{(1)}) + x^{(1)} - x^{(2)}}, \tag{7.2.30a}$$

$$t^{(4)} = \frac{x^{(1)}t^{(2)} - x^{(2)}t^{(1)} + (t^{(2)} - t^{(1)})(\lambda_3^{(4)} t^{(3)} - x^{(3)})}{\lambda_3^{(4)}(t^{(2)} - t^{(1)}) + x^{(1)} - x^{(2)}}. \tag{7.2.30b}$$

Next, we use these values of $x^{(4)}$ and $t^{(4)}$ to compute σ_{14} from (7.2.29d). We then use the expression

$$\lambda_3^{(4)} = \lambda_3^{(1)} + \frac{\sigma_{14}}{\sigma_{12}}(\lambda_3^{(2)} - \lambda_3^{(1)}), \tag{7.2.31}$$

obtained from (7.2.29b), to calculate an improved value for $\lambda_3^{(4)}$, and so on. Once the values of $x^{(4)}$, $t^{(4)}$, and $\lambda_3^{(4)}$ have converged, the final calculation for $s^{(3)}$ follows from the difference form of (7.2.24c):

$$s^{(3)} = s^{(4)} + h^{(4)}(t^{(3)} - t^{(4)}), \tag{7.2.32a}$$

where $s^{(4)}$ is obtained by linear interpolation between the points (1) and (2):

$$s^{(4)} = s^{(1)} + \frac{\sigma_{14}}{\sigma_{12}}(s^{(2)} - s^{(1)}). \tag{7.2.32b}$$

This procedure defines u, p, and s at (3) uniquely as long as characteristics of the same family emerging from points adjacent to (1) or (2) do not intersect on or inside the the triangle (1)-(2)-(3).

7.2.3 A Semilinear Example

The model system

$$\frac{\partial u_1}{\partial t} + x\frac{\partial u_1}{\partial x} + x^2\frac{\partial u_2}{\partial x} = f_1(x, t, u_1, u_2, u_3), \tag{7.2.33a}$$

$$\frac{\partial u_2}{\partial t} + (1+t)^2\frac{\partial u_1}{\partial x} + x\frac{\partial u_2}{\partial x} = f_2(x, t, u_1, u_2, u_3), \tag{7.2.33b}$$

$$\frac{\partial u_3}{\partial t} + x\frac{\partial u_3}{\partial x} = f_3(x, t, u_1, u_2, u_3) \tag{7.2.33c}$$

is similar in structure to the one just discussed, except that the u_i do not occur in the $\{A_{ij}\}$ matrix but only in the right-hand sides. The matrix $\{A_{ij}\}$ is

$$\{A_{ij}\} = \begin{pmatrix} x & x^2 & 0 \\ (1+t)^2 & x & 0 \\ 0 & 0 & x \end{pmatrix}, \tag{7.2.34}$$

and its eigenvalues are

$$\lambda_1 = x(2+t), \quad \lambda_2 = -xt, \quad \lambda_3 = x. \tag{7.2.35}$$

The equations corresponding to (7.2.5b) are now

$$(x - \lambda_i)\ell_{i1} + (1+t)^2 \ell_{i2} = 0, \tag{7.2.36a}$$

$$x^2 \ell_{i1} + (x - \lambda_i)\ell_{i2} = 0, \tag{7.2.36b}$$

$$(x - \lambda_i)\ell_{i3} = 0, \quad i = 1, 2, 3, \tag{7.2.36c}$$

and we choose the eigenvectors $\boldsymbol{\ell}_i$ having the components

$$\boldsymbol{\ell}_i = \frac{1}{2x}\mathbf{b}_1 + \frac{1}{2(1+t)}\mathbf{b}_2, \quad \boldsymbol{\ell}_2 = \frac{1}{2x}\mathbf{b}_1 - \frac{1}{2(1+t)}\mathbf{b}_2, \quad \boldsymbol{\ell}_3 = \mathbf{b}_3, \tag{7.2.37}$$

consistent with (7.2.36). Therefore, the $\{\ell_{ij}\}$ matrix is given by

$$\{\ell_{ij}\} = \begin{pmatrix} \frac{1}{2x} & \frac{1}{2(1+t)} & 0 \\ \frac{1}{2x} & -\frac{1}{2(t+1)} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.2.38}$$

It follows from the expression for $\{\ell_{ij}\}$ that

$$\left\{\frac{\partial \ell_{ij}}{\partial t}\right\} = \begin{pmatrix} 0 & -\frac{1}{2(1+t)^2} & 0 \\ 0 & \frac{1}{2(1+t)^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \left\{\frac{\partial \ell_{ij}}{\partial x}\right\} = \begin{pmatrix} -\frac{1}{2x^2} & 0 & 0 \\ -\frac{1}{2x^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the system (7.2.12) becomes

$$\frac{\partial U_1}{\partial t} + x(2+t)\frac{\partial U_1}{\partial x} + \frac{t^2+3t+3}{2(1+t)}U_1 + \frac{t^2+3t+1}{2(1+t)}U_2 = F_1, \tag{7.2.39a}$$

$$\frac{\partial U_2}{\partial t} - xt\frac{\partial U_2}{\partial x} - \frac{(t^2+t+1)}{2(1+t)}U_1 - \frac{(t^2+t-1)}{2(1+t)}U_2 = F_2, \tag{7.2.39b}$$

$$\frac{\partial U_3}{\partial t} + x\frac{\partial U_3}{\partial x} = F_3. \tag{7.2.39c}$$

The right-hand sides in (7.2.39) are given by

$$F_1 = \frac{1}{2x}f_1 + \frac{1}{2(1+t)}f_2, \quad F_2 = \frac{1}{2x}f_1 - \frac{1}{2(1+t)}f_2, \quad F_3 = f_3, \tag{7.2.40}$$

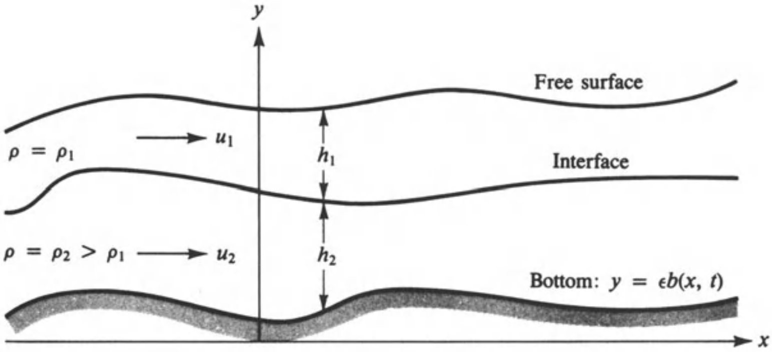

FIGURE 7.3. Two-layer shallow-water flow

and the arguments of the f_i in (7.2.40) are as follows:

$$f_i = f_i(x, t, xU_1 + xU_2, (1+t)U_1 - (1+t)U_2, U_3). \tag{7.2.41}$$

Problems

7.2.1 Consider shallow-water flow over a variable bottom for two layers of fluid with constant densities ρ_1 and ρ_2 for the stable case where the lighter fluid is on top ($\rho_1 < \rho_2$). See Figure 7.3.

We denote the vertically averaged horizontal speed in each layer by $u_i(x, t)$ and the height of the layer by $h_i(x, t)$. Using appropriate dimensionless variables, show that the laws of mass and momentum conservation in each layer reduce to the following system of four equations:

$$\frac{\partial h_1}{\partial t} + \frac{\partial}{\partial x}(u_1 h_1) = 0, \tag{7.2.42a}$$

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + \frac{\partial h_1}{\partial x} + \frac{\partial h_2}{\partial x} = -\epsilon \frac{\partial b}{\partial x}, \tag{7.2.42b}$$

$$\frac{\partial h_2}{\partial t} + \frac{\partial}{\partial x}(u_2 h_2) = 0, \tag{7.2.42c}$$

$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + (1-\beta)\frac{\partial h_1}{\partial x} = -\epsilon \frac{\partial b}{\partial x}, \tag{7.2.42d}$$

where $\beta \equiv (\rho_2 - \rho_1)/\rho_2 > 0$, and $\epsilon b(x, t)$ is the height of the bottom measured from some reference level.

Derive the quartic that determines the eigenvalues of the $\{A_{ij}\}$ matrix for this example. Assuming $h_1 > 0$, $h_2 > 0$, show that the λ_i are real and distinct. Without solving for the λ_i explicitly, transform (7.2.42) to the normal form (7.2.9).

7.2.2 Study the transformation of the system (7.2.33) from the point of view of Section 4.3.3i.

a. Show that the right eigenvectors of the $\{A_{ij}\}$ matrix defined by $A\mathbf{w}_i = \lambda_i \mathbf{w}_i$, and having components $\mathbf{w}_i = \sum_{k=1}^{3} W_{ki}\mathbf{b}_k$, satisfy

$$(x - \lambda_i)W_{1i} + x^2 W_{2i} = 0, \tag{7.2.43a}$$

$$(1 + t)^2 W_{1i} + (x - \lambda_i)W_{2i} = 0, \tag{7.2.43b}$$

$$(x - \lambda_i)W_{3i} = 0. \tag{7.2.43c}$$

b. Verify that $\{W_{ij}^{-1}\} = \alpha\{\ell_{ij}\}$, where α is an arbitrary constant. In particular, show that for the choice

$$\{W_{ij}\} = \begin{pmatrix} x & x & 0 \\ 1+t & -(1+t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7.2.44}$$

consistent with (7.2.43), $\{W_{ij}^{-1}\} = \{\ell_{ij}\}$ as given in (7.2.38).

c. Verify that using (7.2.44) in (4.3.26b) and (4.3.26c) gives the same expressions for the C_{ij} and F_i that were derived in (7.2.39).

d. For a general semilinear hyperbolic problem, show that the transformation (4.3.21) using right eigenvectors is equivalent to the transformation (7.2.10) using left eigenvectors.

7.3 Systems of Two First-Order Hyperbolic Equations

If (7.2.1) is hyperbolic and $n = 2$, there are two distinct families of characteristic curves for a given solution. These two families of curves may be regarded as a curvilinear coordinate system in terms of which the normal form (7.2.9) will simplify further.

7.3.1 Characteristic Independent Variables

The eigenvalues of the 2×2 matrix $\{A_{ij}\}$ are (see (4.3.16b))

$$\lambda_1 = \frac{1}{2}\{A_{11} + A_{22} + [(A_{11} - A_{22})^2 + 4A_{12}A_{21}]^{1/2}\}, \tag{7.3.1a}$$

$$\lambda_2 = \frac{1}{2}\{A_{11} + A_{22} - [(A_{11} - A_{22})^2 + 4A_{12}A_{21}]^{1/2}\}. \tag{7.3.1b}$$

These eigenvalues are real and distinct if

$$(A_{11} - A_{22})^2 + 4A_{12}A_{21} > 0. \tag{7.3.2}$$

Along the characteristic with $(dx/dt) = \lambda_1$, we have

$$\phi(x, t) = \xi = \text{constant}, \quad \lambda_1 = -\frac{\phi_t}{\phi_x}. \tag{7.3.3a}$$

Similarly, along the characteristic with $(dx/dt) = \lambda_2$, we have

$$\psi(x,t) = \eta = \text{constant}, \ \lambda_2 = -\frac{\psi_t}{\psi_x}. \tag{7.3.3b}$$

Now, the λ_i are given functions of x, t, u_1, and u_2, so the curves defined in (7.3.3) depend on the solution and not just on x and t.

Let us regard u_1, u_2, x, and t as functions of ξ and η for a given solution. We shall use the notation

$$u_1 = U(\xi,\eta), \ u_2 = V(\xi,\eta), \ x = X(\xi,\eta), \ t = T(\xi,\eta). \tag{7.3.4}$$

It follows from (7.3.3) that the characteristic curves satisfy the pair of partial differential equations (see (7.1.7))

$$X_\eta = \lambda_1(X,T,U,V)T_\eta, \tag{7.3.5a}$$

$$X_\xi = \lambda_2(X,T,U,V)T_\xi. \tag{7.3.5b}$$

To solve these, we need to derive the equations that follow from (7.2.6) for the evolution of U and V along the characteristic curves. We compute

$$\frac{\partial u_1}{\partial t} = U_\xi\phi_t + U_\eta\psi_t, \quad \frac{\partial u_2}{\partial t} = V_\xi\phi_t + V_\eta\psi_t,$$

$$\frac{\partial u_1}{\partial x} = U_\xi\phi_x + U_\eta\psi_x, \quad \frac{\partial u_2}{\partial x} = V_\xi\phi_x + V_\eta\psi_x.$$

Thus,

$$\begin{aligned}\frac{\partial u_1}{\partial t} + \lambda_1\frac{\partial u_1}{\partial x} &= U_\xi(\phi_t + \lambda_1\phi_x) + U_\eta(\psi_t + \lambda_1\psi_\xi), \\ &= (\lambda_1 - \lambda_2)\psi_x U_\eta\end{aligned} \tag{7.3.6a}$$

because $\phi_t + \lambda_1\phi_x = 0$ and $\psi_t = -\lambda_2\psi_x$. Similarly,

$$\frac{\partial u_1}{\partial t} + \lambda_2\frac{\partial u_1}{\partial x} = (\lambda_2 - \lambda_1)U_\xi\phi_x, \tag{7.3.6b}$$

$$\frac{\partial u_2}{\partial t} + \lambda_1\frac{\partial u_2}{\partial x} = (\lambda_1 - \lambda_2)V_\eta\psi_x, \tag{7.3.6c}$$

$$\frac{\partial u_2}{\partial t} + \lambda_2\frac{\partial u_2}{\partial x} = (\lambda_2 - \lambda_1)V_\xi\phi_x. \tag{7.3.6d}$$

Substituting (7.3.6) into (7.2.9) gives the system

$$\ell_{11}U_\eta + \ell_{12}V_\eta = \frac{\ell_{11}f_1 + \ell_{12}f_2}{(\lambda_1 - \lambda_2)\psi_x}, \tag{7.3.7a}$$

$$\ell_{21}U_\xi + \ell_{22}V_\xi = \frac{\ell_{21}f_1 + \ell_{22}f_2}{(\lambda_2 - \lambda_1)\phi_x}. \tag{7.3.7b}$$

We will now show that the right-hand sides of (7.3.7) simplify further. Since $x = X(\xi,\eta)$, $t = T(\xi,\eta)$, we have

$$\begin{pmatrix} dx \\ dt \end{pmatrix} = \begin{pmatrix} X_\xi & X_p \\ T_\xi & T_\eta \end{pmatrix}\begin{pmatrix} d\xi \\ d\eta \end{pmatrix}. \tag{7.3.8a}$$

We also have $\xi = \phi(x, t)$, $\eta = \psi(x, t)$. Therefore,

$$\begin{pmatrix} d\xi \\ d\eta \end{pmatrix} = \begin{pmatrix} \phi_x & \phi_t \\ \psi_x & \psi_t \end{pmatrix} \begin{pmatrix} dx \\ dt \end{pmatrix}. \tag{7.3.8b}$$

Substituting (7.3.8b) for the column vector on the right-hand side of (7.3.8a) shows that

$$\begin{pmatrix} X_\xi & X_\eta \\ T_\xi & T_\eta \end{pmatrix} \begin{pmatrix} \phi_x & \phi_t \\ \psi_x & \psi_t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{7.3.9}$$

Thus, the two Jacobian matrices appearing in (7.3.8a) and (7.3.8b) are each other's inverse. If we left multiply (7.3.9) by the inverse of the first Jacobian matrix and use the familiar formula for this inverse we obtain

$$\begin{pmatrix} \phi_x & \phi_t \\ \psi_x & \psi_t \end{pmatrix} = J \begin{pmatrix} T_\eta & -X_\eta \\ -T_\xi & X_\xi \end{pmatrix}, \tag{7.3.10}$$

where we have used the notation

$$J \equiv \phi_x\psi_t - \psi_x\phi_t, \quad \tilde{J} = X_\xi T_\eta - T_\xi X_\eta$$

for the Jacobians, and have recognized that $J = 1/\tilde{J}$. Identifying components in (7.3.10) gives

$$\phi_x = JT_\eta, \quad \phi_t = -JX_\eta, \quad \psi_x = -JT_\xi, \quad \psi_t = JX_\xi. \tag{7.3.11}$$

Now consider the factor $1/(\lambda_1 - \lambda_2)\psi_x$ in (7.3.7a). Using the result for ψ_x given in (7.3.11), we obtain

$$\frac{1}{(\lambda_1 - \lambda_2)\psi_x} = \frac{1}{(\lambda_1 - \lambda_2)(-J)T_\xi} = \frac{\tilde{J}}{(\lambda_2 - \lambda_1)T_\xi}.$$

We use (7.3.5) to express X_ξ and X_η in terms of T_ξ and T_η to obtain

$$\frac{1}{(\lambda_1 - \lambda_2)\psi_x} = \frac{\lambda_2 T_\xi T_\eta - T_\xi \lambda_1 T_\eta}{(\lambda_2 - \lambda_1)T_\xi} = T_\eta.$$

Similarly, we obtain

$$\frac{1}{(\lambda_2 - \lambda_1)\phi_x} = T_\xi,$$

and (7.3.7) simplifies to

$$\ell_{11}U_\eta + \ell_{12}V_\eta = (\ell_{11}f_1 + \ell_{12}f_2)T_\eta, \tag{7.3.12a}$$

$$\ell_{21}U_\xi + \ell_{22}V_\xi = (\ell_{21}f_1 + \ell_{22}f_2)T_\xi. \tag{7.3.12b}$$

The system of four equations (7.3.5) and (7.3.12) (together with the subsidiary equations (7.3.1) that define the λ_i, and (7.2.5b) that define the ℓ_{ij}) governs the solution for X, T, U, and V. A numerical solution using forward differences along the local characteristic directions can be easily implemented for given Cauchy data—that is, with U and V specified along a noncharacteristic spacelike arc. The procedure is analogous to that discussed in Section 7.1.2. See Problem 7.3.1.

7.3.2 The Hodograph Transformation

A hodograph transformation is one that reverses the role of dependent and independent variables. Let us apply this transformation formally to (7.2.1) for the special case where $n = 2$, the A_{ij} do not depend on x, t, and the $f_i = 0$. To simplify the notation we write this special case of (7.2.1) as

$$u_t + a(u, v)u_x + b(u, v)v_x = 0, \tag{7.3.13a}$$

$$v_t + c(u, v)u_x + d(u, v)v_x = 0. \tag{7.3.13b}$$

Thus, we want to regard x and t as functions of u and v. For the time being, let us assume that this is possible and proceed with the formal transformation; later on, we shall examine the conditions under which this procedure is feasible. Let x and t be functions of u and v defined by

$$x = \widetilde{X}(u, v), \quad t = \widetilde{T}(u, v). \tag{7.3.14}$$

The inverse transformation has $u(x, t)$ and $v(x, t)$.

Applying the results derived in (7.3.11) to this case ($\phi \to u, \psi \to v, X \to \widetilde{X}$, $T \to \widetilde{T}$) gives

$$u_x = J\widetilde{T}_v, \quad u_t = -J\widetilde{X}_v, \quad v_x = -J\widetilde{T}_u, \quad v_t = J\widetilde{X}_u, \tag{7.3.15}$$

where

$$J \equiv \widetilde{X}_u\widetilde{T}_v - \widetilde{T}_u\widetilde{X}_v, \quad \widetilde{J} \equiv u_x v_t - v_x u_t. \tag{7.3.16}$$

When we use (7.3.15) in (7.3.13) and cancel out J, we obtain the following *linear* system in the hodograph plane:

$$-\widetilde{X}_v + a(u, v)\widetilde{T}_v - b(u, v)\widetilde{T}_u = 0, \tag{7.3.17a}$$

$$\widetilde{X}_u + c(u, v)\widetilde{T}_v - d(u, v)\widetilde{T}_u = 0. \tag{7.3.17b}$$

Note that if f_1 and f_2 are not absent from (7.3.13), then J does not cancel out of (7.3.17), and the transformed system remains quasilinear. This is also the outcome if the A_{ij} depend on x and t.

The linear system (7.3.17) has the same type (hyperbolic, elliptic, or parabolic) as (7.3.13). To show this we write (7.3.17) in matrix form as

$$\begin{pmatrix} 1 & -d \\ 0 & -b \end{pmatrix}\begin{pmatrix} \widetilde{X}_u \\ \widetilde{T}_u \end{pmatrix} + \begin{pmatrix} 0 & c \\ -1 & a \end{pmatrix}\begin{pmatrix} \widetilde{X}_v \\ \widetilde{T}_v \end{pmatrix} = 0. \tag{7.3.18}$$

Then assuming $b \not\equiv 0$, we multiply this equation by the inverse of the first matrix to obtain

$$\begin{pmatrix} \widetilde{X}_u \\ \widetilde{T}_u \end{pmatrix} + \begin{pmatrix} \dfrac{d}{b} & \dfrac{bc - ad}{b} \\ \dfrac{1}{b} & -\dfrac{a}{b} \end{pmatrix}\begin{pmatrix} \widetilde{X}_v \\ \widetilde{T}_v \end{pmatrix} = 0. \tag{7.3.19}$$

The type of the system in (7.3.19) depends on the sign of $(d - a)^2/b^2 + 4c/b$, which is the same as the sign of $(d - a)^2 + 4cb$, which characterizes the type of the system (7.2.13). If $b \equiv 0$, but $c \not\equiv 0$, we multiply (7.3.18) by the inverse of the second matrix to arrive at the same conclusion. If $b \equiv c \equiv 0$, both (7.3.13) and (7.3.17) are in diagonal form and hyperbolic. Thus, a hodograph transformation is possible as long as $J \not\equiv 0$, regardless of the type of the system (7.3.13).

We show next that if (7.3.13) is hyperbolic, its characteristics map onto characteristics of the hodograph system (7.3.19). To prove this result, recall first that the two families of characteristic curves for (7.3.13) have $(dx/dt) = \lambda_1$ and $(dx/dt) = \lambda_2$, where λ_1 and λ_2 are the two roots of

$$(a - \lambda)(d - \lambda) - bc = 0, \tag{7.3.20}$$

that is,

$$\lambda_1 = \frac{d + a + \sqrt{\Delta}}{2}, \quad \lambda_2 = \frac{d + a - \sqrt{\Delta}}{2}, \tag{7.3.21}$$

where

$$\Delta \equiv (d - a)^2 + 4bc. \tag{7.3.22}$$

The two families of characteristic curves associated with (7.3.19) have $(dv/du) = \mu_1$, and $(dv/du) = \mu_2$ in the hodograph plane, where μ_1 and μ_2 are the two roots of

$$\left(\frac{d}{b} - \mu\right)\left(-\frac{a}{b} - \mu\right) - \frac{bc - ad}{b^2} = 0, \tag{7.3.23}$$

that is,

$$\mu_1 = \frac{d - a + \sqrt{\Delta}}{2b}, \quad \mu_2 = \frac{d - a - \sqrt{\Delta}}{2b}. \tag{7.3.24}$$

Let C be a charcteristic curve with $(dx/dt) = \lambda$ in the physical plane, and let C^* be its image in the hodograph plane. Along C^*, we have the slope

$$\frac{dv}{du} = \frac{v_x dx + v_t dt}{u_x dx + u_t dt} = \frac{v_x \lambda + v_t}{u_x \lambda + u_t},$$

and using (7.3.13), this becomes

$$\frac{dv}{du} = \frac{-cu_x + (\lambda - d)v_x}{(\lambda - a)u_x - bv_x}. \tag{7.3.25}$$

It follows from (7.3.20) that

$$(\lambda - a) = \frac{bc}{\lambda - d}.$$

Substituting this into (7.3.25) and simplifying gives

$$\frac{dv}{du} = \frac{d - \lambda}{b}. \tag{7.3.26}$$

In particular, according to (7.3.21) and (7.3.24), the right-hand side of (7.3.26) for $\lambda = \lambda_1$ and $\lambda = \lambda_2$ becomes

$$\frac{d-\lambda_1}{b} = \frac{d-a-\sqrt{\Delta}}{2b} = \mu_2, \quad \frac{d-\lambda_2}{b} = \frac{d-a+\sqrt{\Delta}}{2b} = \mu_1.$$

Therefore, we have shown that the characteristics of (7.3.13) with slope $(dx/dt) = \lambda_1$ map to characteristics of (7.3.19) with slope $(dv/du) = \mu_2$. Similarly, the λ_2 characteristics in the physical plane map to the μ_1 characteristics in the hodograph plane. This result has a very important implication that is explored in the next section.

The requirement that $J \not\equiv 0$ excludes use of a hodograph transformation in regions where either u or v is constant, or when u can be expressed in terms of v. This latter special case corresponds to a simple wave solution of (7.3.13) and is discussed in Section 7.3.4. However, as we shall see, one can solve (7.3.13) exactly in such regions. A more serious drawback of a solution in the hodograph plane is that, as boundaries are unknown in general, it is difficult to satisfy boundary conditions given in terms of u and v. Also, the interpretation of discontinuities in u and v becomes troublesome.

Steady two-dimensional transonic flow

This is an example where the hodograph transformation is quite useful, and the reader is referred to Sections 3.5–3.8 of [10] for a detailed discussion of various problems. Here, we demonstrate only how the Tricomi equation (see (4.1.24)) arises in the hodograph plane for this problem.

As shown in [10], the governing equations for steady two-dimensional transonic flow with small disturbances are (see (3.5.1) of [10])

$$u_y - v_x = 0, \quad \text{irrotational flow}, \tag{7.3.27a}$$

$$v_y - uu_x = 0, \quad \text{mass conservation}, \tag{7.3.27b}$$

where u and v are rescaled dimensionless velocity components, and x, y are Cartesian coordinates. This system can also be expressed as a quasilinear second-order equation for the velocity potential.

Comparing (7.3.13) with (7.3.27), we identify $x \to x, t \to y, u \to u, v \to v$, $a \to 0, b \to -1, c \to -u$, and $d \to 0$. We also denote $\widetilde{T}$ in (7.3.17) by $\widetilde{Y}$ and obtain the following pair of equations in the hodograph plane:

$$-\widetilde{X}_v + \widetilde{Y}_u = 0, \quad \widetilde{X}_u - u\widetilde{Y}_v = 0. \tag{7.3.28}$$

Eliminating $\widetilde{X}$ from this pair results in the Tricomi equation for $\widetilde{Y}$:

$$u\widetilde{Y}_{vv} - \widetilde{Y}_{uu} = 0.$$

We also verify from (7.3.21) and (7.3.24) that $\lambda_1 = u^{1/2} = \mu_2$ and $\lambda_2 = -u^{1/2} = \mu_1$.

7.3.3 The Riemann Invariants

In this section we study the implications of the property that if (7.3.13) is hyperbolic, then its characteristics map onto the characteristics of (7.3.17). Let us introduce the following notation for the characteristic curves in the hodograph plane:

$$S(u, v) = s = \text{ constant on the characteristics } \frac{dv}{du} = \mu_1, \tag{7.3.29a}$$

$$R(u, v) = r = \text{ constant on the characteristics } \frac{dv}{du} = \mu_2. \tag{7.3.29b}$$

Recall that in the physical plane we have

$$\phi(x, t) = \xi = \text{ constant on the characteristics } \frac{dx}{dt} = \lambda_1, \tag{7.3.30a}$$

$$\psi(x, t) = \eta = \text{ constant on the characteristics } \frac{dx}{dt} = \lambda_2. \tag{7.3.30b}$$

Let us express the system (7.3.13) in terms of the characteristic independent variables (cf. (7.3.7))

$$A(U, V)U_\eta + B(U, V)V_\eta = 0, \tag{7.3.31a}$$

$$C(U, V)U_\xi + D(U, V)V_\xi = 0, \tag{7.3.31b}$$

where we have set $\ell_{11} = A(U, V)$, $\ell_{12} = B(U, V)$, $\ell_{21} = C(U, V)$, and $\ell_{22} = D(U, V)$.

We have shown that the curves $\phi(x, t) = $ constant map onto the curves $R(u, v) = $ constant, and that the curves $\psi(x, t) = $ constant map onto the curves $S(u, v) = $ constant. Thus, for any solution of (7.3.13), if we evaluate the function $R(u, v)$ along a characteristic $\phi(x, t) = \xi = $ constant, we will find that this function remains constant. Similarly, $S(u, v)$ will remain constant along a characteristic $\psi(x, t) = \eta = $ constant. This means that the system (7.3.31) must reduce to

$$\frac{\partial R(U, V)}{\partial \eta} = 0, \tag{7.3.32a}$$

$$\frac{\partial S(U, V)}{\partial \xi} = 0, \tag{7.3.32b}$$

that is,

$$R(U, V) = r = \text{ constant on } \xi = \text{ constant}, \tag{7.3.33a}$$

$$S(U, V) = s = \text{ constant on } \eta = \text{ constant}. \tag{7.3.33b}$$

(i) *Derivation of the Riemann invariants*

The functions $R(u, v)$ and $S(u, v)$ are called the Riemann invariants of the system (7.3.13) and are most directly derived by solving the two first-order ordinary differential equations

$$\frac{dv}{du} = \mu_1, \quad \frac{dv}{du} = \mu_2, \tag{7.3.34}$$

respectively. A second, less direct, approach, which provides insight into their role as integrals of the system (7.3.31), is discussed next.

The idea is to identify (7.3.31a) with (7.3.32a), and (7.3.31b) with (7.3.32b) for appropriate functions R and S. We have

$$\frac{\partial R}{\partial \eta} = R_U U_\eta + R_V V_\eta,$$

and if we identify $(\partial R/\partial \eta) = 0$ in the preceding expression with (7.3.31a), we conclude that we must set

$$R_U = G(U, V)A(U, V), \quad R_V = G(U, V)B(U, V), \tag{7.3.35}$$

for some function $G(U, V)$ analogous to an integrating factor. To specify G, we require the two equations in (7.3.35) to be consistent—that is, $(R_U)_V = (R_V)_U$—which gives the following quasilinear first-order equation for G:

$$BG_U - AG_V = G(A_V - B_U). \tag{7.3.36}$$

Any nontrivial solution of this equation will define $G(U, V)$, which can then be used in (7.3.35) to obtain R by quadrature. It is easily seen that being able to solve (7.3.36) is exactly equivalent to being able to calculate the solution of the corresponding equation in (7.3.34).

Similarly, in order that (7.3.31b) reduce to (7.3.32b), we must have

$$S_U = K(U, V)C(U, V), \quad S_V = K(U, V)D(U, V), \tag{7.3.37}$$

which implies that K must satisfy

$$DK_U - CK_V = K(C_V - D_U). \tag{7.3.38}$$

It is important to note that Riemann invariants need not exist if $n > 2$. To illustrate this point let $n = 3$ and consider (7.2.6) with $f_1 = f_2 = f_3 = 0$, where the ℓ_{ij} do not depend on x and t. If we attempt to identify this equation for a given i with the directional derivative ∂_i of a function $R_i(u_1, u_2, u_3)$, we obtain the following three conditions analogous to (7.3.35) involving the integrating factor $G_i(u_1, u_2, u_3)$:

$$\frac{\partial R_i}{\partial u_1} = G_i \ell_{i1}, \quad \frac{\partial R_i}{\partial u_2} = G_i \ell_{i2}, \quad \frac{\partial R_i}{\partial u_3} = G_i \ell_{i3}.$$

Consistency of these conditions introduces the three requirements

$$\frac{\partial}{\partial u_2}(G_i \ell_{i1}) = \frac{\partial}{\partial u_1}(G_i \ell_{i2}), \quad \frac{\partial}{\partial u_2}(G_i \ell_{i3}) = \frac{\partial}{\partial u_3}(G_i \ell_{i2}),$$

$$\frac{\partial}{\partial u_1}(G_i \ell_{i3}) = \frac{\partial}{\partial u_3}(G_i \ell_{i1}).$$

This implies that each G_i must satisfy three independent first-order equations analogous to (7.3.36), and this is not possible in general.

The availability of the Riemann invariants (7.3.33) simplifies the solution of the system (7.3.5), (7.3.31) considerably; we shall explore this further in Section 7.3.4. Here, we note that we no longer need to consider (7.3.31); these two equations are already integrated in the form (7.3.33), and the solution is actually governed by the four relations

$$X_\eta - \lambda^+(U, V)T_\eta = 0 \text{ on } \xi = \text{ constant}, \tag{7.3.39a}$$

$$R(U, V) = \text{ constant on } \xi = \text{ constant}, \tag{7.3.39b}$$

$$X_\xi - \lambda^-(U, V)T_\xi = 0 \text{ on } \eta = \text{ constant}, \tag{7.3.39c}$$

$$S(U, V) = \text{ constant on } \eta = \text{ constant}, \tag{7.3.39d}$$

where we have set $\lambda_1 = \lambda^+$ and $\lambda_2 = \lambda^-$.

(ii) *Riemann invariants as independent variables*

Now, suppose that we use

$$r = R(U, V), \quad s = S(U, V), \tag{7.3.40}$$

as independent variables instead of ξ and η. This is a generalized hodograph transformation in the sense that the independent variables are certain functions of u and v. The preceding is possible as long as neither R nor S is identically constant in a given solution domain (see the discussion of simple waves in Section 7.3.4). Let us denote

$$X(\xi, \eta) \equiv \overline{X}(r, s), \quad T(\xi, \eta) \equiv \overline{T}(r, s), \tag{7.3.41}$$

and we compute

$$X_\eta = \overline{X}_r R_\eta + \overline{X}_s S_\eta = \overline{X}_s S_\eta, \quad T_\eta = \overline{T}_r R_\eta + \overline{T}_s S_\eta = \overline{T}_s S_\eta,$$

because $R_\eta = 0$. Similarly, we have

$$X_\xi = \overline{X}_r R_\xi, \quad T_\xi = \overline{T}_r R_\xi.$$

Therefore, (7.3.39a) and (7.3.39c) reduce to the following pair of linear equations, which are simpler than (7.3.17):

$$\overline{X}_s - \overline{\lambda}^+(r, s)\overline{T}_s = 0, \tag{7.3.42a}$$

$$\overline{X}_r - \overline{\lambda}^-(r, s)\overline{T}_r = 0. \tag{7.3.42b}$$

Here we have solved (7.3.41) for U and V in terms of r and s in the form

$$U = \overline{U}(r, s), \quad V = \overline{V}(r, s),$$

and have substituted these expressions into the definitions (7.3.1) for $\lambda_1(U, V)$ and $\lambda_2(U, V)$, where we have denoted

$$\overline{\lambda}^+(r, s) \equiv \lambda_1(\overline{U}(r, s), \overline{V}(r, s)), \quad \overline{\lambda}^-(r, s) = \lambda_2(\overline{U}(r, s), \overline{V}(r, s)). \tag{7.3.43}$$

We can combine the two equations in (7.3.42) into a single linear second-order equation for $\overline{T}$ by differentiating (7.3.42a) with respect to r and (7.3.42b) with respect to s, and subtracting the result to obtain

$$\overline{T}_{rs} + \frac{1}{\overline{\lambda}^- - \overline{\lambda}^+}(\overline{\lambda}_s^- \overline{T}_r - \overline{\lambda}_r^+ \overline{T}_s) = 0. \tag{7.3.44}$$

(iii) *An example, transonic flow*

To illustrate these ideas, let us consider the transonic flow equations (7.3.27) for the hyperbolic case ($u > 0$) and where we identify y with t. The $\{A_{ij}\}$ matrix elements are $A_{11} = 0$, $A_{12} = -1$, $A_{21} = -u$, and $A_{22} = 0$, and we compute $\lambda_1 = u^{1/2}$, $\lambda_2 = -u^{1/2}$ from (7.3.1). Therefore, the system (7.2.5) becomes

$$-\lambda_i \ell_{i1} - u\ell_{i2} = 0, \quad -\ell_{i1} - \lambda_i \ell_{i2} = 0, \tag{7.3.45}$$

and we choose

$$\{\ell_{ij}\} = \begin{pmatrix} u^{1/2} & -1 \\ u^{1/2} & 1 \end{pmatrix}. \tag{7.3.46}$$

The homogeneous system (7.3.31) for this example then takes the form

$$U^{1/2}U_\eta - V_\eta = 0, \quad U^{1/2}U_\xi + V_\xi = 0, \tag{7.3.47}$$

and (7.3.36) becomes

$$-G_U - U^{1/2}G_V = -\frac{1}{2U}G. \tag{7.3.48}$$

The solution of (7.3.48) may be written in the form (chosen for convenience)

$$G(U, V) = 2U^{1/2}\Gamma'(\frac{2}{3}U^{3/2} - V), \tag{7.3.49}$$

where Γ' is the derivative of an arbitrary function. Using (7.3.49) in (7.3.35) gives

$$R_U = U^{1/2}\Gamma'(\frac{2}{3}U^{3/2} - V), \quad R_V = -\Gamma'(\frac{2}{3}U^{3/2} - V), \tag{7.3.50}$$

which implies that $R = \Gamma$. Similarly, we find that S is an arbitrary function of $(\frac{2}{3}U^{3/2} + V)$. Thus, as expected, each Riemann invariant is constant along the corresponding characteristic of the hodograph equations. For simplicity let us choose $\Gamma(z) = z$ and write (7.3.40) as

$$r = \frac{2}{3}U^{3/2} - V, \quad s = \frac{2}{3}U^{3/2} + V. \tag{7.3.51}$$

We also obtain (7.3.51) more directly by integrating the two characteristic equations

$$\frac{dv}{du} = \mu_2 = u^{1/2}, \quad \frac{dv}{du} = \mu_1 = -u^{1/2} \tag{7.3.52}$$

in the hodograph plane.

To use the Riemann invariants as independent variables, we invert (7.3.51)

$$U = \left(\frac{3s+3r}{4}\right)^{2/3}, \quad V = \frac{s-r}{2}, \tag{7.3.53}$$

and substitute this result into (7.3.42),

$$\overline{X}_s - \left(\frac{3s+3r}{4}\right)^{1/3} \overline{Y}_s = 0, \tag{7.3.54a}$$

$$\overline{X}_r + \left(\frac{3s+3r}{4}\right)^{1/3} \overline{Y}_r = 0. \tag{7.3.54b}$$

We can also derive this result by transforming the hodograph equations (7.3.28) to the diagonal characteristic form (4.3.25). Thus, we introduce $r = (2/3)u^{3/2} - v$ and $s = (2/3)u^{3/2} + v$ as independent variables instead of u and v in (7.3.28). If we denote $\tilde{X}(u, v) \equiv \overline{X}(r, s)$, $\tilde{Y}(u, v) \equiv \overline{Y}(r, s)$, we compute

$$\tilde{X}_u = \overline{X}_r r_u + \overline{X}_s s_u = \overline{X}_r u^{1/2} + \overline{X}_s u^{1/2}, \tag{7.3.55a}$$

$$\tilde{X}_v = \overline{X}_r r_v + \overline{X}_s s_v = -\overline{X}_r + \overline{X}_s, \tag{7.3.55b}$$

$$\tilde{Y}_u = \overline{Y}_r r_u + \overline{Y}_s s_u = \overline{Y}_r u^{1/2} + \overline{Y}_s u^{1/2}, \tag{7.3.55c}$$

$$\tilde{Y}_v = \overline{Y}_r r_v + \overline{Y}_s s_v = -\overline{Y}_r + \overline{Y}_s. \tag{7.3.55d}$$

Substituting (7.3.55b) and (7.3.55c) into (7.3.28a) gives

$$\overline{X}_r - \overline{X}_s + u^{1/2}\overline{Y}_r + u^{1/2}\overline{Y}_s = 0. \tag{7.3.56a}$$

Substituting (7.3.55a) and (7.3.55d) into (7.3.28b) and dividing by $u^{1/2}$ gives

$$\overline{X}_r + \overline{X}_s + u^{1/2}\overline{Y}_r - u^{1/2}\overline{Y}_s = 0. \tag{7.3.56b}$$

If we now subtract (7.3.56a) from (7.3.56b) and note the definition for u in (7.3.53), we obtain (7.3.54a). Similarly, if we add (7.3.56a) to (7.3.56b), we obtain (7.3.54b).

This demonstrates that the use of the Riemann invariants as independent variables is equivalent to use of characteristic coordinates in the hodograph plane. The second-order equation (7.3.44) for $\overline{Y}$ is

$$\overline{Y}_{rs} + \frac{1}{6(r+s)}(\overline{Y}_r + \overline{Y}_s) = 0, \tag{7.3.57}$$

and this is just the canonical form of the Tricomi equation (see (4.1.27)).

7.3.4 Applications of the Riemann Invariants

In most applications, it is preferable to work with the physical variables u, v, x, and t as functions of ξ and η; the governing equations are (7.3.39). We now discuss some specfic consequences of the existence of the Riemann invariants.

(i) *Cauchy problem on a spacelike arc*

Consider the solution of the Cauchy problem on the spacelike arc $\mathcal{C}_0$, where U and V are prescribed, as shown in Figure 7.4. If we use the subscript notation of

Section 7.1.2, we prescribe $\mathcal{C}_0$ in the discrete form

$$x = X(\xi_i, \eta_{-i}) \equiv X_{i,-i}, \; t = T(\xi_i, \eta_{-i}) \equiv T_{i,-i}, \quad -N \leq i \leq M.$$

The values of U and V are also prescribed on $\mathcal{C}_0$: $U_{i,-i} \equiv U(\xi_i, \eta_{-i})$, $V_{i,-i} \equiv V(\xi_i, \eta_{-i})$, which means that we know the values of

$$R_{i,-i} \equiv R(U_{i,-i}, V_{i,-i}), \quad S_{i,-i} \equiv S(U_{i,-i}, V_{i,-i}) \text{ for } -N \leq i \leq M.$$

The constancy of the Riemann invariants implies that for any (m, n) such that $-N \leq m \leq M$ and $-M \leq n \leq N$, we have

$$R_{m,n} = R_{m,-m}, \; S_{m,n} = S_{-n,n}. \tag{7.3.58}$$

Thus, $R_{m,n}$ and $S_{m,n}$ are known at all the gridpoints in the zone of influence of the initial arc. This means that solving the two algebraic equations (7.3.39b) and (7.3.39d) gives U and V at each gridpoint in $\mathcal{D}$.

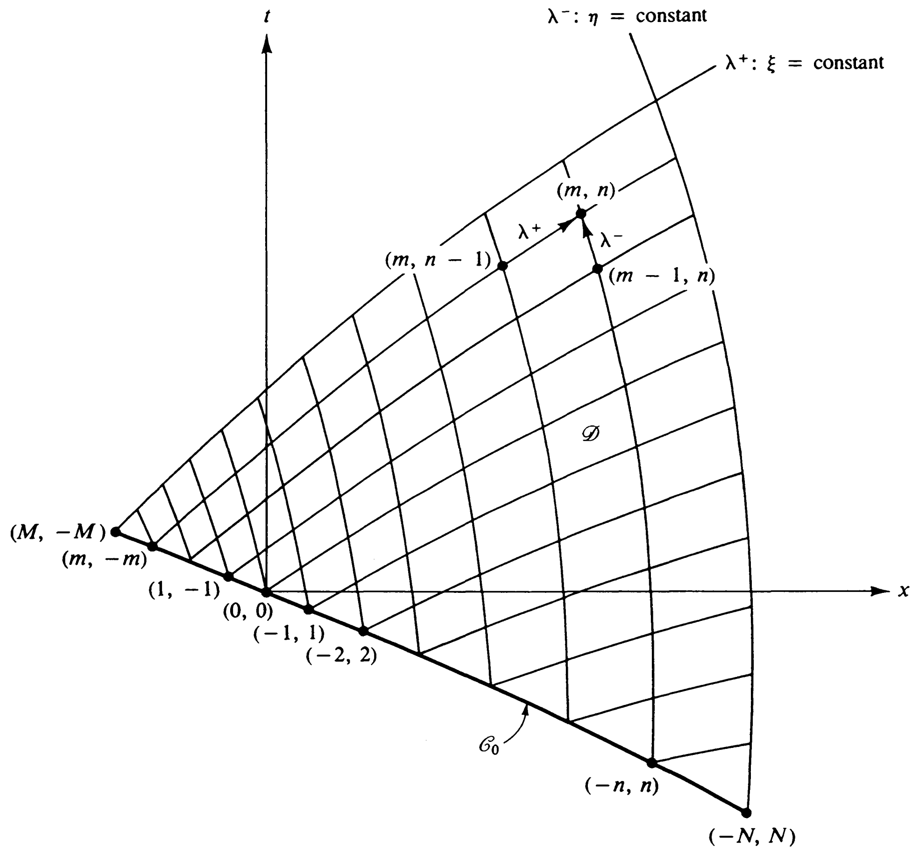

FIGURE 7.4. Cauchy problem

It remains to establish the location of the various gridpoints in the xt-plane, and this is easily accomplished using the forward difference form of (7.3.39a) and (7.3.39c). If we know X, T, U, and V at the two adjacent gridpoints $(m-1, n)$ and $(m, n-1)$, then we also know λ^- and λ^+ there. Solving the two linear algebraic equations that result from the forward difference form of (7.3.39a) and (7.3.39c) then gives (see (7.1.15) with $X \to T$, and $Y \to X$)

$$T_{m,n} = \frac{1}{\lambda^+_{m,n-1} - \lambda^-_{m-1,n}} (X_{m-1,n} - X_{m,n-1} + \lambda^+_{m,n-1} T_{m,n-1} - \lambda^-_{m-1,n} T_{m-1,n}), \tag{7.3.59a}$$

$$X_{m,n} = \frac{1}{\lambda^+_{m,n-1} - \lambda^-_{m-1,n}} \Big[\lambda^+_{m,n-1} X_{m-1,n} - \lambda^-_{m-1,n} X_{m,n-1} + \lambda^+_{m,n-1}\lambda^-_{m-1,n}(T_{m,n-1} - T_{m-1}, n)\Big]. \tag{7.3.59b}$$

(ii) *Boundary condition on a timelike arc*

Consider now the situation illustrated in Figure 7.5, where we prescribe a relationship between U and V on the timelike arc $\mathcal{T}_0$, which is the left boundary of our domain $\mathcal{D}$. Such an arc is characterized by the property that the characteristics on it with slope λ^- all point to the left as t increases. We must keep in mind that since λ^- depends in general on both U and V, we cannot establish whether a given arc in the xt-plane is timelike until we know both U and V along it. Since our boundary condition gives only one relation between U and V, we need to actually solve the problem in order to ascertain whether it is well posed. Therefore, the validity of the remainder of our arguments is contingent on meeting this criterion on λ^- all along $\mathcal{T}_0$ for the solution that we calculate.

The Cauchy data are now given as the set of discrete values $X_{0,0} = 0$, $T_{0,0} = 0$, $U_{0,0}$, $V_{0,0}$; $X_{-1,1}$, $T_{-1,1}$, $U_{-1,1}$, $V_{-1,1}$; ...; $X_{-M,N}$, $T_{-M,M}$, $U_{-M,M}$, $V_{-M,M}$. In addition, the boundary data on $\mathcal{T}_0$ consist of the following set of discrete values $X_{0,0} = 0$, $T_{0,0} = 0$, $\alpha_{0,0}U_{0,0} + \beta_{0,0}V_{0,0} = \gamma_{0,0}$; $X_{1,1}$, $T_{1,1}$, $\alpha_{1,1}U_{1,1} + \beta_{1,1}V_{1,1} = \gamma_{1,1}$; ...; $X_{M,M}$, $T_{M,M}$, $\alpha_{M,M}U_{M,M} + \beta_{M,M}V_{M,M} = \gamma_{M,M}$, for given constants $X_{m,m}$, $T_{m,m}$, $\alpha_{m,m}$, $\beta_{m,m}$, and $\gamma_{m,m}$. We have assumed a linear relation between U and V along $\mathcal{T}_0$ for simplicity; we could equally easily handle a nonlinear boundary condition of the form $G(U, V, T) =$ constant for a given function G on $\mathcal{T}_0$.

The solution procedure for all gridpoints not adjacent to $\mathcal{T}_0$ is as in the Cauchy problem, and we need to discuss only how to specify both U and V on $\mathcal{T}_0$ as well as on the adjacent gridpoints $(0, 1), (1, 2), \ldots, (M-1, M)$.

So far in this chapter, we have restricted attention to smooth solutions. In the present context, the solution in the neighborhood of the origin is smooth if the Cachy data at the origin are consistent with the boundary data in the limit as a point approaches the origin along $\mathcal{T}_0$. More precisely, we assume that using the values $U_{0,0}$ and $V_{0,0}$ specified at the origin on $\mathcal{C}_0$ to compute $(\alpha_{0,0}U_{0,0} + \beta_{0,0}V_{0,0})$ gives the prescribed value $\gamma_{0,0}$. We shall illustrate the case where a shock is introduced at the origin later on for specific examples.

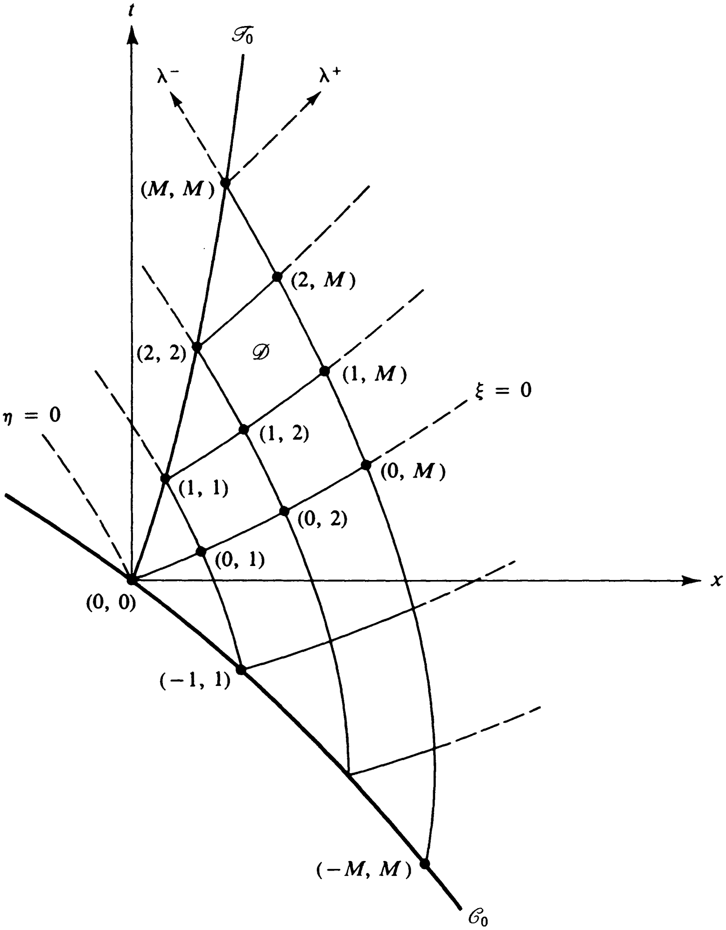

FIGURE 7.5. Boundary condition on a timelike arc

It is easy to see that the solution procedure for U and V at the $(m, m+1)$ gridpoint is no different than the procedure we outlined in discussing the Cauchy problem. In fact, since we know U and V at (0, 0) and (−1, 1), we can compute U and V at (0, 1) using the Riemann invariants $R_{0,1} = R_{0,0}$ and $S_{0,1} = S_{-1,1}$. Thus, we need to discuss only how to compute U and V at each of the (m, m) gridpoints on $\mathcal{T}_0$, assuming that we know their values at the $(m-1, m)$ points.

We have the Riemann invariant $S_{m,m} = S_{-m,m}$, which gives one relation linking $U_{m,m}$ to $V_{m,m}$. The second relation is provided by the boundary condition $\alpha_{m,m}U_{m,m} + \beta_{m,m}V_{m,m} = \gamma_{m,m}$. If the boundary data are well posed, these two conditions will be independent and will define $U_{m,m}$ and $V_{m,m}$. Before proceeding to larger values of t, one must check that $\lambda^-_{m,m}$ is indeed less than $(X_{m+1,m+1} - X_{m,m})/(T_{m+1,m+1} - T_{m,m})$ to verifiy that $\mathcal{T}_0$ is timelike beyond (m, m).

(iii) *Simple waves*

Consider the special case of Figure 7.5, where U and V are constant on $\mathcal{C}_0$. This is illustrated in Figure 7.6.

In the domain $\mathcal{D}_0$ above $\mathcal{C}_0$ and to the right of the λ^+ characteristic emerging from the origin the solution is constant, because the values of the two Riemann invariants remain unchanged throughout this region. Furthermore, since this implies that λ^+ and λ^- are both constant, the ξ and η characteristics are straight lines in $\mathcal{D}_0$.

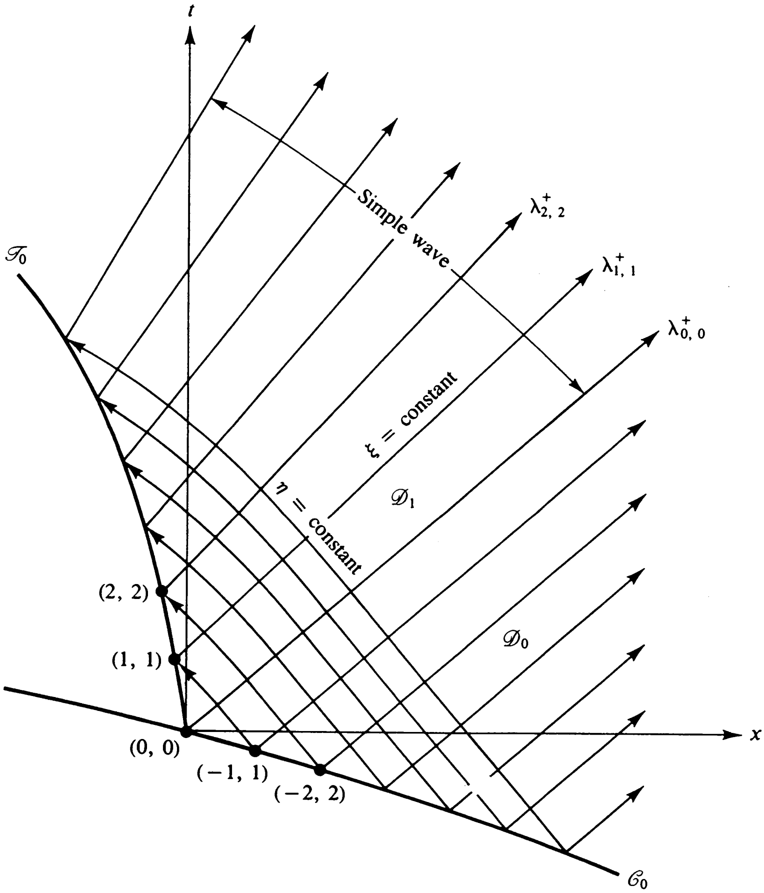

FIGURE 7.6. Simple wave generated by constant Cauchy data on a spacelike arc

Let us assume that $\mathcal{T}_0$ is a timelike arc—that is, that all the η = constant characteristics emerging from $\mathcal{T}_0$ lie to its left, as indicated in Figure 7.6. Thus, the domain $\mathcal{D}_1$ is entirely covered by the η = constant characteristics that emerge from $\mathcal{C}_0$. Therefore, the relation $S(U, V)$ = constant *holds throughout* $\mathcal{D}_1$, and we can, in principle, solve for V as a function on U in $\mathcal{D}_1$, for example, $V = F(U)$. Moreover, all the ξ = constant characteristics that emerge from $\mathcal{T}_0$ into $\mathcal{D}_1$ are straight lines. To see this, note that (7.3.39b) gives $R(U, F(U))$ = constant on ξ = constant; that is, U does not vary on any given ξ = constant characteristic. This, in turn, implies that $\lambda^+(U, F(U))$ = constant on a given ξ = constant characteristic; that is, the ξ = constant characteristics are straight lines, as claimed. The solution is defined by the given boundary values for U that propagate unchanged along the ξ = constant characteristics and the values $V = F(U)$, which also propagate unchanged along each ξ = constant ray. The solution in $\mathcal{D}_1$ is called a *simple wave*

In effect, we can use the expression $v = f(u)$ that results from the constancy of $S(u, v)$ throughout $\mathcal{D}_1$ to eliminate v in favor of u from the governing system to obtain the single first-order equation for u

$$u_t + [a(u, f(u)) + b(u, f(u))f'(u)]u_x = 0. \tag{7.3.60}$$

Note, incidentally, that we cannot use the Riemann invariants as independent variables in a simple wave region (as either R or S is constant throughout) or in a region of constant u and v (where both R and S are constants).

A simple wave solution fails to exist whenever the λ^- that we calculate on $\mathcal{T}_0$ imply that the η = constant characteristics emerge to the right of $\mathcal{T}_0$ for increasing t. We have also tacitly assumed that the values of λ^+ decrease as ξ increases, that is, the ξ = constant characteristics "fan out." Both of these features depend on the boundary values prescribed on $\mathcal{T}_0$ for U. In the first instance, if the η = constant characteristics emerge to the right of $\mathcal{T}_0$, then this arc is spacelike, and we need to specify a second condition there. In the second instance, if the λ^+ characteristics converge and intersect in $\mathcal{D}_1$, a solution with continuous values of U and V is not possible, and we need to introduce a shock.

A limiting case for which a shock is needed has the boundary value of U prescribed to be a constant U_1 on $\mathcal{T}_0$, in addition to having $U = U_0 = \text{constant} \neq U_1$, $V = V_0$ = constant on $\mathcal{C}_0$. If we assume that the solution is continuous across the λ^+ characteristics emerging from the origin, we would conclude that the Riemann invariant S holds in $\mathcal{D}_0$ and $\mathcal{D}_1$. This means that in $\mathcal{D}_1$, V is given by $V = F(U)$ obtained by solving the expression $S(U, V) = \text{constant} = S_0(U_0, V_0)$ for V in terms of U. In particular, since $U = U_1$ in $\mathcal{D}_1$, we must set $V_1 = F(U_1)$. We then compute the slopes $\lambda^+(U_0, V_0)$ and $\lambda^+(U_1, F(U_1))$ for the ξ = constant characteristics emerging from $\mathcal{C}_0$ and $\mathcal{T}_0$, respectively. These characteristics *intersect* if

$$\lambda^+(U_0, V_0) < \lambda^+(U_1, F(U_1)), \tag{7.3.61}$$

and we need to introduce a shock starting at $x = t = 0$. In general, we must introduce a shock whenever characteristics of the same family cross. Special cases

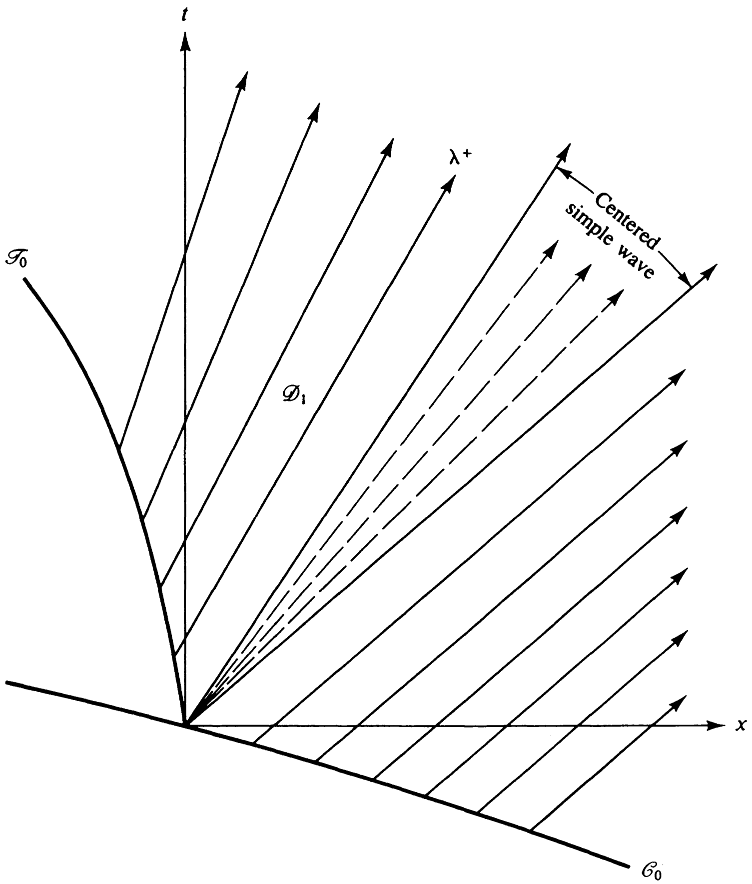

FIGURE 7.7. Centered simple wave

were discussed in Chapter 5 (see Figures 5.10 and 5.12) and will be reconsidered in the next section.

The reverse situation has

$$\lambda^+(U_0, V_0) > \lambda^+(U_1, F(U_1)), \tag{7.3.62}$$

with $U = U_0 = \text{constant} \neq U_1$, $V = V_0 = \text{constant}$ on C_0. This results in the centered simple wave that corresponds to the following limiting case of the smooth problem in Figure 7.6. Assume that U is prescribed as a monotone function of the arc length σ over an ϵ interval above the origin on $\mathcal{T}_0$, with $U(0) = U_0$, $U(\epsilon) = U_1$, and $U = U_1$ for $\sigma \geq \epsilon$. If we solve this problem and then let $\epsilon \to 0$, we obtain the centered simple wave shown in Figure 7.7.

Problem

7.3.1 In this problem we wish to derive the forward difference solution along the local characteristic directions for the system (7.3.5), (7.3.12) of four first-order equations. To reserve subscripts for locating gridpoints, let us write (7.3.5) and (7.3.12) in the following form:

$$X_\eta = \lambda^+(X, T, U, V)T_\eta, \quad (7.3.63a)$$

$$X_\xi = \lambda^-(X, T, U, V)T_\xi, \quad (7.3.63b)$$

$$AU_\eta + BV_\eta = ET_\eta, \quad (7.3.63c)$$

$$CU_\xi + DV_\xi = FT_\xi, \quad (7.3.63d)$$

where λ^+ refers to λ_1 as given in (7.3.5a), and λ^- refers to λ_2 as given in (7.3.5b). We have also set $\ell_{11} = A$, $\ell_{12} = B$, $\ell_{21} = C$, $\ell_{22} = D$, $\ell_{11}f_1 + \ell_{12}f_2 = E$, and $\ell_{21}f_1 + \ell_{22}f_2 = F$. All eight coefficients λ^-, λ^+, A, B, C, D, E, and F are known functions of X, T, U, and V.

a. Consider the initial-value problem where Cauchy data are prescribed on the x-axis and use the subscript notation of Section 7.1.2 to locate gridpoints. Thus, the $X_{i,-i}$, $U_{i,-i}$, $V_{i,-i}$ are all given, and the $T_{i,-i}$ are all equal to zero. Assume also that $\lambda^-_{i,-i} < 0$ and $\lambda^+_{i,-i} > 0$ for all the given values of these four quantities at the two preceding gridpoints $(m-1, n)$ and $(m, n-1)$.

b. Now assume that U is specified along the t-axis in addition to the Cauchy data along the x-axis. Develop the characteristic solutions for $x > 0$, $t > 0$ and indicate under what conditions you can calculate this solution.

7.4 Shallow-Water Waves

In this section we illustrate the results derived in Section 7.3 for the one-dimensional flow of shallow water on a flat bottom. The integral laws of mass and momentum conservation in dimensionless form are (see (3.2.6) and (3.2.10))

$$\frac{d}{dt}\int_{x_1}^{x_2} h(x,t)dx + \{u(x,t)h(x,t)\}\Big|_{x=x_1}^{x=x_2} = 0, \quad (7.4.1a)$$

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)h(x,t)dx + \{u^2(x,t)h(x,t) + \frac{1}{2}h^2(x,t)\}\Big|_{x=x_1}^{x=x_2} = 0, \quad (7.4.1b)$$

where x_1 and x_2 are two arbtirary fixed points with $x_1 < x_2$.

The bore conditions associated with (7.4.1) are (see (5.3.24))

$$V[h] = [uh], \quad V[uh] = [u^2h + \frac{1}{2}h^2], \quad (7.4.2)$$

where [ ] denotes the jump in a quantity across a bore that propagates with speed $(dx/dt) = V$ (see (5.3.3) and (5.3.5)).

For strict solutions (7.4.1) imply (see (5.3.25) in reverse order)

$$u_t + uu_x + h_x = 0, \quad h_t + hu_x + uh_x = 0. \tag{7.4.3}$$

Therefore, for this example the vector $\mathbf{u}$ and the matrix $\{A_{ij}\}$ in (7.2.2) are

$$\mathbf{u} = (u, h), \quad \{A_{ij}\} = \begin{pmatrix} u & 1 \\ h & u \end{pmatrix}. \tag{7.4.4}$$

7.4.1 Characteristic Coordinates; Riemann Invariants

The eigenvalues of $\{A_{ij}\}$ are defined by the vanishing of the determinant

$$\det\{A_{ij} - \delta_{ij}\lambda\} \equiv (u - \lambda)^2 - h = 0, \tag{7.4.5}$$

which gives

$$\lambda_1 = u + \sqrt{h}, \quad \lambda_2 = u - \sqrt{h}. \tag{7.4.6}$$

The following $\{\ell_{ij}\}$ is consistent with (7.2.5) for the given values of the $\{A_{ij}\}$ and λ_i

$$\{\ell_{ij}\} = \begin{pmatrix} -1 & -\frac{1}{\sqrt{h}} \\ -1 & \frac{1}{\sqrt{h}} \end{pmatrix}. \tag{7.4.7}$$

Therefore, the system (7.3.5) and (7.3.31) takes the form

$$X_\eta - (U + H^{1/2})T_\eta = 0, \quad X_\xi - (U - H^{1/2})T_\xi = 0,$$
$$-U_\eta - \frac{1}{H^{1/2}} H_\eta = 0, \quad -U_\xi + \frac{1}{H^{1/2}} T_\xi = 0,$$

where we are denoting $x = X(\xi, \eta)$, $t = T(\xi, \eta)$, $u = U(\xi, \eta)$, and $h = H(\xi, \eta)$.

The hodograph form (7.3.17) of the system (7.4.3) is

$$-\tilde{X}_h + u\tilde{T}_h - \tilde{T}_u = 0, \quad \tilde{X}_u + h\tilde{T}_h - u\tilde{T}_u = 0. \tag{7.4.8}$$

We have argued that the Riemann invariants can be interpreted as the characteristics of (7.4.8)—that is, the curves with slope $(dh/du) = \mu_1$ and $(dh/du) = \mu_2$. Using (7.3.24), we obtain $\mu_1 = \sqrt{h}$ and $\mu_2 = -\sqrt{h}$. Therefore, integrating gives

$$R(u, h) = u + 2\sqrt{h}, \quad S(u, h) = u - 2\sqrt{h}. \tag{7.4.9}$$

This result also follows from the solutions of (7.3.36) and (7.3.38) for G and K, respectively. In particular, (7.3.36) reduces to

$$-\frac{1}{H^{1/2}} G_u + G_H + 0.$$

The solution of this has G equal to an arbitrary function of $(U + 2H^{1/2})$, and for convenience, we choose

$$G(U, H) = -2\Gamma'(U + 2H^{1/2}),$$

which when used in (7.3.35) gives $R = U + 2H^{1/2}$. A similar calculation gives the expression for S in (7.4.9).

7.4.2 *Simple Waves*

In this section we study two examples where the solution is made up of simple wave regions bounded by constant states, so that we can calculate an exact solution explicitly.

(i) *The dam-breaking problem*

The exact solution of this problem was used in Chapter 4 to study linear small-disturbance equations (see (4.3.38) and Figure 4.10).

The initial coditions are

$$u(x,0) = 0; \quad h(x,0) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases} \tag{7.4.10}$$

For the time being, let us consider only the characteristics that emerge from $x < 0$. These characteristics have constant speeds $\lambda^+ = \sqrt{h} = 1$ and $\lambda^- = -\sqrt{h} = -1$ in the domain $\mathcal{D}_0$ to the left of $t = -x$, where the solution is the uniform state $u = 0, h = 1$ (see Figure 7.8). The Riemann invariant on the λ^+ characteristic is

$$R = u + 2\sqrt{h} = u(x,0) + 2\sqrt{h(x,0)} = 2. \tag{7.4.11}$$

Thus, the relation (7.4.11) links u and h in the domain $\mathcal{D}_1$ to the right of $t = -x$ that is covered by the λ^+ characteristics emerging from $x < 0, t = 0$. We do not yet know the extent of this domain.

At this point, we anticipate the occurrence of a centered simple wave at the origin. This simple wave is defined by the λ^- characteristics emerging from the origin and forming the family of rays

$$\frac{x}{t} = u - \sqrt{h}. \tag{7.4.12}$$

The solution of the two algebraic relations (7.4.11) and (7.4.12) for u and h gives the result (see (4.3.38))

$$u = \frac{2}{3}\left(\frac{x}{t} + 1\right), \quad h = \frac{1}{9}\left(2 - \frac{x}{t}\right)^2. \tag{7.4.13}$$

Using these values of u and h in the expressions $(dx/dt) = u + \sqrt{h}$ and integrating gives the curved λ^+ characteristics shown in Figure 7.8 (see also (4.3.47a) and Figure 4.12). The solution (7.4.13) remains valid as long as $h \geq 0$, and we see that $h \to 0$ as $(x/t) \to 2$. Therefore, $\mathcal{D}_1$ consists of the triangular domain $-1 \leq x/t \leq 2$. Since $u = h = 0$ in $\mathcal{D}_2$ and $u = 2, h = 0$ on $x = 2t$, we see that R, S, and u are discontinuous on $x = 2t$, but h remains continuous.

The fact that h must vanish at the right boundary of $\mathcal{D}_1$ is also confirmed by the bore conditions (7.4.2). Let us assume that the right boundary of $\mathcal{D}_1$ is a bore

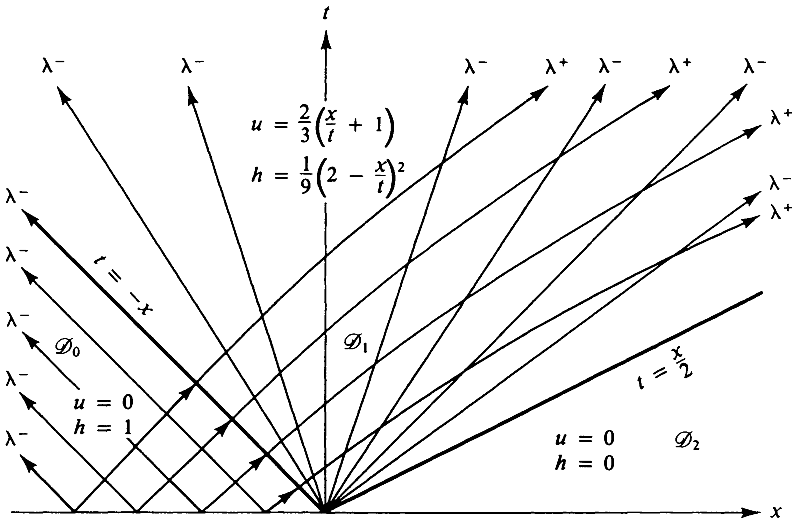

FIGURE 7.8. Characteristics for the dam-breaking problem

traveling with the constant speed V and let $u_2 = h_2 = 0$ to its right, whereas $u_1 > 0, h_1 > 0$ to its left. The bore conditions then give

$$Vh_1 = u_1 h_1, \quad Vu_1 h_1 = u_1^2 h_1 + \frac{1}{2} h_1^2, \tag{7.4.14}$$

and these imply $h_1 = 0$.

Once we note that h remains continuous, it is also possible to derive the solution (7.4.13) by using (7.4.11) to eliminate u from either of the governing equations (7.4.3) to obtain

$$h_t + (2 - 3\sqrt{h})h_x = 0. \tag{7.4.15}$$

This equation is to be solved subject to the initial condition in (7.4.10) for h. The transformation $v = 2 - 3\sqrt{h}$ reduces the problem to

$$v_t + vv_x = 0, \tag{7.4.16a}$$

$$v(x, 0) = \begin{cases} -1 & \text{if } x < 0, \\ 2 & \text{if } x > 0, \end{cases} \tag{7.4.16b}$$

and we recognize this as (5.3.15c) for the special case of the initial condition (5.3.37) with $x_0 = 0, u_1 = -1, u_2 = 2$. The solution in (5.3.42) for this case reduces to

$$v(x, t) = \begin{cases} -1 & \text{if } x \geq -t, \\ x/t & \text{if } -t \leq x \leq 2t, \\ 2 & \text{if } 2t \leq x, \end{cases} \tag{7.4.17a}$$

or

$$h(x,t) = \frac{1}{9}(v-2)^2 = \begin{cases} 1 & \text{if } x \le -t, \\ \frac{1}{9}(2-\frac{x}{t})^2 & \text{if } -t \le x \le 2t, \\ 0 & \text{if } 2t \le x. \end{cases} \qquad (7.4.17b)$$

We use (7.4.17b) in the Riemann invariant (7.4.11), which is valid in $\mathcal{D}_0$ and $\mathcal{D}_1$, to obtain u there in the form

$$u(x,t) = \begin{cases} 0 & \text{if } x \le t, \\ \frac{2}{3}(\frac{x}{t}+1) & \text{if } -t \le x < 2t. \end{cases} \qquad (7.4.18)$$

For $x > 2t$, we have $u = 0$.

(ii) *Retracting piston*

Consider the flow generated in a semi-infinite body of water at rest when a wavemaker (piston) begins to move to the left (away from the fluid) according to $x_p = -\epsilon g(t) < 0$, $g(0) = 0$ (see Figure 3.7). In order to ensure that the flow be free of bores (see the discussion following (7.3.60)), we require $-\epsilon \dot{g}$ to be a monotone nonincreasing function of t for $t > 0$.

Thus, we wish to solve (7.4.3) subject to the initial conditions

$$u(x,0) = 0, \quad h(x,0) = 1 \qquad (7.4.19)$$

and the boundary condition

$$u(-\epsilon g(t), t) = -\epsilon \dot{g}(t) \text{ if } t > 0. \qquad (7.4.20)$$

Here ϵ is the dimensionless ratio of the speeds characterizing the wavemaker and the flow (see (3.2.20) with $\nu = \epsilon$). We do not assume that ϵ is small in the analysis of this section. The geometry is sketched in Figure 7.9a.

In the domain $\mathcal{D}_0$ to the right of the λ^+ characteristic $x = t$, the solution is the constant state $u = 0$, $h = 1$. Therefore, the Riemann invariant

$$S = u - 2\sqrt{h} = -2 \qquad (7.4.21)$$

holds in the domain covered by the λ^- characteristics

$$\frac{dx}{dt} = u - \sqrt{h} \qquad (7.4.22)$$

that emerge from $t = 0$, $x > 0$.

The λ^+ characteristics that originate from the piston curve $x_p = -\epsilon g(t)$ have slope

$$\frac{dx}{dt} = u + \sqrt{h}, \qquad (7.4.23a)$$

and this equals

$$\frac{dx}{dt} = u + \frac{1}{2}(u+2) = \frac{1}{2}(2 - 3\epsilon\dot{g}), \qquad (7.4.23b)$$

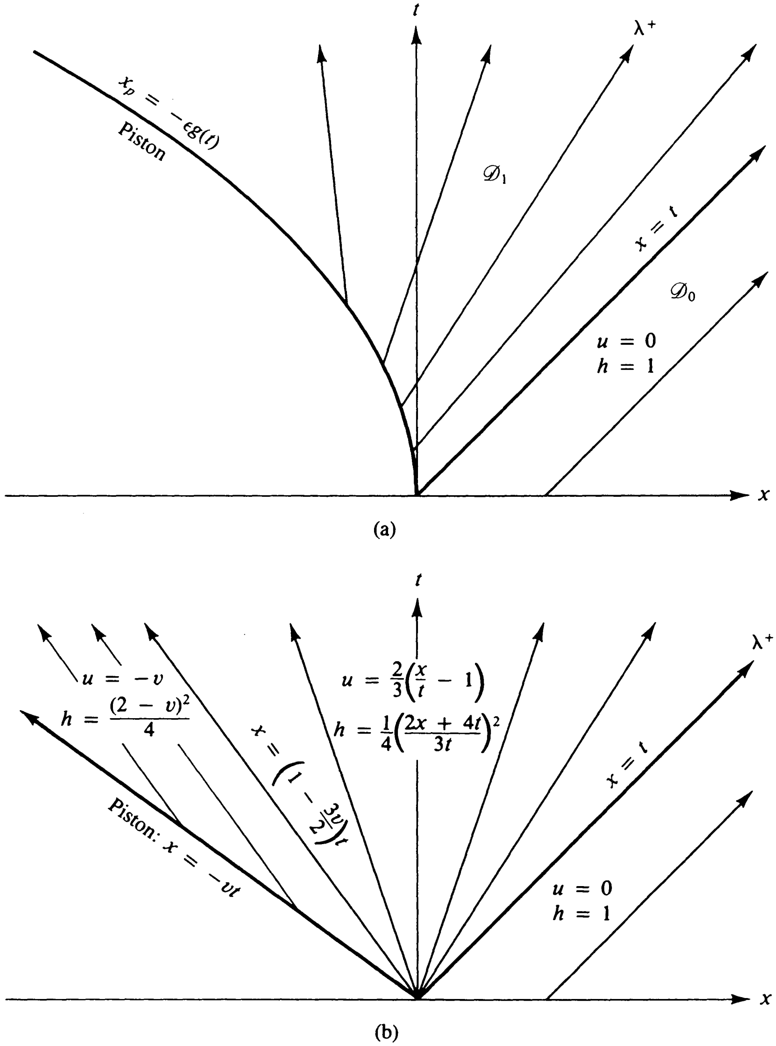

FIGURE 7.9. Characteristics for retracting piston

when we use (7.4.21) to express h in terms of u and impose the boundary condition (7.4.20). If $\epsilon\dot{g}(0^+) = 0$, that is, the piston starts out with zero speed, then the λ^+ characteristic emerging from $x = -\epsilon g(0^+)$, $t = 0^+$ coincides with the λ^+ characteristic emerging from $x = 0^+$, $t = 0$. However, if $\dot{g}(0^+) > 0$, we must insert a centered simple wave in the triangular domain $[(2 - 3\epsilon\dot{g}(0^+))/2]t \leq x \leq t$ bounded by these two characteristics (see (7.4.27)).

The solution for u in $\mathcal{D}_1$ follows from the fact that $R = u + 2\sqrt{h}$ is a constant on each λ^+ characteristic in $\mathcal{D}_1$. Since (7.4.21) holds throughout $\mathcal{D}_1$, we have $\sqrt{h} = (u+2)/2$ there. Therefore, the statement $R =$ constant means that $u + (u+2) = 2u + 2 =$ constant; that is, u is a constant equal to its boundary value on each λ^+ characteristic (7.4.23b). This result may be expressed more precisely in the parametric form

$$u = -\epsilon \dot{g}(\tau) \tag{7.4.24a}$$

on

$$x = \frac{1}{2}(2 - 3\epsilon \dot{g}(\tau))(t - \tau) - \epsilon g(\tau). \tag{7.4.24b}$$

For given $g(\tau)$, we can solve (7.4.24b) for τ as a function of x and t. Substituting this into (7.4.24a) gives u as function of x and t (see Problem 7.4.1). Once u is known, we compute h from (7.4.21):

$$h = \frac{1}{4}(u+2)^2 = \frac{1}{4}[2 - \epsilon \dot{g}(\tau)]^2 \tag{7.4.24c}$$

on the rays defined by (7.4.24b).

The preceding assumes that $\epsilon \dot{g}(0^+) = 0$ and that the piston trajectory is a timelike arc. This is the requirement that λ^- be less than (dx_p/dt) on $x = x_p$, that is,

$$u - \sqrt{h} = u - \frac{1}{2}(u+2) = \frac{1}{2}u - 1 = -\frac{1}{2}\epsilon\dot{g} - 1 < -\epsilon\dot{g}. \tag{7.4.25}$$

This inequality is satisfied as long as $\epsilon\dot{g} < 2$, but if the piston is pulled away with a speed that exceeds 2, the λ^- characteristics will propagate to the right and render $x = -\epsilon g(t)$ a spacelike arc. Note also from (7.4.24a) and (7.4.24c) that as $\epsilon\dot{g} \to 2$, $h \to 0$ and $u \to -2$, so that in this limit, we have the mirror image of the dam-breaking problem. Thus, if the piston is pulled with a speed faster than the critical speed 2, the flow is the same as if the piston were suddenly removed at $t = 0$.

The result (7.4.24a)–(7.4.24b) also follows from the single equation for u that results when (7.4.21) is used to eliminate h from either of the equations in (7.4.3) to obtain

$$u_t + \frac{1}{2}(3u + 2)u_x = 0. \tag{7.4.26}$$

This is to be solved subject to the boundary condition (7.4.20), which we write in parametric form as

$$t = \tau, \quad x = -\epsilon g(\tau), \quad u = -\epsilon\dot{g}(\tau).$$

The characteristic equations for (7.4.26) are (see (5.2.3))

$$\frac{dt}{ds} = 1, \quad \frac{dx}{ds} = \frac{1}{2}(3u + 2), \quad \frac{du}{ds} = 0,$$

which can be immediately solved to give (7.4.24a)–(7.4.24b).

If $\dot{g}(0^+) > 0$, we insert the centered simple wave solution obtained from $x/t = (3u + 2)/2$; that is,

$$u = \frac{2}{3}\left(\frac{x}{t} - 1\right), \tag{7.4.27a}$$

$$h = \frac{1}{9}\left(\frac{x}{t} + 2\right)^2 \tag{7.4.27b}$$

for $[(2 - 3\epsilon\dot{g}(0^+))/2]t \leq x \leq t$.

For the special case where the piston moves with constant speed v, the solution (7.4.27) terminates at the ray $x = (1 - 3v/2)t$. To the left of this ray, we have the constant state $u = -v$, $h = (2 - v)^2/4$, as shown in Figure 7.9b.

(iii) *Interacting simple waves*

Consider the problem of two pistons initially at rest a unit distance apart and retaining quiescent water of unit height. At $t = 0$, the pistons are impulsively retracted with equal and opposite speeds v. By symmetry, this also corresponds to the problem for a single piston with a vertical wall at $x = 0$ (see Figure 7.10).

Each piston motion initially generates a centered simple wave with origin at P: $x = -\frac{1}{2}$, $t = 0$, and Q: $x = \frac{1}{2}$, $t = 0$. These simple waves are unaffected by each other until the fastest λ^+ characteristic from P intersects the fastest λ^- characteristic from Q at the point A. Let us first define the solution before this interaction starts.

Consider the equilibrium domain (0) in which $u = 0$ and $h = 1$. The boundary characteristic between (0) and (2) has slope $\lambda^+ = \sqrt{h} = 1$; that is, it is the straight line $x = t - \frac{1}{2}$, and by symmetry, the bounding characteristic between (0) and (4) is $x = \frac{1}{2} - t$. These characteristics intersect at the point A: $x = 0, t = \frac{1}{2}$.

The Riemann invariant that holds in (2) and (1) is

$$S(u, h) = u - 2\sqrt{h} = -2, \tag{7.4.28a}$$

which gives h in terms of u in the form

$$h = \frac{1}{4}(u + 2)^2 \tag{7.4.28b}$$

in (2) and (1). In particular, $u = -v$ in (1), so that $h = (2 - v)^2/4$ there.

The characteristics emerging from P are given by (see (7.4.12))

$$\frac{x + 1/2}{t} = u + \sqrt{h}. \tag{7.4.29}$$

Upon substituting (7.4.28b) for h into (7.4.29) and solving the result for u, we obtain

$$u(x, t) = -\frac{2}{3} + \frac{2x + 1}{3t} \text{ in (2)}. \tag{7.4.30a}$$

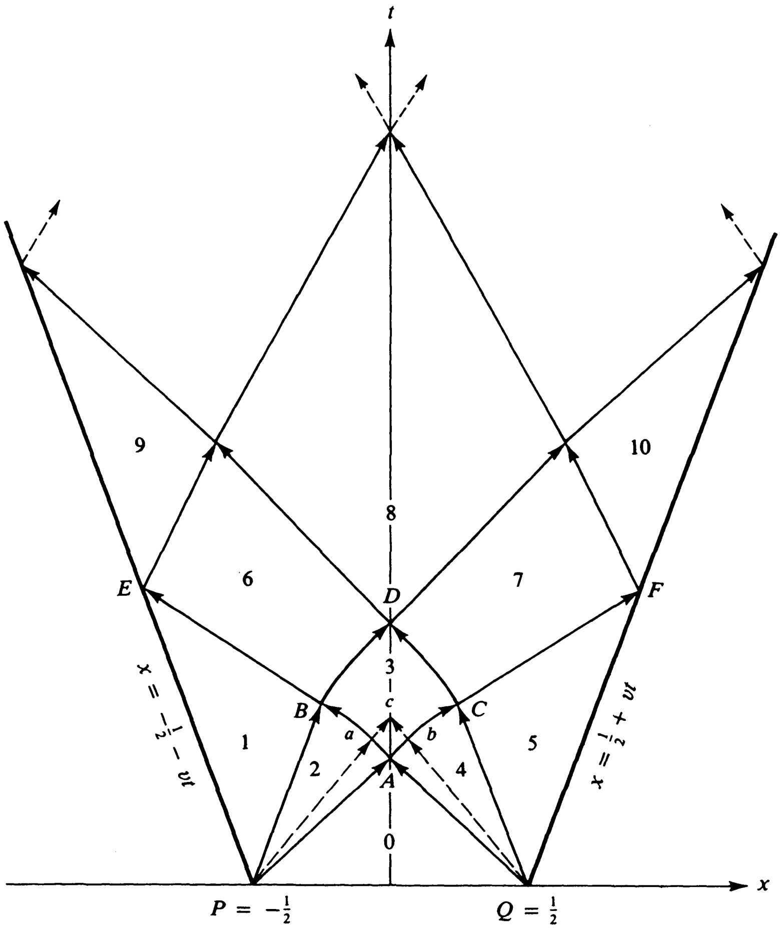

FIGURE 7.10. Characteristics for interacting simple waves

When this is substituted into (7.4.28b), we obtain

$$h(x, t) = \left(\frac{4t + 2x + 1}{6t} \right)^2 \text{ in (2).} \tag{7.4.30b}$$

The ray PB, which is the left boundary of (2), is given by (7.4.29), in which we set $u = -v$ and $\sqrt{h} = (2 - v)/2$, that is,

$$x = -\frac{1}{2} + \frac{1}{2}(2 - 3v)t. \tag{7.4.31}$$

Thus, in the limit $v \to 2$, (2) extends all the way to the piston curve, and if $v > 2$, the flow is the same as if the piston were suddenly removed at $t = 0$ (dam-breaking problem).

Similarly, in region (4), we have

$$u(x, t) = \frac{2}{3} + \frac{2x - 1}{3t}, \quad h(x, t) = \left(\frac{4t - 2x + 1}{6t} \right)^2, \tag{7.4.32}$$

with the boundary ray QC defined by

$$x = \frac{1}{2} + \frac{1}{2}(3v - 2)t. \tag{7.4.33}$$

Now consider the solution in (3) that is bounded by the arcs AB, BD, DC, and CA. We can define the arc AB in terms of its slope $dx/dt = u - \sqrt{h}$, as given in (2):

$$\frac{dx}{dt} = -\frac{2}{3} + \frac{2x + 1}{3t} - \frac{4t + 2x + 1}{6t} = \frac{2x - 8t + 1}{6t}.$$

Solving this linear equation subject to the initial condition $t = \frac{1}{2}$, $x = 0$, gives

$$x = -\frac{1}{2} - 2t + \frac{3}{2^{2/3}} t^{1/3}. \tag{7.4.34}$$

The intersection of this curve with the ray (7.4.31) defines the coordinates of B to be

$$x = -\frac{1}{2} + \frac{2 - 3v}{2^{1/2}(2 - v)^{3/2}}, \quad t = \frac{2^{1/2}}{(2 - v)^{3/2}}. \tag{7.4.35}$$

One can define the arc AC from the preceding by symmetry.

The solution in (3) can be computed numerically in a straightforward way for the given values of u and h on AB and AC using essentially the approach outlined in Section 7.3.4. The fact that AB and AC are intersecting characteristic arcs does not alter the solution procedure. One starts with the values of u and h at the points (a) and (b) to compute the location of (c) and the values of u and h there, and so on. The known data on AB and AC then define the solution in all of (3).

It is also instructive to examine the solution in (3) from the point of view of the general result (7.3.44), which defines t as a function of r and s, regarded as independent variables. For our special case, u and $\sqrt{h}$ are given in terms of r and s by

$$u = \frac{1}{2}(r + s), \quad \sqrt{h} = \frac{1}{4}(r - s). \tag{7.4.36}$$

Therefore (see (7.3.43)),

$$\overline{\lambda}^{+} = \frac{1}{2}(r + s) + \frac{1}{4}(r - s) = \frac{1}{4}(3r + s), \tag{7.4.37a}$$

$$\overline{\lambda}^{-} = \frac{1}{2}(r + s) - \frac{1}{4}(r - s) = \frac{1}{4}(3s + r), \tag{7.4.37b}$$

and (7.3.44) becomes (see also (7.3.57) for the corresponding result for the Tricomi equation)

$$\overline{T}_{rs} + \frac{3}{2(s-r)}(\overline{T}_r - \overline{T}_s) = 0. \tag{7.4.38}$$

This linear second-order hyperbolic equation is to be solved subject to prescribed values of $\overline{T}$ on the two arcs AB and AC. The arc AB is a λ^- characteristic on which $s = -2$. Similarly, AC is a λ^+ characteristic with $r = 2$. But $r =$ constant and $s =$ constant are also characteristics of (7.4.38), so we need to solve a characteristic boundary-value problem for this linear equation (see Section 4.2.2iv).

We can actually compute $\overline{T}$ as a function of r on AB and $\overline{T}$ as a function of s on AC from the two known solutions in (2) and (4). For example, AB is defined by (7.3.42b), which in our case gives

$$(u - \sqrt{h})\overline{T}_r = \overline{X}_r, \tag{7.4.39a}$$

and using (7.4.36), this becomes

$$\frac{1}{4}(r + 3s)\overline{T}_r = \overline{X}_r. \tag{7.4.39b}$$

In region (2), x is given by (7.4.29)—that is, $x = -(1/2) + (u + \sqrt{h})t$. Using (7.4.36), this becomes

$$\overline{X}(r, s) = -\frac{1}{2} + \left(\frac{r+s}{2} + \frac{r-s}{4}\right)\overline{T} = -\frac{1}{2} + \frac{3r+s}{4}\overline{T}.$$

Therefore,

$$\overline{X}_r = \frac{3}{4}\overline{T} + \frac{3r}{4}\overline{T}_r,$$

and (7.4.39b) reduces to the following linear equation for $\overline{T}$:

$$\overline{T}_r + \frac{3}{2(r-s)}\overline{T} = 0. \tag{7.4.40}$$

Solving this on $s = -2$, subject to $\overline{T} = \frac{1}{2}$ at $r = 2$, gives

$$\overline{T}(r, -2) = \frac{4}{(r+2)^{3/2}}. \tag{7.4.41a}$$

Similarly, on AC we obtain

$$\overline{T}(2, s) = \frac{4}{(2-s)^{3/2}}. \tag{7.4.41b}$$

The fundamental solution of (7.4.38) can be calculated in terms of the hypergeometric function, so that we can, in principle, solve for t as a function of r and s in (3). This calculation is outlined in Problem 9, Section 5.1 of [18], and we do not give the details here. From a practical point of view, such a solution, although elegant, is not very useful because we also have to calculate $x(r, s)$ and then invert these expressions for t and x numerically before we can calculate $u(x, t)$ and

$h(x, t)$. A numerical solution by the method outlined in Section 7.3.4 gives u and h directly and efficiently.

Once the solution in (3) has been calculated, we can compute the simple wave solutions in (6) and (7) explicitly. Region (6) is covered by the λ^+ characteristics emerging from BE, on which u and h have the constant values $u = -v$, $h = (2-v)^2/4$. Therefore, the Riemann invariant

$$u + 2\sqrt{h} = -v + (2 - v) = 2(1 - v) \tag{7.4.42}$$

holds throughout (6), and this allows us to express h in terms of u there. Now, each of the λ^- characteristics emerging from BD has the constant slope

$$\frac{dx}{dt} = u_0(\sigma) - \sqrt{h_0(\sigma)}, \tag{7.4.43}$$

where σ is a parameter that varies along the arc BD and $u_0(\sigma)$, $h_0(\sigma)$ are the values of u and h on BD, as computed in (3). On each of the straight lines defined by (7.4.43) for fixed σ, we have $u = u_0(\sigma)$ and $h = h_0(\sigma)$. Similar remarks apply to region (7).

In region (8), we have the uniform flow $u = u_0^*$, $h = h_0^*$, corresponding to the values of u and h at D as computed in (3). One can continue the solution above the λ^+ characteristics emerging from E and the λ^- characteristics emerging from F, and so on. The details of the calculation procedure are left as an exercise in Problem 7.4.2.

7.4.3 Solutions with Bores

In this section we study some examples where bores are needed in order to prevent the crossing of characteristics of the same family.

(i) *The uniformly propagating bore*

In Section 5.3.4ii, we saw that the bore conditions (7.4.2) admit physically realistic solutions consisting of constant-speed bores propagating into uniform flow of lower height. Let us focus on the problem of a piston set impulsively into motion with constant speed $v > 0$ into water of unit height at rest (see Problem 5.3.8). We argued that this left boundary condition produces a bore that propagates to the right with constant speed, say, V, and that the speed of the water u_1 and the height h_1 between the piston and the bore are constants with $u_1 = v$, and h_1 given by the larger positive root, $h_1^{(2)}$, of the cubic

$$h_1^3 - h_1^2 - (1 + 2v^2)h_1 + 1 = 0, \tag{7.4.44}$$

and V given in terms of $h_1^{(2)}$ by

$$V = \left(h_1^{(2)} \frac{h_1^{(2)} + 1}{2}\right)^{1/2}. \tag{7.4.45}$$

To confirm the need for a bore, let us assume a continuous solution across the characteristic $x = t$ separating regions $\mathcal{D}_0$ and $\mathcal{D}_1$ as in Figure 7.9a, except that now the piston curve is the straight line $x = vt$ in the first quadrant of the xt-plane. If the solution is continuous across $x = t$, the Riemann invariant (7.4.21) must hold in both $\mathcal{D}_0$ and $\mathcal{D}_1$, and this expression may be used to set $\sqrt{h} = (u + 2)/2$ in $\mathcal{D}_1$. We then compute the following value for λ_p^+ all along the piston trajectory:

$$\lambda_p^+ = u + \sqrt{h} = u + \frac{u+2}{2} = \frac{3u+2}{2} = 1 + \frac{3}{2}v > 1.$$

But the value of λ^+ in $\mathcal{D}_0$ is $\lambda_0^+ = 1$. Therefore, since $\lambda_p^+ > \lambda_0^+$, the limiting characteristics immediately cross (see (7.3.61)), and we need to introduce a bore at the origin. The fact that the values of u and h on either side of this bore remain the same implies that the bore speed must be constant (see Figure 7.11).

It is instructive to study the solution (7.4.44)–(7.4.45) for the limiting case of a weak disturbance. Assume that the piston speed v is small. Since $h_1 \to 1$ as $v \to 0$, we expect the root $h_1^{(2)}$ of (7.4.44) to be a function of v that may be expanded in powers of v in the form

$$h_1^{(2)} = 1 + c_1 v + c_2 v^2 + c_3 v^3 + O(v^4), \tag{7.4.46}$$

where the c_i are constants to be determined. Substituting this series into (7.4.44) and collecting like powers of v, we obtain

$$v^2(2c_1^2 - 2) + v^3(c_1^3 + 4c_1 c_2 - 2c_1) = O(v^4). \tag{7.4.47}$$

As discussed in Appendix A.3.4, each coefficient of v in this series must vanish, and we calculate $c_1 = 1$, $c_2 = \frac{1}{4}$, for the larger of the two positive roots. The

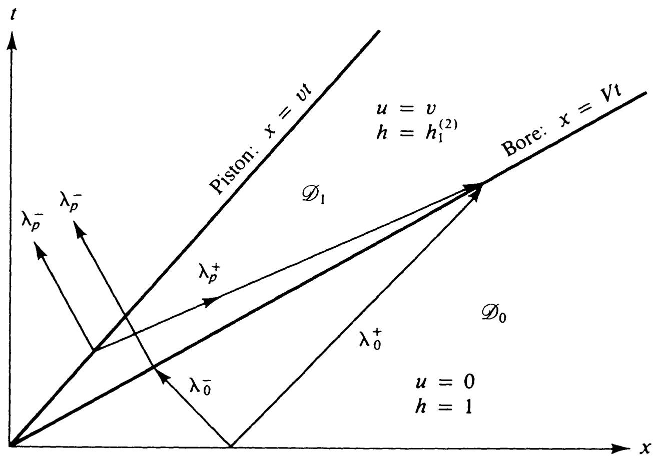

FIGURE 7.11. Constant-speed bore

result $c_1 = 1$ agrees with what we calculated in Chapter 3 for the small disturbance theory (see the second equation in (3.4.35)).

Using this result, we derive the following expansions for the bore speed, the characteristic speeds λ^+, λ^-, and the Riemann invariants S and R behind the bore:

$$V = 1 + \frac{3}{4}v + \frac{5}{32}v^2 + O(v^3), \tag{7.4.48a}$$

$$\lambda^+ = 1 + \frac{3}{2}v + O(v^3), \tag{7.4.48b}$$

$$\lambda^- = -1 + \frac{1}{2}v + O(v^3), \tag{7.4.48c}$$

$$S = -2 + O(v^3), \quad R = 2 + 2v + O(v^3), \tag{7.4.48d}$$

where $\lambda^+ = 1$, $\lambda^- = -1$, $S = -2$, and $R = 2$ ahead of the bore.

Equations (7.4.48a)–(7.4.48b) show that to $O(v)$, the bore speed is the average of the λ^+ speeds on each side. The other noteworthy observation is that although S is indeed discontinuous across the bore, the jump in its value is only $O(v^3)$; the terms proportional to v and v^2 cancel out identically in the expression (7.4.48d).

(ii) *Varible-speed bore*

The case where the piston speed is variable is the counterpart of the problem discussed in Section 7.4.2ii. Now, the piston moves to the right into the fluid at rest according to

$$x_p(\epsilon g(t), t) = \epsilon \dot{g}(t), \ t > 0, \tag{7.4.49}$$

where $g(0) = 0$, and $\epsilon g(t) > 0$; again we do not assume that ϵ is small. We do assume, however, that $\epsilon \dot{g}(0^+)$ is a nondecreasing function of t with $\epsilon \dot{g}(0^+) > 0$ to ensure that we have only one bore starting at the origin. The geometry is sketched in Figure 7.12.

An analytic solution is not possible (unless $\dot{g}$ is constant), and we outline a numerical procedure based on the Riemann invariants. First we need to establish how to start the solution from the origin. We assume that in some small neighborhood of (0, 0) the solution corresponds to a constant-speed bore associated with the initial value of the piston speed $\epsilon \dot{g}(0^+)$. Thus, in Figure 7.12, we regard $u =$ constant $= \epsilon \dot{g}(0)$ along the straight-line segment joining the origin to the known point (1, 1) on the piston curve, and we update the value of u at (1,1).

The successive points on the piston curve at which the value of u is updated are indicated in Figure 7.12 by (1,1), (2,2), . . ., (m, m) without asterisks, to indicate that these are a priori fixed points. The piston curve is approximated by straight-line segments joining these gridpoints, and we assume, for the sake of simplicity, that u varies linearly on these straight-line segments. Points on the piston curve marked by an asterisk (for example, the point (3, 3)* in Figure 7.12) represent the intersections of reflected characteristics that originated at the bore. As mentioned

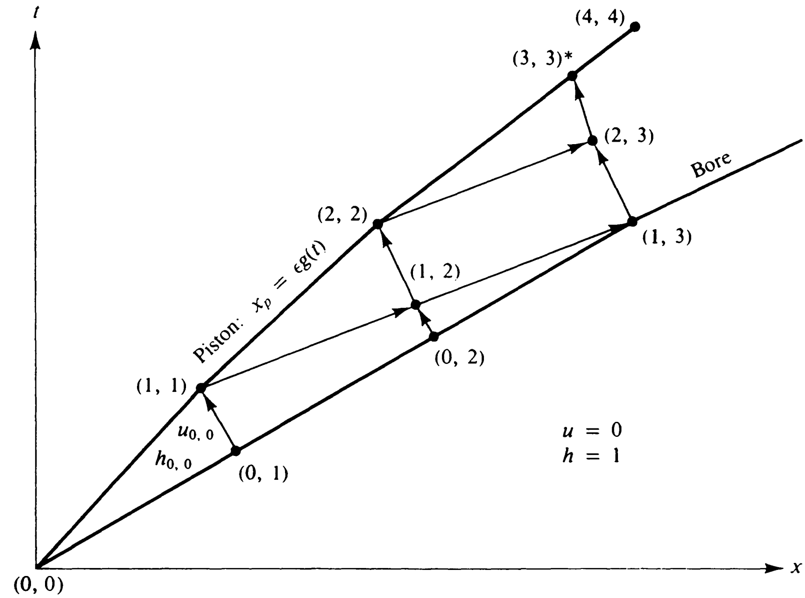

FIGURE 7.12. Variable-speed bore

earlier, we assume that the value of u at these boundary gridpoints is derived by linear interpolation from the given values at the two adjacent gridpoints.

We also approximate the bore trajectory by a piecewise linear curve and update the bore speed (as well as the values of u and h behind the bore) whenever a λ^+ characteristic originating from the piston arrives. Since the bore propagates into a region of uniform flow, the values of u and h behind the bore remain the same on each straight segment. Thus, in Figure 7.12, the gridpoints (0, 0), (0, 1), (0, 2), and (1, 3) all lie on the same straight segment associated with the starting uniform flow solution inside the triangle (0, 0), (1, 3), (1, 1). We first update u, h, and the bore speed at (1, 3), where the λ^+ characteristic from (1,2) intersects with the bore trajectory.

To derive the starting solution, we use the given value of $u_{0,0} = \epsilon \dot{g}(0^+)$ for v in (7.4.44) to compute $h_{0,0}$, and then use this in (7.4.45) to compute $V_{0,0}$. The next calculation concerns $h_{1,1}$ and the location of the point (0, 1) on the bore from which the λ^- characteristic reaches (1, 1). Thus, we have three unknowns: $x_{0,1}$, $t_{0,1}$, and $h_{1,1}$. We know $u_{0,0}$, $h_{0,0}$, $u_{1,1}$, $x_{1,1}$, $t_{1,1}$, $V_{0,0}$ and have the following three conditions:

$$u_{1,1} - 2\sqrt{h_{1,1}} = S_{0,0}, \tag{7.4.50a}$$

$$x_{1,1} - x_{0,1} = \lambda^-_{0,0}(t_{1,1} - t_{0,1}), \tag{7.4.50b}$$

$$x_{0,1} = V_{0,0}t_{0,1}, \tag{7.4.50c}$$

where

$$S_{0,0} = u_{0,0} - 2\sqrt{h_{0,0}}, \quad \lambda^-_{0,0} = u_{0,0} - \sqrt{h_{0,0}}. \tag{7.4.51}$$

Equation (7.4.50a) states that S is constant along the λ^- characteristic emerging from (0,1), where the values of u and h are the same as those at (0, 0). Equations (7.4.50b) and (7.4.50c) are the straight-line approximations of the characteristic (0, 1)-(1, 1) and bore (0, 0)-(0, 1), respectively. Solving these two equations gives

$$t_{0,1} = \frac{x_{1,1} - \lambda^-_{0,0}t_{1,1}}{V_{0,0} - \lambda^-_{0,0}}, \quad x_{0,1} = \frac{V_{0,0}(x_{1,1} - \lambda^-_{0,0}t_{1,1})}{V_{0,0} - \lambda^-_{0,0}}, \tag{7.4.52}$$

and (7.4.50a) defines $h_{1,1}$ as

$$h_{1,1} = \frac{1}{4}(u_{1,1} - S_{0,0})^2. \tag{7.4.53}$$

We can now compute $\lambda^-_{1,1}$, $\lambda^+_{1,1}$, $R_{1,1}$, and $S_{1,1}$ to proceed with the calculation to adjacent points. Note, incidentally, that for this problem the condition $\lambda^-_{m,m} < u_{m,m}$ needed to ensure that the boundary curve is timelike is automatically satisfied, because $\lambda^-_{m,m} = u_{m,m} - \sqrt{h_{m,m}} < u_{m,m}$.

We observe from Figure 7.12 that the domain of interest can be covered entirely by λ^- characteristics. Either these originate from a corner in the bore trajectory (and hence end up, in general, somewhere between two fixed boundary points), or we require them to end up at a fixed boundary point, and therefore they originate somewhere on a straight segment of bore. We shall next illustrate the calculation details for both these types of points using the sequence (0, 2), (1, 2), and (2, 2), followed by the sequence (1, 3), (2, 3), and (3, 3)*. The calculations for the remainder of the solution domain belong to one or the other of these two categories.

Along the sequence of points (0, 2), (1, 2), and (2, 2), the unknowns are $x_{0,2}$, $t_{0,2}$, $x_{1,2}$, $t_{1,2}$, $u_{1,2}$, $h_{1,2}$, and $h_{2,2}$. We know the values of $u_{0,2} = u_{0,0}$, $h_{0,2} = h_{0,0}$, $V_{0,2} = V_{0,0}$, $x_{1,1}$, $t_{1,1}$, $u_{1,1}$, $h_{1,1}$, $x_{2,2}$, $t_{2,2}$, and $u_{2,2}$. Thus, we need seven independent conditions to define our seven unknowns.

Two of these independent conditions are the two Riemann invariants that reach (1, 2):

$$R_{1,2} = R_{1,1}, \quad S_{1,2} = S_{0,2} = S_{0,0}. \tag{7.4.54}$$

They give $u_{1,2} = u_{1,1}$ and $h_{1,2} = h_{1,1}$, which imply that $\lambda^-_{1,2} = \lambda^-_{1,1}$ and $\lambda^+_{1,2} = \lambda^+_{1,1}$. Two other conditions locate (1, 2) from (1, 1) via a λ^+ characteristic, and (2, 2) from (1, 2) via a λ^- characteristic; that is,

$$x_{1,2} = x_{1,1} + \lambda^+_{1,1}(t_{1,2} - t_{1,1}), \quad x_{2,2} = x_{1,2} + \lambda^-_{1,1}(t_{2,2} - t_{1,2}).$$

Solving these two equations defines $x_{1,2}$ and $t_{1,2}$ in terms of known quantities in the form

$$x_{1,2} = \frac{1}{\lambda^+_{1,1} - \lambda^-_{1,1}}\left[\lambda^+_{1,1}(x_{2,2} - \lambda^-_{1,1}t_{2,2}) - \lambda^-_{1,1}(x_{1,1} - \lambda^+_{1,1}t_{1,1})\right], \tag{7.4.55a}$$

$$t_{1,2} = \frac{1}{\lambda_{1,1}^+ - \lambda_{1,1}^-}\left[x_{2,2} - \lambda_{1,1}^- t_{2,2} - x_{1,1} + \lambda_{1,1}^+ t_{1,1}\right]. \tag{7.4.55b}$$

The fifth condition is the invariance of S along (1, 2)-(2, 2), which gives

$$h_{2,2} = \frac{1}{4}(u_{2,2} - S_{0,0})^2. \tag{7.4.56}$$

The sixth and seventh conditions define the slopes of the segments (0, 2)-(1, 2) and (0, 1)-(0, 2):

$$x_{1,2} - x_{0,2} = \lambda_{0,2}^-(t_{1,2} - t_{0,2}) = \lambda_{0,0}^-(t_{1,2} - t_{0,2}),$$
$$x_{0,2} = V_{0,0}t_{0,2}.$$

Solving these gives

$$x_{0,2} = \frac{V_{0,0}(x_{1,2} - \lambda_{0,0}^- t_{1,2})}{V_{0,0} - \lambda_{0,0}^-}, \quad t_{0,2} = \frac{x_{1,2} - \lambda_{0,0}^- t_{1,2}}{V_{0,0} - \lambda_{0,0}^-}. \tag{7.4.57}$$

Such a sequence of calculations can always be implemented along a λ^- characteristic that terminates at a fixed piston gridpoint. Notice that this sequence of calculations always breaks up into a subsequence involving at most the solution of two linear algebraic equations. The converse approach, which will generally not terminate on a fixed gridpoint, is to start the sequence of calculations from the bore. We illustrate this for the points (1, 3), (2, 3), and (3, 3)*.

We begin by fixing the point (1, 3) according to $x_{1,3} = V_{0,0}t_{1,3}$, since this point is the endpoint of the first bore segment. The second condition is that (1,3) lies on the λ^+ characteristic from (1, 2); that is, $x_{1,3} = x_{1,2} + \lambda_{1,2}^+(t_{1,3} - t_{1,2})$. Having already computed $x_{1,2}$, $t_{1,2}$ and $\lambda_{1,2}^+ = u_{1,2} + \sqrt{h_{1,2}}$, we solve these two equations for $x_{1,3}$, and $t_{1,3}$ in the form (see (7.4.57))

$$x_{1,3} = \frac{V_{0,0}(x_{1,2} - \lambda_{1,2}^+ t_{1,2})}{V_{0,0} - \lambda_{1,2}^+}, \quad t_{1,3} = \frac{x_{1,2} - \lambda_{1,2}^+ t_{1,2}}{V_{0,0} - \lambda_{1,2}^+}. \tag{7.4.58}$$

Next we solve the pair of nonlinear algebraic equations

$$u_{1,3} + 2\sqrt{h_{1,3}} = R_{1,2} = R_{1,1}, \tag{7.4.59a}$$

$$u_{1,3} = (h_{1,3} - 1)\sqrt{\frac{h_{1,3} + 1}{2h_{1,3}}}, \tag{7.4.59b}$$

for $u_{1,3}$ and $h_{1,3}$. Equation (7.4.59b) is just the expression we obtain by solving (7.4.2) for the flow speed behind the bore in terms of the height (see (5.3.47a)). One approach for solving this pair is to guess a value of $u_{1,3}^{(1)}$ and calculate $h_{1,3}^{(1)}$ from (7.4.59a),

$$h_{1,3}^{(1)} = \frac{1}{4}(R_{1,1} - u_{1,3}^{(1)})^2. \tag{7.4.60}$$

Then substitute this result into (7.4.59b) to derive

$$\bar{u}_{1,3}^{(1)} = (h_{1,3}^{(1)} - 1)\sqrt{\frac{h_{1,3}^{(1)} + 1}{2h_{1,3}^{(1)}}}. \tag{7.4.61}$$

The iteration converges if we let $u_{1,3}^{(2)}$ be the average

$$u_{1,3}^{(2)} = \frac{1}{2}(u_{1,3}^{(1)} + \bar{u}_{1,3}^{(1)}), \tag{7.4.62}$$

and repeat the process. Once the values of $u_{1,3}$ and $h_{1,3}$ have converged, we compute $\lambda_{1,3}^-$, $\lambda_{1,3}^+$, $S_{1,3}$, $R_{1,3}$; the updated bore speed is given by (7.4.45).

We calculate $u_{2,3}$ and $h_{2,3}$ from

$$u_{2,3} = \frac{1}{2}(S_{1,3} + R_{2,2}), \quad h_{2,3} = \frac{1}{16}(R_{2,2} - S_{1,3})^2. \tag{7.4.63}$$

The values of $x_{2,3}$ and $t_{2,3}$ follow from the solution of

$$x_{2,3} = x_{1,3} + \lambda_{1,3}^-(t_{2,3} - t_{1,3}), \tag{7.4.64a}$$

$$x_{2,3} = x_{2,2} + \lambda_{2,2}^+(t_{2,3} - t_{2,2}). \tag{7.4.64b}$$

Next, we fix the location of the terminal point (3, 3)* using the two conditions

$$x_{3,3} = x_{2,3} + \lambda_{2,3}^-(t_{3,3} - t_{2,3}), \tag{7.4.65a}$$

$$\frac{x_{3,3} - x_{2,2}}{t_{3,3} - t_{2,2}} = \frac{x_{4,4} - x_{2,2}}{t_{4,4} - t_{2,2}}, \tag{7.4.65b}$$

where (7.4.65b) is just the equation for a point $(x_{3,3}, t_{3,3})$ lying on the straight line joining the two fixed points (2, 2) and (4, 4).

Finally, we compute $u_{3,3}$ by linear interpolation:

$$\frac{u_{3,3} - u_{2,2}}{t_{3,3} - t_{2,2}} = \frac{u_{4,4} - u_{2,2}}{t_{4,4} - t_{2,2}}. \tag{7.4.66a}$$

We use this in

$$h_{3,3} = \frac{1}{4}(S_{1,3} - u_{3,3})^2 \tag{7.4.66b}$$

to obtain $h_{3,3}$.

Table 7.1 gives the numerical results that we obtain for the case

$$x_p = \frac{\sqrt{3}}{2}(t + t^2) \tag{7.4.67}$$

for fixed gridpoints at $t = 0, 0.1, 0.2, 0.3, \ldots$ along the piston curve.

(iii) *Dam-breaking problem with water downstream*

The dam-breaking problem discussed in Section 7.4.2i must be modified if there is a body of quiescent water of height $a < 1$ downstream ($x > 0$). The initial

TABLE 7.1. Numerical Solution for Piston Motion (7.4.67)

Point	V	R	S	u	h	x	t	λ^+	λ^-
(0, 0)	1.732	3.694	−1.962	0.866	2.000	0	0	2.280	−0.548
(0, 1)	1.732	3.694	−1.962	0.866	2.000	0.114	0.066	2.280	−0.548
(1, 1)	–	4.040	−1.962	1.039	2.252	0.095	0.100	2.540	−0.462
(0, 2)	1.732	3.694	−1.962	0.866	2.000	0.239	0.138	2.280	−0.548
(1, 2)	–	4.040	−1.962	1.039	2.252	0.230	0.153	2.540	−0.462
(2, 2)	–	4.638	−1.962	1.212	2.519	0.208	0.200	2.799	−0.375
(1, 3)	1.901	4.040	−1.940	1.050	2.235	0.341	0.197	2.545	−0.445
(2, 3)	–	4.386	−1.940	1.223	2.501	0.321	0.241	2.804	−0.358
(3, 3)*	–	4.386	−1.940	1.346	2.699	0.307	0.277	2.988	−0.297

conditions that replace (7.4.10) are now

$$u(x,0) = 0, \quad h(x,0) = \begin{cases} 1 & \text{if } x < 0, \\ a & \text{if } x > 0. \end{cases} \tag{7.4.68}$$

The gas-dynamic counterpart of this problem is the shock-tube problem where a diaphragm separating gases of different densities is suddenly removed at $t = 0$. This problem is discussed in Section 7.5.1v.

We recall that for $a = 0$, we have a discontinuity in u propagating along the bounding λ^+ characteristic, but h is continuous across this characteristic. Now, for $t = 0$, $\lambda^+ = \sqrt{a} < 1$ for $x > 0$, and $\lambda^+ = 1$ for $x = 0$. Actually, we shall see from our results that $\lambda^+ = u_2 + \sqrt{h_2} > 1$ in $\mathcal{D}_2$. Therefore, the λ^+ characteristics emerging from either side of the origin immediately cross, and a bore must start out from there (see Figure 7.13).

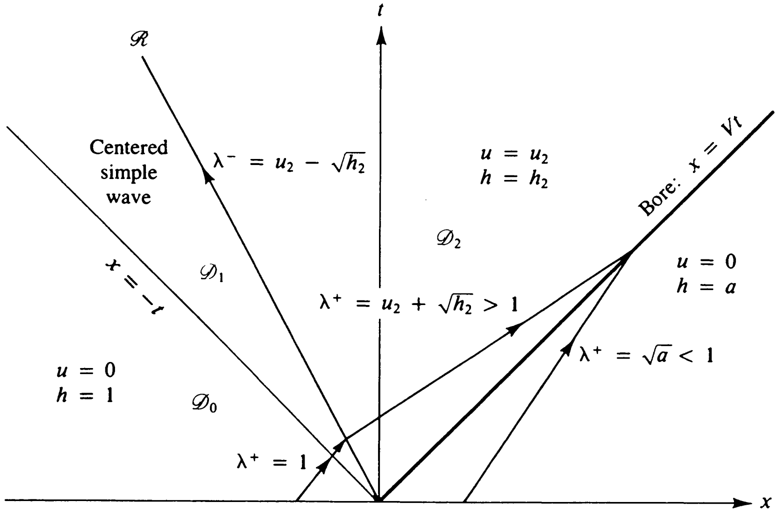

FIGURE 7.13. Dam-breaking problem with water downstream

We also know that the solution in $\mathcal{D}_0$ to the left of $x = -t$ is the quiescent state $u = 0, h = 1$. Therefore, as in the case $a = 0$, the Riemann invariant

$$u + 2\sqrt{h} = 2 \qquad (7.4.69)$$

must hold in the entire domain $\mathcal{D}_0$, $\mathcal{D}_1$, and $\mathcal{D}_2$ to the left of the bore, covered by the λ^+ characteristics emerging from $t = 0, x < 0$. This means that the characteristic $x = -t$ is the left boundary of the centered simple wave domain $\mathcal{D}_1$, which terminates along some ray $\mathcal{R}$ from the origin. In $\mathcal{D}_1$ the solution is exactly the one given in (7.4.13). The ray $\mathcal{R}$ is the left boundary of a uniform flow domain $\mathcal{D}_2$, which extends up to the bore and in which $u = u_2 =$ constant, and $h = h_2 =$ constant. Once $\mathcal{R}$ is identified, we will know u_2 and h_2, since these values must be the same as those predicted by the simple wave solution (7.4.13) on its right boundary $\mathcal{R}$.

The crucial question is to identify $\mathcal{R}$, and we do so by combining the information provided by the two bore conditions (7.4.2) with the requirement (7.4.69), which must persist into $\mathcal{D}_2$. For our case, the bore conditions are

$$V = \frac{u_2 h_2}{h_2 - a}, \quad V = \frac{u_2^2 h_2 + h_2^2/2 - a^2/2}{u_2 h_2}. \qquad (7.4.70)$$

Eliminating V gives

$$h_2^3 - a h_2^2 - (a^2 + 2a u_2^2) h_2 + a^3 = 0. \qquad (7.4.71)$$

Now, using (7.4.69) to express u_2 in terms of h_2 in (7.4.71) gives

$$h_2^3 - 9a h_2^2 + 16a h_2^{3/2} - (a^2 + 8a) h_2 + a^3 = 0. \qquad (7.4.72)$$

We note that as $a \to 1$, (7.4.72) has a root $h_2 = 1$, as expected. Also, as $a \to 0$, $h_2 \to 0$, in agreement with the result shown in Figure 7.8. The appropriate root of (7.4.72) must be larger than a for $a > 0$, and we compute this root using Newton's method for a range of values of a. Having h_2, we obtain u_2 from (7.4.69) and the slope of the ray $\mathcal{R}$ from

$$\lambda^- = u_2 - \sqrt{h_2}. \tag{7.4.73}$$

The bore speed is defined by either equation in (7.4.70). These results are listed in Table 7.2 for a range of values of a.

In the last two columns of Table 7.2, we compare our nonlinear results with those obtained in Chapter 3 using the linear theory (see Figures 3.16–3.18). We set $\tilde{h}_2 = \tilde{u}_2 = \tilde{u}_1 = 0$ in the results given for u and h and consider region (5) of Figure 3.17 to obtain

$$u = \frac{\epsilon}{2}\tilde{h}_1, \; h = a + \frac{\epsilon}{2}\tilde{h}_1, \tag{7.4.74}$$

corresponding to the normalized initial conditions

$$u(x,0) = 0, \;\; h(x,0) = \begin{cases} a & \text{if } x > 0, \\ a + \epsilon\tilde{h}_1 & \text{if } x < 0. \end{cases} \tag{7.4.75}$$

The normalization used here has $a + \epsilon\tilde{h} = 1$. Therefore, we must set $\epsilon\tilde{h} = 1 - a$ in the linear results (7.4.74) for comparison. This gives

$$u_\ell = \frac{1}{2}(1-a), \;\; h_\ell = \frac{1}{2}(1+a), \tag{7.4.76}$$

where we use the subscript ℓ to indicate the results of the linear theory.

As seen in Table 7.2, the linear result for u and h deteriorate rapidly for $a > \frac{1}{2}$. This is not surprising because $(1-a)/a$ measures the perturbations, and this parameter tends to 1 as $a \to \frac{1}{2}$. In the linear theory, the domain $\mathcal{D}_1$ gets squeezed into the characteristic $x = -t$, across which u and h jump from the values given in (7.4.76) to the equilibrium values $u = 0$, $h = 1$. Although this result is correct in the limit $a \to 1$, the linear theory does not give an indication of the simple wave behavior. In Figure 7.14 we compare the "exact" profile for h with that predicted by the linear theory (dashed profile) at $t = 1$ and the extreme case $a = 0.6$ in order to highlight the features pointed out earlier.

Problems

7.4.1 Calculate u and h as functions of x and t for the piston problem discussed in Section 7.4.2ii for the special case where $\epsilon = \frac{1}{2}$ and

$$g(t) = \begin{cases} \frac{t}{2} + \frac{t^2}{4} & \text{if } 0 < t \le 1, \\ t - \frac{1}{4} & \text{if } 1 \le t. \end{cases} \tag{7.4.77}$$

TABLE 7.2. Solution of the Initial-Value Problem (7.4.68)

a	h_2	u_2	V	λ^-	h_ℓ	u_ℓ
0.9	0.94933	0.05132	0.98763	−0.92302	0.95	0.05
0.8	0.89715	0.10564	0.97555	−0.84154	0.90	0.10
0.7	0.84309	0.16360	0.96394	−0.67949	0.85	0.15
0.6	0.78661	0.22618	0.95340	−0.56043	0.80	0.20
0.5	0.72692	0.29480	0.94437	−0.43212	0.75	0.25
0.4	0.66268	0.37190	0.93822	−0.29078	0.70	0.30
0.3	0.59143	0.46192	0.93742	−0.12951	0.65	0.35
0.2	0.50787	0.57470	0.94804	0.06683	0.60	0.40
0.1	0.39617	0.74116	0.99141	0.34499	0.55	0.45
0.06	0.33080	0.84970	1.03796	0.51890		
0.03	0.25811	0.98390	1.11330	0.72579		
0.01	0.17118	1.17252	1.24527	1.0013		
0	0	2	2	2		

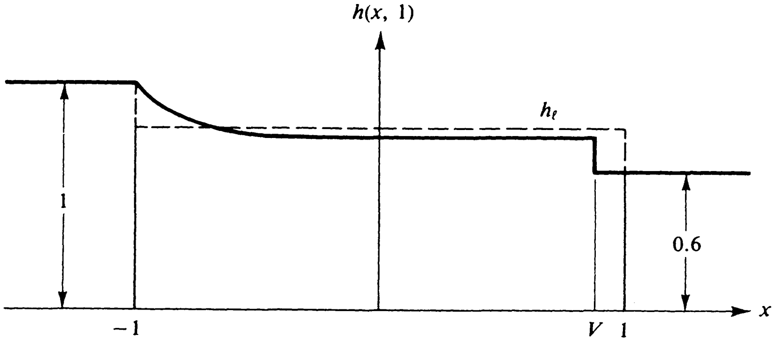

FIGURE 7.14. Comparison of nonlinear and linear theories for dam-breaking problem with water downstream

7.4.2 Assume that the solution in region (3) of Figure 7.10 has been worked out and, in particular, that we know the characteristic arc BD in parametric form,

$$x = x_0(\sigma),\ t = t_0(\sigma), \tag{7.4.78}$$

and that we also know u and h on BD in the form,

$$u = u_0(\sigma), \quad h = h_0(\sigma). \tag{7.4.79}$$

a. Derive the solution in region (6) of Figure 7.10 in parametric form.
b. Use the results in part (a) to compute the solution in (9).

7.5 Compressible Flow Problems

7.5.1 One-Dimensional Unsteady Flow

We return to the problem of one-dimensional unsteady compressible flow for an ideal, inviscid, non-heat-conducting gas considered in Chapters 3 and 5 to illustrate a second particular application area for the general results derived in Section 7.3. For nonisentropic flows, we must keep track of three dependent variables, as in our calculations in Section 7.2.2. Here, our first goal is to establish the conditions under which one can describe the flow in terms of two variables; in this case, the results are analogous to those worked out for shallow water-flow in Section 7.4.

(i) *Problem formulation*

We begin as in Section 7.4 with a summary of the governing equations. Using dimensionless variables, the divergence relations for mass, momentum, and energy conservation are (see (5.3.50))

$$\rho_t + (u\rho)_x = 0, \tag{7.5.1a}$$

$$(\rho u)_t + (\rho u^2 + p/\gamma)_x = 0, \tag{7.5.1b}$$

$$\left(\frac{\rho u^2}{2} + \frac{p}{\gamma(\gamma - 1)}\right)_t + \left(\frac{\rho u^3}{2} + \frac{pu}{\gamma - 1}\right) = 0. \tag{7.5.1c}$$

The temperature θ for an ideal gas is defined in terms of the pressure and density by the equation of state (3.3.16).

The shock conditions associated with (7.5.1) are

$$V[\rho] = [\rho u], \tag{7.5.2a}$$

$$V[\rho u] = [\rho u^2 + p/\gamma], \tag{7.5.2b}$$

$$V\left[\frac{\rho u^2}{2} + \frac{p}{\gamma(\gamma - 1)}\right] = \left[\frac{\rho u^3}{2} + \frac{pu}{\gamma - 1}\right], \tag{7.5.2c}$$

where $V \equiv (dx/dt)$ is the shock speed.

For strict solutions, equations (7.5.1) simplify to

$$\rho_t + (u\rho)_x = 0, \tag{7.5.3a}$$

$$u_t + uu_x + \frac{1}{\gamma\rho} p_x = 0, \tag{7.5.3b}$$

$$\left(\frac{p}{\rho^\gamma}\right)_t + u\left(\frac{p}{\rho^\gamma}\right)_x = 0. \tag{7.5.3c}$$

Again, we point out that the factor $1/\gamma$ multiplying the dimensionless pressure in the above equations is due to our choice of the ambient speed of sound $a_0 \equiv \sqrt{\gamma p_0/\rho_0}$ as the velocity scale.

Equation (7.5.3c) states that (p/ρ^γ) remains constant on particle paths; these are curves in the xt-plane along which $(dx/dt) = u$. Since the entropy is a function of p/ρ^γ, (7.5.3c) also implies that the entropy remains constant on particle paths. But (7.5.3c) is valid only if the solution is strict; that is, if the flow is free of shocks. Therefore, the statement that the entropy remains constant along particle paths is correct only as long as these paths do not cross a shock. For the special case where p/ρ^γ is constant over some intitial arc, a shock-free flow has the same entropy in the domain swept out by all the particle paths emerging from this arc. This is called isentropic flow, and if the ambient state $u = 0$ is prescribed along this initial arc, we have $p/\rho^\gamma = 1$ for our choice of dimensionless variables. Therefore, we can replace p with ρ^γ in (7.5.3b) to obtain (see (3.3.21))

$$u_t + uu_x + \rho^{\gamma-2}\rho_x = 0. \tag{7.5.4}$$

The pair of equations (7.5.3a) and (7.5.4) govern u and ρ for isentropic flow. An alternative description involves the dimensionless speed of sound (see (3.3.22))

$$a \equiv \left(\frac{p}{\rho}\right)^{1/2} = \rho^{(\gamma-1)/2} \tag{7.5.5}$$

instead of ρ as the second variable, and the governing equations become (see (3.3.23))

$$u_t + uu_x + \frac{2}{\gamma - 1} aa_x = 0, \tag{7.5.6a}$$

$$a_t + \frac{\gamma - 1}{2} au_x + ua_x = 0. \tag{7.5.6b}$$

(ii) *Flows with shocks*

Under what conditions is it correct to use the pair or equations (7.5.3a), (7.5.4) or (7.5.6a)–(7.5.6b) for describing flows that contain shocks?

To begin with, consider a flow in which we have one constant-speed shock; that is, the flow is uniform on either side of the shock, so entropy is a different constant on either side. We can still use the isentropic equations to describe the flow on either side of the shock as long as we relate the two states through the shock conditions (7.5.2). This case was discussed in Section 5.3.4iii, where we viewed the solution in a reference frame moving with the speed of the flow in front of the shock. Thus, our dimensionless variables have $u = 0$, $p = 1$, $\rho = 1$ in front of the shock. The shock speed V, density ρ, and pressure p behind the shock can then be expressed in terms of the speed $u > 0$ behind the shock in the form (see (5.3.54))

$$V = \frac{\gamma + 1}{4} u + \frac{1}{4} \left[(\gamma + 1)^2 u^2 + 16 \right]^{1/2}, \tag{7.5.7a}$$

$$\rho = \frac{4 + u[(\gamma + 1)^2 u^2 + 16]^{1/2} + (\gamma + 1) u^2}{4 + 2(\gamma - 1) u^2}, \tag{7.5.7b}$$

$$p = 1 + \frac{\gamma(\gamma + 1)}{4} u^2 + \frac{\gamma u}{4} [(\gamma + 1)^2 u^2 + 16]^{1/2}. \tag{7.5.7c}$$

Equations (7.5.7) also define the local conditions behind a variable-speed shock propagating into a uniform region. Such a flow could be generated by impulsively setting a piston in motion with a prescribed variable speed $v(t)$ into a gas at rest. In this case, $V(t)$ is the local shock speed and $\rho(t)$, $p(t)$ are the local values just behind the shock. It must be kept in mind that now u, the flow speed just behind the shock, is a function of t that is not equal to $v(t)$ unless v is constant. In Section 5.3.4iii we noted the remarkable fact that whereas $p - 1$, $\rho - 1$, and $V - 1$ are all of order u for small u, the entropy change across a shock is of order u^3. This means that if the piston speed is of order ϵ (see (3.3.17)), u will also be of order ϵ, and we can still use (7.5.6) to describe the flow behind the shock correctly up to terms of order ϵ^2. A more precise statement is that the $O(\epsilon)$ and $O(\epsilon^2)$ perturbation equations that we derive from (7.5.6) remain correct; we need to account for entropy changes only in the equations governing the $O(\epsilon^3)$ terms. In Section 8.3.4 we shall show for a similar problem that the governing equations correct to $O(\epsilon^2)$ are sufficient to define the solution to $O(\epsilon)$ in the far field (that is, for $x = O(\epsilon^{-1})$ and $t = O(\epsilon^{-1})$).

Now suppose that the piston speed is not small, but the piston acceleration (or deceleration) is small. The entropy jump across the shock is not negligible, but the difference in the entropy between different particle paths behind the shock is of third order. We can therefore still use (7.5.6) to describe the flow behind a strong shock as long as the curvature of this shock is small.

In view of this wide range of applicability for (7.5.6), and the mathematical similarity between this system and (7.4.3) that we have discussed thoroughly, we shall briefly review the corresponding results and outline some sample problems with little further discussion.

(iii) *Characteristic coordinates; the hodograph transformation; Riemann invariants*

The vector **u** and matrix $\{A_{ij}\}$ in (7.2.2) are now

$$\mathbf{u} = (u, a), \quad \{A_{ij}\} = \begin{pmatrix} u & \dfrac{2}{\gamma - 1}a \\ \dfrac{\gamma - 1}{2}a & u \end{pmatrix}, \tag{7.5.8}$$

so the eigenvalues of the $\{A_{ij}\}$ matrix are

$$\lambda_1 = u + a, \quad \lambda_2 = u - a. \tag{7.5.9}$$

We use the following $\{\ell_{ij}\}$ matrix satisfying (7.2.5) for this case,

$$\{\ell_{ij}\} = -\frac{1}{2a}\begin{pmatrix} \dfrac{\gamma - 1}{2} & 1 \\ \dfrac{\gamma - 1}{2} & -1 \end{pmatrix}, \tag{7.5.10}$$

and obtain the system (7.3.5) and (7.3.7) in the form

$$x_\eta - (u + a)t_\eta = 0, \quad x_\xi - (u - a)t_\xi = 0, \tag{7.5.11}$$

$$\frac{\gamma - 1}{2}u_\eta + a_\eta = 0, \quad \frac{\gamma - 1}{2}u_\xi - a_\xi = 0. \tag{7.5.12}$$

In (7.5.11)–(7.5.12) and for the remainder of this chapter, we shall use the same lowercase symbols for functions of (x, t), (ξ, η), (u, v), or (r, s) to simplify the notation. The particular choice of independent variables will be clear from the context.

The hodograph form of (7.5.6) is

$$-x_a + ut_a - \frac{2}{\gamma - 1}at_u = 0, \quad x_u + \frac{\gamma - 1}{2}at_a - ut_u = 0. \tag{7.5.13}$$

The characteristics of (7.5.13) have slopes $\mu_1 \equiv (da/du) = (\gamma - 1)/2$ and $\mu_2 \equiv (da/du) = -(\gamma - 1)/2$. Therefore, integrating these simple expressions gives the Riemann invariants

$$r = a + \frac{\gamma - 1}{2}u, \quad s = a - \frac{\gamma - 1}{2}u. \tag{7.5.14}$$

The system (7.5.11)–(7.5.12) thus reduces to the statements

$$a + \frac{\gamma - 1}{2} u = \text{ constant on } \xi = \text{ constant,} \tag{7.5.15a}$$

$$a - \frac{\gamma - 1}{2} u = \text{ constant on } \xi = \text{ constant.} \tag{7.5.15b}$$

(iv) *Centered simple wave*

Consider the analogue for the problem discussed in Section 7.4.2ii with solution given by (7.4.27) and Figure 7.9b. Now we have a gas at rest ($u = 0, a = 1$), and we study the special case of a piston that is impulsively retracted (to the left) with constant speed v. The solution domains in the xt-plane are qualitatively the same as those in Figure 7.9b. The two conditions that determine the flow in the centered simple wave region are

$$a - \frac{\gamma - 1}{2} u = 1, \quad \frac{x}{t} = u + a. \tag{7.5.16}$$

The first equation is the Riemann invariant along the λ_2 characteristics, and the second equation relates the λ_1 characteristic slopes to the rays from the origin. Solving these expressions for u and a gives

$$u = \frac{2}{\gamma + 1}\left(\frac{x}{t} - 1\right), \quad a = \frac{x}{t}\left(\frac{\gamma - 1}{\gamma + 1}\right) + \frac{2}{\gamma + 1} \tag{7.5.17}$$

for $[1 - v(\gamma + 1)/2]t < x < t$, where the boundary ray $x/t = 1 - v(\gamma + 1)/2$ is obtained by setting $u = -v$ in the first equation in (7.5.17). To the left of this ray we have the uniform flow $u = -v, a = 1 - (\gamma - 1)v/2$. For $x > t$ we have the ambient state $u = 0, a = 1$.

(v) *The shock-tube problem*

In this problem we have a stationary gas at a given pressure p_1 and density ρ_1 (hence temperature $\theta_1 = p_1/R\rho_1$) in the domain $x < 0$, which is separated by a thin diaphragm from $x > 0$, where there is a stationary gas with properties $p_0 < p_1, \rho_0 < \rho_1$. The flow is initiated by suddenly removing the diaphragm at time $t = 0$. For simplicity we shall discuss the case where the two gases are the same, and we nondimensionalize pressures, densities, and speeds using p_0, ρ_0, and $a_0 \equiv (\gamma p_0/\rho_0)^{1/2}$, respectively. Thus, we have the dimensionless quantities $u_0 = 0$, $p_0 = 1$, $\rho_0 = 1$, $a_0 = 1$, and $p_1 > 1$, $\rho_1 > 1$. The case where the gases are different is easy to work out, but it is more convenient to use dimensional quantities in the calculations (for example, see Section 6.13 of [42]).

Although the shock-tube problem is analogous to the problem of a dam breaking over water downstream (see Section 7.4.3iii), there is one important distinction due to the fact that we now have to keep track of three variables (u, p, ρ) here instead of two (u, h). When the diaphragm is removed, the interface separating the two gases can no longer support a pressure difference while this interface moves to the

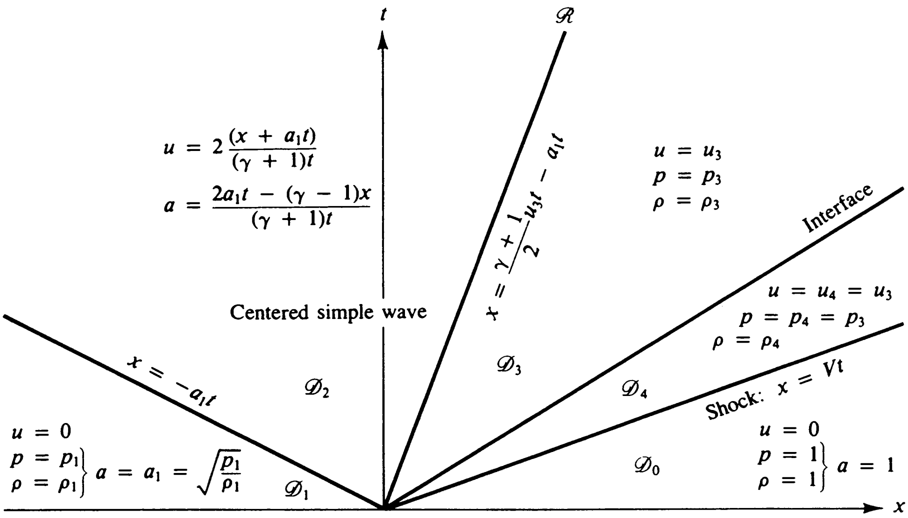

FIGURE 7.15. The shock-tube problem

right at the constant speed of the gases on either side. The flow therefore consists of the five regions shown in Figure 7.15.

We have a shock propagating with speed V to the right into the gas at rest, and we have a centered simple wave propagating to the left into the high-pressure state. In addition, and unlike the dam-breaking problem, we also have the interface that is the boundary between the two initial states propagating to the right with speed $u = u_3 = u_4$. Across the interface $p_3 = p_4$, but $\rho_3 \neq \rho_4$ (hence $\theta_3 \neq \theta_4$). The interface, also called the *contact surface*, is kinematically equivalent to a piston moving with speed u_3 in the sense that it produces a shock to its right and a centered simple wave to its left.

The solution in the centered simple wave region $\mathcal{D}_2$ is obtained from the Riemann invariant

$$a + \frac{\gamma - 1}{2}u = \text{constant} = a_1 = \left(\frac{p_1}{\rho_1}\right)^{1/2} \tag{7.5.18}$$

and the equation for the rays through the origin

$$\frac{x}{t} = u - a. \tag{7.5.19}$$

Solving these two equations for u and a gives

$$u = \frac{2(x + a_1t)}{(\gamma + 1)t}, \quad a = \frac{2a_1t - (\gamma - 1)x}{(\gamma + 1)t}. \tag{7.5.20}$$

Again, (7.5.20) is valid to the right of the known boundary $x = -a_1t$, but at this stage we do not know the right boundary ray $\mathcal{R}$ for $\mathcal{D}_2$. To determine $\mathcal{R}$, we must establish the uniform flow solution $u = u_3$, $p = p_3$, $\rho = \rho_3$ in $\mathcal{D}_3$. To do so, we first list all the information for flow quantities in the various domains.

First, the Riemann invariant (7.5.18) holds on $\mathcal{R}$; therefore,

$$a_3 + \frac{\gamma - 1}{2} u_3 = a_1. \tag{7.5.21a}$$

Secondly, p/ρ^γ is constant throughout $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$; in particular,

$$\frac{p_1}{\rho_1^\gamma} = \frac{p_3}{\rho_3^\gamma}. \tag{7.5.21b}$$

The speed of sound in $\mathcal{D}_3$ and $\mathcal{D}_4$ is given by

$$a_3^2 = \frac{p_3}{\rho_3}, \quad a_4^2 = \frac{p_4}{\rho_4}. \tag{7.5.22}$$

The interface conditions are

$$u_3 = u_4, \quad p_3 = p_4. \tag{7.5.23}$$

The density and pressure in $\mathcal{D}_4$ behind the shock are given by (7.5.7b)–(7.5.7c)

$$\rho_4 = \frac{4 + u_4[(\gamma + 1)^2 u_4^2 + 16]^{1/2} + (\gamma + 1)u_4^2}{4 + 2(\gamma - 1)u_4^2}, \tag{7.5.24a}$$

$$p_4 = 1 + \frac{\gamma(\gamma + 1)}{4} u_4^2 + \frac{\gamma u_4}{4}[(\gamma + 1)^2 u_4^2 + 16]^{1/2}. \tag{7.5.24b}$$

It follows from (7.5.22) and (7.5.23), that

$$a_4^2 = \frac{a_3^2 \rho_3}{\rho_4}. \tag{7.5.25}$$

But using (7.5.21a) and $u_3 = u_4$ we have

$$a_3^2 = \left(a_1 - \frac{\gamma - 1}{2} u_4\right)^2. \tag{7.5.26}$$

Also, according to (7.5.21b) and $p_3 = p_4$ we have

$$\rho_3 = \rho_1 \left(\frac{p_4}{p_1}\right)^{1/\gamma}. \tag{7.5.27}$$

Therefore, (7.5.25) becomes

$$a_4^2 = \frac{\rho_1}{\rho_4}\left(\frac{p_4}{p_1}\right)^{1/\gamma}\left(a_1 - \frac{\gamma - 1}{2} u_4\right)^2. \tag{7.5.28a}$$

We also have the expression (7.5.24b) for p_4, and the second equation in (7.5.22) becomes

$$a_4^2 = \frac{1}{\rho_4}\left\{1 + \frac{\gamma(\gamma + 1)}{4} u_4^2 + \frac{\gamma u_4}{4}\left[(\gamma + 1)^2 u_4^2 + 16\right]^{1/2}\right\}. \tag{7.5.28b}$$

Equating these two expressions for a_4 and simplifying gives the following implicit relation defining u_4 in terms of known quantities:

$$\frac{\rho_1^{(\gamma-1)/2\gamma}}{a_1^{1/\gamma}}\left(a_1 - \frac{\gamma-1}{2}u_4\right)$$
$$= \left\{1 + \frac{\gamma(\gamma+1)}{4}u_4^2 + \frac{\gamma u_4}{4}\left[(\gamma+1)^2u_4^2 + 16\right]^{1/2}\right\}^{(\gamma-1)/2\gamma}. \quad (7.5.29)$$

Thus, given ρ_1 and a_1 (or p_1) we can calculate u_4 from (7.5.29). Here we choose to express the final result for u_4 in terms of given values of ρ_1 and a_1. The corresponding result can also be derived for p_4 in terms of p_1 and a_1 in a somewhat simpler form (see Section 6.13 of [42]). Once u_4 is known, we compute the shock speed from (7.5.7a). The speed of the interface is u_4, and the ray $\mathcal{D}$ is obtained from the first equation (7.5.20) with $u = u_4$; that is,

$$\frac{x}{t} = \frac{\gamma+1}{2}u_4 - a_1. \quad (7.5.30)$$

The density ρ_4 behind the shock is given by (7.5.24a) and a_3 by (7.5.26), and so on. This completes the solution.

The following numerical example illustrates our results. We choose $p_1 = 40$ and $\rho_1 = 10$, that is, $a_1 = 2$, and find that (7.5.29) gives $u_4 = u_3 = 1.9756$. With this value of u_4 we compute $\rho_4 = 3.5974$ from (7.5.24a), $p_4 = p_3 = 8.5679$ from (7.5.24b), and $V = 2.7362$ from (7.5.7a). The ray $\mathcal{R}$ is given by $x/t = 0.7307$ according to (7.5.30). Equation (7.5.27) gives $\rho_3 = 3.3267$. It is also interesting to note that $p_1/\rho_1^\gamma = p_3/\rho_3^\gamma = 1.5924$, whereas $p_4/\rho_4^\gamma = 1.4272$. Thus, the entropy rises in going from $\mathcal{D}_0$ to $\mathcal{D}_4$ across the shock, and it rises again from $\mathcal{D}_4$ to $\mathcal{D}_3$ across the interface. Note that $p_4/\rho_4^\gamma - p_0/\rho_0^\gamma = 0.4272$, a number that is less than $\frac{1}{2}$ even though u_4 is nearly equal to 2.

In an actual shock tube we have end walls at some positive x_r and negative x_ℓ, so that our results are valid only for $t < x_r/V$ when $x > 0$, and $t < -x_\ell/a$ when $x < 0$. The problem of a shock reflecting from an end wall was outlined in Problem 5.3.12. The reflection process from an end wall for a centered simple wave is analogous to the case for water waves discussed in Section 7.4.2iii. See also Section 6.12 of [42].

(vi) *Spherically symmetric isentropic flow*

We use the radial distance r and the time t as independent variables and find that (3.3.33a), (3.3.38) reduce to

$$\rho_t + u\rho_r + \rho u_r + \frac{2\rho}{r}u = 0 \text{ (mass)}, \quad (7.5.31a)$$

$$u_t + uu_r + \frac{1}{\gamma\rho}p_r = 0 \text{ (momentum)}, \quad (7.5.31b)$$

if the flow is spherically symmetric. Here ρ, u, and p are made dimensionless as in (7.5.3). For isentropic flow, we can dispense with the energy equation (3.3.33c), which for smooth solutions implies that $p = \rho^\gamma$. Using this in (7.5.31b) and introducing the dimensionless speed of sound $a = (p/\rho^\gamma)^{1/2}$, we obtain the system

$$u_t + uu_r + \frac{2}{\gamma - 1} aa_r = 0, \tag{7.5.32a}$$

$$a_t + \frac{\gamma - 1}{2} au_r + ua_r = -(\gamma - 1)\frac{ua}{r}, \tag{7.5.32b}$$

which generalizes (7.5.6) to the case of smooth spherically symmetric flows. We do not discuss solutions with shocks here. The interested reader can find an account of certain aspects of spherically symmetric flows with shocks in Section 6.16 of [42].

The extra term on the right-hand side of (7.5.32b), which arises from the expression for the divergence with spherical symmetry, now complicates the solution considerably because Riemann invariants do not exist. The best we can do to simplify this system is to transform it to the form (7.3.7) in terms of the characteristic independent variables.

Equations (7.5.8)–(7.5.10) still hold, since they involve only the coefficients of the left-hand sides of (7.5.32), which are the same as those in (7.5.6). In addition, the two components of f in (7.2.2b) are given by

$$f_1 = 0, \quad f_2 = -(\gamma - 1)\frac{ua}{r}. \tag{7.5.33}$$

Therefore, the system (7.3.5), (7.3.7) becomes

$$r_\eta = (u + a)t_\eta, \quad r_\xi = (u - a)t_\xi, \tag{7.5.34a}$$

$$\frac{\gamma - 1}{2} u_\eta + a_\eta = -(\gamma - 1)\frac{uat_\eta}{r}, \quad \frac{\gamma - 1}{2} u_\xi - a_\xi = (\gamma - 1)\frac{uat_\xi}{r}. \tag{7.5.34b}$$

These equations are in a form convenient for solution by the method of characteristics.

7.5.2 *Steady Irrotational Two-Dimensional Flow*

The formulation of this problem was outlined in Problem 7.1.1, and the reader is referred to Section 2.4 of [10] for more details. Here we shall restrict our discussion to shock-free solutions for brevity. The treatment of flows with shocks is entirely analogous to the cases discussed in Sections 7.4 and 7.5.1. In fact, the shock conditions are (5.3.30) in an appropriate frame (see also Section 6.17 of [42]).

(i) *Characteristics*

We use the velocity components $u = \phi_x$ and $v = \phi_y$ to write (7.1.21a) as

$$u_x + \frac{2uv}{u^2 - a^2} u_y + \frac{v^2 - a^2}{u^2 - a^2} v_y = 0, \tag{7.5.35a}$$

after dividing by $(u^2 - a^2)$. The irrotationality condition is just

$$v_x - u_y = 0, \tag{7.5.35b}$$

and (7.1.21b), the equation for the speed of sound, becomes

$$a^2 \equiv 1 - \frac{\gamma - 1}{2}(q^2 - M^2), \tag{7.5.36}$$

where q is the dimensionless local speed $q \equiv (u^2 + v^2)^{1/2}$, and M is the Mach number at $x = -\infty$.

The system (7.5.35) is in the standard form (7.3.13) if we identify (t, x) in (7.3.13) with (x, y) in (7.5.35). Therefore,

$$\{A_{ij}\} = \begin{pmatrix} \dfrac{2uv}{u^2 - a^2} & \dfrac{v^2 - a^2}{u^2 - a^2} \\ -1 & 0 \end{pmatrix}. \tag{7.5.37}$$

The eigenvalues of $\{A_{ij}\}$ are defined by the roots of the quadratic

$$\lambda^2 - \frac{2uv}{u^2 - a^2}\lambda + \frac{v^2 - a^2}{u^2 - a^2} = 0,$$

that is (see (7.1.23)),

$$\lambda_1 = \frac{uv + a(u^2 + v^2 - a^2)^{1/2}}{u^2 - a^2}, \tag{7.5.38a}$$

$$\lambda_2 = \frac{uv - a(u^2 + v^2 - a^2)^{1/2}}{u^2 - a^2}. \tag{7.5.38b}$$

These are real and distinct if the local flow is supersonic—that is, if $u^2 + v^2 > a^2$—and we assume that this is the case for the remainder of this section.

To understand the geometrical meaning of the characteristic curves defined by the two slopes $(dy/dx) = \lambda_1$, $(dy/dx) = \lambda_2$, we consider the following simple example. Suppose that a two-dimensional point disturbance is moving with constant speed M in the negative x direction in a gas at rest. Thus, $x = -Mt$, $y = 0$ locates this point disturbance, which may be thought of as a distribution of mass sources of constant strength along the infinite straight line $x = -Mt$, $y = 0$ in xyz-space. The disturbance that was generated by this point source at time t_0 will be located on the circle of radius $(t - t_0)$ at the time t (recall that the ambient sound speed equals one in our dimesionless variables). The envelope of disturbances then consists of the two straight lines at the angles $\pm\alpha$ relative to the x-axis, where

$$\alpha \equiv \sin^{-1}\frac{1}{M}, \quad 0 < \alpha < \frac{\pi}{2}, \tag{7.5.39}$$

as shown in Figure 7.16a. The angle α is called the Mach angle. This result was also derived in Chapter 6 using the eikonal equation (see (6.3.23)).

An observer moving with the point source experiences a steady flow with speed M to the right and sees the fixed straight lines, called Mach lines, with the constant

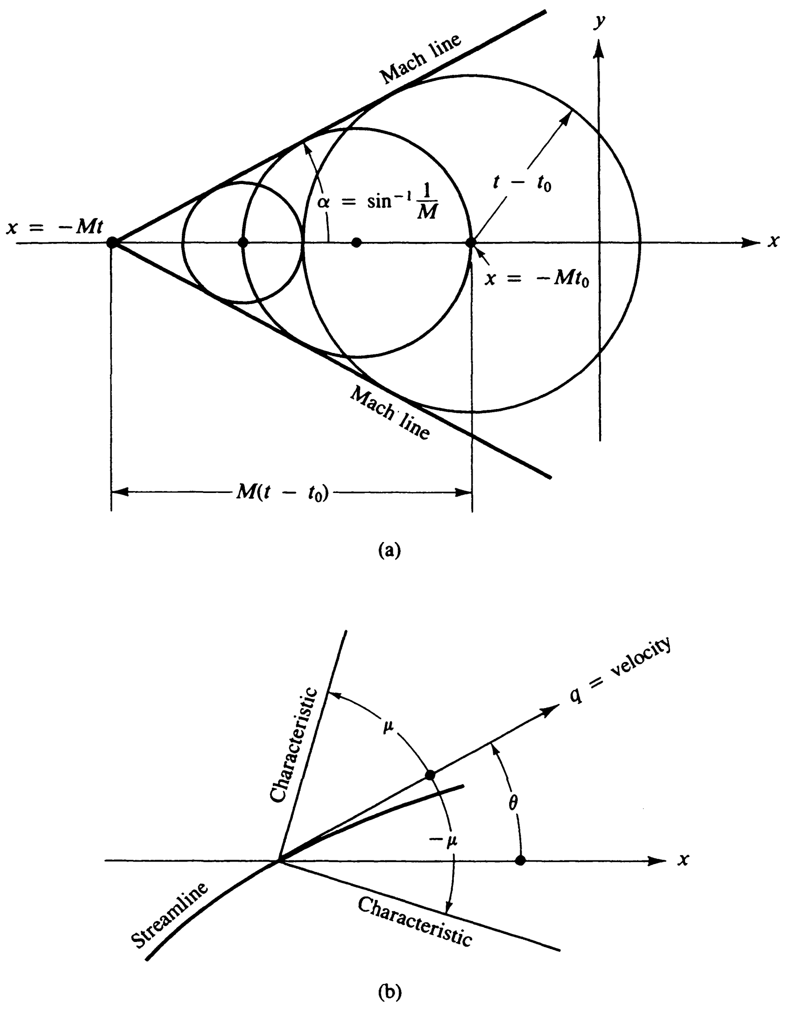

FIGURE 7.16. Characteristics are local Mach lines

slopes $(dy/dx) = \pm \tan \alpha$. More generally, if the disturbance is moving along a straight line inclined at the angle θ relative to the x-axis, then the Mach lines have slopes $(dy/dx) = \tan(\theta \pm \alpha)$.

We shall now show that the characteristic curves have a local slope equal to the slope of the local Mach lines. The local velocity vector is tangent to the local streamline, as shown in Figure 7.16b, and we introduce the polar representation for the velocity components

$$u = q \cos \theta, \quad v = q \sin \theta, \tag{7.5.40}$$

where θ is the angle measured in the counterclockwise sense from the positive x direction to the velocity vector. Guided by the discussion for the case $q = M =$ constant, we denote

$$a \equiv q \sin \mu, \tag{7.5.41}$$

where μ is the local Mach angle (see (7.5.39)). If we now express u, v, and a in (7.5.38) in terms of q, θ, and μ, we obtain

$$\begin{aligned}\lambda_{1,2} &= \frac{q^2 \sin\theta \cos\theta \pm q \sin\mu (q^2 - q^2 \sin^2\mu)^{1/2}}{q^2 \cos^2\theta - q^2 \sin^2\mu} \\ &= \frac{\sin\theta\cos\theta \pm \sin\mu\cos\mu}{\cos^2\theta - \sin^2\mu} = \frac{\sin 2\theta \pm \sin 2\mu}{\cos 2\theta + \cos 2\mu} \\ &= \frac{\sin(\theta \pm \mu)\cos(\theta \mp \mu)}{\cos(\theta + \mu)\cos(\theta - \mu)} = \tan(\theta \pm \mu),\end{aligned} \tag{7.5.42}$$

using trigonometric identities. Thus, the angle between the streamline and the characteristics is just the local Mach angle μ.

(ii) *The Riemann invariants*

The hodograph form (7.3.17) for (7.5.35) is

$$-y_v + \frac{2uv}{u^2 - a^2} x_v - \frac{v^2 - a^2}{u^2 - a^2} x_u = 0, \tag{7.5.43a}$$

$$y_u - x_v = 0. \tag{7.5.43b}$$

Using (7.3.24), we compute the following characteristic slopes for (7.5.43):

$$\frac{dv}{du} = \frac{-uv \pm a(u^2 + v^2 - a^2)^{1/2}}{v^2 - a^2}; \tag{7.5.44}$$

and as we have argued all along, the integration of (7.5.44) gives the two Riemann invariants. This integration is awkward in terms of the u and v variables, so in view of the simplification that was introduced in (7.5.42) when we used θ and μ, let us attempt to solve (7.5.44) in terms of these variables. Note that (7.5.36), (7.5.40), and (7.5.41) define a transformation of variables $(u, v) \longleftrightarrow (\theta, \mu)$.

First, we compute (dv/du) using (7.5.40). This gives

$$\frac{dv}{du} = \frac{\dfrac{dq}{d\theta}\sin\theta + q\cos\theta}{\dfrac{dq}{d\theta}\cos\theta - q\sin\theta}. \tag{7.5.45a}$$

But according to (7.5.40)–(7.5.41), the right-hand side of (7.5.44) is

$$\begin{aligned}\frac{-uv \pm a(u^2 + v^2 - a^2)^{1/2}}{v^2 - a^2} &= \frac{-\sin\theta\cos\theta \pm \sin\mu\cos\mu}{\sin^2\theta - \sin^2\mu} \\ &= -\cot(\theta \pm \mu).\end{aligned} \tag{7.5.45b}$$

Therefore, equating the right-hand sides of (7.5.45a)–(7.5.45b) and solving for $(dq/d\theta)$, we obtain

$$\frac{dq}{d\theta} = \mp \tan\mu. \tag{7.5.46a}$$

To express $(dq/d\theta)$ in terms of $(d\mu/d\theta)$, we use (7.5.36) and (7.5.41) to obtain

$$\frac{dq}{d\theta} = -\frac{q\sin\mu\cos\mu}{\sin^2\mu + (\gamma-1)/2}\frac{d\mu}{d\theta}. \tag{7.5.46b}$$

We now equate the right-hand sides of (7.5.46a)–(7.5.46b) and solve for $(d\mu/d\theta)$ in the form

$$\frac{d\mu}{d\theta} = \pm\frac{\sin^2\mu + (\gamma-1)/2}{\cos^2\mu}. \tag{7.5.47}$$

This expression can be integrated explicitly to define the Riemann invariants

$$\theta + \nu(\mu) = \text{constant on } \frac{dy}{dx} = \tan(\theta+\mu), \tag{7.5.48a}$$

$$\theta - \nu(\mu) = \text{constant on } \frac{dy}{dx} = \tan(\theta-\mu), \tag{7.5.48b}$$

where ν is the Prandtl–Meyer function:

$$\begin{aligned}\nu(\mu) &\equiv \int_0^\mu \frac{\cos^2\sigma}{\sin^2\sigma + (\gamma-1)/2}\,d\sigma = -\mu + \frac{\gamma+1}{2}\int_0^{2\mu}\frac{d\sigma}{\gamma-\cos\sigma}\\ &= \left(\frac{\gamma+1}{\gamma-1}\right)^{1/2}\tan^{-1}\left[\left(\frac{\gamma+1}{\gamma-1}\right)^{1/2}\tan\mu\right] - \mu.\end{aligned} \tag{7.5.49}$$

A solution by the method of characteristics using (7.5.48) is now easy to implement and defines θ, μ at each gridpoint. To obtain u and v there, we use (7.5.36), (7.5.40), and (7.5.41). Examples can be found in standard texts in gas dynamics. For example, see Chapter 4 of [36] and Chapter 4 of [11].

Problems

7.5.1 Consider steady supersonic flow in the corner domain $r \geq 0, \beta \leq \theta \leq \pi$, where r and θ are polar coordinates in the plane, and β is a negative constant with $-\pi < \beta < 0$. The boundary condition is that the velocity vector is tangent to the boundary. Given $u > 1$ and $v = 0$ as $r \to \infty$ with $\pi/2 < \theta \leq \pi$, compute the flow everywhere. For a given Mach number at upstream infinity, what is the maximum turning angle for which the flow downstream of the corner has infinite Mach number?

7.5.2 For axially symmetric supersonic flow, the equations corresponding to (7.5.35) are

$$u_x + \frac{2uv}{u^2-a^2}u_r + \frac{v^2-a^2}{u^2-a^2}v_r = -\frac{a^2v}{r(u^2-a^2)}, \tag{7.5.50a}$$

where u and v are the axial (x) and radial (r) components of the velocity vector, respectively.

Show that

$$\{\ell_{ij}\} = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \dfrac{1}{\lambda_2} & 1 \\ -\dfrac{1}{\lambda_1} & -1 \end{pmatrix} \tag{7.5.51}$$

is consistent with (7.2.5) for this example, with λ_1 and λ_2 given by (7.5.38). Therefore, the system corresponding to (7.3.5), (7.3.7) is

$$r_\eta = \lambda x_\eta, \quad r_\xi = \lambda_2 x_\xi, \tag{7.5.52a}$$

$$u_\eta + \lambda_2 v_\eta = \frac{a^2 v x_\eta}{r(a^2 - u^2)}, \quad u_\xi + \lambda_1 v_\xi = \frac{a^2 v x_\xi}{r(a^2 - u^2)}. \tag{7.5.52b}$$

8

Approximate Solutions by Perturbation Methods

In this chapter we consider differential equations (together with prescribed initial and/or boundary data, as appropriate) involving a small parameter ϵ. Such problems can usually be solved approximately by a perturbation procedure if the limiting case for $\epsilon = 0$ is known. By a suitable definition of ϵ, we can also perturb the solution about any given value of a dimensionless parameter.

The basic concepts that underlie perturbation methods are discussed in detail in texts such as [8.1], devoted entirely to the subject. As it is not possible to cover this vast material in one chapter, we shall give only a sampling of methods (via example problems) that are useful for approximating the solution of partial differential equations. Other methods, such as the method of averaging near identity transformations, that are appropriate only for ordinary differential equations, will not be discussed.

8.1 Regular Perturbations

In Appendix A.3 we review some techniques for constructing the asymptotic expansion of a function $u(\mathbf{x}; \epsilon)$ as $\epsilon \to 0$ for various settings including the case where u is defined in integral form. Here we broaden our scope to include the case where u is defined by a differential equation. The small parameter ϵ may then occur either in the equation itself or in the initial (boundary) conditions or both. Further generalizations to integral equations, difference equations, and differential–delay equations are possible but are not discussed.

Regular perturbation problems (as opposed to singular perturbation problems discussed in Sections 8.2 and 8.3) are characterized by the property that the solution can be expressed by a single asymptotic expansion of the form (A.3.8), and that this expansion remains uniformly valid in the entire domain of interest. Of course, if the solution of the differential equation is known explicitly, the construction of its asymptotic expansion is both trivial and unnecessary. We will be mainly concerned with problems that have an easily calculated solution if $\epsilon = 0$ but that are either very difficult or impossible to solve explicitly if $\epsilon \neq 0$. Often, the equation becomes linear if $\epsilon = 0$; in other cases, the boundaries or the boundary

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data simplify and the problem is solvable for $\epsilon = 0$. We shall illustrate various possibilities by means of examples.

The perturbation approximation of a problem governed by a differential equation relies on the following fundamental assumption: The limit process that defines the asymptotic expansion of the exact solution (if this were known) produces a consistent set of perturbation differential equations and boundary conditions governing each term of the expansion when this limit process is applied directly to the exact governing system. Moreover, we assume that the successive solution of these perturbation equations and boundary conditions gives the same asymptotic expansion that we would calculate from the exact solution. A similar assumption is made implicitly whenever we expand a function $R(x;\epsilon) = 0$ in order to find its roots (see (A.3.18)). In Section 8.2 we shall extend this idea to include different expansions (associated with different limit processes) applied to the same differential equation.

8.1.1 Green's Function for an Ordinary Differential Equation

Consider the problem of a string on an elastic support under load. The string is assumed to have zero deflection at $x = 0$ and $x = 1$. If we restrict attention to the vertical displacement v and assume that the elastic support $k(x;\epsilon)$ varies weakly in the x direction, we have the dimensionless wave equation (see (3.1.15))

$$v_{tt} - v_{xx} + [k_0^2 + \epsilon k_1(x)]v = p(x,t). \tag{8.1.1}$$

Here k_0^2 is the predominantly constant part of k, $0 < \epsilon \ll 1$, and $k_1(x)$ determines the variation of k with x. The string has a uniform density $r(x) = 1$, and the applied load is given by $p(x,t)$. For the static problem, $v(x;\epsilon)$ satisfies

$$-v'' + [k_0^2 + \epsilon k_1(x)]v = p(x), \tag{8.1.2a}$$

$$v(0;\epsilon) = 0, \quad v(1;\epsilon) = 0. \tag{8.1.2b}$$

Green's function $g(x,\xi;\epsilon)$ for this problem satisfies

$$-g'' + [k_0^2 + \epsilon k_1(x)]g = \delta(x-\xi), \tag{8.1.3a}$$

$$g(0,\xi;\epsilon) = 0, \quad g(1,\xi;\epsilon) = 0, \tag{8.1.3b}$$

where ξ is a constant, $0 < \xi < 1$, primes denote derivatives with respect to x, and δ is the Dirac delta function.

In view of the variable coefficient $\epsilon k_1(x)$ in (8.1.3a), an exact solution is out of reach, and we seek an asymptotic expansion for g in the form

$$g(x,\xi;\epsilon) = g_0(x,\xi) + \phi_1(\epsilon)g_1(x,\xi) + o(\phi_1), \tag{8.1.4}$$

where $\phi_1 \ll 1$ and remains to be specified. Substituting (8.1.4) into (8.1.3a) gives

$$-g_0'' + k_0^2 g_0 - \delta(x-\xi) - \phi_1 g_1'' + \epsilon k_1 g_0 + \phi_1 k_0^2 g_1 = o(\phi_1), \tag{8.1.5a}$$

and the boundary conditions (8.1.3b) give

$$g_0(0,\xi) + \phi_1 g_1(0,\xi) = o(\phi_1), \quad g_0(1,\xi) + \phi_1 g_1(1,\xi) = o(\phi_1). \tag{8.1.5b}$$

Regardless of the choice of ϕ_1, as long as $\phi_1 \ll 1$, (8.1.5b) implies the following homogeneous boundary conditions for g_0 and g_1

$$g_0(0,\xi) = g_0(1,\xi) = 0, \quad g_1(0,\xi) = g_1(1,\xi) = 0. \tag{8.1.6}$$

It follows from (8.1.5a) that the leading term g_0 satisfies

$$-g_0'' + k_0^2 g_0 = \delta(x-\xi), \tag{8.1.7}$$

whereas the perturbation term g_1 satisfies

$$-g_1'' + k_0^2 g_1 = \begin{cases} 0 & \text{if } \epsilon \ll \phi_1, \\ -k_1(x) g_0 & \text{if } \phi_1 = O_s(\epsilon). \end{cases} \tag{8.1.8}$$

The choice $\epsilon \ll \phi_1$ in (8.1.8), combined with the boundary conditions for g_1 in (8.1.6), implies that $g_1 \equiv 0$. Therefore, we set $\phi_1 = \epsilon$. Henceforth, we shall omit this detailed justification for the choice of the various terms in the asymptotic sequence and shall anticipate the final result directly. The operating rule is to choose ϕ_1 so as to obtain the most general possible equation for g_1.

For $x \neq \xi$, (8.1.7) has zero right-hand side and has the solution

$$g_0(x,\xi) = \begin{cases} A(\xi)e^{k_0 x} + B(\xi)e^{-k_0 x} & \text{if } x < \xi, \\ C(\xi)e^{k_0 x} + D(\xi)e^{-k_0 x} & \text{if } x > \xi, \end{cases} \tag{8.1.9}$$

where A, B, C, and D are functions of ξ to be determined by the boundary conditions at the two ends and the jump conditions at $x = \xi$. Imposing the boundary conditions determines B in terms of A, and D in terms of C, to give

$$g_0(x,\xi) = \begin{cases} A(\xi)[e^{k_0 x} - e^{-k_0 x}] & \text{if } x < \xi, \\ C(\xi)[e^{k_0 x} - e^{k_0(2-x)}] & \text{if } x > \xi. \end{cases} \tag{8.1.10}$$

To determine A and C, we impose the jump conditions

$$g_0(\xi^-,\xi) = g_0(\xi^+,\xi), \tag{8.1.11a}$$

$$\frac{\partial g_0}{\partial x}(\xi^-,\xi) - \frac{\partial g_0}{\partial x}(\xi^+,\xi) = 1, \tag{8.1.11b}$$

which follow from integrating (8.1.7) with respect to x fom $x = \xi^-$ to $x = \xi^+$. Using (8.1.10) in (8.1.11) gives the linear system

$$A(\xi)[e^{k_0\xi} - e^{-k_0\xi}] - C(\xi)[e^{k_0\xi} - e^{k_0(2-\xi)}] = 0,$$
$$A(\xi)k_0[e^{k_0\xi} - e^{-k_0\xi}] - C(\xi)k_0[e^{k_0\xi} + e^{k_0(2-\xi)}] = 1,$$

which is easily solved in the form

$$A(\xi) = \frac{e^{k_0\xi} - e^{k_0(2-\xi)}}{2k_0(1-e^{2k_0})}, \quad C(\xi) = \frac{e^{k_0\xi} - e^{-k_0\xi}}{2k_0(1-e^{2k_0})}. \tag{8.1.12}$$

Thus, the leading term in the expansion for Green's function is defined by (8.1.10) and (8.1.12). Note that for $k_0 \to 0$, this result tends to the piecewise linear pofile

$$g_0(x, \xi) = \begin{cases} x(1-\xi) & \text{if } x < \xi, \\ \xi(1-x) & \text{if } x > \xi, \end{cases} \tag{8.1.13}$$

for an unsupported string.

The right-hand side of (8.1.8) is now known, and we can calculate g_1. In fact, since the homogeneous operator for g_1 is the same as that governing g_0, a convenient approach is to use Green's function for g_0, which we have just calculated. Thus,

$$g_1(x, \xi) = \int_0^1 g_0(x, \zeta)[-k_1(\zeta) g_0(\xi, \zeta)] d\zeta, \tag{8.1.14}$$

and this defines g_1 by quadrature for a given k_1. The procedure can be extended to higher orders to define subsequent terms in the expansion for g. Notice that if k_1 is a well-behaved function, the expansion (8.1.4) will be uniformly valid on $0 \le x \le 1$. The proof that (8.1.4) is indeed the asymptotic expansion of the exact solution of (8.1.3) is not interesting and is omitted.

Knowing the expansion of Green's function to $O(\epsilon)$, we can write the expansion for the solution of (8.1.2) directly in the form

$$\begin{aligned} v(x; \epsilon) &= \int_0^1 g(x, \xi; \epsilon) p(\xi) d\xi \\ &= \int_0^1 g_0(x, \xi) p(\xi) d\xi \\ &\quad + \epsilon \int_0^1 p(\xi) \left\{ \int_0^1 g_0(x, \zeta)[-k_1(\zeta) g_0(\xi, \zeta)] d\zeta \right\} d\xi + O(\epsilon^2) \end{aligned} \tag{8.1.15}$$

This result also follows from a direct perturbation solution of (8.1.2). We expand $v(x; \epsilon)$ in the form

$$v(x; \epsilon) = v_0(x) + \epsilon v_1(x) + O(\epsilon^2), \tag{8.1.16}$$

and substitute this into (8.1.2) to derive the following two problems for v_0 and v_1:

$$L(v_0) \equiv -v_0'' + k_0^2 v_0 = p(x), \tag{8.1.17a}$$

$$v_0(0) = 0, \quad v_0(1) = 0, \tag{8.1.17b}$$

$$L(v_1) = -k_1(x) v_0(x), \tag{8.1.18a}$$

$$v_1(0) = 0, \quad v_1(1) = 0. \tag{8.1.18b}$$

Once Green's function g_0 for the unperturbed problem is known, we can express the solutions for v_0 and v_1 by quadrature as

$$v_0(x) = \int_0^1 g_0(x, \xi) p(\xi) d\xi, \tag{8.1.19a}$$

$$v_1(x) = \int_0^1 g_0(x, \xi)[-k_1(\xi) v_0(\xi)] d\xi. \tag{8.1.19b}$$

The result in (8.1.19a) agrees with the $O(1)$ term in (8.1.15). The result in (8.1.19b) also agrees with the $O(\epsilon)$ term in (8.1.15) once we note that $g_0(\xi, \zeta) = g_0(\zeta, \xi)$ and interchange the order of integration. There is no particular advantage in using one approach in favor of the other.

8.1.2 Perturbed Self-Adjoint Operator

(i) *One-dimensional wave equation*

To motivate the discussion for the general case, we first consider the wave equation

$$u_{tt} - u_{xx} + \epsilon xu = 0, \quad 0 \le x \le \pi, \quad 0 \le t, \tag{8.1.20a}$$

which may again be regarded as the equation governing the vertical displacement (now denoted by $u(x, t; \epsilon)$) of a string on a weak $(0 < \epsilon \ll 1)$, x-dependent elastic support (ϵxu). We assume the homogeneous boundary conditions

$$u(0, t; \epsilon) = 0, \quad u(\pi, t; \epsilon) = 0, \text{ if } t > 0, \tag{8.1.20b}$$

and the general initial conditions

$$u(x, 0; \epsilon) = f(x; \epsilon), \quad u_t(x, 0; \epsilon) = g(x; \epsilon). \tag{8.1.20c}$$

We studied the unperturbed $(\epsilon = 0)$ problem in Section 3.6 using Green's function. As indicated by the discussion in Section 3.6.2, this problem can also be solved using eigenfunction expansions. This is the approach that we shall follow here for the perturbed problem.

Assuming the separated form

$$u(x, t; \epsilon) = X(x; \epsilon)T(t; \epsilon)$$

for the solution and substituting this into (8.1.20a) gives

$$-\frac{\ddot{T}}{T} = -\frac{X''}{X} + \epsilon x = \lambda > 0.$$

Thus, the eigenvalue problem associated with (8.1.20) is

$$-\frac{d^2 X_n}{dx^2} + \epsilon x X_n = \lambda_n X_n, \tag{8.1.21a}$$

$$X_n(0; \epsilon) = 0, \quad X_n(\pi; \epsilon) = 0. \tag{8.1.21b}$$

Later on, we shall consider a more general eigenvalue problem for which the above is a one-dimensional special case. Although (8.1.21) can be solved exactly, we shall study its asymptotic expansion for $0 < \epsilon \ll 1$ to illustrate ideas. First, we note that if $\epsilon = 0$, the eigenvalues are $\lambda_n^{(0)} = n^2$, and the normalized eigenfunctions are

$$\xi_n^{(0)}(x) = \left(\frac{2}{\pi}\right)^{1/2} \sin nx, \quad n = 1, 2, \ldots. \tag{8.1.22}$$

These eigenfunctions are orthogonal in the sense that the inner product of any pair $\xi_n^{(0)}$ and $\xi_m^{(0)}$ vanishes if $m \neq n$. The inner product is defined as

$$\left\langle \xi_n^{(0)}, \xi_m^{(0)} \right\rangle \equiv \int_0^\pi \xi_n^{(0)}(x)\xi_m^{(0)}(x)dx = 0 \text{ if } m \neq n, \tag{8.1.23}$$

and we have normalized the eigenfunctions by the choice of the multiplier $(2/\pi)^{1/2}$ in (8.1.22), so that $\langle \xi_n^{(0)}, \xi_n^{(0)} \rangle = 1$.

We assume that the eigenfunctions $X_n(x;\epsilon)$ for $\epsilon \neq 0$ have the expansion

$$X_n(x;\epsilon) = \left(\frac{2}{\pi}\right)^{1/2} \sin nx + \epsilon\xi_n^{(1)}(x) + O(\epsilon^2), \tag{8.1.24}$$

and that the eigenvalues $\lambda_n(\epsilon)$ in (8.1.21a) have the expansion

$$\lambda_n(\epsilon) = n^2 + \epsilon\lambda_n^{(1)} + O(\epsilon^2). \tag{8.1.25}$$

Substituting these expansions into (8.1.21a) gives

$$\frac{d^2\xi_n^{(1)}}{dx^2} + n^2\xi_n^{(1)} = -\lambda_n^{(1)}\left(\frac{2}{\pi}\right)^{1/2} \sin nx + \left(\frac{2}{\pi}\right)^{1/2} x \sin nx. \tag{8.1.26}$$

Rather than solving (8.1.26) explicitly, which is certainly possible in this case, and then imposing the boundary conditions $\xi_n^{(1)}(0) = \xi_n^{(1)}(\pi) = 0$, let us express the solution for $\xi_n^{(1)}$ in the form of a series of the eigenfunctions $\xi_n^{(0)}$, that is, a normalized Fourier sine series

$$\xi_n^{(1)}(x) = \left(\frac{2}{\pi}\right)^{1/2} \sum_{j=1}^{\infty} a_{nj} \sin jx. \tag{8.1.27}$$

Thus, the boundary conditions at $x = 0$ and $x = \pi$ are automatically satisfied.

Equation (8.1.26) then becomes

$$\left(\frac{2}{\pi}\right)^{1/2} \sum_{j=1}^{\infty}(-j^2 + n^2)a_{nj} \sin jx = \left(\frac{2}{\pi}\right)^{1/2} [-\lambda_n^{(1)} \sin nx + x \sin nx]. \tag{8.1.28}$$

We now multiply (8.1.28) by $\xi_k^{(0)}$, integrate the result over $(0,\pi)$, and use orthogonality to simplify the left-hand side to obtain

$$a_{nk}(-k^2 + n^2) = -\frac{2}{\pi}\lambda_n^{(1)} \int_0^\pi \sin nx \sin kx dx + \frac{2}{\pi}\int_0^\pi x \sin nx \sin kx dx, \tag{8.1.29}$$

valid for arbitrary integers n and k.

If $k \neq n$, the first integral on the right-hand side of (8.1.29) vanishes, and we have

$$a_{nk} = \frac{2}{\pi(n^2 - k^2)}\int_0^\pi x \sin nx \sin kx dx = \frac{4kn[(-1)^{k+n} - 1]}{\pi(n^2 - k^2)^3}. \tag{8.1.30}$$

If $k = n$, the left-hand side of (8.1.29) vanishes and the first term on the right-hand side just equals $-\lambda_n^{(1)}$, so we obtain

$$\lambda_n^{(1)} = \frac{2}{\pi}\int_0^{\pi} x \sin^2 nx dx = \frac{\pi}{2}. \tag{8.1.31}$$

It remains to compute a_{nn}. Recall that if X_n is an eigenfunction, then any constant times X_n is also an eigenfunction. If we choose to normalize the X_n, we have

$$\langle X_n, X_n\rangle = 1 = \left\langle \xi_n^{(0)}, \xi_n^{(0)}\right\rangle + 2\epsilon\left\langle \xi_n^{(0)}, \xi_n^{(1)}\right\rangle + O(\epsilon^2). \tag{8.1.32}$$

Since $\langle \xi_n^{(0)}, \xi_n^{(0)}\rangle = 1$, we must set

$$\langle \xi_n^{(0)}, \xi_n^{(1)}\rangle = 0 = \int_0^{\pi} \sin n\xi \left(\sum_{j=1}^{\infty} a_{nj} \sin jx\right) dx,$$

in order to satisfy (8.1.32) to $O(\epsilon)$, and this implies that all the a_{nn} vanish.

To complete the solution of the vibration problem (8.1.20), we expand $u(x, t; \epsilon)$ in a series form

$$u(x, t; \epsilon) = \sum_{n=1}^{\infty} \alpha_n(t; \epsilon) X_n(x; \epsilon), \tag{8.1.33}$$

where α_n is the amplitude of the nth eigenfunction. Substituting this series into (8.1.20a) shows that the α_n are governed by the decoupled oscillator equations

$$\frac{d^2\alpha_n}{dt^2} + \lambda_n(\epsilon)\alpha_n = 0, \tag{8.1.34}$$

so the frequency of each eigenfunction is $\lambda_n^{1/2}$ (for which an asymptotic expansion can be found to any desired order). The amplitudes are given by

$$\alpha_n(t; \epsilon) = A_n(\epsilon) \sin \lambda_n^{1/2}(\epsilon)t + B_n(\epsilon) \cos \lambda_n^{1/2}(\epsilon)t, \tag{8.1.35}$$

where the A_n and B_n are constants that can be defined to any desired order in terms of the initial data (8.1.20c). The details are outlined in Problem 8.1.1.

The fact that (8.1.20) is linear is reflected by the result (8.1.34) that the amplitude equation for each mode decouples from the others. In a weakly nonlinear problem, this is no longer the case. In fact, this approach is not suitable if the perturbation term in nonlinear (for example, if it is $\epsilon x u^2$); we can no longer separate the x-dependence of the solution into an exact eigenvalue problem as in (6.1.21). However, we can expand u itself in the form

$$u(x, t; \epsilon) = \sum_{n=1}^{\infty} q_n(t; \epsilon) \sin nx, \tag{8.1.36}$$

and derive weakly nonlinear, weakly coupled oscillator equations for the q_n as outlined in Problem 3.6.7. These oscillator equations can then be solved asymptotically using various techniques. For a detailed account see Chapters 4 and 5 of

[26]. In Section 8.3.1 we consider a single weakly nonlinear oscillator and discuss its asymptotic approximation using the method of multiple scales.

(ii) *The general problem*

Perturbed linear or weakly nonlinear eigenvalue problems arise in contexts other than vibration problems, and we now consider the following generalization of (8.1.21):

$$L(u_n) + \epsilon F(\mathbf{x}, u_n) = \lambda_n u_n \text{ in } \mathcal{D}, \tag{8.1.37a}$$

$$u_n(\mathbf{x}; \epsilon) = 0 \text{ on the boundary of } \mathcal{D}. \tag{8.1.37b}$$

Here $\mathcal{D}$ is a given N-dimensional domain and $\mathbf{x}$ denotes $x_1, \ldots, x_N$. The nth eigenfunction is denoted by $u_n(x; \epsilon)$, and the associated eigenvalue is $\lambda_n(\epsilon)$, where $0 < \epsilon \ll 1$. The linear differential operator L is assumed to be self-adjoint with no repeated eigenvalues, and we assume that F is continuous in $\mathcal{D}$. We do not discuss the more interesting case where the perturbation term in (8.1.37a) is $F_n(\mathbf{x}, u_1, u_2, \ldots, u_N)$; that is, it is a different function for each n and may also depend on all the u_n. In this case, the n equations (8.1.37a) are coupled.

We define the inner product beteween two eigenfunctions u_m and u_n by

$$\langle u_m, u_n \rangle \equiv \int_{\mathcal{D}} \cdots \int u_m(\mathbf{x}; \epsilon) u_n(\mathbf{x}; \epsilon) dV, \tag{8.1.38}$$

where dV is the volume element in $\mathcal{D}$. Since L is self-adjoint, we have

$$\left\langle u_m^{(0)}, L(u_n^{(0)}) \right\rangle = \left\langle L(u_m^{(0)}), u_n^{(0)} \right\rangle \text{ for all } m \text{ and } n, \tag{8.1.39}$$

where $u_m^{(0)}$ and $u_n^{(0)}$ are the eigenfunctions of the unperturbed problem; that is, $u_m^{(0)}$ and $u_n^{(0)}$ satisfy $L(u_m^{(0)}) = \lambda_m^{(0)} u_m^{(0)}$, $L(u_n^{(0)}) = \lambda_n^{(0)} u_n^{(0)}$ and (8.1.37b). As we have assumed that the eigenvalues associated with different eigenfunctions are distinct, (8.1.39) implies the orthogonality condition

$$\left\langle u_m^{(0)}, u_n^{(0)} \right\rangle = 0 \text{ if } m \neq n. \tag{8.1.40a}$$

As in the example problem (8.1.21), we shall normalize the $u_n^{(0)}$ by requiring

$$\langle u_n^{(0)}, u_n^{(0)} \rangle = 1. \tag{8.1.40b}$$

To illustrate ideas, let L be

$$L \equiv \Delta + r(\mathbf{x}), \tag{8.1.41}$$

where Δ is the Laplacian in three dimensions, and r is a given continuous function of x_1, x_2, x_3 in some domain $\mathcal{D}$ with boundary Γ. Using the symmetric form of Green's formula (2.5.4) to simplify the expression that results for $\langle u, L(v) \rangle - \langle v, L(u) \rangle$, it is easily seen that if u and v vanish on Γ, then L is self-adjoint.

We expand the eigenfunctions and eigenvalues of the perturbed problem (8.1.37) with $\epsilon \neq 0$ in the form

$$u_n(\mathbf{x}; \epsilon) = u_n^{(0)}(\mathbf{x}) + \epsilon u_n^{(1)}(\mathbf{x}) + O(\epsilon^2), \tag{8.1.42a}$$

$$\lambda_n(\epsilon) = \lambda_n^{(0)} + \epsilon \lambda_n^{(1)} + O(\epsilon^2). \tag{8.1.42b}$$

For suitable F, (8.1.42a) implies that $\epsilon F(\mathbf{x}; u_n)$ has the expansion

$$\epsilon F(\mathbf{x}, u_n) = \epsilon F_n^{(0)}(\mathbf{x}) + O(\epsilon^2), \quad F_n^{(0)}(\mathbf{x}) \equiv F(\mathbf{x}, u_n^{(0)}(\mathbf{x})). \tag{8.1.43}$$

Since L is linear, we have

$$L(u_n^{(0)} + \epsilon u_n^{(1)} + O(\epsilon^2)) = L(u_n^{(0)}) + \epsilon L(u_n^{(1)}) + O(\epsilon^2).$$

Therefore, the $u_n^{(1)}$ satisfy

$$L(u_n^{(1)}) - \lambda_n^{(0)} u_n^{(1)} = -F_n^{(0)}(\mathbf{x}) + \lambda_n^{(1)} u_n^{(0)}(\mathbf{x}). \tag{8.1.44}$$

We assume that each $u_n^{(1)}$ may be expressed as an expansion in terms of the unperturbed eigenfunctions in the form (cf. (8.1.27))

$$u_n^{(1)}(\mathbf{x}) = \sum_{j=1}^{\infty} a_{nj} u_j^{(0)}(\mathbf{x}). \tag{8.1.45a}$$

Therefore, in view of (8.1.40), the constants a_{nj} are given by

$$a_{nj} = \int_{\mathcal{D}} \cdots \int u_n^{(1)}(\mathbf{x}) u_j^{(0)}(\mathbf{x}) dV. \tag{8.1.45b}$$

Similarly, we assume that we may express the known functions $F_n^{(0)}(\mathbf{x})$ in the series form

$$F_n^{(0)}(\mathbf{x}) = \sum_{j=1}^{\infty} f_{nj} u_j^{(0)}, \tag{8.1.46a}$$

in which the f_{nj} are the known coefficients

$$f_{nj} = \int_{\mathcal{D}} \cdots \int F_n^{(0)}(\mathbf{x}) u_j^{(0)}(\mathbf{x}) dV. \tag{8.1.46b}$$

In order to calculate the a_{nj} and $\lambda_n^{(1)}$, we proceed as in part (i) and multiply (8.1.44) by $u_k^{(0)}$ for some arbitrary integer k, then integrate the result over $\mathcal{D}$ to find

$$\int_{\mathcal{D}} \cdots \int [u_k^{(0)} L(u_n^{(1)}) - \lambda_n^{(0)} u_n^{(1)} u_k^{(0)}] dV = -f_{nk} + \lambda_n^{(1)} \delta_{nk}, \tag{8.1.47}$$

where δ_{nk} is the Kronecker delta. Now multiply $L(u_k^{(0)}) - \lambda_k^{(0)} u_k^{(0)} = 0$ by $u_n^{(1)}$ and integrate the result over $\mathcal{D}$ to obtain

$$\int_{\mathcal{D}} \cdots \int [u_n^{(1)} L(u_k^{(0)}) - \lambda_k^{(0)} u_k^{(0)} u_n^{(1)}] dV = 0. \tag{8.1.48}$$

The second term on the left-hand side of (8.1.47) and (8.1.48) is a multiple of

$$\int_{\mathcal{D}} \cdots \int u_k^{(0)} u_n^{(1)} dV = \int_{\mathcal{D}} \cdots \int u_k^{(0)} \sum_{j=1}^{\infty} a_{nj} u_j^{(0)} dV = a_{nk}.$$

Therefore, subtracting (8.1.48) from (8.1.47) gives

$$\int_{\mathcal{D}} \cdots \int [u_k^{(0)} L(u_n^{(1)}) - u_n^{(1)} L(u_k^{(0)})] dV = a_{nk}(\lambda_n^{(0)} - \lambda_k^{(0)}) - f_{nk} + \lambda_n^{(1)} \delta_{nk}. \tag{8.1.49a}$$

It follows from (8.1.45a), the linearity of L, and the fact that this operator is self-adjoint (exhibited by (8.1.39)) that the left-hand side of (8.1.49a) vanishes, and we are left with (cf. (8.1.29)

$$a_{nk}(\lambda_n^{(0)} - \lambda_k^{(0)}) = f_{nk} - \lambda_n^{(1)} \delta_{nk}. \tag{8.1.49b}$$

If $k \neq n$, then $\delta_{nk} = 0$, and (8.1.49b) gives

$$a_{nk} = \frac{f_{nk}}{\lambda_n^{(0)} - \lambda_k^{(0)}}. \tag{8.1.50a}$$

If $k = n$, the left-hand side of (8.1.49b) vanishes, and we have

$$\lambda_n^{(1)} = f_{nn}. \tag{8.1.50b}$$

To complete the solution to $O(\epsilon)$, we note that if we normalize the u_n, we must have

$$\langle u_n, u_n \rangle = 1 = \langle u_n^{(0)}, u_n^{(0)} \rangle + 2\epsilon \langle u_n^{(0)}, u_n^{(1)} \rangle + O(\epsilon^2);$$

that is, $\langle u_n^{(0)}, u_n^{(1)} \rangle = 0$, and this implies that $a_{nn} = 0$. Problems 8.1.2 and 8.1.3 illustrate the ideas.

8.1.3 A Boundary Perturbation Problem

We showed in Chapter 2 that once Green's function for Laplace's equation is known for a given domain and the case where the potential vanishes on the boundary, we can solve the Dirichlet problem for this domain (see Section 2.6.3). Let us study the special case of the two-dimensional Laplacian over a domain nearly equal to the interior of the unit circle; that is, we wish to solve

$$\Delta_P K(r, \theta, \rho, \phi; \epsilon) = \delta_2(P, Q) \text{ in } \mathcal{D}, \tag{8.1.51a}$$

$$K(1 + \epsilon f(\theta), \theta, \rho, \phi; \epsilon) = 0, \tag{8.1.51b}$$

where $\mathcal{D}$ is the domain $0 \leq r \leq 1 + \epsilon f(\theta)$ for a given 2π-periodic function $f(\theta)$. Here r and θ are the polar coordinates of the observer point P, ρ and ϕ are the polar coordinates of the source point Q, δ_2 is the two-dimensional delta function, and Δ_P denotes the two-dimensional Laplacian with respect to the r and θ variables.

We showed in (2.6.29) that for $\epsilon = 0$, we have

$$K(r, \theta, \rho, \phi; 0) = \frac{1}{4\pi} \log \frac{r^2 + \rho^2 - 2r\rho\cos(\theta - \phi)}{\rho^2 r^2 + 1 - 2r\rho\cos(\theta - \phi)} \equiv K^{(0)}(r, \theta, \rho, \phi). \tag{8.1.52}$$

If we expand K in the form

$$K(r, \theta, \rho, \phi; \epsilon) = K^{(0)}(r, \theta, \rho, \phi) + \epsilon K^{(1)}(r, \theta, \rho, \phi) + O(\epsilon^2), \tag{8.1.53}$$

we see that $\Delta_P K^{(1)} = 0$. To compute the boundary condition for $K^{(1)}$, we first expand (8.1.51b) near $r = 1$,

$$K(1, \theta, \rho, \phi; \epsilon) + \epsilon f(\theta) \frac{\partial K}{\partial r}(1, \theta, \rho, \phi; \epsilon) = O(\epsilon^2),$$

then use (8.1.53) to obtain

$$K^{(0)}(1, \theta, \rho, \phi) + \epsilon K^{(1)}(1, \theta, \rho, \phi) + \epsilon f(\theta) \frac{\partial K^{(0)}}{\partial r}(1, \theta, \rho, \phi) = O(\epsilon^2).$$

Since $K^{(0)}(1, \theta, \rho, \phi) = 0$, we must have

$$K^{(1)}(1, \theta, \rho, \phi) = -f(\theta) \frac{\partial K^{(0)}}{\partial r}(1, \theta, \rho, \phi). \tag{8.1.54}$$

When we use (8.1.52) to evaluate the right-hand side, we find the following boundary condition:

$$K^{(1)}(1, \theta, \rho, \phi) = \frac{f(\theta)(\rho^2 - 1)}{1 + \rho^2 - 2\rho\cos(\theta - \phi)} \equiv g(\theta, \rho, \phi). \tag{8.1.55}$$

The solution for $K^{(1)}$ is given by Poisson's formula (2.6.36)

$$K^{(1)}(r, \theta, \rho, \phi) = \frac{1 - r^2}{2\pi} \int_0^{2\pi} \frac{g(\theta', \rho, \phi)}{1 + r^2 - 2r\cos(\theta - \theta')} d\theta'. \tag{8.1.56}$$

An application of this result is outlined in Problem 8.1.4. The above idea generalizes to any operator and bounded domain for which Green's function is known.

Problems

8.1.1 Expand the initial data in (8.1.20c) first with respect to ϵ in the form

$$f(x; \epsilon) = f^{(0)}(x) + \epsilon f^{(1)}(x) + O(\epsilon^2), \tag{8.1.57a}$$

$$g(x; \epsilon) = g^{(0)}(x) + \epsilon g^{(1)}(x) + O(\epsilon^2), \tag{8.1.57b}$$

and then expand each of the functions of x in a series of the unperturbed eigenfunctions. Thus, for example, we have

$$f^{(0)}(x) = \sum_{n=1}^{\infty} f_n^{(0)} \xi_n^{(0)}(x), \tag{8.1.58a}$$

where the constants $f_n^{(0)}$ are defined by

$$f_n^{(0)} = \int_0^{\pi} f^{(0)}(x)\xi_n^{(0)}(x)dx. \qquad (8.1.58b)$$

Also expand the $A_n(\epsilon)$ and $B_n(\epsilon)$ in (8.1.35) with respect to ϵ in the form

$$A_n(\epsilon) = A_n^{(0)} + \epsilon A_n^{(1)} + O(\epsilon^2), \qquad (8.1.59a)$$

$$B_n(\epsilon) = B_n^{(0)} + \epsilon B_n^{(1)} + O(\epsilon^2). \qquad (8.1.59b)$$

Show that the constants $A_n^{(i)}$ and $B_n^{(i)}$ for $i = 1, 2$ are given by

$$A_n^{(0)} = \frac{g_n^{(0)}}{n}, \quad A_n^{(1)} = \frac{g_n^{(1)}}{n} - \frac{g_n^{(0)}\lambda_n^{(1)}}{2n^3} - \sum_{j=1}^{\infty} a_{jn} g_j^{(0)}, \qquad (8.1.60a)$$

$$B_n^{(0)} = f_n^{(0)}, \quad B_n^{(1)} = f_n^{(1)} - \sum_{j=1}^{\infty} a_{jn} f_j^{(0)}. \qquad (8.1.60b)$$

8.1.2 Consider the weakly nonlinear eigenvalue problem

$$u_n'' + \lambda_n u_n - \epsilon x u_n^2 = 0, \quad 0 \le x \le \pi, \qquad (8.1.61a)$$

$$u_n(0; \epsilon) = 0, \quad u_n(\pi; \epsilon) = 0. \qquad (8.1.61b)$$

Assume that $0 < \epsilon \ll 1$, and that the $u_n(x; \epsilon)$ and $\lambda_n(\epsilon)$ have the expansions

$$u_n(x; \epsilon) = \left(\frac{2}{\pi}\right)^{1/2} \sin nx + \epsilon u_n^{(1)}(x) + O(\epsilon^2), \qquad (8.1.62a)$$

$$\lambda_n(\epsilon) = n^2 + \epsilon\lambda_n^{(1)} + O(\epsilon^2). \qquad (8.1.62b)$$

a. Show that if we expand $u_n^{(1)}$ as in (8.1.45a) we obtain

$$a_{nk} = \begin{cases} \frac{(-1)^k}{n^2-k^2}\left(\frac{2}{\pi}\right)^{1/2} \frac{4n^2}{k(k^2-4n^2)} & \text{if } k \neq n, k \neq 2n, \\ 0 & \text{if } k = n, \\ \frac{1}{8n^3}\left(\frac{2}{\pi}\right)^{1/2} & \text{if } k = 2n, \end{cases} \qquad (8.1.63a)$$

$$\lambda_n^{(1)} = \frac{4}{3n}\left(\frac{2}{\pi}\right)^{1/2} (-1)^{n+1}. \qquad (8.1.63b)$$

b. For $n = 2$, calculate the expansion of $x_0(\epsilon)$ to two terms, where x_0 is defined as the point where u_2 vanishes; that is, $u_2(x_0; \epsilon) = 0$.

8.1.3 Calculate the eigenfunctions corresponding to the first two eigenvalues λ_1 and λ_2 (where $0 < \lambda_1^{(0)} < \lambda_2^{(0)}$) for the following linear problem inside the unit circle:

$$\frac{\partial^2 u_n}{\partial r^2} + \frac{1}{r}\frac{\partial u_n}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_n}{\partial \theta^2} + (\lambda_n - \epsilon r^2 \sin 2\theta)u_n = 0, \qquad (8.1.64a)$$

$$u_n(1, \theta) = 0, \quad u_n(r, \theta) = \text{ finite as } r \to 0. \qquad (8.1.64b)$$

8.1.4 We wish to solve

$$\Delta u = 0 \text{ in } \mathcal{D}, \tag{8.1.65a}$$

$$u(1 + \epsilon f(\theta), \theta) = h(\theta), \tag{8.1.65b}$$

where $\mathcal{D}$ is the domain $0 \leq r \leq 1 + \epsilon f(\theta)$, $0 \leq \theta \leq 2\pi$, and f and h are prescribed 2π-periodic functions of θ.

a. Calculate the solution directly to $O(\epsilon)$ in integral form assuming that u has the expansion

$$u(r, \theta; \epsilon) = u^{(0)}(r, \theta) + \epsilon u^{(1)}(r, \theta) + O(\epsilon^2). \tag{8.1.66}$$

Thus, $u^{(0)}$ is given by (2.6.36):

$$u^{(0)}(r, \theta) = \frac{1 - r^2}{2\pi} \int_0^{2\pi} \frac{h(\theta')}{1 + r^2 - 2r\cos(\theta - \theta')} d\theta'. \tag{8.1.67a}$$

Show that $u^{(1)}$ is given by

$$u^{(1)}(r, \theta) = -\frac{1 - r^2}{2\pi} \int_0^{2\pi} \frac{f(\theta') \dfrac{\partial u^{(0)}}{\partial r}(1, \theta')}{1 + r^2 - 2r\cos(\theta - \theta')} d\theta', \tag{8.1.67b}$$

and evaluate $(\partial u^{(0)}/\partial r)(1, \theta)$ using (8.1.67a).

b. Use Green's function $K^{(0)} + \epsilon K^{(1)}$ calculated in (8.1.52) and (8.1.56) in the generalized Poisson formula (2.6.12) to calculate the solution of (8.1.65) to $O(\epsilon)$. Show that this agrees with the result you found in part (a).

8.2 Matched Asymptotic Expansions

In this section we discuss a class of problems characterized by the property that the solution $u(\mathbf{x}; \epsilon)$ has a different asymptotic expansion in certain subdomains of the $\mathbf{x}$-space. These subdomains have boundaries that depend on ϵ and can be established either trivially from knowledge of the exact solution or indirectly from the governing differential equations. Typically, the order of magnitude of certain terms in the differential equation will depend on the solution domain; in certain thin layers near the boundaries or along particular interior regions, a term multiplied by ϵ in the differential equation becomes important because the term itself is large there. By considering all the possible limiting forms of the governing equation, we can usually establish the location and nature of these layers. An example of this behavior was discussed in Section 5.3.6, where we studied shocks and corner layers for Burgers' equation.

8.2.1 An Ordinary Differential Equation

We begin our discussion with the following linear boundary-value problem for a second-order equation of some historical interest:

$$\epsilon u'' + u' = \frac{1}{2}, \quad 0 < \epsilon \ll 1, \quad 0 \leq x \leq 1 \tag{8.2.1a}$$

$$u(0; \epsilon) = 0, \quad u(1; \epsilon) = 1. \tag{8.2.1b}$$

A slightly more general version of this problem was first introduced by K.O. Friedrichs in 1942 to illustrate boundary-layer behavior. See [30] for the original reference and a more detailed discussion.

The solution is

$$u(x; \epsilon) = \frac{1 - e^{-x/\epsilon}}{2(1 - e^{-1/\epsilon})} + \frac{x}{2}, \tag{8.2.2}$$

and we shall examine the asymptotic behavior of this solution in order to motivate our later discussion concerning a perturbation analysis based only on (8.2.1).

(i) *Outer and inner limits*

For any fixed x in $0 < x \leq 1$, the limiting value as $\epsilon \to 0$ of u derived from (8.2.2) is

$$\lim_{\substack{\epsilon \to 0 \\ x \text{ fixed} \neq 0}} u(x; \epsilon) = \frac{1 + x}{2} \equiv u_0(x). \tag{8.2.3}$$

We shall refer to the limit process $\epsilon \to 0$ with x fixed $\neq 0$ as the *outer limit process*, and to $u_0(x)$ as the *outer limit* of $u(x; \epsilon)$. We note that $u_0(x)$, which is the only term in the asymptotic expansion of $u(x; \epsilon)$ in the outer limit, does not satisfy the boundary condition at $x = 0$. In fact, the statement

$$u(x; \epsilon) = u_0(x) + \text{T.S.T.} \tag{8.2.4}$$

is not uniformly valid if $x = O(\epsilon)$. Here, T.S.T. denotes transcendentally small terms—that is, terms that are $o(\epsilon^\alpha)$ for arbitrarily large positive α (see (A.3.6)).

The difficulty at $x = 0$ is clear; we ignored $e^{-x/\epsilon}$ in deriving (8.2.3), but this term is important if $x = O(\epsilon)$. Thus, in a thin *boundary layer* over the interval $0 \leq x \leq \epsilon x^*$ (where x^* is an arbitrary positive constant), this term equals $e^{-x^*} = O(1)$. This behavior suggests that we should consider an *inner limit process* where we let $\epsilon \to 0$ for a fixed value of the boundary-layer variable $x^* \equiv x/\epsilon$. In this case, we replace x in (8.2.2) with ϵx^* and obtain

$$u(\epsilon x^*; \epsilon) \equiv u^*(x^*; \epsilon) = \frac{1 - e^{-x^*}}{2(1 - e^{-1/\epsilon})} + \epsilon \frac{x^*}{2}. \tag{8.2.5}$$

We define the *inner limit* of (8.2.2) to be

$$\lim_{\substack{\epsilon \to 0 \\ x^* \text{fixed} \neq \infty}} u(\epsilon x^*; \epsilon) = \frac{1 - e^{-x^*}}{2} \equiv u_0^*(x^*). \tag{8.2.6}$$

Now, we notice that $u_0^*(x^*)$ does not satisfy the boundary condition at $x = 1$; that is, at $x^* = 1/\epsilon$. In fact, $u_0^*(x^*)$ is the leading term of the following asymptotic expansion of $u^*(x^*;\epsilon)$, constructed in the limit $\epsilon \to 0$ with x^* fixed and not equal to infinity:

$$u^*(x^*;\epsilon) = \frac{1 - e^{-x^*}}{2} + \epsilon\frac{x^*}{2} + \text{T.S.T.} \tag{8.2.7}$$

This *inner expansion*, which terminates after two terms for this example, is not uniformly valid if $x^* = O(\epsilon^{-1})$.

(ii) *Extended domains of validity; matching*

We shall now examine in what sense the outer and inner limits approximate the function in (8.2.2). The outer limit (8.2.3) is formally derived by letting $\epsilon \to 0$ for arbitrary fixed x in the half-open interval $0 < x \leq 1$. We shall demonstrate that (8.2.3) actually remains valid in an *extended* domain that corresponds to $\epsilon \to 0$ *with* $x \to 0$ at some maximal rate relative to ϵ. To be more precise, we set $x \equiv \eta(\epsilon)x_\eta$ for a function $\eta(\epsilon) \ll 1$ to be specified and a *fixed* $x_\eta > 0$. Thus, as $\epsilon \to 0$ with x_η fixed, x tends to zero "at the rate" $\eta(\epsilon)$.

Observe now that the statement

$$\lim_{\substack{\epsilon\to 0\\ x_\eta \text{ fixed}\neq 0}} \{u(\eta x_\eta;\epsilon) - u_0(\eta x_\eta)\} = 0 \tag{8.2.8}$$

holds as long as $e^{-\eta x_\eta/\epsilon} \to 0$ and $\eta \to 0$, as postulated. Actually, $e^{-\eta x_\eta/\epsilon}$ must be transcendentally small as $\epsilon \to 0$, and this implies that $\epsilon|\log\epsilon| \ll \eta$; if $\eta = O_s(\epsilon|\log\epsilon|)$, then $e^{-\eta x_\eta/\epsilon} = O_s(\epsilon^{x_\eta})$, which does vanish as $\epsilon \to 0$ but is not transcendentally small. The result (8.2.8) remains true if $\eta = O_s(1)$; therefore the *extended domain of validity* of the outer limit is defined by the set of functions $\eta(\epsilon)$ satisfying

$$\epsilon|\log\epsilon| \ll \eta(\epsilon) \underline{\ll} 1. \tag{8.2.9}$$

Here we have introduced the notation

$$\phi(\epsilon)\underline{\ll}\psi(\epsilon) \text{ if } \phi \ll \psi \text{ or } \phi = O_s(\psi). \tag{8.2.10}$$

The shaded region in Figure 8.1a represents this domain in the $x\epsilon$-plane. This diagram should not be interpreted literally to mean that x must be in the shaded region as $\epsilon \to 0$, since x_η is an arbitrary positive constant. Moreover, $\eta_0(\epsilon)$ is also arbitrary as long as $\eta_0 = O_s(1)$. Roughly speaking, the left boundary of the extended domain restricts the admissible class of functions $\eta(\epsilon)$ to those that tend to zero "more slowly" than $\epsilon|\log\epsilon|$.

One final point concerning this very special example is that all our statements concerning the outer limit remain true if the bracketed expression in (8.2.8) is divided by ϵ^α for any $\alpha > 0$. This is because the outer expansion terminates after one term with a transcendentally small remainder.

Let us now examine the inner limit defined by (8.2.6). This limit corresponds formally to having $x^* \equiv x/\epsilon$ fixed as $\epsilon \to 0$. To see how far we can extend the

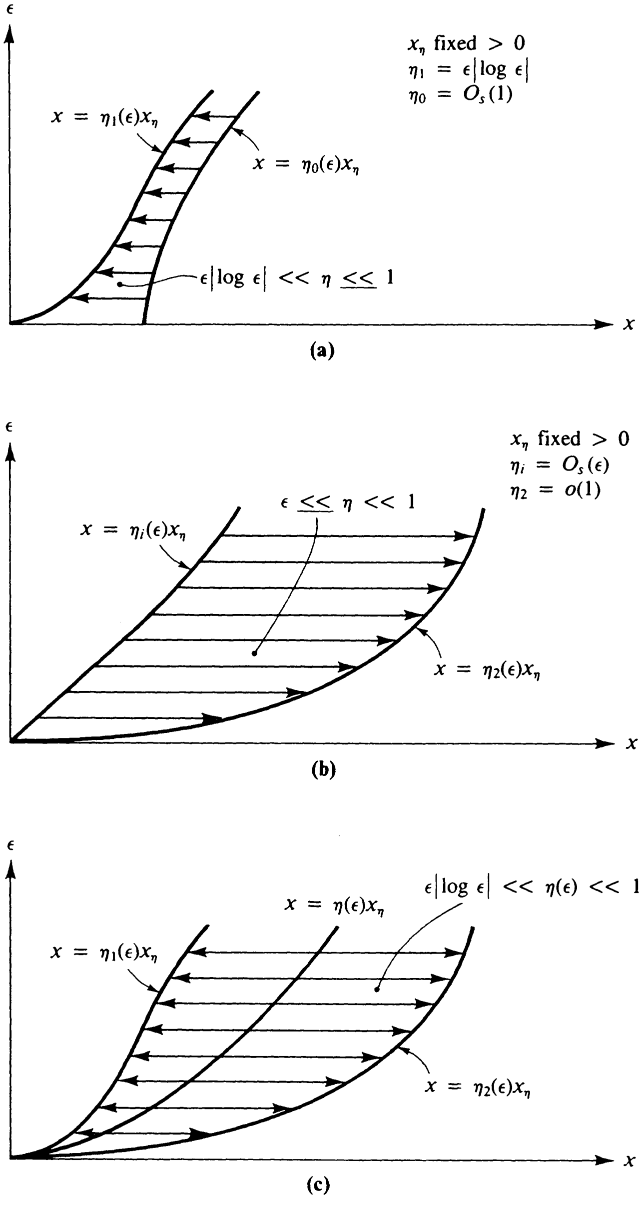

FIGURE 8.1. Extended domain of validity of the outer limit (a), the inner limit (b), and their common overlap domain (c)

domain of validity of this limit, we again set $x = \eta(\epsilon)x_\eta$; that is, $x^* = \eta(\epsilon)x_\eta/\epsilon$. We then find that

$$\lim_{\substack{\epsilon \to 0 \\ x_\eta \text{ fixed} \neq \infty}} \{u^*(\eta x_\eta/\epsilon; \epsilon) - u_0^*(\eta x_\eta/\epsilon)\} = 0 \tag{8.2.11}$$

actually remains true as long as $\eta \ll 1$. Thus, the extended domain of validity of the inner limit is given by the class of functions $\eta(\epsilon)$ with the property that

$$\epsilon \underline{\ll} \eta(\epsilon) \ll 1, \tag{8.2.12}$$

as shown in Figure 8.1b.

We note the following remarkable result for this example. The extended domains of validity of the outer and inner limits overlap in the sense that their intersection consists of the nonempty domain defined by the class of functions $\eta(\epsilon)$ with the property

$$\epsilon |\log \epsilon| \ll \eta(\epsilon) \ll 1 \tag{8.2.13}$$

and described qualitatively in Figure 8.1c. Any $\eta(\epsilon)$ in the class (8.2.13) lies in the extended domain of validity of both the outer and inner limits. For such an $\eta(\epsilon)$, (8.2.8) and (8.2.11) both hold; therefore, their difference vanishes, and we have the *direct matching condition*

$$\lim_{\substack{\epsilon \to 0 \\ x_\eta \text{ fixed} \neq 0, \neq \infty}} \{u_0(\eta x_\eta) - u_0^*(\eta x_\eta/\epsilon)\} = 0 \tag{8.2.14}$$

for any η such that $\epsilon |\log \epsilon| \ll \eta \ll 1$.

This result in the form

$$u_0(0) = u_0^*(\infty) \tag{8.2.15}$$

was first proposed on physical grounds by L. Prandtl in 1905 as the matching condition for the horizontal component of the velocity (u_0) of an inviscid flow evaluated on the boundary of a body ($x = 0$) with the velocity of a viscous boundary-layer flow (u_0^*) evaluated at ∞. It was shown by S. Kaplun in 1957 that the direct matching condition (8.2.14) is a special case of a more general situation where the outer and inner limits can always be matched with an *intermediate limit*. It is beyond the scope of this chapter to give an account of this theory. The reader is referred to [30] and the works cited there for a comprehensive treatment.

The result in (8.2.14) (and its extension to higer order, as discussed in Section 8.2.2) turns out to be sufficient for most applications as long as we retain enough terms in each expansion. We demonstrate this in the examples we present in this section. Other examples can be found in [26] and [30].

(iii) *Uniformly valid result to $O(1)$*

We have seen that the outer limit fails if $x = O_s(\epsilon)$ as $\epsilon \to 0$, and that the inner limit fails if $x = O_s(1)$ as $\epsilon \to 0$. But each of these limits is individually valid in its extended domain. We show next that one can construct a *composite* expression that tends to the outer limit as $\epsilon \to 0$ with x fixed $\neq 0$, and to the inner limit as

$\epsilon \to 0$ with x/ϵ fixed $\neq \infty$. The idea is to add the inner and outer limits and then subtract from this sum the term that is common to both expressions in the overlap domain. In our case, this common term is $\frac{1}{2}$. Therefore, we propose

$$\widetilde{u}_0(x; \epsilon) \equiv u_0(x) + u_0^*(x^*) - \frac{1}{2} = \frac{1 + x - e^{-x/\epsilon}}{2}, \tag{8.2.16}$$

and verify that this expression satisfies the dual requirements

$$\lim_{\substack{\epsilon \to 0 \\ x \text{ fixed} \neq 0}} \widetilde{u}_0(x; \epsilon) = \frac{1 + x}{2} = u_0(x) \tag{8.2.17a}$$

and

$$\lim_{\substack{\epsilon \to 0 \\ x^* \text{ fixed} \neq \infty}} \widetilde{u}_0(\epsilon x^*; \epsilon) = \frac{1 - e^{-x^*}}{2} = u_0^*(x^*). \tag{8.2.17b}$$

We have shown that $\widetilde{u}_0$ tends to u_0 under the outer limit process, and to u_0^* under the inner limit process. It then follows that $\widetilde{u}_0$ approximates $u(x; \epsilon)$ to $O(1)$ as $\epsilon \to 0$ uniformly in $0 \leq x \leq 1$. In fact, for this example $(u - \widetilde{u}_0)$ is transcendentally small as $\epsilon \to 0$.

A sketch of the three functions $u_0(x)$, $u_0^*(x^*)$, and $\widetilde{u}_0(x; \epsilon)$ is shown in Figure 8.2 . The matching condition (8.2.15) corresponds graphically to the fact that the asymptote of the u_0^* curve as $x^* \to \infty$ is at the value $u_0(0)$. A more important observation is that the composite expression $\widetilde{u}_0$ is a smooth function of x; it is not just a "patching" of u_0^* over part of the interval $0 \leq x \leq 1$ with u_0 over the remainder. In fact, for this example u_0^* and u_0 do not intersect, and it is not possible to patch these two functions at any point in the unit interval.

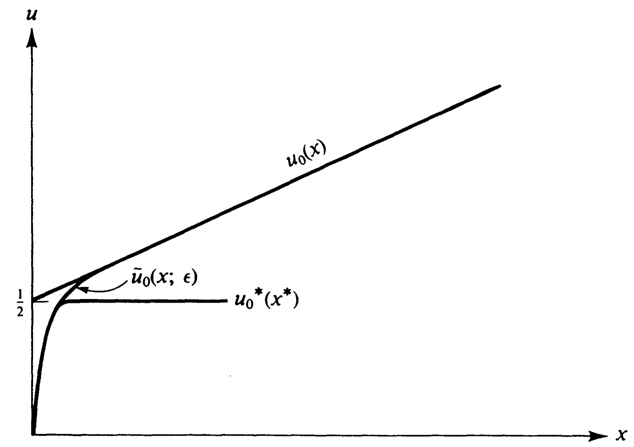

FIGURE 8.2. Inner limit u_0^*, outer limit u_0, and composite approximation $\widetilde{u}_0$

(iv) *Distinguished limits of the differential equation*

In the preceding discussion we derived the outer and inner limits from the exact solution. A useful approximation procedure will be possible for more general cases only if the same information can be derived directly from the governing differential equation in a systematic way without knowledge of the exact solution. We now demonstrate that this is the case for the present example; less trivial examples will be discussed in Sections 8.2.2–8.2.5.

Since the rate at which x approaches the point of nonuniformity is the key feature that distinguishes the two limits that we found, let us investigate what all the possible limit processes are for (8.2.1). We introduce the general transformation of the independent variable given by

$$x_\eta \equiv \frac{x - x_0}{\eta(\epsilon)} \tag{8.2.18}$$

to study how the various terms in (8.2.1) behave under all possible rescalings $\eta(\epsilon)$. Here x_0 is some fixed point in $0 \leq x_0 \leq 1$ and $0 < \eta(\epsilon) \ll 1$ but is otherwise arbitrary. Equation (8.2.1a) transforms to

$$\frac{d^2U}{dx_\eta^2} + \frac{\eta}{\epsilon}\frac{dU}{dx_\eta} = \frac{\eta^2}{2\epsilon} \tag{8.2.19}$$

after multiplication by η^2/ϵ, where we are setting $u(\eta x_\eta; \epsilon) \equiv U(x_\eta; \epsilon)$. Since (8.2.1) is autonomous, the transformed problem does not depend on x_0.

We note that for any $\eta(\epsilon) = O_s(1)$, the second and third terms in (8.2.19) are both $O_s(\epsilon^{-1})$ and dominate as $\epsilon \to 0$. With no loss of generality, we may set $\eta = 1$ to obtain the outer limiting equation

$$\eta_s = O_s(1) : \quad \frac{dU}{dx_\eta} = \frac{1}{2} + O(\epsilon), \quad x_\eta = x - x_0. \tag{8.2.20}$$

Similarly, for any $\eta = O_s(\epsilon)$, the first two terms in (8.2.19) are in dominant balance. We set $\eta = \epsilon$ to obtain the inner limiting equation

$$\eta_s = O_s(\epsilon) : \quad \frac{d^2U}{dx_\eta^2} + \frac{dU}{dx_\eta} = O(\epsilon), \quad x_\eta = \frac{x - x_0}{\epsilon}. \tag{8.2.21}$$

These are the only two distinguished limits for (8.2.19) in the sense that they each correspond to a function $\eta(\epsilon)$ that is of a *specific order* in ϵ and for which a subset of terms in (8.2.1a) are in dominant balance. For example, the limiting equation

$$\frac{dU}{dx_\eta} = o(1) \tag{8.2.22}$$

that results for any $\eta(\epsilon)$ with $\epsilon \ll \eta \ll 1$ is not is not a distinguished limit because it does not correspond to an $\eta(\epsilon)$ having a specific order in ϵ.

We now demonstrate that knowledge of the two limiting equations (8.2.20)–(8.2.21), and our ability to match their solutions, uniquely determines x_0 and the correct limits $u_0(x)$ and $u_0^*(x^*)$.

First we consider (8.2.20) and denote the solution by u_0. We have $u_0 = (x - x_0)/2 + C$, where C is a constant, and with no loss of generality, we absorb the constant $-x_0/2$ in C and write

$$u_0 = \frac{x}{2} + C. \tag{8.2.23}$$

Let us defer the determination of C until we have established the behavior of the inner limit. The general solution of (8.2.21) is

$$u_0^*(x^*) = A + Be^{-x^*}, \tag{8.2.24}$$

where A and B are constants and $x^* \equiv (x - x_0)/\epsilon$.

If $x_0 = 1$, that is, we assume a boundary layer at $x = 1$, the boundary condition $u(1; \epsilon) = 1$ must be satisfied by (8.2.24). This means that we must set $u_0^*(0) = 1$, that is, $B = 1 - A$, and the inner limit has the form

$$u_0^*(x^*) = A + (1 - A)e^{-x^*}. \tag{8.2.25}$$

Now, the matching with a solution outside the boundary layer has $x < x_0$; that is, $x^* \to -\infty$ in the matching domain. But e^{-x^*} grows exponentially large in this limit and must therefore be excluded by setting $A = 1$. In this case the inner limit is simply $u_0^* = 1$, and the outer limit (8.2.23) must satisfy the boundary condition at $x = 0$, which gives $C = 0$, so $u_0 = x/2$. Clearly, the inner and outer limits do not match for the choice $x_0 = 1$. In general, a boundary-layer solution that grows exponentially in the domain of interest is inappropriate (for an exception to this rule, see the example in Problem 8.2.1). Similarly, the choice of any positive x_0 will require that we set $A = 1$, with the result that the inner and outer limits will not match. The only possible location of a boundary layer for this example is at $x = 0$, and we must choose $x_0 = 0$. In this case, $x^* \to \infty$ in the matching domain and (8.2.25) gives $u_0^* \to A$, where A is unknown. However, since we know that we can have a layer only at $x = 0$, the solution (8.2.23) must hold everywhere else. In particular, u_0 must satisfy the boundary condition at $x = 1$, which means that $C = \frac{1}{2}$. Now we can impose the matching condition (8.2.14) or (8.2.15) to conclude the correct result that $A = \frac{1}{2}$.

8.2.2 *A Second Example*

A less trivial problem for which we do not attempt an exact solution is

$$\epsilon u'' + (1 + 2x)u' + u = x, \tag{8.2.26a}$$

$$u(0; \epsilon) = 0, \quad u(1; \epsilon) = 1. \tag{8.2.26b}$$

(i) *Solution to $O(1)$*

The outer limit $u_0(x)$ obeys

$$(1 + 2x)u_0' + u_0 = x \tag{8.2.27}$$

and has the solution

$$u_0(x) = \frac{c_0}{(1+2x)^{1/2}} + \frac{x-1}{3}, \tag{8.2.28}$$

where c_0 is an arbitrary constant. The choice of c_0 depends on which of the two boundary conditions derived from (8.2.26b) is to be satisfied.

In order to answer this question, we study the inner limit of (8.2.26a) for an unspecified layer location to see where such a layer is appropriate. As in the example of the previous section, we introduce the general scaled variable x_η by

$$x_\eta \equiv \frac{x - x_0}{\eta(\epsilon)}, \tag{8.2.29}$$

with x_0 and $\eta(\epsilon)$ as yet unspecified. Equation (8.2.26a) for $U(x_\eta;\epsilon) \equiv u(x_0 + \eta x_\eta;\epsilon)$ becomes

$$\frac{d^2U}{dx_\eta^2} + \frac{\eta}{\epsilon}(1 + 2x_0 + 2\eta x_\eta)\frac{dU}{dx_\eta} + \frac{\eta^2}{\epsilon}U = \frac{\eta^2}{\epsilon}(x_0 + \eta x_\eta). \tag{8.2.30}$$

The terms multiplied by η^2/ϵ are less important than the second term for any $\eta \ll 1$. In fact, the most general limit results for $\eta = O_s(\epsilon)$, and this is a distinguished limit. For simplicity, we choose $\eta = \epsilon$ and denote

$$x^* \equiv \frac{x - x_0}{\epsilon}. \tag{8.2.31}$$

The inner limiting equation is then

$$\frac{d^2u_0^*}{dx^{*2}} + (1 + 2x_0)\frac{du_0^*}{dx^*} = 0 \tag{8.2.32}$$

with solution

$$u_0^*(x^*) = A_0e^{-(1+2x_0)x^*} + B_0. \tag{8.2.33}$$

If $x_0 = 1$, that is, the boundary layer is at the right end, (8.2.33) must satisfy the right boundary condition $u^*(0) = 1$, which gives $A_0 = 1 - B_0$, and (8.2.33) becomes

$$u_0^*(x^*) = (1 - B_0)e^{-3x^*} + B_0. \tag{8.2.34}$$

To prevent exponential growth in the interior of our domain, we must set $B_0 = 1$. In this case, we must choose $c_0 = \frac{1}{3}$ in order to have $u_0(x)$ satisfy the left boundary condition. Again, we see that with $c_0 = \frac{1}{3}$, $u_0(x)$ as given in (8.2.28) does not match with the inner limit $u^* = 1$. Similarly, any choice of x_0 other than $x_0 = 0$ leads to the same difficulty.

Therefore, the boundary layer must be at $x = 0$. Thus, $x_0 = 0$, $x^* = x/\epsilon$, and (8.2.33) reads

$$u_0^*(x^*) = B_0(1 - e^{-x^*}) \tag{8.2.35}$$

after we impose the left boundary condition. We set $c_0 = \sqrt{3}$ to have (8.2.28) satisfy the right boundary condition and determine B_0 from the matching condition to $O(1)$. This is just (8.2.14) with the preceding values of u_0 and u_0^*, and we obtain

$$B_0 = \frac{3\sqrt{3} - 1}{3}. \tag{8.2.36}$$

Thus, matching is possible for all $\eta(\epsilon)$ in the overlap domain

$$\epsilon|\log\epsilon| \ll \eta \ll 1. \tag{8.2.37}$$

By adding the inner and outer limits and subtracting the common part, B_0, we obtain the composite approximation

$$\tilde{u}_0(x;\epsilon) = \frac{\sqrt{3}}{(1+2x)^{1/2}} + \frac{x-1}{3} - \left(\frac{3\sqrt{3}-1}{3}\right)e^{-x/\epsilon}, \tag{8.2.38}$$

which is uniformly valid to $O(1)$ over the entire unit interval. The inner limit less the common part is often called the *boundary-layer correction* because it is a term that is transcendentally small everywhere except inside the boundary layer, where it combines with the outer limit to satisfy the boundary conditon. In this example, the boundary-layer correction to $O(1)$ is

$$-\left(\frac{3\sqrt{3}-1}{3}\right)e^{-x/\epsilon}.$$

(ii) *Solution to* $O(\epsilon)$

Having established the location of the boundary layer, we can proceed to compute the next term in each of the two limit process expansions. As we have seen before, the choice of asymptotic sequence is dictated by our need to have the most general (richest) equations governing each successive term in our expansions. In the present case, it is easily seen that this requires choosing the next term in both the inner and outer expansions to be $O(\epsilon)$.

We substitute the two-term outer expansion ($\epsilon \to 0$, x fixed $\neq 0$)

$$u(x;\epsilon) = u_0(x) + \epsilon u_1(x) + o(\epsilon) \tag{8.2.39}$$

into (8.2.26a) and find that $u_1(x)$ obeys

$$(1+2x)u_1' + u_1 = -u_0'' = -\frac{3\sqrt{3}}{(1+2x)^{5/2}}. \tag{8.2.40}$$

We know that the outer expansion must satisfy the right boundary condition, hence $u_1(1) = 0$. The solution of (8.2.40) subject to this condition is easily found as

$$u_1(x) = \frac{2 - x - x^2}{\sqrt{3}(1+2x)^{5/2}}. \tag{8.2.41}$$

Next, we construct the inner expansion in the form ($\epsilon \to 0$, x^* fixed $\neq \infty$)

$$u^*(x^*;\epsilon) \equiv u(\epsilon x^*;\epsilon) = u_0^*(x^*) + \epsilon u_1^*(x^*) + o(\epsilon). \tag{8.2.42}$$

The change of variable $x \to x^*$ implies that $u^*(x^*; \epsilon)$ obeys

$$\frac{d^2u^*}{dx^{*2}} + (1 + 2\epsilon x^*)\frac{du^*}{dx^*} + \epsilon u^* = \epsilon^2 x^*,$$
$$u^*(0; \epsilon) = 0.$$

We do not express the right boundary condition in terms of x^*, as (8.2.42) is not supposed to hold there. Substituting (8.2.42) into the above gives

$$\frac{d^2u_1^*}{dx^{*2}} + \frac{du_1^*}{dx^*} = -2x^*\frac{du_0^*}{dx^*} - u_0^*$$
$$= -2B_0x^*e^{-x^*} - B_0 + B_0e^{-x^*} \qquad (8.2.43a)$$
$$u_1^*(0) = 0, \qquad (8.2.43b)$$

where B_0 is the known constant in (8.2.36). The solution of (8.2.43) involving an undetermined constant B_1 is found to be

$$u_1^*(x^*) = B_0e^{-x^*}(x^{*2} + x^* - 1) - B_0(x^* - 1) + B_1(1 - e^{-x^*}). \qquad (8.2.44)$$

(iii) *Matching to $O(\epsilon)$; composite expansion*

The direct matching to $O(\epsilon)$ of the two-term outer and two-term inner expansion requires that (see(8.2.14))

$$\lim_{\substack{\epsilon\to 0\\ x_\eta \text{ fixed}\neq 0,\infty}} \frac{1}{\epsilon}\left\{u_0(\eta x_\eta) + \epsilon u_1(\eta x_\eta) - u_0^*\left(\frac{\eta x_\eta}{\epsilon}\right) - \epsilon u_1^*\left(\frac{\eta x_\eta}{\epsilon}\right)\right\} = 0 \qquad (8.2.45)$$

in some overlap domain to be determined.

Note that calculating an outer and inner expansion to some order in ϵ does not always imply that these two expansions can be matched to that particular order in ϵ. Here, for example, we have computed each of the outer and inner expansions to order ϵ, and we propose to match these to order ϵ also. In general, we may need to retain terms of order higher than the order of the matching in one or the other of the two expansions. An example of this is discussed in Section 2.2.3 of [26]. In the more general matching procedure of Kaplun, this situation does not arise. In fact, Kaplun's procedure was formulated to resolve the classical problem of low Reynolds number flow, where the inner and outer limits do not match directly. More details can be found in Section 2.5 of [30].

In preparation for imposing (8.2.45), we expand u_0 and u_1 for x small, and u_0^* and u_1^* for x^* large. This gives

$$u_0(x) = \frac{3\sqrt{3}-1}{3} - \frac{3\sqrt{3}-1}{3}x + O(x^2), \qquad (8.2.46a)$$

$$u_1(x) = \frac{2}{\sqrt{3}} + O(x), \qquad (8.2.46b)$$

$$u_0^*(x^*) = \frac{3\sqrt{3}-1}{3} + \text{T.S.T.}, \qquad (8.2.46c)$$

$$u_1^*(x^*) = \frac{3\sqrt{3}-1}{3} - \left(\frac{3\sqrt{3}-1}{3}\right)x^* + B_1 + \text{T.S.T.} \tag{8.2.46d}$$

We first cancel the constant terms of $O(\epsilon^{-1})$ out of (8.2.45) that have already been matched. Next we note that the term $-(3\sqrt{3}-1)x/3$ in u_0, which produces a singular contribution proportional to $\eta/\epsilon \to \infty$ in (8.2.45), matches identically with a corresponding term proportional to x^* in u_1^*. This provides us with a partial check of the calculations for u_1^*. The next largest terms left are the two constants contributed by u_1 and u_1^*. Matching these requires that we set

$$B_1 + \frac{3\sqrt{3}-1}{3} = \frac{2}{\sqrt{3}},$$

or

$$B_1 = \frac{1-\sqrt{3}}{3}. \tag{8.2.47}$$

The two terms that have not been matched to $O(\epsilon)$ are the $O(x^2)$ remainder in (8.2.46a) and the $O(x)$ remainder in (8.2.46b). These contribute remainders of order η^2/ϵ and η, respectively, in (8.2.45). We need be concerned only about having $\eta^2/\epsilon \to 0$, since $\eta \to 0$ automatically in the overlap domain. To have $\eta^2/\epsilon \to 0$, we must restrict η such that $\eta \ll \epsilon^{1/2}$. Since $\epsilon|\log\epsilon| \ll \epsilon^{1/2}$ and we must have $\epsilon|\log\epsilon| \ll \eta$ in order to ignore the transcendentally small terms in (8.2.46c)–(8.2.46d), we conclude that the overlap domain is defined by

$$\epsilon|\log\epsilon| \ll \eta(\epsilon) \ll \epsilon^{1/2}, \tag{8.2.48}$$

and this is "smaller" than (8.2.37). This is typical for higher-order matching.

Even though the result (8.2.47) could have been directly deduced by comparing the various expansions in (8.2.46), it is essential for exhibiting the overlap domain (8.2.48) that we keep track of all the terms that we ignored in the matching, and that we express these in terms of the x_η variable in the overlap domain. See Chapter 2 of [26] for more details.

The reader should verify that the expression

$$\tilde{u}^{(1)}(x;\epsilon) = \tilde{u}_0(x;\epsilon) + \tilde{u}_1(x;\epsilon) + O(\epsilon^2), \tag{8.2.49a}$$

where $\tilde{u}_0$ is given by (8.2.38), and $\tilde{u}_1$ is defined as

$$\tilde{u}_1(x;\epsilon) = \epsilon\left[\frac{2-x-x^2}{\sqrt{3}(1+2x)^{5/2}} + \left(\frac{3\sqrt{3}-1}{3}\right)\left(\frac{x^2}{\epsilon^2} + \frac{x}{\epsilon} - 1\right)e^{-x/\epsilon} - \frac{1-\sqrt{3}}{3}e^{-x/\epsilon}\right], \tag{8.2.49b}$$

gives the inner and outer expansions to order ϵ in the appropriate limits. Thus, this result is (without proof) the asymptotic expansion of the exact solution to order ϵ, and is uniformly valid in $0 \le x \le 1$. Note that (8.2.49) is in the form of a generalized asymptotic expansion as defined in (A.3.10).

8.2.3 Interior Dirichlet Problems for Elliptic Equations

In some applications modeled by a second-order partial differential equation, the small parameter multiplies the second derivative terms. In this section we study two examples for elliptic equations with Dirichlet boundary conditons. More details can be found in Sections 3.1 and 3.2 of [26] and in Chapter 3 of [30].

(i) *A linear elliptic equation in the unit square*

We wish to solve

$$\epsilon(u_{xx} + u_{yy}) = u_y \tag{8.2.50a}$$

in the interior of the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$ with the following Dirichlet-type boundary conditions

$$u(x, 0; \epsilon) = f(x), \quad u(x, 1; \epsilon) = g(x), \tag{8.2.50b}$$

$$u(0, y; \epsilon) = k(y), \quad u(1, y; \epsilon) = \ell(y), \tag{8.2.50c}$$

where f, g, k, and ℓ are prescribed continuous functions. The choice of a square domain serves to illustrate the basic ideas without inessential complications (see Figure 8.3).

If we set $\epsilon = 0$ in (8.2.50a), we conclude that the outer limit is independent of y; that is, $u =$ constant on vertical lines $x =$ constant. These vertical lines are the characteristics of the lower-order operator that results for $\epsilon = 0$, which is the trivial one $\partial/\partial y = 0$. In a more general problem, this lower-order operator may contain both x and y derivatives as well as variable coefficients; it may also be quasilinear, and so on. In such cases, the characteristics of this operator define

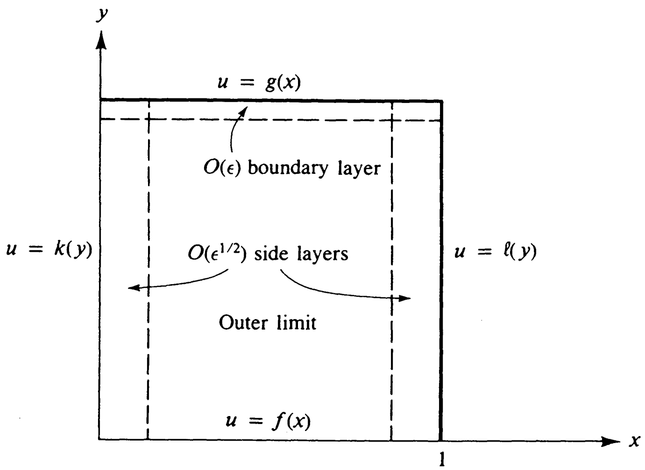

FIGURE 8.3. Solution domain for (8.2.50)

a more complicated family of curves on which u may vary. Some examples are discussed in [26].

For our case, since the outer limit for u does not depend on y, we must decide which of the two boundary conditions in (8.2.50b), if any, is to be imposed on this limit. Along any line x = constant, (8.2.50a) is an ordinary differential equation in terms of the y variable, and we proceed just as in previous examples. We first need to establish the thickness and location of possible layers by introducing the rescaled variable

$$y^* = \frac{y - y_0}{\delta(\epsilon)},$$

where y_0 is a constant in the interval $0 \leq y_0 \leq 1$ and $\delta(\epsilon) \ll 1$ measures the thickness of the layer. If we denote $u(x, y_0 + \delta y^*; \epsilon) \equiv u^*(x, y^*; \epsilon)$, (8.2.50a) becomes

$$\epsilon \frac{\partial^2 u^*}{\partial x^2} + \frac{\epsilon}{\delta^2} \frac{\partial^2 u^*}{\partial y^*} = \frac{1}{\delta} \frac{\partial u^*}{\partial y^*}.$$

The distinguished inner limit corresponds to $\delta = \epsilon$, and the limiting equation is

$$\frac{\partial^2 u_0^*}{\partial y^{*2}} = \frac{\partial u_0^*}{\partial y^*}, \tag{8.2.51}$$

where $u_0^* = u^*(x, y^*; 0)$. The solution is

$$u_0^*(x, y^*) = A(x) + B(x)e^{y^*}. \tag{8.2.52a}$$

Thus, the solution decays as $y^* \to -\infty$. This means that we can have a boundary layer only along the top boundary $y = 1$, that is, $y_0 = 1$, and (8.2.52a) must satisfy the boundary condition at $y = 1$. We then obtain the boundary layer limit

$$u_0^*(x, y^*) = A(x) + [g(x) - A(x)]e^{y^*}. \tag{8.2.52b}$$

This solution must match with the outer limit. Since we introduced the boundary layer at $y = 1$, we expect the outer limit to satisfy the lower boundary conditon. In this case, the outer limit is just

$$u_0(x, y) = f(x). \tag{8.2.53}$$

Now, we must match (8.2.52b) with (8.2.53) in the limit $\epsilon \to 0$ with x and y_η fixed, where y_η is the matching variable

$$y_\eta \equiv \frac{y - 1}{\eta(\epsilon)},$$

and $\eta(\epsilon) \ll 1$ is in the overlap domain that has to be determined. Note that neither the boundary layer limit nor the outer limit is valid near $x = 0$ or $x = 1$, and we exclude neighborhoods of these lines from consideration for the time being. The matching condition to $O(1)$ is see (8.2.14)

$$\lim_{\epsilon \to 0} \{u_0(x, 1 + \eta y_\eta) - u_0^*(x, \eta y_\eta/\epsilon)\} = 0, \tag{8.2.54}$$

with x fixed $\neq 0, 1$, and y_η fixed $\neq 0, \infty$. Using the equations for u_0 and u_0^*, we see that matching requires that we set $A(x) = f(x)$. Also, in order that e^{y^*} be transcendentally small in the matching domain, we must restrict $\eta(\epsilon)$ to the class $\epsilon |\log \epsilon| \ll \eta(\epsilon) \ll 1$, and this defines the overlap domain to this order.

The boundary layer limit is now defined explictly by

$$u_0^*(x, y^*) = f(x) + [g(x) - f(x)]e^{y^*}, \tag{8.2.55}$$

and it contains the outer limit $u_0 = f$. Therefore, (8.2.55) is the uniformaly valid approximation to $O(1)$ (as long as we stay away from $x = 0$ and $x = 1$).

The prescribed boundary conditions (8.2.50c) at $x = 0$ and $x = 1$ can be satisfied only by layers of appropriate thickness along these two vertical lines. Again, we assume a side layer of unknown thickness $\gamma(\epsilon)$ at $x = 0$ or $x = 1$ by introducing the rescaled x variable

$$\overline{x} \equiv \frac{x - x_0}{\gamma(\epsilon)}, \quad x_0 = 0, 1,$$

and we write $u(x_0 + \gamma\overline{x}, y; \epsilon) \equiv \overline{u}(\overline{x}, y; \epsilon)$ to obtain

$$\frac{\epsilon}{\gamma^2}\frac{\partial^2\overline{u}}{\partial\overline{x}^2} + \epsilon\frac{\partial^2\overline{u}}{\partial y^2} = \frac{\partial\overline{u}}{\partial y}.$$

Now, the distinguished limit corresponds to $\gamma = \epsilon^{1/2}$, and the limiting equation is

$$\frac{\partial^2\overline{u}_0}{\partial\overline{x}^2} = \frac{\partial\overline{u}_0}{\partial y}, \quad \overline{u}_0(\overline{x}, y) \equiv \overline{u}(\overline{x}, y; 0). \tag{8.2.56}$$

This is the one-dimensional diffusion equation where y is a timelike variable.

Consider the solution at the left boundary $x_0 = 0$; the solution at the right boundary is entirely analogous. The solution of (8.2.56) must satisfy the left boundary condition, that is, $\overline{u}(0, y) = k(y)$, and matching this solution with the outer limit dictates that $\overline{u}_0(\infty, y) = f(0)$ for $y \neq 0, 1$. We thus have the two needed boundary conditions at $x = 0$ and $x = \infty$ for the diffusion equation (8.2.56). In order to have a well-posed problem, we also need to specify the "initial condition" at $y = 0$. But (8.2.56) is not valid if y is small. In fact, near the origin we have a local layer where both x and y are rescaled as follows: $x^\dagger = x/\epsilon$, $y^\dagger = y/\epsilon$. With $u^\dagger(x^\dagger, y^\dagger; \epsilon) \equiv u(\epsilon x^\dagger, \epsilon y^\dagger; \epsilon)$, we then find that $u^\dagger$ obeys the full equation

$$\frac{\partial^2 u^\dagger}{\partial x^{\dagger 2}} + \frac{\partial^2 u^\dagger}{\partial y^{\dagger 2}} = \frac{\partial u^\dagger}{\partial y^\dagger}, \tag{8.2.57a}$$

subject to the constant boundary conditions

$$u^\dagger(0, y^\dagger) = k(0), \quad u^\dagger(x^\dagger, 0) = f(0) \tag{8.2.57b}$$

to leading order. If the boundary data are continuous at the corners, in particular, if $k(0) = f(0)$, then the solution is just $u^\dagger(x^\dagger, y^\dagger) = f(0)$ to leading order. Matching of the side-layer solution for $\overline{u}$ with the corner-layer solution for $u^\dagger$ then dictates that

$$\overline{u}(\overline{x}, 0) = f(0). \tag{8.2.58}$$

We assert without proof that (8.2.58) is also true if $k(0) \neq f(0)$. Aside from providing this unsurprising result, the corner-layer solution is needed only to describe the solution in the local layer, an $O(\epsilon)$ neighborhood of the origin. A similar local layer is needed near the other three corners of the square. For more details, see [26].

The solution of (8.2.56) subject to the boundary conditions $\bar{u}_0(0, y) = k(y)$, $\bar{u}_0(\infty, y) = f(0)$, and initial condition $\bar{u}_0(x, 0) = f(0)$, is given by combining the results in (1.4.11b) and (1.4.22). We obtain

$$\bar{u}_0(\bar{x}, y) = f(0)\,\mathrm{erf}\left(\frac{\bar{x}}{2y^{1/2}}\right) + \frac{\bar{x}}{2\pi^{1/2}} \int_0^y \frac{k(y-s)e^{-\bar{x}^2/4s}}{s^{3/2}}\,ds. \tag{8.2.59a}$$

In particular, if $k = k_0 =$ constant, this reduces to

$$\bar{u}_0(\bar{x}, y) = k_0 + (f(0) - k_0)\,\mathrm{erf}\left(\frac{\bar{x}}{2y^{1/2}}\right). \tag{8.2.59b}$$

Near the edge $x = 1$, we obtain the limiting solution

$$\hat{u}_0(\hat{x}, y) = f(0)\,\mathrm{erf}\left(\frac{\hat{x}}{2y^{1/2}}\right) + \frac{\hat{x}}{2\pi^{1/2}} \int_0^y \frac{\ell(y-s)e^{-\hat{x}^2/4s}}{s^{3/2}}\,ds, \tag{8.2.60}$$

where $\hat{x} \equiv (1 - x)/\epsilon^{1/2}$.

The uniformly valid approximation to $O(1)$ in the interior of the square (except in $O(\epsilon)$ neighborhoods of the four corners) is obtained by adding (8.2.55), (8.2.59a), and (8.2.60), and subtracting $2f(0)$ because $f(0)$ is the common term in the matching of u_0^* with each of the limits $\bar{u}_0$ and $\hat{u}_0$. Thus, denoting this composite approximation to $O(1)$ by $\tilde{u}_0$, we have

$$\begin{aligned}\tilde{u}_0(x, y; \epsilon) = {} & f(x) + [g(x) - f(x)]e^{y^*} \\ & - f(0)\left[\mathrm{erfc}\left(\frac{\bar{x}}{2y^{1/2}}\right) + \mathrm{erfc}\left(\frac{\hat{x}}{2y^{1/2}}\right)\right] \\ & + \frac{\bar{x}}{2\pi^{1/2}} \int_0^y \frac{k(y-s)e^{-\bar{x}^2/4s}}{s^{3/2}}\,ds \\ & + \frac{\hat{x}}{2\pi^{1/2}} \int_0^y \frac{\ell(y-s)e^{-\hat{x}^2/4s}}{s^{3/2}}\,ds.\end{aligned} \tag{8.2.61a}$$

For the special case $k = k_0 =$ constant, $\ell = \ell_0 =$ constant, this simplifies to

$$\begin{aligned}\tilde{u}_0(x, y; \epsilon) = {} & f(x) + [g(x) - f(x)]e^{y^*} \\ & + [k_0 - f(0)]\,\mathrm{erfc}\left(\frac{\bar{x}}{2y^{1/2}}\right) \\ & + [\ell_0 - f(0)]\,\mathrm{erfc}\left(\frac{\hat{x}}{2y^{1/2}}\right).\end{aligned} \tag{8.2.61b}$$

(ii) *A heat transfer problem*

Consider the steady viscous incompressible flow of a fluid in an infinitely long cylindrical pipe of radius R. The flow, assumed to be laminar, has horizontal speed

$U(1 - r^2)$, where the radial distance r has been normalized by dividing by R. Thus, the flow speed has a parabolic profile that vanishes at the pipe wall and equals U along the pipe axis. The wall temperature is maintained at the constant value T^- for $x < 0$ and $T^+ > T^-$ for $x > 0$. Here the dimensionless variable x has been normalized by dividing by R also. As discussed in Problem 3.1.10 of [26], the dimensionless steady-state temperature distribution $u(x, r; \epsilon)$ obeys the axisymmetric Laplacian with a lower derivative term

$$\epsilon\left(u_{rr} + \frac{1}{r}u_r + u_{xx}\right) = (1 - r^2)u_x, \tag{8.2.62a}$$

with boundary condition

$$u(x, 1; \epsilon) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases} \tag{8.2.62b}$$

The temperature T has been normalized using

$$u \equiv \frac{T - T^-}{T^+ - T^-}, \tag{8.2.63a}$$

and the small parameter is

$$\epsilon \equiv \frac{k}{\rho c U R}, \tag{8.2.63b}$$

where k is the thermal conductivity, ρ is the fluid density, and c is the specific heat. This problem is somewhat simpler than the one discussed in (ii), as boundaries in the x direction are at $\pm\infty$, where the temperature agrees with the wall boundary conditions, so no side layers are needed.

The outer limit has $u_x = 0$, and we conclude that we must have $u_0(x, r) = 0$. This means that the temperature is everywhere equal to the upstream ($x < 0$) value, as is the case if $R \to \infty$, for example; the effect of changing the wall temperature for $x > 0$ is felt only in a thin boundary layer near $r = 1$.

To establish the thickness of this boundary layer, we introduce the rescaled radial variable

$$r^* = \frac{1 - r}{\delta(\epsilon)}, \tag{8.2.64}$$

for some $\delta(\epsilon) \ll 1$ to be determined, and hold x fixed > 0. We denote $u(x, 1 - \delta r^*; \epsilon) \equiv u^*(x, r^*; \epsilon)$ and write (8.2.62a) as

$$\frac{\epsilon}{\delta^2}\left(\frac{\partial^2 u^*}{\partial r^{*2}} - \frac{\delta}{1 - \delta r^*}\frac{\partial u^*}{\partial r^*} + \delta^2\frac{\partial^2 u^*}{\partial x^2}\right) = \delta(2r^* - \delta r^{*2})\frac{\partial u^*}{\partial x}. \tag{8.2.65}$$

We see that we must have $\epsilon/\delta^2 = O_s(\delta)$; that is, $\delta = O_s(\epsilon^{1/3})$ for the distinguished boundary-layer limit. For simplicity, we set $\delta = \epsilon^{1/3}$ in (8.2.64). Then, the limiting solution $u_0^*(x, r^*) \equiv u^*(x, r^*; 0)$ obeys

$$2r^*\frac{\partial u_0^*}{\partial x} = \frac{\partial^2 u_0^*}{\partial r^{*2}}, \quad x > 0, \quad r^* > 0. \tag{8.2.66}$$

The boundary condition at $r = 1$ ($r^* = 0$) is $u_0^*(x, 0) = 1$, and matching with the outer limit requires that we set $u_0^*(x, \infty) = u_0(x, r)$. The solution of (8.2.66), subject to these two conditions, can be calculated using similarity (see Problem 1.2.1). We obtain

$$u_0^*(x, r^*) = 1 - \frac{6^{1/3}}{\Gamma(1/3)} \int_0^{r^*/x^{1/3}} \exp\left(-\frac{2s^3}{9}\right) ds,$$

where Γ denotes the gamma function. As this result contains the outer limit, it is the uniformly valid approximation to $O(1)$ for the temperature everywhere in $x > 0$, except in a small neighborhood of $x = 0$ and $r = 1$. There, we need to introduce a local layer of order ϵ in both x and $r - 1$. The details of this calculation are omitted.

8.2.4 Incompressible Irrotational Flow over an Axisymmetric Slender Body

This is an example that illustrates a rather common occurrence in singular perturbations. In the limit as $\epsilon \to 0$, the order of the equation does not change, but instead, a boundary surface degenerates to a line and introduces a singularity in the outer expansion.

In Section 2.4.3 we derived an integral equation for the source distribution for the incompressible irrotational flow over an axisymmetric body. The velocity potential u satisfies

$$\Delta u \equiv \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial x^2} = 0, \quad \text{in } r \geq \epsilon F(x), \tag{8.2.67a}$$

with the boundary condition at infinity

$$u \to x \text{ as } (x^2 + r^2)^{1/2} \to \infty, \tag{8.2.67b}$$

and the tangency condition on the body surface

$$\frac{u_r(x, \epsilon F(x); \epsilon)}{u_x(x, \epsilon F(x); \epsilon)} = \epsilon F'(x). \tag{8.2.67c}$$

Here $r = \epsilon F(x)$ specifies the body surface for a given $F(x)$ defined on $0 < x < 1$ with $F(0) = F(1) = 0$ and $0 \leq F(x) \leq 1$. The small parameter ϵ (which was not exhibited explicitly in Section 2.4.3) is the ratio of the maximum body radius to the body length.

The outer limit has $\epsilon \to 0$ with x and r fixed at some point off the body surface. In this limit, the body shrinks to a "needle" of zero radius that does not disturb the free stream, that is, $u_0(x, r) = x$. We therefore assume the outer expansion in the form

$$u(x, r; \epsilon) = x + \mu_1(\epsilon) u_1(x, r) + o(\mu_1), \tag{8.2.68}$$

where $\mu_1(\epsilon)$ is to be determined. Since (8.2.67a) is linear, u_1 also satisfies Laplace's equation subject to the zero boundary condition (8.2.67b) at infinity,

$$\Delta u_1 = 0, \tag{8.2.69a}$$

$$u_1 \to 0 \text{ as } (x^2 + r^2)^{1/2} \to \infty. \tag{8.2.69b}$$

If we attempt to impose the tangency condition (8.2.67c), we must have

$$\frac{\mu_1 \dfrac{\partial u_1}{\partial r}(x, \epsilon F) + o(\mu_1)}{1 + \mu_1 \dfrac{\partial u_1}{\partial x}(x, \epsilon F) + o(\mu_1)} = \epsilon F', \tag{8.2.70a}$$

or u_1 must satisfy

$$\mu_1 \frac{\partial u_1}{\partial r}(x, 0) + O(\mu_1 \epsilon) + o(\mu_1) = \epsilon F'. \tag{8.2.70b}$$

For $\epsilon \ll \mu_1$, we conclude from (8.2.70b) that $(\partial u_1/\partial r)(x, 0) = 0$, which combined with (8.2.69) implies that $u_1(x, r) = 0$. Thus, the right-hand side of (8.2.70b) comes into play only if $\mu_1 = O_s(\epsilon)$, $\mu_1 = \epsilon$, for instance. With this choice, we obtain

$$\frac{\partial u_1}{\partial r}(x, 0) = F'(x). \tag{8.2.71}$$

The fact that in (8.2.71) the boundary condition for u_1 is evaluated at $r = 0$ leads to difficulties, as seen from the general solution (see (2.4.15))

$$u_1(x, r) = -\frac{1}{4\pi} \int_0^1 \frac{S_1(\xi)}{\sqrt{(x - \xi)^2 + r^2}} d\xi \tag{8.2.72}$$

for the potential u_1 in terms of an unknown axial distribution of sources of strength/unit length S_1. It is shown in (A.3.60) that this potential has the behavior

$$u_1(x, r) = \frac{S_1(x)}{2\pi} \log r - \frac{S_1(0)}{4\pi} \log 2x - \frac{S_1(1)}{4\pi} \log 2(1 - x)$$
$$+ T_1(x) + O(r^2) \text{ as } r \to 0, \tag{8.2.73a}$$

where

$$T_1(x) \equiv -\frac{1}{4\pi} \int_0^1 S_1'(\xi) \operatorname{sgn}(x - \xi) \log 2|x - \xi| d\xi. \tag{8.2.73b}$$

If $S_1(x) \not\equiv 0$, $(\partial u_1/\partial r)$ has a $1/r$ singularity at $r = 0$ and cannot satisfy (8.2.71). In addition, we have singularities at $x = 0$ and $x = 1$ unless we set $\lim_{x\to 0} S_1(x) \log x = 0$ and $\lim_{x \to 1} S_1(x) \log 2(1 - x) = 0$. We shall discuss the implication of the sigularities at $x = 0$ and $x = 1$ later on. Note that in the exact problem, the singularity in r is still along the $r = 0$ axis, but the boundary condition (8.2.67c) is evaluated at $r = \epsilon F(x)$, which is off this axis for $0 < x < 1$. Thus, the boundary singularity in u_1 is a direct consequence of the nonuniform validity of (8.2.68) near the body. We abandon the requirement that u_1 satisfy (8.2.71) (which means that μ_1 need not equal ϵ), and we look for an inner expansion valid near $r = 0$.

Since the body radius shrinks to zero at a rate proportional to ϵ, we introduce the inner variable $r^* \equiv r/\epsilon$, write $u(x, \epsilon r^*; \epsilon) \equiv u^*(x, r^*; \epsilon)$, and consider the limit as $\epsilon \to 0$ with r^* and x fixed. Equation (8.2.67a) transforms to

$$\frac{\partial^2 u^*}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial u^*}{\partial r^*} + \epsilon^2 \frac{\partial^2 u^*}{\partial x^2} = 0, \tag{8.2.74a}$$

and the boundary condition (8.2.67c) becomes

$$\frac{\dfrac{\partial u^*}{\partial r^*}(x, F; \epsilon)}{\dfrac{\partial u^*}{\partial x}(x, F; \epsilon)} = \epsilon^2 F'(x). \tag{8.2.74b}$$

The inner expansion for u^* has the form

$$u^*(x, r^*; \epsilon) = u_0^*(x, r^*) + \mu_1^*(\epsilon) u_1^*(x, r^*) + o(\mu_1^*). \tag{8.2.75}$$

Substituting this into (8.2.74a) gives the following equations for u_0^* and u_1^*:

$$\frac{\partial^2 u_0^*}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial u_0^*}{\partial r^*} = 0, \tag{8.2.76a}$$

$$\frac{\partial^2 u_1^*}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial u_1^*}{\partial r^*} = \begin{cases} 0 & \text{if } \epsilon^2 \ll \mu_1^*, \\ -\dfrac{\partial^2 u_0^*}{\partial x^2} & \text{if } \mu_1^* = O_s(\epsilon^2). \end{cases} \tag{8.2.76b}$$

The boundary conditions at $r = \epsilon F$, that is at $r^* = F$, that we derive for u_0^* and u_1^* from (8.2.47b) are

$$\frac{\partial u_0^*}{\partial r^*}(x, F) = 0, \tag{8.2.77a}$$

$$\frac{\partial u_1^*}{\partial r^*}(x, F) = \begin{cases} 0 & \text{if } \epsilon^2 \ll \mu_1^*, \\ F'(x)\dfrac{\partial u_0^*}{\partial x}(x, F) & \text{if } \mu_1^* = O_s(\epsilon^2). \end{cases} \tag{8.2.77b}$$

The general solution of (8.2.76a) subject to the condition (8.2.77a) is $u_0^*(x, r^*) = B_0(x)$, and matching this with the outer limit $u_0 = x$ fixes $B_0 = x$. Thus, the inner limit is also $u_0^* = x$.

Since now $(\partial^2 u_0^*/\partial x^2) = 0$, u_1^* satisfies the homogeneous equation (8.2.76b) for any μ_1^*: $\epsilon^2 \underline{\ll} \mu_1^*$. Therefore,

$$u_1^*(x, r^*) = A_1(x) \log r^* + B_1(x). \tag{8.2.78}$$

The boundary condition (8.2.77b) gives $A_1 = 0$ if $\epsilon^2 \ll \mu_1^*$ or $A_1(x) = F(x)F'(x)$ if $\mu_1^* = \epsilon^2$. Therefore, the inner expansion has the form

$$u^*(x, r^*; \epsilon) = x + \mu_{01}^* B_{01}(x) + \epsilon^2[F(x)F'(x) \log r^* + B_1(x)] + o(\epsilon^2), \tag{8.2.79}$$

where we have inserted a solution of $O(\mu_{01}^*)$ with $\epsilon^2 \ll \mu_{01}^* \ll 1$, satisfying the homogeneous case of the boundary condition (8.2.77b), for reasons that will become clear from the matching.

As we have calculated the inner expansion to $O(\epsilon^2)$, let us attempt a matching to this order also. We must satisfy

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\Bigg\{x+\mu_1(\epsilon)u_1(x,\eta r_\eta)-x-\mu_{01}^*(\epsilon)B_{01}(x)$$
$$-\epsilon^2\left[F(x)F'(x)\log\frac{\eta r_\eta}{\epsilon}+B_1(x)\right]\Bigg\}=0, \qquad (8.2.80a)$$

for x and r_η fixed and for η in some overlap domain to be determined. Here u_1 is given in (8.2.72). We use the expansion (8.2.73) for u_1, assume that $\lim_{x\to 0} S(x)\log x = 0$ and $\lim_{x\to 1} S(x)\log 2(1-x) = 0$, and simplify the matching condition to

$$\lim_{\epsilon\to 0}\Bigg\{\frac{\mu_1(\epsilon)}{\epsilon^2}\left[\frac{S_1(x)}{2\pi}\log\eta r_\eta+T_1(x)\right]-\frac{\mu_{01}^*(\epsilon)}{\epsilon^2}B_{01}(x)$$
$$-F(x)F'(x)\log\eta r_\eta+F(x)F'(x)\log\epsilon-B_1(x)\Bigg\}=0. \qquad (8.2.80b)$$

In order to match the singular terms proportional to $\log\eta r_\eta$, we must set $\mu_1=\epsilon^2$ and $S_1(x)/2\pi = F(x)F'(x)$. In order to remove the singular term proportional to $\log\epsilon$, we must set $\mu_{01}^*(\epsilon)/\epsilon^2=\log\epsilon$ and $B_{01}(x)=F(x)F'(x)$. Finally, in order to match the $O(1)$ terms, we must set $B_1(x)=T_1(x)$. The matching to $O(\epsilon^2)$ is then complete and defines all the unknowns

$$\mu_1(\epsilon)=\epsilon^2,\ S_1(x)=2\pi F(x)F'(x), \qquad (8.2.81a)$$
$$\mu_{01}^*(\epsilon)=\epsilon^2\log\epsilon,\ B_{01}(x)=F(x)F'(x), \qquad (8.2.81b)$$
$$B_1(x)=-\frac{1}{4}\int_0^1[F^2(\xi)]''\,\mathrm{sgn}(x-\xi)\log 2|x-\xi|d\xi. \qquad (8.2.81c)$$

We see from (8.2.81a) that the source strength $\epsilon^2 S_1(x)$ equals the rate of change of cross-sectional area of the body. Our result remains uniformly valid near $x=0$ and $x=1$ only if the prescribed body shape satisfies the conditions $\lim_{x\to 0} a'(x)\log x = 0$ and $\lim_{x\to 1} a'(x)\log(1-x)=0$, where $a(x)=\pi F^2(x)$. For example, if $F(x)$ behaves like x^α near $x=0$ (and $(1-x)^\alpha$ near $x=1$), we must restrict α to be larger than $\frac{1}{2}$. A discussion of the behavior of the solution if $\alpha<\frac{1}{2}$ is given in Section 3.3.3 of [26]. The reader can also find there a discussion for the more general problem of a deformable body, $F(x,t)$, and a calculation of the force on the body.

We note that for this example, *every* term in the inner expansion to $O(\epsilon^2)$ is contained in the two-term outer expansion

$$u(x,r;\epsilon)=x-\frac{\epsilon^2}{2}\int_0^1\frac{F(\xi)F'(\xi)}{[(x-\xi)^2+r^2]^{1/2}}d\xi+o(\epsilon^2). \qquad (8.2.82)$$

Thus, (8.2.82) is the uniformly valid exapansion of the solution to $O(\epsilon^2)$; the matching here serves to determine the source strength. Also, if we are interested only in the flow near the body, as in the calculation of the pressure on the body, we do not need (8.2.82) and can use the simpler result

$$u^*(x, r^*; \epsilon) = x + (\epsilon^2 \log \epsilon) F(x) F'(x) + \epsilon^2 [F(x) F'(x) \log r^* + B_1(x)] + o(\epsilon^2), \qquad (8.2.83)$$

where B_1 is given in (8.2.81c).

8.2.5 Burgers' Equation

In Section 5.3.6 we studied a class of exact solutions for Burgers' equation corresponding to piecewise constant initial data on the infinite interval. Here we recalculate their asymptotic expansions by matching appropriate limiting equations and show that the results are in agreement. Boundary-value problems, for which an exact solution is more complicated (as discussed in Section 1.7.3), can also be easily approximated using matched asymptotic expansions. An example is discussed in Section 3.1.3 of [26].

(i) *Intial-value problem with a shock layer*

As pointed out in Section 5.3.6, there is no loss of generality in studying the initial-value problem

$$u_t + u u_x = \epsilon u_{xx}, \qquad (8.2.84a)$$

$$u(x, 0; \epsilon) = \begin{cases} 1 & \text{if } x < 0, \\ -1 & \text{if } x > 0, \end{cases} \qquad (8.2.84b)$$

if the initial condition is piecewise constant on either side of $x = x_0$ with $u(x_0^+, 0) < u(x_0^-, 0)$.

The outer expansion of (8.2.84) terminates with the outer limit $u_0(x, t) \equiv u(x, t; 0)$ given by (see (5.3.36))

$$u_0(x, t) = \begin{cases} 1 & \text{if } x < 0, \\ -1 & \text{if } x > 0, \end{cases} \qquad (8.2.85)$$

and the remainder is transcendentally small. This result is not uniformly valid if x is small, and we look for an appropriate inner limit.

We introduce the inner variable $x^* \equiv x/\delta(\epsilon)$ with $\delta(\epsilon)$ to be determined, and set $u^*(x^*, t; \epsilon) \equiv u(\delta x^*, t; \epsilon)$. Substituting these into (8.2.84a) gives

$$\frac{\partial u^*}{\partial t} + \frac{1}{\delta} u^* \frac{\partial u^*}{\partial x^*} = \frac{\epsilon}{\delta^2} \frac{\partial^2 u^*}{\partial x^{*2}}.$$

We see that the distinguished inner limit corresponds to $\delta(\epsilon) = O_s(\epsilon)$, and we set $\delta = \epsilon$ for simplicity to obtain

$$\epsilon \frac{\partial u^*}{\partial t} + u^* \frac{\partial u^*}{\partial x^*} = \frac{\partial^2 u^*}{\partial x^{*2}}. \qquad (8.2.86)$$

Thus, the inner limit $u_0^*(x^*, t) \equiv u^*(x^*, t; 0)$ obeys

$$u_0^* \frac{\partial u_0^*}{\partial x^*} = \frac{\partial^2 u_0^*}{\partial x^{*2}}. \tag{8.2.87}$$

Integrating with respect to x^* gives

$$\frac{u_0^{*2}}{2} - \frac{\partial u_0^*}{\partial x^*} = \alpha(t). \tag{8.2.88}$$

In order to match with the outer limit, we must have $u_0^* \to \pm 1$ as $x^* \to \mp\infty$ and $(\partial u_0^*/\partial x^*) \to 0$ as $|x^*| \to \infty$. Therefore, $\alpha(t)$ must equal $\frac{1}{2}$. Integrating (8.2.88) gives

$$u_0^* = -\tanh\left(\frac{x^* + x_0^*}{2}\right), \tag{8.2.89a}$$

or

$$u_0^* = -\coth\left(\frac{x^* + x_0^*}{2}\right), \tag{8.2.89b}$$

where x_0^* is an arbitrary function of t. We discard the hyperbolic cotangent solution for this problem because the interior layer must hold over the infinite x^* interval and $t > 0$, and (8.2.89b) becomes singular when $x^* = -x_0^*$. In fact, (8.2.89a) matches with (8.2.85) to all algebraic orders for arbitrary $x_0^*(t)$.

For this example we can use a symmetry argument to conclude that $x_0^* = 0$. We note that (8.2.84) is invariant under the transformation $x \to -x$, $u \to -u$, $t \to t$. Thus, in (8.2.89a) we must have

$$-\tanh\left(\frac{x^* + x_0^*}{2}\right) = \tanh\left(\frac{-x^* + x_0^*}{2}\right)$$

for all x^*; this can hold only if $x_0^* = 0$. In general, for a curved shock, one must account for the transcendentally small terms in order to determine the "shift" x_0^* in the shock location.

The small parameter ϵ is artificial in (8.2.84) in the sense that it can be removed from the governing equation and initial condition (for the special case of (8.2.84b)) through the transformation $u \to u, t \to t/\epsilon, x \to x/\epsilon$. In fact, note that the exact solution given by (5.3.79) is free of ϵ if we use the x/ϵ, t/ϵ variables. Thus, the limit $u_0 = -\tanh x^*/2$, which we also derived from the exact solution in Chapter 5, is the uniformly valid approximation for all x and all $t > 0$ to all algebraic orders in ϵ; it differs from the exact expression only by transcendentally small terms if t is not small. The exact expression is needed only if $t = O_s(\epsilon)$ and $x = O_s(\epsilon)$.

A limiting expression such as (8.2.89a) is easier to derive in general, and it gives the shock structure for the solution directly even if we cannot solve the exact problem in the entire domain. A discussion of the shock structure for one-dimensional viscous heat-conducting flow (governed by (3.3.6)) is given in Section 6.15 of [42]. This solution describes how the flow actually varies in a thin shock layer. In the limit Re $\to \infty$ with x and t fixed, this thin layer shrinks to the shock of zero thickness that we calculated in Sections 5.3.4iii. In Problem 8.2.6, the simpler case of an isothermal shock layer is outlined.

(ii) *Initial-value problem with corner layers*

If we replace (8.2.84b) with

$$u(x, 0; \epsilon) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x > 0, \end{cases} \tag{8.2.90}$$

the outer limit for $t > 0$ is (see (5.3.42))

$$u_0(x, t) = \begin{cases} 1 & \text{if } x \geq t, \\ x/t & \text{if } -x \leq t \leq x, \\ -1 & \text{if } x \leq -t, \end{cases} \tag{8.2.91}$$

and it is easily seen that the outer expansion again terminates with u_0. Although (8.2.91) is valid to $O(1)$ everywhere, the two partial derivatives $(\partial u_0/\partial t)$ and $(\partial u_0/\partial x)$ do not exist if $|x| = t$.

Near $x = \pm t$ we need to introduce a *corner layer* to smoothly join the solutions on either side. We look for rescaled independent variables of the form

$$x_c \equiv \frac{x \mp t}{\delta(\epsilon)}; \ t_c = t. \tag{8.2.92}$$

Since the layers are near $x = \pm t$, where $u = \pm 1$ if $t > 0$, we expand u as follows:

$$u(x, t; \epsilon) = \pm 1 + \mu(\epsilon) u_c(x_c, t_c) + o(\mu). \tag{8.2.93}$$

We compute

$$\frac{\partial u}{\partial t} = \mu \frac{\partial u_c}{\partial t_c} \mp \frac{\mu}{\delta} \frac{\partial u_c}{\partial x_c}, \quad \frac{\partial u}{\partial x} = \frac{\mu}{\delta} \frac{\partial u_c}{\partial x_c}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\mu}{\delta^2} \frac{\partial^2 u_c}{\partial x_c^2}.$$

Therefore, (8.2.84a) transforms to

$$\mu \frac{\partial u_c}{\partial t_c} + \frac{\mu^2}{\delta} u_c \frac{\partial u_c}{\partial x_c} = \frac{\mu \epsilon}{\delta^2} \frac{\partial^2 u_c}{\partial x_c^2} + o(\mu), \tag{8.2.94}$$

and we see that we must set $\mu = \delta = \epsilon^{1/2}$ for the richest limit, which is the full equation

$$\frac{\partial u_c}{\partial t_c} + u_c \frac{\partial u_c}{\partial x_c} = \frac{\partial^2 u_c}{\partial x_c^2} + o(1). \tag{8.2.95}$$

One cannot avoid solving the exact Burgers equation in order to calculate the corner-layer solution. This calculation was discussed in Section 5.3.6ii, where we obtained (see (5.3.88))

$$u_c(x_c, t_c) = -\frac{2}{\sqrt{\pi t_c}} \frac{\exp(-x_c^2/4t_c)}{\operatorname{erfc}(-x_c/2t_c^{1/2})}, \tag{8.2.96}$$

for the corner layer at $x = t$ after we reconcile the notation. It was also pointed out in Section 5.3.6ii that using (8.2.96) for u_c in (8.2.93) and taking the limit as

$x_c \to \infty$ or $x_c \to -\infty$ with t fixed > 0 gives $u \to 1$ or $u \to x/t$, respectively. This is just the matching condition for the corner-layer expansion (8.2.93) with the outer limit on either side. Similar results apply for the corner layer at $x = -t$.

In a sense, the calculation of (8.2.96) is not a perturbation problem. However, we need this result only to demonstrate that u_x and u_t join smoothly across the corner layer and to find a better approximation than $u = \pm 1$ at $x = \pm t$.

Problems

8.2.1 Calculate the exact solution of

$$\epsilon u'' + xu' + u = 0, \quad -1 \le x \le 1, \quad 0 < \epsilon \ll 1, \tag{8.2.97a}$$

$$u(-1;\epsilon) = 1, \quad u(1;\epsilon) = 2 \tag{8.2.97b}$$

in the form

$$u(x;\epsilon) = R(x;\epsilon)\exp\left(\frac{1-x^2}{2\epsilon}\right), \tag{8.2.98a}$$

where

$$R(x;\epsilon) \equiv \frac{3}{2} + \frac{1}{2}\frac{\int_0^x e^{t^2/2\epsilon}dt}{\int_0^1 e^{t^2/2\epsilon}dt}. \tag{8.2.98b}$$

Thus, $R(-1;\epsilon) = 1$, $R(1;\epsilon) = 2$, and $1 \le R \le 2$ for all x in $-1 \le x \le 1$. Equation (8.2.98a) then implies that the two boundary layers at $x = \pm 1$ grow exponentially toward the interior of the interval. In fact, the solution has the very large maximal value $u(0;\epsilon) = \frac{3}{2}\exp(1/2\epsilon)$.

8.2.2 Calculate and match the outer and inner exapansions to $O(\epsilon)$ for

$$\epsilon u'' + \frac{u'}{\sqrt{x}} - u = 0, \quad 0 \le x \le 1, \quad 0 < \epsilon \ll 1, \tag{8.2.99a}$$

$$u(0;\epsilon) = 0, \quad u(1;\epsilon) = e^{2/3}. \tag{8.2.99b}$$

In particular, show that the overlap domain for the matching to $O(\epsilon)$ is $\epsilon^2(\log\epsilon)^2 \ll \eta(\epsilon) \ll \epsilon^{2/3}$ and that the uniformly valid approximation to $O(\epsilon)$ is

$$\begin{aligned}\tilde{u}(x;\epsilon) = {} & \exp\left(\frac{2x^{3/2}}{3}\right) - (2x^{*^{1/2}} + 1)\exp(-2x^{*^{1/2}}) \\ & + \epsilon\left\{\left[\frac{9}{10} - \frac{x}{2} + \frac{2}{5}x^{5/2}\exp\left(\frac{2x^{3/2}}{3}\right)\right]\right. \\ & \left. - \frac{9}{10}(2x^{*^{1/2}} + 1)\exp(-2x^{*^{1/2}})\right\},\end{aligned} \tag{8.2.100}$$

where $x^* \equiv x/\epsilon^2$.

8.2.3 Calculate the next term in the uniformly valid approximation for each of the following problems.

a. The boundary-value problem (8.2.50) for the case $k = k_0 =$ constant, $\ell = \ell_0 =$ constant.

b. The heat transfer problem (8.2.62).

8.2.4 The following is a mathematical model to illustrate the asymptotic behavior of certain collision trajectories that occur in celestial mechanics. We wish to calculate $u(t;\epsilon)$ over the interval $0 \leq t \leq T(\epsilon)$, where $T(\epsilon)$ is the collision time defined by $u(T(\epsilon);\epsilon) = 1$, for the initial-value problem

$$\ddot{u} + u - \frac{\epsilon}{(1-u)^2} = 2\sin t, \tag{8.2.101a}$$

$$u(0;\epsilon) = 0, \ \dot{u}(0;\epsilon) = 0. \tag{8.2.101b}$$

Thus, if $\epsilon = 0$, u is given by

$$u_0(t) = \sin t - t\cos t, \tag{8.2.102}$$

which has $u_0(\pi/2) = 1$. But if $u = 1$, the term in (8.2.101a) that is multiplied by ϵ becomes infinite. Thus, the outer expansion of (8.2.101) is not uniformly valid near $t = \pi/2$.

a. Show that this outer expansion is given by

$$u(t;\epsilon) = u_0(t) + \epsilon \int_0^t \frac{\sin(t-\tau)}{(1-\sin\tau + \tau\cos\tau)^2} d\tau + O(\epsilon^2), \tag{8.2.103}$$

and refer to (A.3.54) for the singular behavior of (8.2.103) as $t \to \pi/2$.

b. For t close to the collision time $T(\epsilon)$, introduce the rescaled dependent and independent variables

$$u \equiv 1 - \epsilon u^*(t^*;\epsilon), \ t^* \equiv \frac{t - T(\epsilon)}{\epsilon}. \tag{8.2.104}$$

Expand $u^*(t^*;\epsilon)$ in the form $u^*(t^*;\epsilon) = u_0^*(t^*) + O(\epsilon)$, and show that u_0^* satisfies

$$\frac{d^2u_0^*}{dt^{*2}} + \frac{1}{u_0^{*2}} = 0. \tag{8.2.105}$$

c. Match the outer and inner expansions for (du/dt) to show that the appropriate integral of (8.2.105) is

$$-\frac{1}{2}\left(\frac{du_0^*}{dt^*}\right)^2 + \frac{1}{u_0^*} = -\frac{\pi^2}{8}. \tag{8.2.106}$$

d. Use the condition $u = 1$ at $t = T$ to express the solution of (8.2.106) as the inverse of

$$t^* = -\frac{2}{\pi}u_0^*\left(1+\frac{8}{\pi^2 u_0^*}\right)^{1/2} + \frac{16}{\pi^3}\log\left\{\left[u_0^*\left(1+\frac{8}{\pi^2 u_0^*}\right)\right]^{1/2} + (u_0^*)^{1/2}\right\} + \frac{8}{\pi^3}\log\frac{\pi^2}{8}. \tag{8.2.107}$$

e. Use (8.2.107) to calculate the asymptotic behavior of u_0^* as $t^* \to -\infty$ in the form

$$u_0^* = -\frac{\pi}{2}t^* + \frac{4}{\pi^2}\log(-t^*) + \frac{4}{\pi^2}\left(\log\frac{\pi^3}{4} - 1\right) + O\left(\frac{1}{t^*}\log(-t^*)\right) \text{ as } t^* \to -\infty. \tag{8.2.108}$$

f. Match the outer and inner expansions of u to $O(\epsilon)$ to show that

$$T(\epsilon) = \frac{\pi}{2} + \frac{8}{\pi^3}\epsilon\log\epsilon - \epsilon\left[\frac{2C_2}{\pi} + \frac{8}{\pi^3}\left(\log\frac{\pi^3}{4} - 1\right)\right] + o(\epsilon), \tag{8.2.109}$$

where C_2 is the constant defined in (A.3.54b).

8.2.5 Consider the following signaling problem for a pair of quasilinear first-order equations:

$$\ell(v)v_t = u, \quad \ell(v)u_x = -u. \tag{8.2.110}$$

We are interested in the solution for $u(x, t; \epsilon)$ and $v(x, t; \epsilon)$ on $0 \le x < \infty$, $0 \le t < \infty$, subject to the initial condition

$$v(x, 0; \epsilon) = \epsilon = \text{constant} > 0 \tag{8.2.111}$$

and the boundary condition

$$u(0, t; \epsilon) = f(t) = \text{prescribed if } t > 0. \tag{8.2.112}$$

Here $\ell(v)$ is a prescribed function of v.

a. Show that we must have

$$\frac{\partial}{\partial t}\left[\ell(v)\frac{\partial v}{\partial x} + v\right] = 0, \tag{8.2.113}$$

and use this result to express the solution for v in the implicit form

$$x = -\int_{v(0,t;\epsilon)}^{v(x,t;\epsilon)} \frac{\ell(s)}{s-\epsilon}ds, \tag{8.2.114}$$

where $v(0, t; \epsilon)$ is obtained from the solution of

$$\int_{\epsilon}^{v(0,t;\epsilon)} \ell(s)ds = \int_0^t f(\tau)d\tau. \tag{8.2.115}$$

Therefore, (8.2.114)–(8.2.115) define $v(x, t; \epsilon)$, and the first equation in (8.2.110) gives $u(x, t; \epsilon)$ by quadrature once v is known.

b. Specialize your results to the case

$$\ell(v) = v^{3/2}, \quad f(t) = 1, \tag{8.2.116}$$

and derive a uniformly valid expression for $v(x, t; \epsilon)$ to $O(1)$ for $0 < \epsilon \ll 1$. Sketch curves of v as a function of x for fixed values of t. This problem is an idealized model that describes the heating ($v =$ temperature) of a semi-infinite plasma column by shining a laser at one end.

8.2.6 In the isothermal one-dimensional flow of a gas, we assume that the pressure is proportional to the density $\overline{\rho}$. The flow is then defined in terms of $\overline{\rho}$ and the speed $\overline{u}$ by the laws of mass and momentum conservation:

$$\overline{\rho}_{\overline{t}} + (\overline{\rho}\,\overline{u})_{\overline{x}} = 0 \quad \text{(mass)}, \tag{8.2.117a}$$

$$(\overline{\rho}\,\overline{u})_{\overline{t}} + (\overline{\rho}\,\overline{u}^2 + \overline{C}_0^2\overline{\rho} - \mu\overline{u}_{\overline{x}})_{\overline{x}} = 0 \quad \text{(momentum)}, \tag{8.2.117b}$$

where $\overline{C}_0$ is the ambient speed of sound, a constant, and μ is the coefficient of viscosity, also a constant.

We wish to study the piston problem analogous to the one discussed in Section 5.3.4iii. A piston is impulsively set into motion with constant speed $\overline{v} > 0$ into gas at rest with ambient properties $\overline{\rho}_0 =$ constant, $\overline{C}_0 =$ constant. The initial conditions are

$$\overline{\rho}(\overline{x}, 0) = \overline{\rho}_0, \quad \overline{u}(\overline{x}, 0) = 0, \tag{8.2.117c}$$

for $x > 0$, and the boundary condition at the piston is

$$\overline{u}(\overline{v}\,\overline{t}, \overline{t}) = \overline{v}. \tag{8.2.117d}$$

a. Introduce the dimensionless variables

$$x \equiv \frac{\overline{x}}{L}, \quad t = \frac{\overline{t}\,\overline{C}_0}{L}, \quad u = \frac{\overline{u}}{\overline{C}_0}, \quad v = \frac{\overline{v}}{\overline{C}_0}, \quad \rho = \frac{\overline{\rho}}{\overline{\rho}_0},$$

where L is a characteristic length. Show that (8.2.117) becomes

$$\rho_t + (u\rho)_x = 0, \tag{8.2.118a}$$

$$(\rho u)_t + (\rho u^2 + \rho - \epsilon u_x)_x = 0, \tag{8.2.118b}$$

$$\rho(x, 0; \epsilon) = 1, \quad u(x, 0; \epsilon) = 0, \tag{8.2.118c}$$

$$u(vt, t; \epsilon) = v, \quad t > 0, \tag{8.2.118d}$$

where $\epsilon \equiv \mu/\overline{\rho}_0\overline{C}_0 L$. Thus, ϵ is an artificial small parameter because the only length scale in the problem is $\mu/\rho_0\overline{C}_0$, and choosing L equal to this scale gives $\epsilon = 1$.

b. For $\epsilon = 0$, solve the piston problem and show that the shock speed U and and density ρ behind the shock are given by

$$U = v/2 + (1 + v^2/4)^{1/2}, \quad \rho = 1 + v^2/2 + v(1 + v^2/4)^{1/2}. \tag{8.2.119}$$

c. For $0 < \epsilon \ll 1$, calculate the shock structure and then verify that ϵ is indeed an artificial parameter. Introduce a coordinate system moving with the shock and show that the shock structure is given by the solution of the pair of ordinary differential equations that result from (8.2.118a)–(8.2.118b). In particular, show that $\rho = -U/w$, and that $w \equiv u - U$ is given by the inversion of the formula

$$\xi - \xi_0 = \frac{1+U^2}{2U(U^2-1)} \log \left| \frac{Uw+1}{Uw+U^2} \right| - \frac{1}{2U} \log \left| Uw^2 + (1+U^2)w + U \right|, \qquad (8.2.120)$$

where $\xi = (x - Ut)/\epsilon$, and ξ_0 fixes the location of the shock. Verify that $w \to -U$ as $\xi \to \infty$, and that $w \to v - U$ as $\xi \to -\infty$.

8.3 Multiple-Scale Expansions

In Section 3.5.4 we studied the signaling problem for a wavemaker in the limiting case of small-amplitude waves on shallow water. Using a regular perturbation expansion for the speed u and height h, we found that our results are not uniformly valid in the far field, $x = O(\epsilon^{-1})$, even though the boundary data are bounded. A similar nonuniformity was encountered in the regular expansion for supersonic flow over a thin airfoil (see (4.2.73)). In an initial-value problem for a weakly nonlinear wave equation on $-\infty < x < \infty$, a regular expansion will result in nonuniformity for $t = O(\epsilon^{-1})$ even if the initial data are bounded.

The reason for these nonuniformities is that a regular expansion is based on the limit $\epsilon \to 0$ with x and t fixed. Typically, small nonlinear terms in the governing equations introduce slow modulations that are exhibited by bounded functions of ϵx and ϵt in the solution, such as $\sin \epsilon x$ or $e^{-\epsilon t}$. Such functions cannot be uniformly approximated by the limit process $\epsilon \to 0$, with x or t fixed, if x or t are allowed to become $O(\epsilon^{-1})$. In particular, the solution of an intial-value problem for a weakly nonlinear wave equation with bounded initial data depends simultaneously on a "fast time" t and a "slow time" $t_1 \equiv \epsilon t$ in the form $f(x, t, t_1; \epsilon)$. For example, we may have a damped traveling wave of the form $u(x, t; \epsilon) = e^{-\epsilon t} \sin(x - t)$. In such cases, we cannot use a limit process expansion to express u in terms of x and t_1 because $\lim_{\epsilon \to 0} \sin t_1/\epsilon$, with t_1 fixed $\neq 0$, does not exist. We need to introduce a multiple-scale expansion involving x and both t and t_1. This expansion will be in the form of a generalized asymptotic expansion (see (A.3.10)). For a signaling problem with zero initial data and bounded boundary data at $x = 0$, the solution depends simultaneously on x and ϵx. Problem 8.3.1 illustrates some of these ideas.

We shall restrict attention to initial-value problems in this section. More details and a discussion of the signaling problem can be found in Section 6.2.4. of [26].

8.3.1 A Weakly Nonlinear Oscillator; Slowly Varying Amplitude and Phase

We begin our discussion with an ordinary differential equation example to illustrate the failure of the regular expansion and to motivate the need for an expansion involving multiple time scales.

Consider the initial-value problem

$$\ddot{u} + u + \epsilon\dot{u}|\dot{u}| = 0, \ 0 \le t < \infty, \ 0 < \epsilon \ll 1, \tag{8.3.1a}$$

$$u(0;\epsilon) = a > 0, \quad \dot{u}(0;\epsilon) = 0. \tag{8.3.1b}$$

(i) *Regular expansion*

If we look for a regular expansion of the form

$$u(t;\epsilon) = u_0(t) + \epsilon u_1(t) + O(\epsilon^2) \tag{8.3.2}$$

—that is, $u_0 = \lim_{\epsilon\to 0} u(t;\epsilon)$, $u_1 = \lim_{\epsilon\to 0}\{u(t;\epsilon) - u_0(t)\}/\epsilon$, and so on—we find that u_0 and u_1 satisfy

$$\ddot{u}_0 + u_0 = 0, \tag{8.3.3a}$$

$$u_0(0) = a, \quad \dot{u}_0(0) = 0, \tag{8.3.3b}$$

$$\ddot{u}_1 + u_1 = -\dot{u}_0|\dot{u}_0|, \tag{8.3.4a}$$

$$u_1(0) = 0, \quad \dot{u}_1(0) = 0. \tag{8.3.4b}$$

Thus,

$$u_0(t) = a\cos t, \tag{8.3.5}$$

and (8.3.4a) becomes

$$\ddot{u}_1 + u_1 = a^2 \sin t|\sin t|. \tag{8.3.6}$$

The right-hand side of (8.3.6) is the odd 2π-periodic function that equals $a^2\sin^2 t$ on $0 \le t \le \pi$ and $-a^2\sin^2 t$ on $-\pi \le t \le 0$. Therefore, we can develop this right-hand side in the Fourier sine series

$$a^2\sin t|\sin t| = \sum_{n=1}^{\infty} b_n \sin nt, \tag{8.3.7a}$$

where

$$b_n = \frac{2a^2}{\pi}\int_0^{\pi} \sin^2 t \sin nt dt = \frac{4a^2}{\pi}\frac{[(-1)^n - 1]}{n(n^2-4)}, \ n \neq 2. \tag{8.3.7b}$$

Thus, $b_n = 0$ for n even, and we can write (8.3.7a) as

$$a^2\sin t|\sin t| = \frac{4a^2}{\pi}\sum_{k=0}^{\infty} b_{2k+1}\sin(2k+1)t, \tag{8.3.8a}$$

where

$$b_{2k+1} = \frac{-2}{(2k-1)(2k+1)(2k+3)}. \tag{8.3.8b}$$

The solution of (8.3.4) is then easily calculated in the form

$$u_1(t) = -\frac{4}{3\pi}a^2 t\cos t - \frac{4a^2}{\pi}\sum_{k=1}^{\infty}\frac{b_{2k+1}(2k+1)}{4k(k+1)}\sin(2k+1)t$$
$$+\left[\frac{4a^2}{3\pi} + \frac{4a^2}{\pi}\sum_{k=1}^{\infty}\frac{b_{2k+1}(2k+1)}{4k(k+1)}\right]\sin t. \tag{8.3.9}$$

The first term on the right-hand side is proportional to $t\cos t$. Therefore, ϵu_1 has a term proportional to $\epsilon t\cos t$. This term becomes $O(1)$ for $t = O(\epsilon^{-1})$, so the expansion (8.3.2) is not uniformly valid for t in the interval $0 \leq t \leq T(\epsilon)$, for any $T(\epsilon) = O(\epsilon^{-1})$; the expansion is uniformly valid if $T(\epsilon) = O(1)$. A periodic function of t that is multiplied by t is called "mixed secular."

(ii) *Multiple-scale expansion*

We know that the small damping term in (8.3.1a) must introduce a slow decay in the amplitude. Thus, instead of $u_0 = a\cos t$ with $a =$ constant, we must have a leading approximation $u_0 = A\cos t$, where the amplitude A decreases slowly with time and satisfies $A(0) = a$. The term $-(4/3\pi)a^2\epsilon t\cos t$ is evidently the second term in a nonuniform expansion of A and shows that A must depend on $t_1 = \epsilon t$. But we have argued that we cannot express a periodic function by a limit process where $\epsilon \to 0$ with t fixed. Therefore, we are led to look for an expansion that depends *simultaneously* on $t_0 \equiv t$ and $t_1 \equiv \epsilon t$, a so-called multiple-scale expansion. A discussion of the various source references for this method is given is Section 4.2 of [26].

Basically, we assume that the solution of (8.3.1) has a certain structure—it involves oscillations that occur over the "fast scale" t_0, and these oscillations are modulated over the "slow scale" t_1. Dimensional analysis can be used to argue that the small parameter ϵ is the ratio of two characteristic times: the time T_0 associated with the oscillatory behavior and T_1 associated with the damping. At any rate, we assume that the solution has the form $u(t;\epsilon) = U(t_0, t_1;\epsilon)$, and we regard t_0 and t_1 as being *independent variables*. We expand U with respect to its ϵ-dependence in the form

$$U = U_0(t_0, t_1) + \epsilon U_1(t_0, t_1) + O(\epsilon^2). \tag{8.3.10}$$

We emphasize the fact that (8.3.10) is not a limit process expansion because it does not correspond to having either t_0 or t_1 fixed as $\epsilon \to 0$. In fact, we regard (8.3.10) as a generalized asymptotic expansion. We shall demonstrate that when the limit process $\epsilon \to 0$, t_0 fixed, is applied to (8.3.10), we obtain (8.3.2), but the limit process $\epsilon \to 0$, t_1 fixed, does not exist.

The assumed form of U implies that derivatives are given by

$$\dot{u} = \frac{\partial U}{\partial t_0} + \epsilon \frac{\partial U}{\partial t_1} = \frac{\partial U_0}{\partial t_0} + \epsilon \left(\frac{\partial U_1}{\partial t_0} + \frac{\partial U_0}{\partial t_1} \right) + O(\epsilon^2), \tag{8.3.11a}$$

$$\begin{aligned} \ddot{u} &= \frac{\partial^2 U}{\partial t_0^2} + 2\epsilon \frac{\partial^2 U}{\partial t_0 \partial t_1} + \epsilon^2 \frac{\partial U}{\partial t_1^2} \\ &= \frac{\partial^2 U_0}{\partial t_0^2} + \epsilon \left(\frac{\partial^2 U_1}{\partial t_0^2} + 2 \frac{\partial U_0}{\partial t_0 \partial t_1} \right) + O(\epsilon^2). \end{aligned} \tag{8.3.11b}$$

Using these expressions in (8.3.1) leads to the following equations and initial conditions for U_0 and U_1.

$$\frac{\partial^2 U_0}{\partial t_0^2} + U_0 = 0, \tag{8.3.12a}$$

$$U_0(0, 0) = a, \quad \frac{\partial U_0}{\partial t_0}(0, 0) = 0, \tag{8.3.12b}$$

$$\frac{\partial^2 U_1}{\partial t_0^2} + U_1 = -2 \frac{\partial^2 U_0}{\partial t_0 \partial t_1} - \frac{\partial U_0}{\partial t_0} \left| \frac{\partial U_0}{\partial t_0} \right|, \tag{8.3.13a}$$

$$U_1(0, 0) = 0, \quad \frac{\partial U_1}{\partial t_0}(0, 0) = -\frac{\partial U_0}{\partial t_1}(0, 0). \tag{8.3.13b}$$

In (8.3.12a), t_1 occurs as a parameter (there are no derivatives with respect to t_1), so that this equation is actually an ordinary differential equation with a solution where the integration "constants" depend on t_1. We express the solution in the form

$$U_0(t_0, t_1) = A_0(t_1) \cos[t_0 + \phi_0(t_1)], \tag{8.3.14}$$

where A_0 and ϕ_0 are the leading-order slowly varying amplitude and phase shift, which are unknown at this stage. We know only that $A_0(0) = a$ and $\phi_0(0) = 0$, according to (8.3.12b).

To calculate U_1 we use (8.3.14) to express the right-hand side of (8.3.13a) explicitly. This gives (see (8.3.8) for the definition of the Fourier coefficients)

$$\begin{aligned} \frac{\partial^2 U_1}{\partial t_0^2} + U_1 =& 2A_0' \sin(t_0 + \phi_0) + 2A_0\phi_0' \cos(t_0 + \phi_0) + \frac{8}{3\pi} A_0^2 \sin(t_0 + \phi_0) \\ &+ \frac{4A_0^2}{\pi} \sum_{k=1}^{\infty} b_{2k+1} \sin(2k + 1)(t_0 + \phi_0), \end{aligned} \tag{8.3.15}$$

where a prime denotes differentiation with respect to t_1.

Again, t_1 is just a parameter as far as the integration of (8.3.15) with respect to t_0 is concerned. We know that the terms proportional to $\sin(t_0 + \phi_0)$ and $\cos(t_0 + \phi_0)$ on the right-hand side of (8.3.15) will give rise to mixed secular terms proportional to $t_0 \cos(t_0 + \phi_0)$ and $t_0 \sin(t_0 + \phi_0)$, respectively, in the solution for U_1. If we now regard the expansion as a generalized asymptotic expansion depending on t and ϵ, mixed secular terms introduce a nonuniformity when $t = O(\epsilon^{-1})$. The idea is to choose A_0 and ϕ_0 so that the solution is free of mixed secular terms. In this case, we must set

$$2A_0' + \frac{8}{3\pi} A_0^2 = 0, \quad A_0\phi_0' = 0. \tag{8.3.16}$$

The solution for A_0 subject to $A_0(0) = a$ is

$$A_0(t_1) = \frac{3a\pi}{4at_1 + 3\pi}, \tag{8.3.17a}$$

and the solution for ϕ_0 (with $a \neq 0$) subject to $\phi_0(0) = 0$ is

$$\phi_0(t_1) = 0. \tag{8.3.17b}$$

Therefore, we have determined U_0 as a function of t_0 and t_1 completely:

$$U_0(t_0, t_1) = \frac{3a\pi}{4at_1 + 3\pi} \cos t_0. \tag{8.3.18}$$

Note that in the limit as $\epsilon \to 0$ with t fixed, we obtain

$$U_0 = a \cos t - \frac{4}{3\pi} \epsilon a^2 t \cos t + O(\epsilon^2 t^2), \tag{8.3.19}$$

but the limit of (8.3.18) as $\epsilon \to 0$ with t_1 fixed does not exist. Equation (8.3.19) gives precisely the leading-order and mixed secular term of $O(\epsilon)$ that we calculated by regular perturbations; the bounded part of the $O(\epsilon)$ result in (8.3.9) is contained in U_1. For this example, the amplitude is a decreasing function of t_1, as is physically obvious. We have, in fact, shown that to leading order, the amplitude decays like $(\epsilon t)^{-1}$ for large t if the damping is quadratic. In contrast, if the damping term in (8.3.1) is linear ($2\epsilon\dot{u}$), the amplitude decays like $e^{-\epsilon t}$.

With the troublesome terms removed from the right-hand side of (8.3.15), we can solve this equation to obtain

$$U_1 = A_1(t_1) \cos[t_0 + \phi_1(t_1)] - \frac{A_0^2}{\pi} \sum_{k=1}^{\infty} \frac{b_{2k+1}}{k(k+1)} \sin(2k+1)t_0, \tag{8.3.20}$$

where A_1 and ϕ_1 are functions of t_1 to be determined by requiring the solution to $O(\epsilon^2)$ to be uniformly valid. This is as far as we shall proceed with the calculation of the expansion of (8.3.10). The procedure is straightforward and can be implemented, at least in principle, to higher orders. Actually, as pointed out in [26], we need to refine the assumed form (8.3.10) of the expansion to depend also on $t_2 = \epsilon^2 t$ if we wish to compute the solution to $O(\epsilon)$ and to include a dependence on $t_3 = \epsilon^3 t$ for a result correct to $O(\epsilon^2)$, and so on. Thus, the dependence of U_0 on t_2 is determined by imposing a consistency condition on U_1 with respect to t_1, and so on. If the perturbation term in (8.3.1) is of the form $\epsilon f(u, \dot{u})$, then U depends only on the "strained coordinate" $t^+ \equiv (1 + \epsilon^2\omega_2 + \epsilon^3\omega_3 + \ldots)t$, and t_1. Here $\omega_1, \omega_2, \ldots$ are constants that are derived by consistency conditions on the $O(\epsilon^2)$, $O(\epsilon^3), \ldots$ terms in the expansion. For more details on this and other aspects of the method of multiple scales for ordinary differential equations, the reader is referred to Chapter 4 of [26].

8.3.2 Small-Amplitude Shallow-Water Flow; Evolution Equations

In this section we use shallow-water flow to illustrate multiple-scale expansions for approximating the solution of weakly nonlinear partial differential equations in a form that remains uniformly valid in the far field. It is suggested that the reader first study Problem 8.3.1, where the simpler solution for a scalar quasilinear equation is outlined.

The Boussinesq approximation for shallow-water flow is given by the dimensionless system

$$h_t + (uh)_x = O(\epsilon^4), \tag{8.3.21a}$$

$$u_t + uu_x + h_x = -\frac{\kappa^2}{3}\epsilon h_{xtt} + O(\epsilon^3). \tag{8.3.21b}$$

We are interested in the solution over $-\infty < x < \infty$, $t \geq 0$ for a general small-amplitude disturbance in the form

$$u(x, 0; \epsilon) = \epsilon v(x), \quad h(x, 0; \epsilon) = 1 + \epsilon \ell(x). \tag{8.3.22}$$

It is shown in Section 5.2 of [27] that the system (8.3.21) is a consistent approximation for shallow water, up to the orders indicated, that follows from the more accurate equations governing an incompressible, irrotational, and inviscid flow. The term $-(\kappa^2/3)\epsilon h_{xtt}$ is the first correction term to the system (3.2.12) that we derived assuming hydrostatic balance in the vertical direction. The constant κ^2 is a similarity parameter, $\kappa^2 \equiv \delta^2/\epsilon$, and is held fixed as $\delta \to 0$ and $\epsilon \to 0$. Here, $\delta \equiv H/L$, where H is the undisturbed water height and L is a characteristic wavelength of the initial disturbance. Thus, $\delta \ll 1$ corresponds to shallow water (or long waves). The second small parameter is $\epsilon \equiv A/H$, where A is a characteristic amplitude for the initial disturbance. It is shown in [27] that the choice $\delta = \kappa\epsilon^{1/2}$ leads to the richest limiting equations. In our discussions of shallow-water flow so far in this book, we have set $\kappa \equiv 0$, which corresponds to having $\delta \ll \epsilon^{1/2}$.

(i) *Expansion procedure*

We assume a multiple-scale expansion in the form

$$u(x, t; \epsilon) = \epsilon u_1(x, t_0, t_1) + \epsilon^2 u_2(x, t_0, t_1) + O(\epsilon^3), \tag{8.3.23a}$$

$$h(x, t; \epsilon) = 1 + \epsilon h_1(x, t_0, t_1) + \epsilon^2 h_2(x, t_0, t_1) + O(\epsilon^3), \tag{8.3.23b}$$

where $t_0 \equiv t$, $t_1 \equiv \epsilon t$, and as mentioned in Section 8.3.2, we need not include a dependence on $t_2 \equiv \epsilon^2 t$ in the expansion because we are concerned only with the solution for u_1 and h_1.

Derivatives with respect to t become

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \epsilon \frac{\partial}{\partial t_1}, \tag{8.3.24}$$

and we obtain the following equations governing the terms of order ϵ and ϵ^2:

$$\frac{\partial h_1}{\partial t_0} + \frac{\partial u_1}{\partial x} = 0, \quad \frac{\partial u_1}{\partial t_0} + \frac{\partial h_1}{\partial x} = 0, \tag{8.3.25}$$

$$\frac{\partial h_2}{\partial t_0} + \frac{\partial u_2}{\partial x} = -\frac{\partial h_1}{\partial t_1} - u_1\frac{\partial h_1}{\partial x} - h_1\frac{\partial u_1}{\partial x}, \tag{8.3.26a}$$

$$\frac{\partial u_2}{\partial t_0} + \frac{\partial h_2}{\partial x} = -\frac{\partial u_1}{\partial t_1} - u_1\frac{\partial u_1}{\partial x} - \frac{\kappa^2}{3}\frac{\partial^3 h_1}{\partial x \partial t_0^2}. \tag{8.3.26b}$$

The intitial conditions (8.3.22) imply that u_1, h_1, u_2, and h_2 must satisfy

$$u_1(x, 0, 0) = v(x), \quad h_1(x, 0, 0) = \ell(x), \tag{8.3.27}$$

$$u_2(x, 0, 0) = 0, \quad h_2(x, 0, 0) = 0. \tag{8.3.28}$$

Notice that the system (8.3.25) is hyperbolic, and that each of the higher-order systems is also hyperbolic once the right-hand sides involving lower-order terms are known. Thus, it is convenient to introduce the characteristic dependent variables (see (4.3.22))

$$S_i \equiv h_i + u_i, \quad R_i \equiv h_i - u_i, \quad i = 1, 2, \tag{8.3.29}$$

and regard the S_i and R_i as functions of the slow time t_1 and the two fast characteristic independent variables

$$\xi \equiv x - t, \quad \eta \equiv x + t. \tag{8.3.30}$$

It is easily seen that (8.3.25)–(8.3.26) transform to the following equations for the $S_i(\xi, \eta, t_1)$ and $R_i(\xi, \eta, t_1)$:

$$2\frac{\partial S_1}{\partial \eta} = 0, \quad -2\frac{\partial R_1}{\partial \xi} = 0, \tag{8.3.31}$$

$$\begin{aligned} 2\frac{\partial S_2}{\partial \eta} = &-\frac{\partial S_1}{\partial t_1} - \left(\frac{S_1 - R_1}{2}\right)\left(\frac{\partial S_1}{\partial \xi} + \frac{\partial S_1}{\partial \eta}\right) \\ &- \left(\frac{S_1 + R_1}{4}\right)\left(\frac{\partial S_1}{\partial \xi} + \frac{\partial S_1}{\partial \eta} - \frac{\partial R_1}{\partial \xi} - \frac{\partial R_1}{\partial \eta}\right) \\ &- \frac{\kappa^2}{6}[D(S_1) + D(R_1)] \end{aligned} \tag{8.3.32a}$$

$$\begin{aligned} -2\frac{\partial R_2}{\partial \xi} = &-\frac{\partial R_1}{\partial t_1} - \left(\frac{S_1 - R_1}{2}\right)\left(\frac{\partial R_1}{\partial \xi} + \frac{\partial R_1}{\partial \eta}\right) \\ &- \left(\frac{S_1 + R_1}{4}\right)\left(\frac{\partial S_1}{\partial \xi} + \frac{\partial S_1}{\partial \eta} - \frac{\partial R_1}{\partial \xi} - \frac{\partial R_1}{\partial \eta}\right) \\ &+ \frac{\kappa^2}{6}[D(S_1) + D(R_1)], \end{aligned} \tag{8.3.32b}$$

where D is the third-order differential operator

$$D \equiv \frac{\partial^3}{\partial \xi^3} - \frac{\partial^3}{\partial \xi^2 \partial \eta} - \frac{\partial^3}{\partial \xi \partial \eta^2} + \frac{\partial^3}{\partial \eta^3}. \tag{8.3.33}$$

The initial conditions (8.3.27)–(8.3.28) transform to

$$S_1(x,0,0) = \ell(x) + v(x), \quad R_1(x,0,0) = \ell(x) - v(x), \tag{8.3.34}$$

$$S_2(x,0,0) = 0, \quad R_2(x,0,0) = 0. \tag{8.3.35}$$

An entirely equivalent approach is first to rescale the dependent variables in (8.3.21) as in (3.2.23), then to introduce the characteristic dependent variables to derive the weakly nonlinear system in the form (4.3.25) with additional $O(\epsilon)$ nonlinear terms. The characteristic dependent variables are then assumed to depend on the three scales ξ, η, t_1, and are expanded. This is the approach used in Section 6.2.4 of [26].

(ii) *Consistency conditions to $O(\epsilon)$; Korteweg–de Vries equations*

We conclude from (8.3.31) that S_1 is independent of η and R_1 is independent of ξ; that is,

$$S_1 = f_1(\xi, t_1), \quad R_1 = g_1(\eta, t_1), \tag{8.3.36}$$

where, according to (8.3.34), f_1 and g_1 satisfy the initial conditions

$$f_1(\xi, 0) = \ell(\xi) + v(\xi), \quad g_1(\eta, 0) = \ell(\eta) - v(\eta). \tag{8.3.37}$$

This is as far as we can go in defining S_1 and R_1; we need to consider the solution to $O(\epsilon^2)$ to derive conditions on f_1 and g_1.

Using the result in (8.3.36) simplifies the right-hand sides of (8.3.32) considerably, and we obtain

$$2\frac{\partial S_2}{\partial \eta} = -\left(\frac{\partial f_1}{\partial t_1} + \frac{3}{4} f_1 \frac{\partial f_1}{\partial \xi} + \frac{\kappa^2}{6}\frac{\partial^3 f_1}{\partial \xi^3}\right)$$
$$+ \frac{g_1}{4}\frac{\partial f_1}{\partial \xi} + \frac{f_1 + g_1}{4}\frac{\partial g_1}{\partial \eta} - \frac{\kappa^2}{6}\frac{\partial^3 g_1}{\partial \eta^3}, \tag{8.3.38a}$$

$$-2\frac{\partial R_2}{\partial \xi} = -\left(\frac{\partial g_1}{\partial t_1} - \frac{3}{4} g_1 \frac{\partial g_1}{\partial \eta} - \frac{\kappa^2}{6}\frac{\partial^3 g_1}{\partial \eta^3}\right) \tag{8.3.38b}$$
$$+ \frac{f_1}{4}\frac{\partial g_1}{\partial \eta} - \frac{f_1 + g_1}{4}\frac{\partial f_1}{\partial \xi} + \frac{\kappa^2}{6}\frac{\partial^3 f_1}{\partial \xi^3}. \tag{8.3.38b}$$

We can now integrate these equations and obtain

$$S_2 = -\frac{1}{2}\left(\frac{\partial f_1}{\partial t_1} + \frac{3}{4} f_1 \frac{\partial f_1}{\partial \xi} + \frac{\kappa^2}{6}\frac{\partial^3 f_1}{\partial \xi^3}\right)\eta + \frac{1}{8}\frac{\partial f_1}{\partial \xi}\int^{\eta} g_1(s, t_1)ds$$
$$+ \frac{1}{8} f_1 g_1 + \frac{1}{16} g_1^2 - \frac{\kappa^2}{12}\frac{\partial^2 g_1}{\partial \eta^2} + f_2(\xi, t_1), \tag{8.3.39a}$$

$$R_2 = \frac{1}{2}\left(\frac{\partial g_1}{\partial t_1} - \frac{3}{4} g_1 \frac{\partial g_1}{\partial \eta} - \frac{\kappa^2}{6}\frac{\partial^3 g_1}{\partial \eta^3}\right)\xi - \frac{1}{8}\frac{\partial g_1}{\partial \eta}\int^{\xi} f_1(s, t_1)ds$$

$$+ \frac{1}{8} f_1 g_1 + \frac{1}{16} f_1^2 - \frac{\kappa^2}{12} \frac{\partial^2 f_1}{\partial \xi^2} + g_2(\eta, t_1), \tag{8.3.39b}$$

where f_2 and g_2 are functions to be determined at the next stage.

The first group of terms multiplied by η on the right-hand side of (8.3.39a) must be eliminated because it contributes a component to S_2 that becomes infinite as $|\eta| \to \infty$ (that is, as $t \to \infty$ with x fixed, or as $x \to \infty$ with t fixed). This behavior implies that the expansions (8.3.23) fail to be uniform for $x = O_s(\epsilon^{-1})$ or $t = O_s(\epsilon^{-1})$. The equation that results by eliminating this troublesome term is the following evolution equation for f_1:

$$\frac{\partial f_1}{\partial t_1} + \frac{3}{4} f_1 \frac{\partial f_1}{\partial \xi} + \frac{\kappa^2}{6} \frac{\partial^3 f_1}{\partial \xi^3} = 0, \tag{8.3.40a}$$

which must be solved subject to the first equation in (8.3.37). Similarly, the boundedness of R_2 for $|\xi| \to \infty$ requires that we set

$$\frac{\partial g_1}{\partial t_1} - \frac{3}{4} g_1 \frac{\partial g_1}{\partial \eta} - \frac{\kappa^2}{6} \frac{\partial^3 g_1}{\partial \eta^3} = 0, \tag{8.3.40b}$$

and this is to be solved for g_1 subject to the second condition in (8.3.37).

Equations (8.3.40a) and (8.3.40b) are formally identical; the transformation $\xi \to -\eta$ takes the first to the second. One form of these equations (see (8.3.44)–(8.3.45)) was first derived by D.G. Korteweg and G. de Vries in 1895, and the reader is referred to Sections 13.11–13.15 of [42] and to [37] for a survey of results. In particular, the Korteweg–de Vries equation has bounded solutions for a large class of initial conditions.

An interesting feature of the result (8.3.40) is that the equations for f_1 and g_1 are decoupled and can be solved individually. This is a direct consequence of the fact that the conservation laws leading to (8.3.21) do not contain any source terms (or equivalently, that undifferentiated terms are absent from (8.3.21)). The solution for f_1 defines a disturbance that propagates to the right (as exhibited by the dependence of f_1 on $\xi \equiv x - t$), whereas g_1 defines a disturbance that propagates to the left; both of these disturbances evolve slowly with time (as exhibited by their dependence on t_1). Once we have found f_1 and g_1, the solution for u and h will be known to $O(\epsilon)$ in the form

$$u(x, t; \epsilon) = \frac{\epsilon}{2} \left[f_1(x - t, \epsilon t) - g_1(x + t, \epsilon t) \right] + O(\epsilon^2), \tag{8.3.41a}$$

$$h(x, t; \epsilon) = 1 + \frac{\epsilon}{2} \left[f_1(x - t, \epsilon t) + g_1(x + t, \epsilon t) \right] + O(\epsilon^2) \tag{8.3.41b}$$

For bounded f_1 and g_1, this result is uniformly valid if $|x|$ and t are of order ϵ^{-1}.

To compute S_2 and R_2 completely, we need to examine the terms of order ϵ^3 in the expansion to derive evolution equations for f_2 and g_2, and to account for a possible dependence on t_2. We shall not discuss these calculations, but observe from (8.3.39) that S_2 and R_2 are no longer functions of (ξ, t_1) and (η, t_1), respectively; they contain products of such functions.

Often, a Korteweg–de Vries equation is derived directly from (8.3.21) by looking for small-amplitude traveling waves. It is emphasized here that equations (8.3.40) apply more generally. In fact, these equations describe arbitrary flows as long as the disturbance amplitude is $O(\epsilon)$.

For the special case of unidirectional flows (for example, $g_1 \equiv 0$), our results also specialize to a Korteweg–de Vries equation for u or h, as shown next. In this case, we have

$$u = h - 1 = \frac{\epsilon}{2} f_1(\xi, t_1) + O(\epsilon^2). \tag{8.3.42a}$$

Thus,

$$f_1 = \frac{2}{\epsilon} u + O(\epsilon) = \frac{2}{\epsilon}(h-1) + O(\epsilon). \tag{8.3.42b}$$

Let us now transform (8.3.40a) to the original independent variables x and t according to

$$x \equiv \xi + \frac{1}{\epsilon} t_1, \quad t \equiv \frac{1}{\epsilon} t_1, \tag{8.3.43a}$$

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial t_1} = \frac{1}{\epsilon}\frac{\partial}{\partial x} + \frac{1}{\epsilon}\frac{\partial}{\partial t}. \tag{8.3.43b}$$

We obtain

$$\frac{2}{\epsilon^2}(u_x + u_t) + \frac{3}{4}\left(\frac{2u}{\epsilon}\right)\left(\frac{2u_x}{\epsilon}\right) + \frac{\kappa^2}{6}\frac{2}{\epsilon} u_{xxx} = 0,$$

or

$$\frac{2}{\epsilon^2}(h_x + h_t) + \frac{3}{4}\left[\frac{2(h-1)}{\epsilon}\right]\left[\frac{2h_x}{\epsilon}\right] + \frac{\kappa^2}{6}\frac{2}{\epsilon} h_{xxx} = 0,$$

to leading order. Therefore, to leading order, u and h obey (see (3.8.43))

$$u_t + u_x + \frac{3}{2} u u_x + \frac{\delta^2}{6} u_{xxx} = 0, \tag{8.3.44a}$$

$$h_t + h_x + \frac{3}{2}(h-1) h_x + \frac{\delta^2}{6} h_{xxx} = 0, \tag{8.3.44b}$$

where we have set $\delta^2 \equiv \kappa^2 \epsilon$. This is the form usually given in the literature for the Korteweg–de Vries equation for u or $h - 1$ for unidirectional shallow water flow. The transformation $3u + 2 = 3w$ or $3h - 1 = 3w$ takes (8.3.44) to the generic form

$$w_t + \frac{3}{2} w w_x + \frac{\delta^2}{6} w_{xxx} = 0. \tag{8.3.45}$$

The exact solution of (8.3.45) (and hence of (8.3.40)) can be derived for initial conditions $w(x, 0)$ that decay sufficiently fast as $|x| \to \infty$. The procedure, known as inverse scattering theory, was developed in 1967 (see [19]) and has since been studied extensively for the Kortewg–de Vries and other equations. Deriving $w(x, t)$ reduces to being able to solve a certain linear integral equation, and this is not explicit for general initial data. A discussion of this theory is beyond the scope of this book. The interested reader is referred to Chapter 17 of [42] and to [2]. In Chapter 3 we outlined the calculation of the solitary traveling wave solution of (8.3.45) (see (3.8.46)). The periodic traveling wave solution can also be easily found in terms of elliptic functions. For the details, see [38].

(iii) *Solution for $\kappa \equiv 0$; shock conditions for the evolution equations*

Explicit solutions of the evolution equations (8.3.40) can be calculated for arbitrary initial values of ℓ and v in (8.3.37) if we set $\kappa \equiv 0$, that is, $\delta \ll \epsilon^{1/2}$. The basic problem now satisfies the divergence relations (see (5.3.23))

$$h_t + (uh)_x = 0, \tag{8.3.46a}$$

$$(uh)_t + \left(u^2 h + \frac{h^2}{2}\right)_x = 0. \tag{8.3.46b}$$

The associated bore conditions are (see (5.3.24))

$$U[h] = [uh], \quad U[uh] = [u^2 h + h^2/2], \tag{8.3.47}$$

where $U \equiv dx/dt$ is the bore speed.

Since the evolution equations

$$\frac{\partial f_1}{\partial t_1} + \frac{3}{4} f_1 \frac{\partial f_1}{\partial \xi} = 0, \quad \frac{\partial g_1}{\partial t_1} - \frac{3}{4} g_1 \frac{\partial g_1}{\partial \eta} = 0, \tag{8.3.48}$$

which are inviscid Burgers' equations (see (1.7.1) with $\epsilon = 0$), now admit shocks, it is important to derive the shock conditions for (8.3.48) consistent with the exact conditions (8.3.47).

We denote $[u] = u^+ - u^-$, $[h] = h^+ - h^-$ and eliminate u from the two bore conditions (8.3.47). This gives the quadratic expression for U

$$U^2 - 2u^- U + (u^-)^2 - \frac{h^+(h^+ + h^-)}{2h^-} = 0 \tag{8.3.49}$$

that generalizes the result in (5.3.47b) calculated for the special case $u_2 = 0$ and $h_2 = 1$. We look for an expansion for U in the form

$$U = U_0 + \epsilon U_1 + O(\epsilon^2). \tag{8.3.50a}$$

But by definition we have

$$U \equiv \frac{dx}{dt} = \begin{cases} \dfrac{d}{dt}(\xi + t) = 1 + \epsilon \dfrac{d\xi}{dt_1} & \text{for } f_1, \\ \dfrac{d}{dt}(\eta - t) = -1 + \epsilon \dfrac{d\eta}{dt_1} & \text{for } g_1. \end{cases} \tag{8.3.50b}$$

Comparing these two expressions for U shows that $U_0 = \pm 1$, as expected, and that we must identify

$$U_1 = \begin{cases} \dfrac{d\xi}{dt_1} & \text{for } f_1, \\ \dfrac{d\eta}{dt_1} & \text{for } g_1. \end{cases} \tag{8.3.51}$$

Since f_1 and g_1 evolve independently, we can set $g_1 = 0$ to calculate the jump condition for f_1. In this case, (8.3.41) becomes

$$u^- = \frac{\epsilon f_1^-}{2} + O(\epsilon^2), \; h^{\pm} = 1 + \frac{\epsilon f^{\pm}}{2} + O(\epsilon^2), \tag{8.3.52}$$

and $U = 1 + \epsilon d\xi/dt_1$. Substituting these expressions into (8.3.49) gives

$$1 + 2\epsilon \frac{d\xi}{dt_1} - 2\epsilon \left(\frac{f_1^-}{2} \right) - \frac{(1 + \epsilon f_1^+/2)(2 + \epsilon f_1^+/2 + \epsilon f_1^-/2)}{2(1 + \epsilon f_1^-/2)} = O(\epsilon^2),$$

and this simplifies to

$$\epsilon \left(2\frac{d\xi}{dt_1} - \frac{3}{4}(f_1^+ + f_1^-) \right) = O(\epsilon^2). \tag{8.3.53}$$

Therefore, the jump condition for f_1 is

$$\frac{d\xi}{dt_1} = \frac{3}{8}(f_1^+ + f_1^-). \tag{8.3.54a}$$

A similar calculation starting with $f_1 \equiv 0$ gives the jump condition

$$\frac{d\eta}{dt_1} = -\frac{3}{8}(g_1^+ + g_1^-) \tag{8.3.54b}$$

for the evolution equation for g_1.

The solution to $O(1)$ of the initial-value problem (8.3.21)–(8.3.22) for the case $\delta \ll \epsilon^{1/2} \ll 1$ thus reduces to the relatively simple solution of the decoupled evolution equations (8.3.48) subject to the jump conditions (8.3.54). An example is outlined in Problem 8.3.2. In Problem 8.3.3 we study the effect of an isolated bottom disturbance on an initially uniform shallow-water flow.

8.3.3 Elastic Waves in a Heterogeneous Medium; Homogenization

Many problems of physical interest obey partial differential equations with coefficients that fluctuate rapidly in space; these fluctuations generally model heterogeneous materials. In this section we use multiple-scale expansions to study one-dimensional elastic waves in a material with rapid density variations. A number of other applications that are modeled by hyperbolic conservation laws with rapid spatial fluctuations are listed in [25], which also has a discussion of the general weakly nonlinear problem and references to related work. The problem of steady one- or two-dimensional heat conduction in a material with rapidly fluctuating thermal conductivity is discussed using multiple-scale expansions in Section 6.3 of [26].

The divergence relations for one-dimensional elastic waves in a solid are (for example, see p. 413 of [4])

$$\frac{\partial}{\partial t}(\rho(x^*)V) - \frac{\partial}{\partial x}(S(F)) = 0, \quad \frac{\partial F}{\partial t} - \frac{\partial V}{\partial x} = 0. \tag{8.3.55}$$

Here $x^* \equiv x/\epsilon$, where $0 < \epsilon \ll 1$. Thus, the density ρ is assumed to vary rapidly with x, and ϵ measures the length scale of these variations relative to the length scale of an initial disturbance. The displacement is denoted by F, and V is the velocity. The relation $T = S(F)$ gives the dependence of the stress T on F. Thus, we are assuming that T does not depend on x^*; the formulation for the case $T = S(F, x^*; \epsilon)$ is given in [25].

For $\rho =$ constant, the hyperbolic system (8.3.55) can be easily solved using the Riemann invariants (see Section 7.3.3), but this is no longer possible for variable ρ.

We wish to study (8.3.55) for small initial disturbances, that is, by perturbing the solution about the rest state $F = 0$, $V = 0$. Let ϵ also measure the amplitude of initial disturbances so that

$$V(x, 0; \epsilon) = \epsilon v_1(x; \epsilon) = O(\epsilon), \quad F(x, 0, \epsilon) = \epsilon v_2(x; \epsilon) = O(\epsilon). \tag{8.3.56}$$

If we introduce the rescaled variables

$$V(x, t; \epsilon) = \epsilon u_1(x, t; \epsilon), \quad F(x, t; \epsilon) = \epsilon u_2(x, t; \epsilon), \tag{8.3.57}$$

where u_1 and u_2 are both $O(1)$ as $\epsilon \to 0$, we obtain the system

$$\frac{\partial u_1}{\partial t} - \frac{S'(0)}{\rho(x^*)}\frac{\partial u_2}{\partial x} = \epsilon \frac{S''(0)}{\rho(x^*)} u_2 \frac{\partial u_2}{\partial x} + O(\epsilon^2), \tag{8.3.58a}$$

$$\frac{\partial u_2}{\partial t} - \frac{\partial u_1}{\partial x} = 0, \tag{8.3.58b}$$

and initial conditions

$$u_1(x, 0; \epsilon) = v_1(x; \epsilon), \quad u_2(x, 0; \epsilon) = v_2(x; \epsilon). \tag{8.3.59}$$

(i) *A scalar problem*

To motivate our discussion of the solution of the linear part of (8.3.58) and the role of initial data, let us first study the scalar problem

$$u_t + \frac{1}{1 + \delta \cos x^*} u_x = 0, \tag{8.3.60}$$

where $|\delta| < 1$ is a constant.

We note that if the initial value for u depends on $x^*(u(x, 0; \epsilon) = f(x^*))$, we may introduce the rescaled independent variables x^* and $t^* \equiv t/\epsilon$ and see that

$u^*(x^*, t^*) \equiv u(\epsilon x^*, \epsilon t^*; \epsilon)$ obeys

$$\frac{\partial u^*}{\partial t^*} + \frac{1}{1+\delta \cos x^*}\frac{\partial u^*}{\partial x^*} = 0, \quad u^*(x^*, 0) = f(x^*). \tag{8.3.61}$$

The solution is thus a function of x^* and t^* and does not depend on ϵ. Although we can calculate $u^*(x^*, t^*)$ exactly, this is no longer a perturbation problem. The corresponding situation for (8.3.58)—that is, initial data of the form $u_i(x, 0; \epsilon) = v_i(x^*)$ for $i = 1, 2$—leads to the linear variable-coefficient system

$$\frac{\partial u_1^*}{\partial t^*} - \frac{S'(0)}{\rho(x^*)}\frac{\partial u_2^*}{\partial x^*} = O(\epsilon), \quad \frac{\partial u_2^*}{\partial t^*} - \frac{\partial u_1^*}{\partial x^*} = 0 \tag{8.3.62}$$

to leading order. Now u_1^* and u_2^* depend on x^* and t^* to leading order, and we cannot solve (8.3.62) exactly for general ρ. In fact, if the solution depends on both x^* and t^* to any order in ϵ, then the governing system to that order has the homogeneous operator (8.3.62) and cannot be solved exactly. To be able to calculate a perturbation solution to any given order in ϵ, we must therefore require that this solution be independent of t^* to that order. This restricts the class of initial-value problems that we can handle, as can be seen from the solution of the model equation (8.3.60) that we will consider next for the initial data

$$u(x, 0; \epsilon) = f^{(0)}(x) + \epsilon f^{(1)}(x^*, x). \tag{8.3.63}$$

The role of the $f^{(1)}$ term will become clear presently.

The characteristics of (8.3.60) satisfy (see (5.2.3))

$$\frac{dt}{ds} = 1, \quad \frac{dx}{ds} = \frac{1}{1+\delta\cos(x/\epsilon)}, \quad \frac{du}{ds} = 0. \tag{8.3.64}$$

The solution for u is then easily obtained in the implicit form

$$u(x, t; \epsilon) = f^{(0)}(\xi) + \epsilon f^{(1)}\left(\frac{\xi}{\epsilon}, \xi\right), \tag{8.3.65a}$$

where ξ is constant on a characteristic and $\xi(x^*, t^*, x, t; \epsilon)$ is given by

$$\xi = x - t + \epsilon g(x^*, t^*). \tag{8.3.65b}$$

The function $g(x^*, t^*)$ is defined by the implicit relation

$$g - \delta \sin x^* + \delta \sin(x^* - t^* + g) = 0, \tag{8.3.65c}$$

and is therefore 2π-periodic in both x^* and t^*.

This result shows that to $O(1)$, u does not depend on x^* or t^*; these dependencies first arise in the $O(\epsilon)$ term. In fact, we compute the following expansion for the solution

$$u(x, t; \epsilon) = f^{(0)}(x-t) + \epsilon[f^{(0)'}(x-t)g(x^*, t^*) + f^{(1)}(x^* - t^* + g(x^*, t^*), x - t)] + O(\epsilon^2). \tag{8.3.66}$$

Note that even if the initial condition were independent of x^* to $O(\epsilon)$, for example, if $f^{(1)} \equiv 0$, the solution to $O(\epsilon)$ still involves t^* through the $f^{(0)'}g$ term. To have a solution independent of t^* to $O(\epsilon)$ (and to subsequent orders), we need to choose $f^{(1)}$ (and the higher-order terms $\epsilon^2 f^{(2)}$, $\epsilon^3 f^{(3)}, \ldots$) appropriately. In particular, we must set

$$f^{(1)}(x^*, x) = \delta f^{(0)'}(x) \sin x^* + \phi^{(1)}(x) \tag{8.3.67}$$

in order to have t^* absent from the $O(\epsilon)$ term in (8.3.66). This choice is unique with regard to the x^* dependence of $f^{(1)}$; the added $\phi^{(1)}$ term is arbitrary. With this choice, (8.3.66) becomes

$$u(x, t; \epsilon) = f^{(0)}(x-t) + \epsilon[\delta f^{(0)'}(x-t) \sin x^* + \phi^{(1)}(x-t)] + O(\epsilon^2). \tag{8.3.68}$$

We call the x^*-independent part of (8.3.68) the *homogenized* solution to $O(\epsilon)$.

Let us now compute this result using the multiple-scale expansion

$$u(x, t; \epsilon) = \sum_{n=0}^{N} \epsilon^n u^{(n)}(x^*, x, t) + O(\epsilon^{N+1}) \tag{8.3.69}$$

and the governing equation without the use of the exact result. The equations for $n = 1, 2, 3$ that follow from (8.3.60) are

$$\frac{1}{1+\delta \cos x^*} \frac{\partial u^{(0)}}{\partial x^*} = 0, \tag{8.3.70a}$$

$$\frac{1}{1+\delta \cos x^*} \frac{\partial u^{(1)}}{\partial x^*} = -\left(\frac{\partial u^{(0)}}{\partial t} + \frac{1}{1+\delta \cos x^*} \frac{\partial u^{(0)}}{\partial x} \right), \tag{8.3.70b}$$

$$\frac{1}{1+\delta \cos x^*} \frac{\partial u^{(2)}}{\partial x^*} = -\left(\frac{\partial u^{(1)}}{\partial t} + \frac{1}{1+\delta \cos x^*} \frac{\partial u^{(1)}}{\partial x} \right). \tag{8.3.70c}$$

The initial conditions on $u^{(0)}$ and $u^{(1)}$ are

$$u^{(0)}(x^*, x, 0) = f^{(0)}(x), \; u^{(1)}(x^*, x, 0) = f^{(1)}(x^*, x). \tag{8.3.71}$$

Equation (8.3.70a) shows that $u^{(0)}$ does not depend on x^*, and we have

$$u^{(0)} = \phi^{(0)}(x, t), \; \phi^{(0)}(x, 0) = f^{(0)}(x). \tag{8.3.72}$$

Multiplying (8.3.70b) by $(1 + \delta \cos x^*)$ and using $u^{(0)} = \phi^{(0)}(x, t)$ in the right-hand side gives

$$\frac{\partial u^{(1)}}{\partial x^*} = -\left(\frac{\partial \phi^{(0)}}{\partial t} + \frac{\partial \phi^{(0)}}{\partial x} \right) - \delta \frac{\partial \phi^{(0)}}{\partial t} \cos x^*. \tag{8.3.73}$$

The first term in parentheses is independent of x^* and will contribute an inconsistent term proportional to x^* to the solution for $u^{(1)}$. Such a term is inconsistent in the sense that the expansion is not uniformly valid in the interval $0 < x^* < X^*(\epsilon)$, where $X^* = O(\epsilon^{-1})$—that is, in an interval of $O(1)$ extent in x. Therefore, we set $\phi_t^{(0)} + \phi_x^{(0)} = 0$ and obtain $\phi^{(0)} = f^{(0)}(x - t)$ once we impose the initial condition. This is just the result we found to $O(1)$ in (8.3.66).

We next integrate what remains in (8.3.73) to calculate

$$u^{(1)} = \delta f^{(0)'}(x-t)\sin x^* + \underline{u}^{(1)}(x,t), \tag{8.3.74}$$

where $\underline{u}^{(1)}$ is to be determined. We now mutliply (8.3.70c) by $(1+\delta\cos x^*)$ and use (8.3.74) for $u^{(1)}$ on the right-hand side to obtain

$$\frac{\partial u^{(2)}}{\partial x^*} = -\left(\frac{\partial \underline{u}^{(1)}}{\partial t} + \frac{\partial \underline{u}^{(1)}}{\partial x}\right) + \delta f^{(0)''}\cos x^* \sin x^* - \frac{\partial \underline{u}^{(1)}}{\partial t}\cos x^*. \tag{8.3.75}$$

Again, to avoid an inconsistent term proportional to x^* in $u^{(2)}$, we must set the terms in parentheses on the right-hand side equal to zero, and this gives $\underline{u}^{(1)} = \phi^{(1)}(x-t)$, where $\phi^{(1)}$ is arbitrary. Thus, (8.3.74) agrees with the $O(\epsilon)$ term in (8.3.68). Notice that the initial condition to $O(\epsilon)$ cannot be specified arbitrarily; we have

$$\delta f^{(0)'}(x)\sin x^* + \phi^{(1)}(x) = f^{(1)}(x^*, x).$$

This is just (8.3.67), and we see that the x^* dependence in $f^{(1)}$ cannot be prescribed arbitrarily; only $\phi^{(1)}$ is arbitrary.

To summarize, we have derived a class of solutions that are independent of t^* to $O(\epsilon)$. In order to do so, we may prescribe arbitrary x^*-independent initial data to $O(1)$, but to $O(\epsilon)$ (and all higher orders) the initial data must have a specific x^* dependence. This dependence can be derived a posteriori once the assumed form (8.3.69) of the multiple-scale expansion has been calculated.

(ii) *Solution of (8.3.58)*

We assume that $\mathbf{u} = (u_1, u_2)$ has the expansion

$$\mathbf{u}(x,t;\epsilon) = \underline{\mathbf{u}}^{(0)}(x,t,t_1) + \epsilon\mathbf{u}^{(1)}(x^*,x,t,t_1) + \epsilon^2\mathbf{u}^{(2)}(x^*,x,t,t_1) + O(\epsilon^3), \tag{8.3.76}$$

where $x^* \equiv x/\epsilon$ and $t_1 = \epsilon t$. Note that the leading term is assumed not to depend on x^*. This follows directly from the fact that the initial condition to $O(1)$ does not depend on x^*:

$$\underline{\mathbf{u}}^{(0)}(x,0,0) = \mathbf{f}^{(0)}(x). \tag{8.3.77}$$

Henceforth, we shall use an underbar to indicate a function that does not depend on x^*. The absence of t^* from this expansion is to be accomplished, as for the scalar problem, by an appropriate a posteriori choice of the initial data to higher order. The t_1 dependence in the expansion is included to account for the weakly nonlinear term in (8.3.58a). Thus, as in the example of Section 8.3.2, we expect the solution of the initial-value problem to evolve slowly in time.

Derivatives have the expansions

$$\mathbf{u}_t = \underline{\mathbf{u}}_t^{(0)} + \epsilon(\mathbf{u}_t^{(1)} + \underline{\mathbf{u}}_{t_1}^{(0)}) + \epsilon^2(\mathbf{u}_t^{(2)} + \mathbf{u}_{t_1}^{(1)}) + O(\epsilon^3), \tag{8.3.78a}$$

$$\mathbf{u}_x = (\mathbf{u}_{x^*}^{(1)} + \underline{\mathbf{u}}_x^{(0)}) + \epsilon(\mathbf{u}_{x^*}^{(2)} + \mathbf{u}_x^{(1)}) + O(\epsilon^2). \tag{8.3.78b}$$

Substituting these expansions and (8.3.76) into (8.3.58) gives

$$A(x^*)\mathbf{u}_{x^*}^{(1)} = -A(x^*)\underline{\mathbf{u}}_x^{(0)} - \underline{\mathbf{u}}_t^{(0)}, \tag{8.3.79a}$$

$$A(x^*)\mathbf{u}^{(2)}_{x^*} = -A(x^*)\mathbf{u}^{(1)}_x - \mathbf{u}^{(1)}_t + \underline{\mathbf{u}}^{(0)}_{t_1} + D(\underline{\mathbf{u}}^{(0)}, x^*)\underline{\mathbf{u}}^{(0)}_x, \tag{8.3.79b}$$

where

$$A(x^*) \equiv \begin{pmatrix} 0 & -c^2(x^*) \\ -1 & 0 \end{pmatrix}, \; c^2(x^*) = \frac{S'(0)}{\rho(x^*)} > 0, \tag{8.3.79c}$$

$$D(\underline{\mathbf{u}}^{(0)}, x^*) \equiv \begin{pmatrix} 0 & -a(x^*)\underline{u}^{(0)}_2 \\ 0 & 0 \end{pmatrix}, \; a(x^*) = \frac{S''(0)}{\rho(x^*)}. \tag{8.3.79d}$$

Let us denote the average value of a function h of x^* by $\langle h \rangle$. Thus,

$$\langle h \rangle \equiv \lim_{P\to\infty} \frac{1}{P} \int_{-P}^{P} h(x^*)dx^*. \tag{8.3.80a}$$

And let us restrict attention to functions for which this limit exists. If h is a $2P$-periodic function of x^*, then

$$\langle h \rangle \equiv \frac{1}{2P} \int_{-P}^{P} h(x^*)dx^*. \tag{8.3.80b}$$

The fluctuating part of $h(x^*)$ will be denoted by $\{h\}$. Thus,

$$\{h\} = h(x^*) - \langle h \rangle, \tag{8.3.80c}$$

and we note that $\langle \{h\} \rangle = 0$.

We multiply (8.3.79a) by A^{-1} (the inverse of the matrix in (8.3.79c)) and isolate the average and fluctuating parts of the resulting right-hand side to obtain

$$\mathbf{u}^{(1)}_{x^*} = -\left(\langle A^{-1}\rangle\underline{\mathbf{u}}^{(0)}_t + \underline{\mathbf{u}}^{(0)}_x\right) - \{A^{-1}\}\underline{\mathbf{u}}^{(0)}_t, \tag{8.3.81a}$$

where

$$\langle A^{-1}\rangle = \begin{pmatrix} 0 & -1 \\ -\langle 1/c^2\rangle & 0 \end{pmatrix}, \; \{A^{-1}\} = \begin{pmatrix} 0 & 0 \\ -\{1/c^2\} & 0 \end{pmatrix}. \tag{8.3.81b}$$

Consistency of $\mathbf{u}^{(1)}$ with respect to x^* demands

$$L(\underline{\mathbf{u}}^{(0)}) \equiv \langle A^{-1}\rangle\underline{\mathbf{u}}^{(0)}_t + \underline{\mathbf{u}}^{(0)}_x = 0, \tag{8.3.82a}$$

or, multiplying by $\langle A^{-1}\rangle^{-1}$,

$$\mathcal{L}(\underline{\mathbf{u}}^{(0)}) \equiv \underline{\mathbf{u}}^{(0)}_t + \mathcal{A}\underline{\mathbf{u}}^{(0)}_x = 0, \tag{8.3.82b}$$

where

$$\mathcal{A} = \langle A^{-1}\rangle^{-1} = \begin{pmatrix} 0 & -k^2 \\ -1 & 0 \end{pmatrix}, \; k^2 = \frac{1}{\langle 1/c^2\rangle}. \tag{8.3.82c}$$

Now we can integrate what remains of (8.3.81a) to calculate

$$\mathbf{u}^{(1)} = -\{\{A^{-1}\}\}\underline{\mathbf{u}}^{(0)}_t + \underline{\mathbf{u}}^{(1)}(x, t, t_1), \tag{8.3.83}$$

where, for a given fluctuating function $\{h\}$ with zero average, $\{\{h\}\}$ denotes the integral

$$\{\{h\}\} \equiv \int_{x_0^*}^{x^*} \{h(s)\}ds, \tag{8.3.84a}$$

and x_0^* is chosen such that $\langle\{\{h\}\}\rangle = 0$. Thus,

$$\{\{A^{-1}\}\} = \begin{pmatrix} 0 & 0 \\ \{\{1/c^2\}\} & 0 \end{pmatrix}. \tag{8.3.84b}$$

The homogenized solution to $O(\epsilon)$ is given by $\underline{\mathbf{u}}^{(0)} + \epsilon\underline{\mathbf{u}}^{(1)}$, and the function $\underline{\mathbf{u}}^{(1)}$ is unknown at this stage.

We now evaluate $\mathbf{u}_t^{(1)}$ and $\mathbf{u}_x^{(1)}$ using (8.3.83) and susbstitute the results into (8.3.79b) after we multiply this equation by A^{-1}. The result is

$$\begin{aligned} \mathbf{u}_{x^*}^{(2)} = &- L(\underline{\mathbf{u}}^{(1)}) - A^{-1}\underline{\mathbf{u}}_{t_1}^{(0)} + \{\{A^{-1}\}\}\underline{\mathbf{u}}_{xt}^{(0)} \\ &+ A^{-1}D\underline{\mathbf{u}}_x^{(0)} + A^{-1}\{\{A^{-1}\}\}\underline{\mathbf{u}}_{tt}^{(0)}. \end{aligned} \tag{8.3.85}$$

Again, we decompose the terms on the right-hand side into average and fluctuating parts, then remove the average terms to maintain consistency of $\mathbf{u}^{(2)}$ with respect to x^* and obtain

$$L(\underline{\mathbf{u}}^{(1)}) = -\langle A^{-1}\rangle\underline{\mathbf{u}}_{t_1}^{(0)} + \langle A^{-1}D\rangle\underline{\mathbf{u}}_x^{(0)}, \tag{8.3.86a}$$

or

$$\mathcal{L}(\underline{\mathbf{u}}^{(1)}) = -\underline{\mathbf{u}}_{t_1}^{(0)} + \mathcal{D}\underline{\mathbf{w}}_x^{(0)}, \quad \mathcal{D} = \mathcal{A}\langle A^{-1}D\rangle. \tag{8.3.86b}$$

For the special matrix $A(x^*)$ in (8.3.79c) we obtain

$$A^{-1}\{\{A^{-1}\}\} = \begin{pmatrix} \{\{1/c^2\}\} & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, $\langle A^{-1}\{\{A^{-1}\}\}\rangle = 0$, and the $\underline{\mathbf{u}}_{tt}^{(0)}$ term in (8.3.85) does not contribute to (8.3.86). For a general matrix $A(x^*)$, this is not the case, and the right-hand side of (8.3.86b) will also involve the term $\mathcal{A}\langle A^{-1}\{\{A^{-1}\}\}\rangle\underline{\mathbf{u}}_{tt}^{(0)}$. The implications of such a term on the evolution of $\underline{\mathbf{u}}^{(0)}$ are explored in [25]. For our example, the $\mathcal{D}$ matrix is just

$$\mathcal{D} = \begin{pmatrix} 0 & \alpha\underline{u}_2^{(0)} \\ 0 & 0 \end{pmatrix}, \quad \alpha \equiv \frac{\langle a/c^2\rangle}{\langle 1/c^2\rangle}. \tag{8.3.87}$$

Equations (8.3.82b) and (8.3.86b) define the average parts of $\mathbf{u}^{(0)}$ and $\underline{\mathbf{u}}^{(1)}$. Actually, $\mathbf{u}^{(0)}$ does not have a fluctuating part, but $\mathbf{u}^{(1)}$ does, as seen from (8.3.83). If we ignore the fluctuating terms to a given order, say ϵ^n, then the sum $\underline{\mathbf{u}}^{(0)} + \epsilon\underline{\mathbf{u}}^{(1)} + \ldots + \epsilon^n\underline{\mathbf{u}}^{(n)}$ is often referred to as the "homogenized" solution to order ϵ^n. This is just the average solution. For our example, we have the following system governing the two components of $\underline{\mathbf{u}}^{(0)}$ and $\underline{\mathbf{u}}^{(1)}$:

$$\frac{\partial\underline{u}_1^{(0)}}{\partial t} - k^2\frac{\partial\underline{u}_2^{(0)}}{\partial x} = 0, \quad \frac{\partial\underline{u}_2^{(0)}}{\partial t} - \frac{\partial\underline{u}_1^{(0)}}{\partial x} = 0, \tag{8.3.88}$$

$$\frac{\partial\underline{u}_1^{(1)}}{\partial t} - k^2\frac{\partial\underline{u}_2^{(1)}}{\partial x} = -\frac{\partial\underline{u}_1^{(0)}}{\partial t_1} + \alpha\underline{u}_2^{(0)}\frac{\partial\underline{u}_2^{(0)}}{\partial x}, \tag{8.3.89a}$$

$$\frac{\partial\underline{u}_2^{(1)}}{\partial t} - \frac{\partial\underline{u}_1^{(1)}}{\partial x} = -\frac{\partial\underline{u}_2^{(0)}}{\partial t_1}. \tag{8.3.89b}$$

Let us introduce the characteristic dependent variables (see (4.3.21))

$$U_1^{(i)} = \frac{1}{2k^2}\underline{u}_1^{(i)} - \frac{1}{2k}\underline{u}_2^{(i)}, \tag{8.3.90a}$$

$$U_2^{(i)} = \frac{1}{2k^2}\underline{u}_1^{(i)} + \frac{1}{2k}\underline{u}_2^{(i)}, \; i = 1, 2, \tag{8.3.90b}$$

regarded as functions of the characteristic independent variables

$$\xi_1 = x - kt, \quad \xi_2 = x + kt \tag{8.3.91}$$

and the slow time t_1. Equations (8.3.88) then simplify to

$$2k\frac{\partial U_1^{(0)}}{\partial \xi_2} = 0, \quad -2k\frac{\partial U_2^{(0)}}{\partial \xi_1} = 0. \tag{8.3.92}$$

Therefore,

$$U_1^{(0)} = \phi_1^{(0)}(\xi_1, t_1), \quad U_2^{(0)} = \phi_2^{(0)}(\xi_2, t_1), \tag{8.3.93}$$

where according to (8.3.63b) and (8.3.90), $\phi_1^{(0)}$ and $\phi_2^{(0)}$ must satisfy the initial conditions

$$\phi_1^{(0)}(\xi_1, 0) = f_1^{(0)}(\xi_1), \quad \phi_2^{(0)}(\xi_2, 0) = f_2^{(0)}(\xi_2). \tag{8.3.94}$$

Transforming (8.3.89) to characteristic dependent and independent variables gives

$$2k\frac{\partial U_1^{(1)}}{\partial \xi_2} = -\frac{\partial \phi_1^{(0)}}{\partial t_1} + \frac{\alpha}{2}\left(\phi_1^{(0)} - \phi_2^{(0)}\right)\left(\frac{\partial \phi_1^{(0)}}{\partial \xi_1} - \frac{\partial \phi_2^{(0)}}{\partial \xi_2}\right), \tag{8.3.95a}$$

$$-2k\frac{\partial U_2^{(1)}}{\partial \xi_1} = -\frac{\partial \phi_2^{(0)}}{\partial t_1} + \frac{\alpha}{2}\left(\phi_1^{(0)} - \phi_2^{(0)}\right)\left(\frac{\partial \phi_1^{(0)}}{\partial \xi_1} - \frac{\partial \phi_2^{(0)}}{\partial \xi_2}\right). \tag{8.3.95b}$$

The terms on the right-hand side of (8.3.95a) that do not depend on ξ_2 will contribute inconsistent terms proportional to ξ_2 in the solution for $U_1^{(1)}$. Similarly, terms independent of ξ_1 on the right-hand side of (8.3.95b) must be eliminated. Removing these terms gives the two decoupled evolution equations

$$\frac{\partial \phi_1^{(0)}}{\partial t_1} - \frac{\alpha}{2}\phi_1^{(0)}\frac{\partial \phi_1^{(0)}}{\partial \xi_1} = 0, \quad \frac{\partial \phi_2^{(0)}}{\partial t_1} - \frac{\alpha}{2}\phi_2^{(0)}\frac{\partial \phi_2^{(0)}}{\partial \xi_2} = 0. \tag{8.3.96}$$

These are inviscid Burgers' equations to be solved subject to the initial conditions in (8.3.94). The straightforward details are omitted. Once $\phi_1^{(0)}$ and $\phi_2^{(0)}$ have been calculated, the solution of $\mathbf{u}$ to $O(1)$ is given by the inverse of (8.3.90),

$$\underline{u}_1^{(0)} = k^2[\phi_1^{(0)}(\xi_1, t_1) + \phi_2^{(0)}(\xi_2, t_1)], \; \underline{u}_2^{(0)} = k[-\phi_1^{(0)}(\xi_1, t_1) + \phi_2^{(0)}(\xi_2, t_1)]. \tag{8.3.97}$$

A specific example is worked out in [25], where it is shown that the asymptotic result in (8.3.97) is in excellent agreement with the numerical solution of (8.3.58) in a time interval of $O(\epsilon^{-1})$. Shock formation degrades the accuracy because the discontinuity across the shock contradicts the assumed form of the expansion. For more details concerning this and other issues regarding multiple-scale homognization, see [25] and the cited references.

Problems

8.3.1 The following simple scalar problem that can be solved exactly illustrates the use of multiple-scale expansions for weakly nonlinear hyperbolic systems:

$$u_t + (1 + \epsilon u)u_x = 0, \quad 0 < \epsilon = \text{ constant.} \tag{8.3.98}$$

a. Consider the initial-value problem for (8.3.98) on $-\infty < x < \infty$ with initial condition

$$u(x, 0; \epsilon) = f(x), \tag{8.3.99}$$

where f is a bounded continuous nondecreasing function of x. Show that the exact solution is given by

$$u = f(\xi), \quad \xi + \epsilon t f(\xi) = x - t. \tag{8.3.100}$$

Thus, ξ is a function of $z \equiv x - t$ and $t_1 \equiv \epsilon t$, and therefore u is also a function of z and t_1.

b. Now let $0 < \epsilon \ll 1$ and assume a multiple-scale expansion of the solution in the form

$$u(x, t; \epsilon) = u_0(x, t, t_1) + \epsilon u_1(x, t, t_1) + O(\epsilon^2). \tag{8.3.101}$$

Show that solving the equation governing u_0 gives

$$u_0 = \phi(z, t_1), \quad \phi(x, 0) = f(x), \tag{8.3.102}$$

where $\phi(z, t_1)$ is to be determined. Consider the equation for u_1 and require the expansion (8.3.101) to be uniformly valid to $O(\epsilon)$ for all t in the interval $0 \le t \le T(\epsilon)$, where $T = O(\epsilon^{-1})$, to derive the evolution equation

$$\frac{\partial \phi}{\partial t_1} + \phi \frac{\partial \phi}{\partial z} = 0. \tag{8.3.103}$$

The solution of (8.3.103) subject to $\phi(z, 0) = f(z)$ gives the exact result (8.3.100) for u_0. Verify that the expansion (8.3.101) does indeed terminate with the first term in this case.

c. Now consider (8.3.98) on $0 \le x \le \infty$ with the initial condition $u(x, 0) = 1$, and the boundary condition

$$u(0, t) = g(t), \quad t > 0, \quad g(0) = 1, \tag{8.3.104}$$

where g is a continuous nonnegative nonincreasing function of t. The solution for $x \ge 0$, $t \ge 0$ is unchanged if we extend the definition of

$g(t)$ to negative t by choosing $g(t) = 1$ if $t \leq 0$ and solving (8.3.98) for $x \geq 0$ and all t. Show that this solution is

$$u = g(\tau), \quad \tau - \frac{\epsilon x g(\tau)}{1 + \epsilon g(\tau))} = t - x. \tag{8.3.105}$$

Thus, u is a function of $y \equiv t - x$ and $x_1 \equiv \epsilon x$.

d. Now assume a multiple-scale expansion of the solution in the form

$$u(x, t; \epsilon) = v_0(x, t, x_1) + \epsilon v_1(x, t, x_1) + O(\epsilon^2), \tag{8.3.106}$$

and show that the solution for the equation governing v_0 gives

$$v_0 = \psi(y, x_1), \quad \psi(t, 0) = g(t). \tag{8.3.107}$$

Derive the evolution equation

$$\frac{\partial \psi}{\partial x_1} - \psi \frac{\partial \psi}{\partial y} = 0 \tag{8.3.108}$$

by requiring the expansion (8.3.106) to be uniformly valid to $O(\epsilon)$ for all x in the interval $0 \leq x \leq X(\epsilon)$, where $X = O(\epsilon^{-1})$. Solve (8.3.108) subject to $\psi(y, 0) = g(y)$ to obtain

$$\psi = g(\tau), \quad \tau - x_1 g(\tau) = y, \tag{8.3.109}$$

which is just the $O(1)$ term of the exact solution (8.3.105).

8.3.2 Consider the initial-value problem for (8.3.46) with

$$u(x, 0; \epsilon) = 0, \tag{8.3.110a}$$

$$h(x, 0; \epsilon) = 1 + \epsilon \begin{cases} 2x - 1 & \text{if } 0 \leq x \leq \dfrac{1}{2}, \\ -1 - 2x & \text{if } -\dfrac{1}{2} < x \leq 0, \\ 0 & \text{if } |x| \geq \dfrac{1}{2}. \end{cases} \tag{8.3.110b}$$

a. Show that the solution of the evolution equation in (8.3.48) for $f_1(\xi, t_1)$ is

$$f_1 = \begin{cases} (4\xi - 2)/(2 + 3t_1) & \text{if } -\dfrac{3}{4}t_1 \leq \xi \leq \dfrac{1}{2}, t_1 < \dfrac{2}{3}, \\ -(4\xi + 2)/(2 - 3t_1) & \text{if } -\dfrac{1}{2} \leq \xi \leq -\dfrac{3}{4}t_1, t_1 < \dfrac{2}{3}, \\ 0 & \text{if } |\xi| \geq \dfrac{1}{2}, \end{cases} \tag{8.3.111}$$

and use symmetry to show that

$$g_1(\eta, t_1) = f_1(-\eta, t_1). \tag{8.3.112}$$

b. Show that the characteristics for f_1 all intersect at the point $\xi = -\frac{1}{2}$, $t_1 = \frac{2}{3}$, and that we must introduce a bore

$$\xi = \frac{1}{2} - \frac{1}{2}(2 + 3t_1)^{1/2} \tag{8.3.113}$$

for $t_1 > \frac{2}{3}$. Show that in the xt-plane, the bores in f_1 and g_1 translate to the following expressions that are correct to $O(1)$ only (Why?):

$$x = t + \frac{1}{2} - \frac{1}{2}(2 + 3\epsilon t)^{1/2} + O(\epsilon) \text{ if } t \geq \frac{2}{3\epsilon}, \tag{8.3.114a}$$

$$x = -t - \frac{1}{2} + \frac{1}{2}(2 + 3\epsilon t)^{1/2} + O(\epsilon) \text{ if } t \geq \frac{2}{3\epsilon}. \tag{8.3.114b}$$

8.3.3 Reconsider Problem 4.3.2, that is,

$$h_t + (uh)_x = 0, \tag{8.3.115a}$$

$$u_t + uu_x + (h + \epsilon B)_x = 0 \tag{8.3.115b}$$

for the case

$$B(x) = \begin{cases} 1 - 4x^2 & \text{if } |x| \leq \frac{1}{2}, \\ 0 & \text{if } |x| \geq \frac{1}{2}, \end{cases} \tag{8.3.116}$$

and the initial condition

$$u(x, 0; \epsilon) = F = \text{constant} \neq 1, \tag{8.3.117a}$$

$$h(x, 0; \epsilon) = 1 - \epsilon B(x). \tag{8.3.117b}$$

The bore conditions for (8.3.115) are the same as those for a flat bottom as long as $B'(x)$ is finite. (Why?)

a. Look for a multiple-scale expansion of the form

$$u(x, t; \epsilon) = F + \epsilon u_1(x, t, t_1) + O(\epsilon^2), \tag{8.3.118a}$$

$$h(x, t; \epsilon) = 1 + \epsilon h_1(x, t, t_1) + O(\epsilon^2), \tag{8.3.118b}$$

and show that S_1 and R_1, defined in terms of u_1 and h_1 by (8.3.29), are now given by

$$S_1 = -\frac{B(x)}{F + 1} + f_1(\xi, t_1), \quad R_1 = \frac{B(x)}{F - 1} + g_1(\eta, t_1), \tag{8.3.119}$$

where

$$\xi = x - (F + 1)t, \quad \eta = x - (F - 1)t. \tag{8.3.120}$$

b. Show that the evolution equations governing f_1 and g_1 are still (8.3.48).

c. Solve the initial-value problem for f_1 to obtain

$$f_1(\xi, t_1) = \frac{1 + 8c\xi t_1 - [1 + 16ct_1(\xi + ct_1)]^{1/2}}{6ct_1^2}, \tag{8.3.121a}$$

where

$$c \equiv \frac{3F}{4(F+1)}. \qquad (8.3.121b)$$

d. Show that a bore starts at $\xi = -\frac{1}{2}$, $t_1 = 1/4c$, and satisfies

$$\frac{d\xi}{dt_1} = \frac{3}{8} f_1(x, t_1), \qquad (8.3.122)$$

where f_1 is given by (8.3.121). Sketch the shape of the bore and show that $\xi \sim \sqrt{t_1}$ as $t_1 \to \infty$ along the bore.

e. Consider the case $F \approx 1$, and assume

$$F = 1 + \epsilon^{\lambda} \overline{F}, \qquad (8.3.123)$$

where $\lambda > 0$ is to be determined and $\overline{F}$ is a fixed constant independent of ϵ. Look for a multiple-scale expansion for u and h in the form

$$u(x, t; \epsilon) = 1 + \epsilon^{\lambda} \overline{F} + \epsilon^{\beta} \overline{u}_1(x, t, \overline{t}) + \epsilon^{2\beta} \overline{u}_2(x, t, \overline{t}) + O(\epsilon^{3\beta}), \qquad (8.3.124a)$$

$$h(x, t; \epsilon) = 1 + \epsilon^{\beta} \overline{h}_1(x, t, \overline{t}) + \epsilon^{2\beta} \overline{h}_2(x, t, \overline{t}) + O(\epsilon^{3\beta}), \qquad (8.3.124b)$$

where $\beta > 0$ is to be determined and $\overline{t} = \epsilon^{\beta} t$.

Show that the richest approximation corresponds to $\lambda = \beta = \frac{1}{2}$ and that $\overline{u}_1$ and $\overline{h}_1$ have the form

$$\overline{u}_1 = -\frac{1}{2} \overline{g}_1(x, \overline{t}), \quad \overline{h}_1 = \frac{1}{2} \overline{g}_1(x, \overline{t}). \qquad (8.3.125)$$

By requiring the terms of order ϵ in (8.3.124) to be consistent, derive the following evolution equation for $\overline{g}_1$,

$$\frac{\partial \overline{g}_1}{\partial \overline{t}} + \left(\overline{F} - \frac{3}{4} \overline{g}_1\right) \frac{\partial \overline{g}_1}{\partial x} = B'(x), \qquad (8.3.126)$$

and initial condition $\overline{g}_1(x, 0) = 0$. The solution of (8.3.126) and other results are given in [28].

8.3.4 Maxwell's equations in dimensionless form for plane-polarized waves propagating in a one-dimensional medium in which the dielectric constant ε, permeability μ, and conductivity σ are all rapidly varying in x are

$$E_t + \frac{1}{\varepsilon(x^*)} H_x + \sigma(x^*) E = \epsilon \gamma(x^*) E^3, \qquad (8.3.127a)$$

$$H_t + \frac{1}{\mu(x^*)} E_x = 0. \qquad (8.3.127b)$$

Here $x^* \equiv x/\epsilon$, E is the y-component of the electric field, H is the z-component of the magnetic field, and we have assumed that the relation linking the current density to the electric field has a weak (order ϵ) cubic nonlinearity. Derive the equations corresponding to (8.3.82b) and (8.3.86b) for this application.

Appendix

A.1 Review of Green's Function for ODEs Using the Dirac Delta Function

In the first four chapters of this book we study linear second-order partial differential equations. A recurrent theme in these chapters is that a wide class of problems can be solved in compact form by superposition once Green's function is known. A simple and intuitively obvious derivation of Green's function proceeds from the solution of the governing equation and homogeneous boundary conditions using the Dirac delta function. Therefore, it is important to review the essential ideas in the simple setting of ordinary differential equations in this appendix. We begin with a discussion of the Dirac delta function.

A.1.1. The Dirac Delta Function

For a *fixed* value of $x = \xi_i$ and a *positive* $\Delta\xi$, we define the unit pulse U centered at $x = \xi_i$ and having width $\Delta\xi$ by

$$U(x - \xi_i, \Delta\xi) \equiv \begin{cases} 1 & \text{if } -\frac{\Delta\xi}{2} \le x - \xi_i \le \frac{\Delta\xi}{2}, \\ 0 & \text{if } |x - \xi_i| > \frac{\Delta\xi}{2}. \end{cases} \tag{A.1.1}$$

This function is sketched in Figure A.1.

Given a bounded continuous function $f(x)$ defined on $-\infty < x < \infty$, an approximate description of $f(x)$ is obtained by regarding $f(x)$ to be piecewise constant over each of the small intervals $\xi_i - \Delta\xi/2 < x < \xi_i + \Delta x/2$, with the constant value given by $f(\xi_i)$. To implement this approximation, we consider the discrete set of values of x consisting of $x = \xi_i \equiv i\Delta\xi$, where i ranges over all the integers, and set

$$f(x) \approx \sum_{i=-\infty}^{\infty} f(\xi_i)U(x - \xi_i, \Delta\xi). \tag{A.1.2}$$

The sense in which (A.1.2) approximates $f(x)$ is illustrated in Figure A.2.

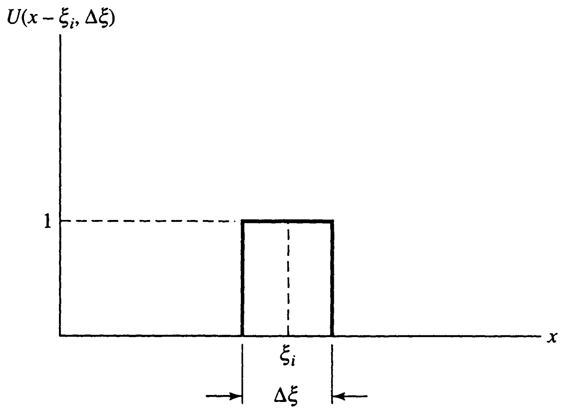

FIGURE A.1. The unit pulse

Instead of the unit pulse $U(x - \xi_i, \Delta\xi)$, let us introduce the *area-preserving pulse* $D(x - \xi_i, \Delta\xi)$ defined by

$$D(x - \xi_i, \Delta\xi) \equiv \frac{1}{\Delta\xi} U(x - \xi_i, \Delta\xi) \qquad \text{(A.1.3)}$$

and sketched in Figure A.3. The sequence of rectangles defined by D for successively smaller values of $\Delta\xi$ all have unit area because their base widths equal $\Delta\xi$ and their heights equal $1/\Delta\xi$. Thus,

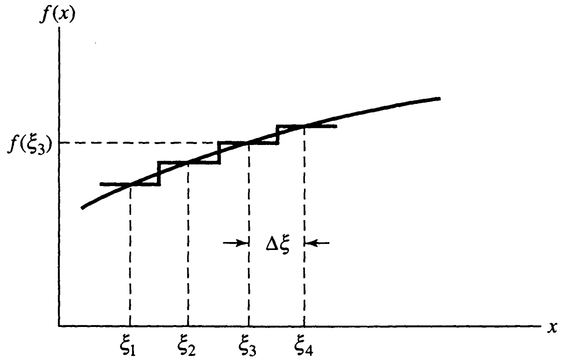

FIGURE A.2. Piecewise constant approximation of a function

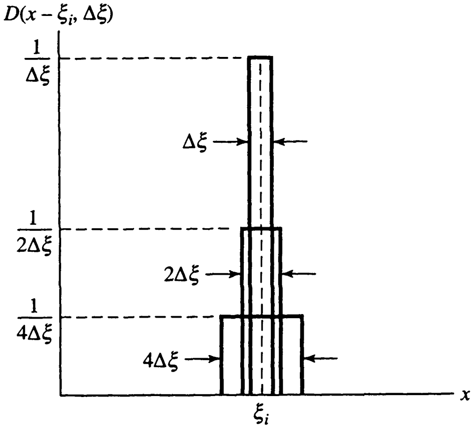

FIGURE A.3. Sequence of area-preserving pulses

$$\int_{-\infty}^{\infty} D(x - \xi_i, \Delta\xi)dx = 1. \tag{A.1.4}$$

Using D instead of U to approximate $f(x)$ in (A.1.2) gives

$$f(x) \approx \sum_{i=-\infty}^{\infty} f(\xi_i)D(x - \xi_i, \Delta\xi)\Delta\xi, \tag{A.1.5}$$

a result that is very reminiscent of the expression

$$f(x) = \int_{-\infty}^{\infty} f(\xi)\delta(x - \xi)d\xi \tag{A.1.6}$$

if we were to let $\Delta\xi \to d\xi$ and assume that $\lim_{\Delta\xi\to 0} D(x - \xi_i, \Delta\xi) = \delta(x - \xi)$, the delta function. Unfortunately, this limit does not exist. Furthermore, since $\delta(x - \xi)$ is zero everywhere except at $x = \xi$, where it is infinite, the integral in (A.1.6) must equal zero regardless of how we choose to define it.

Nevertheless, we note that if we evaluate the integral in (A.1.6) for *finite* $\Delta\xi$, then let $\Delta\xi \to 0$, we obtain the desired result, that is,

$$\lim_{\Delta\xi\to 0} \int_{-\infty}^{\infty} f(\xi)D(x - \xi, \Delta\xi)d\xi = f(x). \tag{A.1.7}$$

To show this, we first observe that for a fixed x and a fixed $\Delta\xi > 0$, $D(x - \xi, \Delta\xi)$ is zero everywhere outside the interval $-\Delta\xi/2 < x - \xi < \Delta\xi/2$ and equals $1/\Delta\xi$ in this interval. Therefore, the left-hand side of (A.1.7), which we denote by $f^*(x)$, is just

$$f^*(x) = \lim_{\Delta\xi\to 0} \frac{1}{\Delta\xi} \int_{x-\Delta\xi/2}^{x+\Delta\xi/2} f(\xi)d\xi. \tag{A.1.8}$$

Since $f(\xi)$ is continuous, we know from the mean value theorem that

$$\int_{x-\Delta\xi/2}^{x+\Delta\xi/2} f(\xi)d\xi = f(\xi_1)\Delta\xi, \tag{A.1.9}$$

where ξ_1 is some point in the interval $x - \Delta\xi/2 \leq \xi \leq x + \Delta\xi/2$. Using (A.1.9) in (A.1.8) gives

$$f^*(x) = \lim_{\Delta\xi\to 0} f(\xi_1) = f(x), \tag{A.1.10}$$

because $\xi_1 \to x$ as $\Delta\xi \to 0$.

It is also easy to show that the actual shape of $D(x - \xi, \Delta\xi)$ is irrelevant, and (A.1.7) holds as long as D is a nonnegative function that satisfies (A.1.4) and $D \to 0$ as $\Delta\xi \to 0$ with $x \neq \xi$, whereas $D \to \infty$ as $\Delta\xi \to 0$ with $x = \xi$. For example,

$$D_1(x - \xi, \Delta\xi) \equiv \frac{\Delta\xi}{\pi[(\Delta\xi)^2 + (x - \xi)^2]}, \tag{A.1.11a}$$

$$D_2(x - \xi, \Delta\xi) \equiv \frac{1}{\sqrt{\pi\Delta\xi}} \exp\left[-\frac{(x - \xi)^2}{\Delta\xi}\right] \tag{A.1.11b}$$

are two other possible choices. We shall refer to functions such as (A.1.3), (A.1.11a), or (A.1.11b) as *representations of the delta function*.

In this book we shall use (A.1.6) as the definition of the delta function and not worry about the strict validity of this expression, because in all of our applications we may use a representation of the delta function and defer taking $\Delta\xi \to 0$ until this limit exists as in (A.1.7). By working with the delta function directly, our calculations will be greatly simplified. The delta function may be viewed as a mathematical description of certain idealized physical entities such as a concentrated source of heat, a concentrated force, an impulse, a point electrostatic charge, or a point source of mass in a flow. In all these idealizations we are passing to the limit of some physical action concentrated at a *point* in space or time; one could equally well study the process with a representation of the delta function at a considerable cost in complexity. An example that illustrates these points is worked out in Section A.1.4 (see (A.1.51)).

The reader should now verify the validity of the following results, which follow easily from the definition (A.1.6) if we regard $\delta(x - \xi)$ as an ordinary function.

(i) The delta function is an even function of its argument

$$\delta(x - \xi) = \delta(\xi - x). \tag{A.1.12}$$

(ii) We may write (A.1.1) in the form

$$U(x - \xi_i, \Delta\xi) = H\left[x - (\xi_i - \frac{\Delta\xi}{2})\right] - H\left[x - (\xi_i + \frac{\Delta\xi}{2})\right], \tag{A.1.13}$$

where $H(x)$ is the Heaviside function

$$H(x) \equiv \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{A.1.14}$$

It then follows that

$$\delta(x - \xi) = \frac{d}{dx} H(x - \xi). \tag{A.1.15}$$

(iii) The effect of multiplying the argument of the delta function by a constant α is equivalent to rescaling it as follows:

$$\delta[\alpha(x - \xi)] = \frac{1}{|\alpha|} \delta(x - \xi). \tag{A.1.16}$$

(iv) The derivative of the delta function, $\delta'(x - \xi)$, has the following property:

$$\int_{-\infty}^{\infty} f(\xi)\delta'(x - \xi)d\xi = -f'(x). \tag{A.1.17}$$

(v) Let $f(x)$ have discontinuities $f(x_n^+) - f(x_n^-) = a_n$ at the points $x = x_1$, $x_2, \ldots, x_N$, and let $f(x)$ have a piecewise continuous derivative at all other values of x. We denote the *symbolic derivative* of $f(x)$ by $f_s'(x)$ and define it by

$$f_s'(x) = f'(x) + \sum_{n=1}^{N} a_n \delta(x - x_n), \tag{A.1.18}$$

where $f'(x)$ is the ordinary derivative of $f(x)$, which exists for all x except $x = x_1, \ldots, x_N$. Show that the standard result

$$\int_{x_0}^{x} f_s'(\xi)d\xi = f(x) - f(x_0), \tag{A.1.19}$$

which is true for a continuous $f(x)$, remains valid for all x and any x_0.

A.1.2 Green's Function for a Linear Second-Order Equation, Superposition

Let $L(u)$ denote a linear second-order differential operator on functions $u(x)$ defined over $0 \leq x \leq 1$. For given homogeneous boundary conditions, e.g., $u(0) = 0$, $u(1) = 0$, Green's function $u = G(x, \xi)$ satisfies

$$L(u) = \delta(x - \xi), \tag{A.1.20}$$

$$u(0) = 0; \quad u(1) = 0 \tag{A.1.21}$$

for a fixed ξ in the interval $0 < \xi < 1$. Green's function has the important property that the solution of the inhomogeneous differential equation

$$L(u) = f(x) \tag{A.1.22}$$

with the same homogeneous boundary conditions (A.1.21) is given by the *superposition integral*

$$u(x) = \int_0^1 f(\xi)G(x,\xi)d\xi. \tag{A.1.23}$$

Thus, once Green's function is known, the solution for an arbitrary forcing $f(x)$ is given by a simple quadrature.

To derive (A.1.23), let us subdivide the interval $0 \leq x \leq 1$ into N equal segments of width $\Delta\xi = 1/N$. We denote the midpoints of each segment by ξ_i, $i = 1, \ldots, N$. Thus,

$$\xi_1 = \frac{1}{2N},\ \xi_2 = \frac{3}{2N}, \ldots,\ \xi_n = \frac{2n-1}{2N}, \ldots, \xi_N = \frac{2N-1}{2N}.$$

Let $\Gamma(x, \xi_n, \Delta\xi)$ be the solution of

$$L(u) = D(x - \xi_n, \Delta\xi) \tag{A.1.24}$$

satisfying the homogeneous boundary conditions (A.1.21), that is,

$$\Gamma(0, \xi_n, \Delta\xi) = \Gamma(1, \xi_n, \Delta\xi) = 0, \tag{A.1.25}$$

for each $n = 1, \ldots, N$. Thus, Γ approximates Green's function G if we use D instead of δ on the right-hand side of (A.1.20). Let us also express $f(x)$ in (A.1.22) by the piecewise constant approximation (A.1.5). Now consider the solution u_j of

$$L(u_j) = f(\xi_j)D(x - \xi_j, \Delta\xi)\Delta\xi \tag{A.1.26}$$

that satisfies the homogeneous boundary condition (A.1.21) for a fixed integer j, $1 \leq j \leq N$. Note that the right-hand side of (A.1.26) is zero everywhere except on the interval $\xi_j - \Delta\xi/2 \leq x \leq \xi_j + \Delta\xi/2$, where it equals $f(\xi_j)$. Comparing (A.1.24) with (A.1.26), we see that u_j is the following constant multiple of $\Gamma(x, \xi_j, \Delta\xi)$:

$$u_j = \Gamma(x, \xi_j, \Delta\xi)f(\xi_j)\Delta\xi. \tag{A.1.27}$$

Since each of the above incremental contributions u_j to the solution vanishes at $x = 0$ and $x = 1$, we may sum these up and still satisfy the homogeneous boundary conditions (A.1.21) because L is linear. Therefore, the approximate solution of (A.1.22) is

$$u \approx \sum_{j=1}^{N} u_j = \sum_{j=1}^{N} f(\xi_j)\Gamma(x, \xi_j, \Delta\xi)\Delta\xi. \tag{A.1.28}$$

In the "limit" $\Delta\xi \to d\xi$, $\Gamma \to G$, $\sum_{j=1}^{N} \to \int_0^1$, we obtain (A.1.23).

For a given L, the result (A.1.23) remains valid for more general homogeneous boundary conditions (see Problem 1.5.1) or for certain inhomogeneous boundary conditions (see the second example in Section A.1.3). The superposition idea also generalizes to the case where L is a partial differential operator, and this is discussed in numerous settings in Chapters 1–3.

A.1.3 Examples of Green's Functions

(i) *Steady-state temperature distribution*

Consider the steady-state temperature distribution in a finite one-dimensional conductor that has a given heat source distribution and whose boundaries are maintained at a constant temperature. In appropriate dimensionless variables (see (1.1.9) and (1.1.10) with $(\partial/\partial t) = 0$), we have

$$L(u) \equiv -\frac{d^2u}{dx^2} = f(x), \tag{A.1.29}$$

$$u(0) = 0; \quad u(1) = 0. \tag{A.1.30}$$

This is a special case of (A.1.22). If the given boundary values of u at $x = 0$ and $x = 1$ are different constants, say $u(0) = a$, $u(1) = b$, we can transform the problem to one with homogeneous boundary conditions by an appropriate change of dependent variable. For example, we may look for a transformation $u \to w$ that is linear in x of the form

$$u(x) = w(x) + \alpha + \beta x, \tag{A.1.31}$$

where α and β are constants to be determined by requiring $w(x)$ to satisfy homogeneous boundary conditions at $x = 0$ and $x = 1$. We have

$$u(0) = a = w(0) + \alpha. \tag{A.1.32}$$

Requiring $w(0) = 0$ implies that we must set $\alpha = a$. Similarly,

$$u(1) = b = w(1) + a + \beta, \tag{A.1.33}$$

and we must set $\beta = b - a$ in order that $w(1) = 0$. Thus, the homogenizing transformation is given by

$$u(x) = w(x) + a + (b - a)x. \tag{A.1.34}$$

Since the second derivative of a linear function vanishes, it follows that w satisfies the same equation (A.1.24) as u. The general idea of homogenizing a given nonzero boundary condition will be invoked often in Chapters 1–3.

A second physical interpretation of (A.1.29)–(A.1.30), is that it represents the static deflection of a string that is clamped at two points and subjected to a given loading $f(x)$. See Section 3.1.

The solution of (A.1.29)–(A.1.30) is easily computed using elementary techniques for ordinary differential equations. For example, given the homogeneous solution $u_h = Ax + B$, we seek a particular solution using the method of variation of parameters in the form

$$u_p = xu_1(x) + u_2(x) \tag{A.1.35}$$

for unknown functions u_1 and u_2. Substituting (A.1.35) into (A.1.29) and setting $u_1'x + u_2' = 0$ (where $' \equiv d/dx$) gives

$$u_1' = -f(x), \quad u_2' = xf(x). \tag{A.1.36}$$

We integrate the expressions in (A.1.36) for u_1 and u_2 and set the solution $u = u_h + u_p$ to obtain

$$u(x) = Ax + B + \int_0^x (\xi - x) f(\xi) d\xi. \tag{A.1.37}$$

The boundary conditions (A.1.30) imply that

$$B = 0; \quad A = \int_0^1 (1 - \xi) f(\xi) d\xi.$$

Therefore, the solution is

$$u(x) = \int_0^1 x(1 - \xi) f(\xi) d\xi + \int_0^x (\xi - x) f(\xi) d\xi. \tag{A.1.38}$$

If we express the first integral on the right-hand side of (A.1.38) as the sum of two integrals, one from 0 to x and the other from x to 1, we obtain

$$u(x) = \int_0^x \xi(1 - x) f(\xi) d\xi + \int_x^1 x(1 - \xi) f(\xi) d\xi. \tag{A.1.39}$$

For any fixed value of x in the interval $0 \leq x \leq 1$, this result may be expressed in the compact form (A.1.23) for the function $G(x, \xi)$ defined by

$$G(x, \xi) = \begin{cases} \xi(1 - x) & 0 < \xi \leq x, \\ x(1 - \xi) & x \leq \xi < 1. \end{cases} \tag{A.1.40}$$

Let us now derive Green's function (A.1.40) directly from its definition (A.1.20)–(A.1.21). We have

$$-\frac{d^2 u}{dx^2} = \delta(x - \xi), \quad 0 < \xi < 1, \tag{A.1.41}$$

with

$$u(0) = 0; \quad u(1) = 0. \tag{A.1.42}$$

For a fixed ξ, consider the solution of (A.1.41) on either side of the delta function. Since $\delta(x - \xi) = 0$ if $x \neq \xi$, we have

$$u = \begin{cases} Ax + B & \text{if } x < \xi, \\ Cx + D & \text{if } x > \xi, \end{cases} \tag{A.1.43}$$

where the constants C and D are not necessarily equal to A and B, respectively. The boundary condition $u(0) = 0$ is to be satisfied by the first expression in (A.1.43), whereas $u(1) = 0$ is to be satisfied by the second. Applying these conditions gives $B = 0$, $D = -C$, and the homogeneous solutions on either side of the delta function reduce to

$$u = \begin{cases} Ax, & x < \xi, \\ C(x - 1), & x > \xi. \end{cases} \tag{A.1.44}$$

To determine the remaining two constants A and C, we make the following observations: (A.1.41) implies that $-u''$ is infinite at the location of the delta function

$x = \xi$. It then follows (see (A.1.18)–(A.1.19)) that u' must have a *finite* jump and that u must be *continuous* there. The continuity of u at $x = \xi$ gives the condition

$$A\xi = C(\xi - 1). \tag{A.1.45}$$

To calculate the appropriate jump in u' across $\xi = x$ we integrate (A.1.41) over the small interval $\xi - \epsilon \leq x \leq \xi + \epsilon$ for some small positive ϵ. We have

$$-\int_{x=\xi-\epsilon}^{x=\xi+\epsilon} u''(x)dx = \int_{x=\xi-\epsilon}^{x=\xi+\epsilon} \delta(x - \xi)dx = 1. \tag{A.1.46}$$

Therefore, in the limit as $\epsilon \to 0$, we have the jump condition

$$-u'(\xi^+) + u'(\xi^-) = 1, \tag{A.1.47}$$

where the notation $u'(\xi^+)$ and $u'(\xi^-)$ denotes $\lim_{x\downarrow\xi} u'(x)$ and $\lim_{x\uparrow\xi} u'(x)$, respectively.

To evaluate $u'(\xi^+)$ we use the second expression in (A.1.44) and obtain $u'(\xi^+) = C$. The first expression in (A.1.44) gives $u'(\xi^-) = A$. Therefore, the jump condition (A.1.47) becomes

$$-C + A = 1. \tag{A.1.48}$$

Solving (A.1.45) and (A.1.48) for A and C gives $A = 1 - \xi$, $C = -\xi$, and this result, when used in (A.1.44), verifies the previously computed expression, (A.1.40), for Green's function.

In general, for the second-order linear operator

$$L(u) \equiv a_0(x)\frac{d^2u}{dx^2} + a_1(x)\frac{du}{dx} + a_2(x)u = \delta(x - \xi), \tag{A.1.49}$$

with coefficients a_0, a_1, and a_i that are continuous on $0 \leq x \leq 1$, we have the continuity condition

$$u(\xi^+) - u(\xi^-) = 0 \tag{A.1.50a}$$

and the jump condition

$$a_0(\xi)[u'(\xi^+) - u'(\xi^-)] = 1. \tag{A.1.50b}$$

We now demonstrate for this example the fact that the expansion for $G(x, \xi)$, just obtained very efficiently using a delta function on the right-hand side of (A.1.41), can also be derived in a more laborious way by using a representation of the delta function in the calculations and then taking the limit $\Delta\xi \to 0$ in the final result. Let us use the representation $D(x - \xi, \Delta\xi)$ given by (A.1.3). Thus, we need to solve

$$-\frac{d^2u}{dx^2} = D(x - \xi, \Delta\xi), \tag{A.1.51}$$

$$u(0) = 0; \quad u(1) = 0, \tag{A.1.52}$$

where D is given by (see (A.1.1) and (A.1.3))

$$D = \begin{cases} \frac{1}{\Delta\xi} & \text{if } -\frac{\Delta\xi}{2} \le x - \xi \le \frac{\Delta\xi}{2}, \\ 0 & \text{if } |x - \xi| > \frac{\Delta\xi}{2}. \end{cases} \tag{A.1.53}$$

For a fixed ξ in $0 < \xi < 1$, the solution of (A.1.51) subject to the zero boundary conditions (A.1.52) at $x = 0$ and $x = 1$ has the form

$$u = \begin{cases} Ax & \text{if } 0 \le x < \xi - \frac{\Delta\xi}{2}, \\ Ex + F - \frac{x^2}{2\Delta\xi} & \text{if } \xi - \frac{\Delta\xi}{2} \le x \le \xi + \frac{\Delta\xi}{2}, \\ C(x-1) & \text{if } \xi + \frac{\Delta\xi}{2} < x \le 1. \end{cases} \tag{A.1.54}$$

Since the right-hand side has a discontinuity in the second derivative at the points $x = \xi \pm \Delta\xi/2$, u' and u are both continuous there. These continuity conditions determine A, E, F, and C. In particular, continuity of u and u' at $x = \xi - \Delta\xi/2$ gives the two conditions

$$A\left(\xi - \frac{\Delta\xi}{2}\right) = E\left(\xi - \frac{\Delta\xi}{2}\right) + F - \left(\xi - \frac{1}{2\Delta\xi}\right)\left(\xi - \frac{\Delta\xi}{2}\right)^2 \tag{A.1.55a}$$

$$A = E - \frac{1}{\Delta\xi}\left(\xi - \frac{\Delta\xi}{2}\right). \tag{A.1.55b}$$

Continuity of u and u' at $x = \xi + \Delta\xi/2$ gives the two conditions

$$C\left(\xi + \frac{\Delta\xi}{2} - 1\right) = E\left(\xi + \frac{\Delta\xi}{2}\right) + F - \frac{1}{2\Delta\xi}\left(\xi + \frac{\Delta\xi}{2}\right)^2, \tag{A.1.55c}$$

$$C = E - \frac{1}{\Delta\xi}\left(\xi + \frac{\Delta\xi}{2}\right). \tag{A.1.55d}$$

The solution of the linear system (A.1.55) gives *exactly* the same values $A = 1 - \xi$, $C = -\xi$ as before, and we also obtain $F = -\left(\xi - \Delta\xi/2\right)^2/2\Delta\xi$ and $E = \left(\xi + \Delta\xi/2 - \xi\Delta\xi\right)/\Delta\xi$. In the limit $\Delta\xi \to 0$, the interval $\xi - \Delta\xi/2 \le x \le \xi - \Delta\xi/2$ shrinks to the point $x = \xi$, where the solution is $u = \xi(1 - \xi)$, as before.

(ii) *Variable thermal diffusivity*

Suppose the thermal diffusivity κ^2 in (1.1.9) varies linearly with x. Using appropriate dimensionless variables, Green's function for the steady-state temperature in a conductor over $0 \le x \le 1$ obeys

$$-\frac{d}{dx}\left(x\frac{du}{dx}\right) = \delta(x - \xi); \quad 0 < \xi < 1. \tag{A.1.56}$$

The end $x = 0$ where the diffusivity vanishes has a *finite* temperature, whereas the temperature is zero at $x = 1$:

$$u(0) = \text{finite}, \quad u(1) = 0. \tag{A.1.57}$$

Equation (A.1.56) also describes the lateral (horizontal) deflection of a hanging chain under the influence of a concentrated horizontal load at $x = \xi$.

As in the previous example, we solve (A.1.56) with zero right-hand side over the two intervals $0 \leq x < \xi$ and $\xi < x \leq 1$, for a fixed ξ in $0 < \xi < 1$. We have

$$u = \begin{cases} A \log x + B & \text{if } 0 \leq x < \xi, \\ C \log x + D & \text{if } \xi < x \leq 1. \end{cases} \tag{A.1.58}$$

The requirement that u be finite at $x = 0$ gives $A = 0$, and $u(1) = 0$ implies $D = 0$. Thus,

$$u = \begin{cases} B & \text{if } 0 \leq x < \xi, \\ C \log x & \text{if } \xi < x \leq 1. \end{cases} \tag{A.1.59}$$

The continuity of u at $x = \xi$ gives $B = C \log \xi$, and the jump condition is given by (A.1.50b) with $a_0(x) = -x$:

$$-\xi[u'(\xi^+) - u'(\xi^-)] = 1. \tag{A.1.60}$$

This reduces to $-\xi(C/\xi - 0) = 1$, or $C = -1$; hence $B = -\log \xi$. Thus, Green's function is given by

$$G(x, \xi) = \begin{cases} -\log \xi & \text{if } 0 \leq x < \xi, \\ -\log x & \text{if } \xi < x \leq 1. \end{cases} \tag{A.1.61}$$

In this example, the boundary condition $u(0) =$ finite is superposable, and the solution of

$$-\frac{d}{dx}\left(x \frac{du}{dx}\right) = f(x) \tag{A.1.62}$$

with

$$u(0) = \text{finite}; \quad u(1) = 0 \tag{A.1.63}$$

is still given by (A.1.23).

We note that for the two examples we have studied, Green's function has the symmetry property $G(x, \xi) = G(\xi, x)$. The proof of this result for the general linear second-order operator follows from the fact that $\delta(x - \xi) = \delta(\xi - x)$. Thus, $G(x, \xi)$ and $G(\xi, x)$ satisfy the same equation and boundary conditions and must therefore be equal. See also Section 2.6.1 for a proof of the symmetry of Green's function for the Laplacian.

(iii) *An initial-value problem*

Consider the following initial-value problem for a second-order differential operator

$$L(u) = f(t) \tag{A.1.64}$$

on $0 \leq t < \infty$ for a prescribed $f(t)$ and zero initial conditions

$$u(0) = 0; \quad u'(0) = 0. \tag{A.1.65}$$

Once Green's function $G(t, \tau)$ that satisfies

$$L(u) = \delta(t - \tau) \tag{A.1.66}$$

is known, the solution of (A.1.64)–(A.1.65) is given by the superposition integral

$$u(t) = \int_0^t G(t, \tau) f(\tau) d\tau. \tag{A.1.67}$$

The proof of this result is very similar to the one used in deriving (A.1.23) and is not repeated. We note that the upper limit in (A.1.67) is now t, the time at which the solution u is to be evaluated. This feature is obvious, as only values of $f(\tau)$ in the interval $0 \leq \tau \leq t$ can affect the result at time t.

For the special case $L(u) \equiv u'' + u$ in (A.1.66), we have the two homogeneous solutions

$$u = \begin{cases} 0 & \text{if } 0 \leq t < \tau, \\ A \sin t + B \cos t & \text{if } \tau < t < \infty \end{cases} \tag{A.1.68}$$

on either side of the delta function, and we have already imposed the two initial conditions $u(0) = u'(0) = 0$ in the definition of the solution for $0 \leq t < \tau$.

The continuity of the solution across $t = \tau$ gives

$$0 = A \sin \tau + B \cos \tau, \tag{A.1.69}$$

and the jump condition (A.1.50b) becomes

$$u'(\tau^+) - u'(\tau^-) = 1. \tag{A.1.70}$$

Since $u(t) \equiv 0$ if $0 \leq t < \tau$, it follows that $u'(\tau^-) = 0$, and (A.1.70) reduces to

$$A \cos \tau - B \sin \tau = 1. \tag{A.1.71}$$

Solving (A.1.69) and (A.1.71) for A and B gives $A = \cos \tau$ and $B = -\sin \tau$. Therefore, Green's function for this example is

$$G(t, \tau) = \begin{cases} 0 & \text{if } 0 \leq t < \tau, \\ \sin(t - \tau) & \tau < t < \infty, \end{cases} \tag{A.1.72}$$

and the solution of the initial-value problem (A.1.64)–(A.1.65) is

$$u(t) = \int_0^t \sin(t - \tau) f(\tau) d\tau. \tag{A.1.73}$$

This concludes our review of Green's function for ordinary differential equations using delta functions. The reader is referred to Chapter 3 of [17] for further discussion of this topic including original references.

A.2 Review of Fourier and Laplace Transforms

The brief review in this appendix is presented to highlight the basic results. A detailed discussion can be found in Chapter 7 of [8] and Chapters 10 and 13 of [22]. As the reader will first encounter the use of transforms in Chapter 1, the illustrative examples we introduce in this section all involve the diffusion equation. Other examples arise in Chapters 2–3.

A.2.1 Formal Derivation of the Fourier Integral Theorem

Consider a function $f(x)$ defined on the interval $-L < x < L$. We may represent $f(x)$ by the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right), \tag{A.2.1}$$

where the coefficients a_n and b_n are given by

$$a_n = \frac{1}{L}\int_{-L}^{L} f(\xi)\cos\frac{n\pi\xi}{L}d\xi,\ b_n = \frac{1}{L}\int_{-L}^{L} f(\xi)\sin\frac{n\pi\xi}{L}d\xi, \tag{A.2.2}$$

for each $n = 0, 1, 2, \ldots$. Substituting these expressions into (A.2.1) gives

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(\xi)d\xi + \sum_{n=1}^{\infty}\frac{1}{L}\int_{-L}^{L} f(\xi)\cos\frac{n\pi}{L}(\xi - x)d\xi, \tag{A.2.3}$$

and since $\cos(n\pi/L)(\xi - x) = \cos(-n\pi/L)(\xi - x)$, we may write the second term on the right-hand side of (A.2.3) as $\frac{1}{2}\left(\sum_{n=1}^{\infty}\cdots + \sum_{n=-\infty}^{1}\cdots\right)$. Combining this with the first term on the right-hand side gives

$$f(x) = \sum_{n=-\infty}^{\infty}\frac{1}{2L}\int_{-L}^{L} f(\xi)\cos\frac{n\pi}{L}(\xi - x)d\xi. \tag{A.2.4}$$

This is an identity for any $f(x)$ that has a convergent Fourier series. The next step is more daring and involves taking the "limit" of this result as $L \to \infty$. To do so, we set $(n\pi/L) = k_n$. Thus, $k_{n+1} - k_n = (\pi/L)(n + 1 - n) = \pi/L \equiv \Delta k_n$, or $1/2L = \Delta k_n/2\pi$. Now, as $L \to \infty$, $\Delta k_n \to dk$, the sum tends to an integral, and (A.2.4) formally becomes

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} f(\xi)\cos k(\xi - x)d\xi. \tag{A.2.5}$$

A more symmetric form of (A.2.5) is obtained by using the identity $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$. This gives

$$f(x) = \frac{1}{4\pi}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} f(\xi)e^{ik(\xi-x)}d\xi + \frac{1}{4\pi}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} f(\xi)e^{-ik(\xi-x)}d\xi. \tag{A.2.6}$$

Now, replacing k by $-k$ in the second integral shows that it equals the first integral on the right-hand side. Therefore, (A.2.6) reduces to

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} f(\xi)e^{ik(\xi-x)}d\xi, \tag{A.2.7a}$$

or, again replacing k by $-k$, we have the equivalent result

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} f(\xi)e^{ik(x-\xi)}d\xi. \tag{A.2.7b}$$

Each of equations (A.2.5), (A.2.7) is a version of the Fourier integral theorem. This result can be proved as long as $f(x)$ satisfies certain conditions. For example, the reader can find a proof in [8] for the case where f is a piecewise smooth function that is absolutely integrable on $(-\infty, \infty)$. In this case, we must interpret $f(x_0) = \frac{1}{2}[f(x_0^+) + f(x_0^-)]$ at any point x_0 where f is discontinuous. There are less restrictive conditions for $f(x)$ for which (A.2.7) still holds; examples can be found in [8].

A.2.2 The Fourier Transform

The identity (A.2.7) can be decomposed into the following *transform pair*:

$$\overline{f}(k) \equiv \frac{\gamma}{2\pi}\int_{-\infty}^{\infty} f(\xi)e^{aik\xi}d\xi, \tag{A.2.8a}$$

$$f(x) = \frac{1}{\gamma}\int_{-\infty}^{\infty} \overline{f}(k)e^{-aikx}dk, \tag{A.2.8b}$$

for an arbitrary constant γ and $a = \pm 1$. For a given $f(x)$, the function $\overline{f}(k)$ defined by (A.2.8a) is called the Fourier transform of f. The inversion formula (A.2.8b) is a consequence of the identity (A.2.7). Unfortunately, there is no standard choice for the constant γ or $a = +1$ or -1 in the literature. In this text we will adopt the definition with $\gamma = \sqrt{2\pi}$ and $a = 1$, i.e.,

$$\overline{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)e^{ik\xi}d\xi, \tag{A.2.9a}$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{f}(k)e^{-ikx}dk. \tag{A.2.9b}$$

The notation $\mathcal{F}(f)$ for the Fourier transform is also useful, particularly in relating the Fourier transform of a derivative to the transform of the function. (See (A.2.12).)

If $f(x)$ is even, the expressions in (A.2.9) reduce to

$$\overline{f}(k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(\xi)\cos k\xi\, d\xi \equiv \overline{f}_c(k), \tag{A.2.10a}$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \overline{f}(k)\cos kx\, dk. \tag{A.2.10b}$$

Equation (A.2.10a) defines the Fourier cosine transform of $f(x)$ and will be denoted by $\overline{f}_c(k)$. The notation $\mathcal{F}_c(f)$ for $\overline{f}_c(k)$ is also useful.

If $f(x)$ is odd, we find the corresponding expressions for the Fourier sine transform

$$\overline{f}_s(k) \equiv \mathcal{F}_s(f) \equiv \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(\xi)\sin k\xi\, d\xi, \tag{A.2.11a}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \overline{f}_s(k) \sin kx \, dk. \tag{A.2.11b}$$

A.2.3 Formulas for Derivatives

The Fourier transform of $f'(x)$, the derivative of $f(x)$, is defined by

$$\mathcal{F}(f') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(\xi) e^{ik\xi} d\xi. \tag{A.2.12}$$

Integration by parts gives

$$\mathcal{F}(f') = -ik\mathcal{F}(f). \tag{A.2.13}$$

Similarly,

$$\mathcal{F}(f'') = -ik\mathcal{F}(f') = -k^2\mathcal{F}(f). \tag{A.2.14}$$

The corresponding formulas for the derivatives of the cosine and sine transforms are given by

$$\mathcal{F}_c(f') = -\sqrt{\frac{2}{\pi}} f(0) + k\mathcal{F}_s(f), \tag{A.2.15a}$$

$$\mathcal{F}_s(f') = -k\mathcal{F}_c(f), \tag{A.2.15b}$$

$$\mathcal{F}_c(f'') = -\sqrt{\frac{2}{\pi}} f'(0) - k^2\mathcal{F}_c(f), \tag{A.2.16a}$$

$$\mathcal{F}_s(f'') = \sqrt{\frac{2}{\pi}} kf(0) - k^2\mathcal{F}_s(f). \tag{A.2.16b}$$

A.2.4 The Convolution Theorem

Let $\overline{f}(k)$ and $\overline{g}(k)$ be the Fourier transforms of $f(x)$ and $g(x)$, respectively. The inverse transform of the product $\overline{f}(k)\overline{g}(k)$ is a function $H(x)$ given by (see (A.2.9b))

$$H(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f}(k)\overline{g}(k) e^{-ikx} dk. \tag{A.2.17}$$

Using the definition (A.2.9a) for $\overline{g}(k)$ gives

$$H(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f}(k) e^{-ikx} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\xi) e^{ik\xi} d\xi \right] dk.$$

We now interchange the order of integration to obtain

$$\begin{aligned} H(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\xi) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f}(k) e^{ik(\xi - x)} dk \right] d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\xi) f(x - \xi) d\xi \end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) g(x-\xi) d\xi. \tag{A.2.18}$$

The integral of a product of the form $g(\xi) f(x-\xi)$ is called a convolution integral. The result (A.2.18) may be written in the form (using the notation $\mathcal{F}^{-1}$ for the inverse transform)

$$\mathcal{F}^{-1}(\overline{f}\overline{g}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) g(x-\xi) d\xi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\xi) f(x-\xi) d\xi. \tag{A.2.19}$$

Equivalently, we have

$$\mathcal{F}\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) g(x-\xi) d\xi\right) = \mathcal{F}\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\xi) f(x-\xi) d\xi\right)$$
$$= \overline{f}(k)\overline{g}(k). \tag{A.2.20}$$

A.2.5 Examples of Solution by Fourier Transforms

We now illustrate the use of Fourier transforms for solving various problems for the diffusion equation. Applications for other equations abound and are discussed throughout Chapters 1–3.

(i) *Fundamental solution of the diffusion equation*

The problem is defined by (1.2.5)–(1.2.7). We denote the Fourier transform of the solution $u(x, t)$ by $\overline{u}(k, t)$ defined by (see (A.2.9a))

$$\overline{u}(k, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(\xi, t) e^{ik\xi} d\xi. \tag{A.2.21a}$$

The inversion formula (A.2.9b) becomes

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{u}(k, t) e^{-ikx} dk. \tag{A.2.21b}$$

We multiply (1.2.5) by $e^{ikx}/\sqrt{2\pi}$, and integrate the result from $-\infty$ to ∞ with respect to x to obtain

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [u_t(x, t) - u_{xx}(x, t)] e^{ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t)\delta(x) e^{ikx} dx. \tag{A.2.22}$$

The first term on the left-hand side is just $\overline{u}_t(k, t)$ because the partial differentiation with respect to t may be moved outside the integral sign. The second term on the left-hand side is $k^2\overline{u}(k, t)$ according to (A.2.14), and the right-hand side is just $\delta(t)/\sqrt{2\pi}$ from (A.1.6). The boundary conditions (1.2.7) at $\pm\infty$ ensure that $\overline{u}(k, t)$ exists, and the initial condition gives $\overline{u}(k, 0^-) = 0$. Thus, we need to solve the following first-order ordinary differential equation for the Fourier transform $\overline{u}(k, t)$:

$$\overline{u}_t + k^2\overline{u} = \frac{1}{\sqrt{2\pi}}\delta(t), \tag{A.2.23}$$

$$\bar{u}(k, 0^-) = 0. \tag{A.2.24}$$

Integrating (A.2.23) with respect to t from $t = 0^-$ to $t = 0^+$ shows that $\bar{u}(k, 0^+) = 1/\sqrt{2\pi}$. Thus, for $t > 0$, $\bar{u}$ is governed by (A.2.23) with zero right-hand side and the initial condition $\bar{u}(k, 0^+) = 1/\sqrt{2\pi}$. The solution is

$$\bar{u}(k, t) = \frac{1}{\sqrt{2\pi}} e^{-k^2 t}. \tag{A.2.25}$$

Substituting this result into the inversion formula (A.2.21b) gives the integral representation

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx - k^2 t} dk. \tag{A.2.26}$$

If we split the interval $(-\infty, \infty)$ into the two intervals $(-\infty, 0)$ and $(0, \infty)$ and replace k by $-k$ over the interval $(-\infty, 0)$, we obtain

$$u(x, t) = \frac{1}{\pi} \int_{0}^{\infty} e^{-k^2 t} \cos kx \, dk. \tag{A.2.27}$$

To evaluate (A.2.27), we set $k = \xi/\sqrt{t}$ to obtain

$$u(x, t) = \frac{1}{2\pi t^{1/2}} \int_{-\infty}^{\infty} e^{-\xi^2} \cos 2\alpha\xi \, d\xi, \tag{A.2.28}$$

where $\alpha \equiv x/2t^{1/2}$. Now let ζ denote the complex variable $\zeta = \xi + i\eta$. Since $e^{-\zeta^2}$ is analytic everywhere, its integral around any closed contour vanishes. In particular,

$$\oint_C e^{-\zeta^2} d\zeta = 0, \tag{A.2.29}$$

where C is the rectangular contour indicated in Figure A.4.

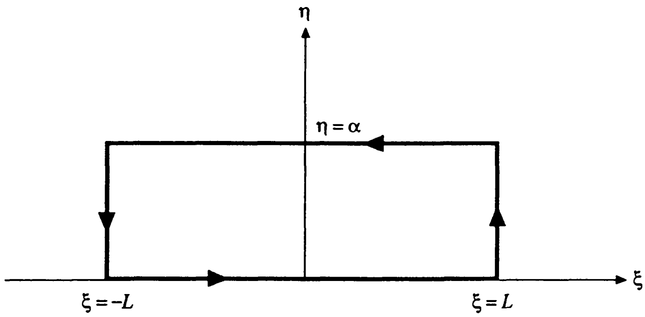

FIGURE A.4. Integration contour for (A.2.29)

Decomposing (A.2.29) into the four contributions from the sides of the rectangle gives

$$\int_{-L}^{L} e^{-\xi^2} d\xi - \int_{-L}^{L} e^{-(\xi+i\alpha)^2} d\xi + \int_0^{\alpha} e^{-(L+i\eta)^2} i\, d\eta$$

$$- \int_0^{\alpha} e^{-(-L+i\eta)^2} i\, d\eta = 0. \qquad \text{(A.2.30)}$$

In the limit as $L \to \infty$, the third and fourth terms in (A.2.30) decay exponentially. The first term tends to $\sqrt{\pi}$, and the second term equals $e^{\alpha^2} \int_{-\infty}^{\infty} e^{-\xi^2} \cos 2\alpha\xi \, d\xi$. Therefore,

$$\int_{-\infty}^{\infty} e^{-\xi^2} \cos 2\alpha\xi \, d\xi = \sqrt{\pi} e^{-\alpha^2}. \qquad \text{(A.2.31)}$$

Substituting this result into (A.2.28) and setting $\alpha = x/2t^{1/2}$ gives the expression (1.2.20) calculated using similarity.

(ii) *Initial-value problem for the diffusion equation*

We consider the initial-value problem

$$u_t - u_{xx} = 0, \quad -\infty < x < \infty, \qquad \text{(A.2.32)}$$

$$u(x, 0) = f(x). \qquad \text{(A.2.33)}$$

Taking Fourier transforms gives

$$\overline{u}_t + k^2 \overline{u} = 0, \qquad \text{(A.2.34)}$$

$$\overline{u}(k, 0) = \overline{f}(k), \qquad \text{(A.2.35)}$$

where $\overline{f}(k)$ is the Fourier transform of $f(x)$. The solution of the initial-value problem for the ordinary differential equation problem (A.2.34)–(A.2.35) is

$$\overline{u}(k, t) = \overline{f}(k) e^{-k^2 t}. \qquad \text{(A.2.36)}$$

We now use the convolution theorem (A.2.19) and note that according to (A.2.25), $e^{-k^2 t}/\sqrt{2\pi}$ is just the Fourier transform of the fundamental solution. Therefore, $u(x, t)$ is given by (1.3.9).

A.2.6 The Laplace Transform

In this section we derive the basic formulas for Laplace transforms as a special case of the general result (A.2.8) for Fourier transforms. We are interested in functions $g(t)$ defined for $0 \leq t < \infty$ that may grow at worst exponentially with t as $t \to \infty$, i.e., there exists a positive constant c such that $e^{-ct} g(t) \to 0$ as $t \to \infty$. Given such a function $g(t)$, we define $f(t)$ on $-\infty < t < \infty$ as follows:

$$f(t) = \begin{cases} 0 & \text{if } t < 0, \\ e^{-ct} g(t) & \text{if } t \geq 0. \end{cases} \qquad \text{(A.2.37)}$$

We also assume that $f(t)$ is absolutely integrable on $-\infty < t < \infty$, that is,

$$\int_0^\infty e^{-ct}|g(t)|dt < \infty. \tag{A.2.38}$$

The Fourier transform pair (A.2.8a) for f with $\gamma = 2\pi, a = 1, x \to t$ becomes

$$\overline{f}(k) = \int_0^\infty g(t)e^{-(-ik+c)t}dt, \tag{A.2.39a}$$

$$g(t)e^{-ct} = \frac{1}{2\pi}\int_{-\infty}^\infty \overline{f}(k)e^{-ikt}dk. \tag{A.2.39b}$$

Let us introduce the complex variable $s = c - ik$ and denote

$$\overline{f}(k) = \overline{f}\left(\frac{s-c}{-i}\right) \equiv G(s). \tag{A.2.40}$$

Using this notation in (A.2.39a), changing the variable of integration from k to s in (A.2.39b), and multiplying this equation by e^{ct} gives the Laplace transform pair

$$G(s) = \int_0^\infty g(t)e^{-st}dt, \tag{A.2.41a}$$

$$g(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(s)e^{st}ds. \tag{A.2.41b}$$

Equation (A.2.41a) defines the Laplace transform $G(s)$ of the function $g(t)$. The inversion formula (A.2.41b) shows that for a given $G(s)$, the inversion involves the contour integral consisting of the vertical line a distance c to the right of the origin in the complex plane.

One may prove (see Section 7.3 of [8]) that if (A.2.41a) converges for a given $s = c$, then it also converges for all complex s such that $\operatorname{Re} s > c$. Thus, the singularities of $G(s)$, i.e., poles and branch points, lie in the half-plane to the left of this vertical contour.

It is also useful to introduce the notation

$$\mathcal{L}(g) \equiv G(s) = \int_0^\infty g(t)e^{-st}dt. \tag{A.2.42}$$

The following formulas are easily derived for the Laplace transform of the first derivative $\dot{g}(t)$ and second derivative $\ddot{g}(t)$ of a given function $g(t)$ by integrations by parts

$$\mathcal{L}(\dot{g}) = s\mathcal{L}(g) - g(0^+), \tag{A.2.43a}$$

$$\mathcal{L}(\ddot{g}) = s^2\mathcal{L}(g) - sg(0^+) - \dot{g}(0^+). \tag{A.2.43b}$$

The convolution theorem for Laplace transforms is also easily derived as for Fourier transforms. We obtain

$$\mathcal{L}\left(\int_0^t f(t-\tau)g(\tau)d\tau\right) = F(s)G(s) \tag{A.2.44}$$

for any two functions $f(t)$ and $g(t)$ having Laplace transforms $F(s)$ and $G(s)$, respectively.

A.2.7 Examples of Solution by Laplace Transforms

(i) *Fundamental solution of the diffusion equation*

To solve (1.2.5)–(1.2.7) by Laplace transforms, we multiply (1.2.5) by e^{-st} and integrate the result with respect to t over $(0, \infty)$. In view of the initial condition (1.2.6), the transform of u_t is just $sU(x, s)$. The transform of $-u_{xx}$ is $-U_{xx}(x, s)$ because we may move the second partial outside the integral sign. Finally, the right-hand side transforms to $\delta(x)$. Thus, we need to solve the ordinary differential equation

$$sU - U_{xx} = \delta(x) \tag{A.2.45}$$

subject to the boundary conditions

$$U(x, s) \to 0 \quad \text{as} \quad |x| \to \infty \tag{A.2.46}$$

that follow from (1.2.7).

The solution $U(x, s)$ has the form

$$U = \begin{cases} A(s)e^{-\sqrt{s}x} & \text{if } x > 0, \\ B(s)e^{\sqrt{s}x} & \text{if } x < 0, \end{cases} \tag{A.2.47}$$

after we impose the boundary conditions (A.2.46). Continuity of u across $x = 0$ gives

$$U(0^+, s) - U(0^-, s) = 0. \tag{A.2.48}$$

The jump condition at $x = 0$ is obtained by integrating (A.2.45) from $x = 0^-$ to $x = 0^+$. We obtain

$$-U_x(0^+, s) + U_x(0^-, s) = 1. \tag{A.2.49}$$

Using (A.2.47) in (A.2.48)–(A.2.49) gives $A = B = 1/2\sqrt{s}$. Therefore, we have the Laplace transform of the fundamental solution in the form

$$U(x, s) = \frac{1}{2\sqrt{s}} e^{-\sqrt{s}|x|}. \tag{A.2.50}$$

The inversion formula (A.2.41b) becomes

$$u(x, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{st-\sqrt{s}|x|}}{2\sqrt{s}} ds. \tag{A.2.51}$$

The only singularities of U are the branch points at $s = 0$ and $s = \infty$. To have U analytic in a right half-plane, we cut the s-plane along the negative real axis; thus, in the inversion formula c is any positive constant. See Figure A.5. Implicit in our expression (A.2.47) is the fact that $\sqrt{s}$ is real and positive if s is real and

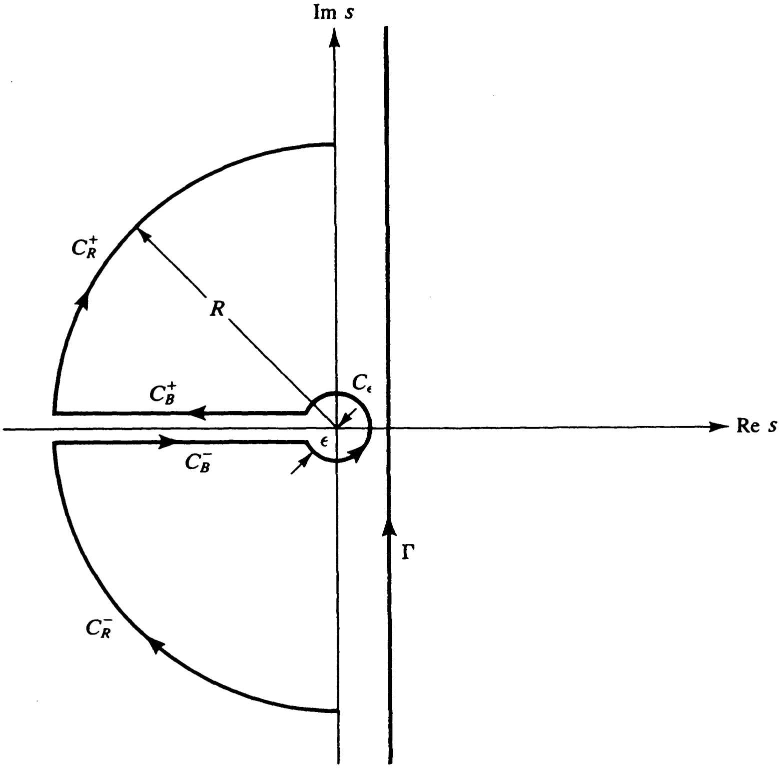

FIGURE A.5. Complex s-plane

positive. Therefore, if we set $s = re^{i\theta}$, the appropriate branch for $\sqrt{s}$ is the one with $-\pi < \theta < \pi$.

As the integrand in (A.2.51) is analytic everywhere in the s-plane except along the branch cut, the path of integration consisting of the segments C_R^-, C_B^-, C_ϵ, C_B^+, and C_R^+, traversed in the directions indicated in Figure A.5, is equivalent to the vertical path $c - i\infty$ to $c + i\infty$. Thus, we may write (A.2.51) as

$$u(x,t) = \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \frac{1}{2\pi i}\left(I_{R^+} + I_{R^-} + I_\epsilon + I_{B^+} + I_{B^-}\right), \tag{A.2.52}$$

where

$$I_{R^-} = \int_{-\pi/2}^{-\pi} \frac{\exp(tRe^{i\theta} - |x|R^{1/2}e^{i\theta/2})iRe^{i\theta}}{2R^{1/2}e^{i\theta/2}}\,d\theta, \tag{A.2.53}$$

$$I_{R^+} = \int_{\pi}^{\pi/2} \frac{\exp(tRe^{i\theta} - |x|R^{1/2}e^{i\theta/2})iRe^{i\theta}}{2R^{1/2}e^{i\theta/2}}\,d\theta, \tag{A.2.54}$$

$$I_\epsilon = \int_{-\pi}^{\pi} \frac{\exp(t\epsilon e^{i\theta} - |x|\epsilon^{1/2}e^{i\theta/2})i\epsilon e^{i\theta}}{2\epsilon^{1/2}e^{i\theta/2}} d\theta, \qquad (A.2.55)$$

$$I_{B^+} = \int_\epsilon^R \frac{\exp(-rt - ir^{1/2}|x|)}{2ir^{1/2}} (-dr), \qquad (A.2.56)$$

$$I_{B^-} = \int_R^\epsilon \frac{\exp(-rt + ir^{1/2}|x|)}{-2ir} (-dr). \qquad (A.2.57)$$

Note carefully that on C_B^+, $\sqrt{s} = r^{1/2}e^{i\pi/2} = ir^{1/2}$, whereas on C_{B^-}, $\sqrt{s} = r^{1/2}e^{-i\pi/2} = -ir^{1/2}$.

Consider the integral I_{R^+}. We have

$$|I_{R^+}| \leq \frac{1}{2}\int_{\pi/2}^{\pi} |\exp(tRe^{-i\theta} - |x|R^{1/2}e^{-i\theta/2})iR^{1/2}e^{-i\theta/2}|d\theta$$
$$\leq \frac{1}{2}\int_{\pi/2}^{\pi} \exp(tR\cos\theta - |x|R^{1/2}\cos\theta/2)R^{1/2}d\theta.$$

Since $\cos\theta < 0$ and $\cos(\theta/2) > 0$ if $\pi/2 < \theta < \pi$, both terms in the exponent in the integrand of (A.2.57) are negative. Therefore, the integrand decays exponentially, and we have $I_{R^+} \to 0$ as $R \to \infty$. Similarly, $I_{R^-} \to 0$ as $R \to \infty$.

An analogous calculation shows that

$$|I_\epsilon| \leq \frac{1}{2}\int_{-\pi}^{\pi} \exp(t\epsilon\cos\theta - |x|\epsilon^{1/2}\cos\theta/2)\epsilon^{1/2}d\theta, \qquad (A.2.58)$$

and the exponential now tends to unity as $\epsilon \to 0$. The occurrence of the $\epsilon^{1/2}$ term in the integrand then ensures that $I_{\epsilon\to 0}$ as $\epsilon \to 0$.

Thus, the only contribution in the limit $R \to \infty$, $\epsilon \to 0$ comes from I_{B^+} and I_{B^-}. These combine to give

$$u(x,t) = \frac{1}{2\pi}\int_0^\infty \frac{e^{-rt}\cos r^{1/2}|x|}{r^{1/2}} dr, \qquad (A.2.59)$$

or changing the variable of integration from r to $k = r^{1/2}$,

$$u(x,t) = \frac{1}{\pi}\int_0^\infty e^{-k^2 t}\cos kx\, dk. \qquad (A.2.60)$$

This is the same integral (A.2.27) that we obained earlier (and much more efficiently) using Fourier transforms.

(ii) *Boundary-value problem for the diffusion equation*

We consider the diffusion equation over the semi-infinite domain $0 \leq x < \infty$, $t \geq 0$, with a prescribed time-dependent boundary condition at $x = 0$ and zero initial value

$$u_t - u_{xx} = 0; \quad 0 \leq x < \infty; \quad 0 \leq t < \infty, \qquad (A.2.61)$$

$$u(x, 0) = 0, \qquad (A.2.62)$$

$$u(0, t) = g(t), \quad t > 0, \qquad (A.2.63)$$

$$u(\infty, t) = 0. \tag{A.2.64}$$

Taking Laplace transforms with respect to t gives

$$sU - U_{xx} = 0, \tag{A.2.65}$$

$$U(0, s) = G(s), \quad U(\infty, s) = 0, \tag{A.2.66}$$

where $G(s)$ is the Laplace transform of $g(t)$.

The solution of (A.2.65)–(A.2.66) is

$$U(x, s) = G(s)e^{-\sqrt{s}x}. \tag{A.2.67}$$

Let $h(x, t)$ have the Laplace transform $e^{-\sqrt{s}x}$, i.e.,

$$h(x, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st-\sqrt{s}} ds. \tag{A.2.68}$$

As in the previous example, the only singularities of $e^{-\sqrt{s}x}$ are the branch points at $s = 0$ and $s = \infty$. We cut the complex s-plane along the negative real axis and let $c = 0^+$. The contour integral over $0^+ - i\infty$ to $0^+ + i\infty$ is equivalent to the contribution we obtain by deforming this contour onto the branch cut. This gives

$$h(x, t) = \frac{1}{2\pi i} \left[\int_0^\infty e^{-rt-i\sqrt{r}x}(-dr) + \int_\infty^0 e^{-rt+i\sqrt{r}x}(-dr) \right]. \tag{A.2.69}$$

Simplifying gives

$$\begin{aligned} h(x, t) &= \frac{1}{2\pi i} \int_0^\infty e^{-rt}(e^{ir^{1/2}x} - e^{-ir^{1/2}x})dr \\ &= \frac{1}{\pi} \int_0^\infty e^{-rt} \sin r^{1/2}x \, dr. \end{aligned} \tag{A.2.70}$$

Integrating by parts gives

$$h(x, t) = \frac{1}{\pi} \left\{ \left[-\frac{1}{t} e^{-rt} \sin r^{1/2}x \right]_0^\infty + \frac{x}{2t} \int_0^\infty \frac{e^{-rt} \cos r^{1/2}x}{r^{1/2}} dr \right\}. \tag{A.2.71}$$

The boundary contributions vanish because $e^{-rt} \to 0$ as $r \to \infty$ and $\sin r^{1/2}x \to 0$ as $r \to 0$. We change the integration variable in the second term of (A.2.71) from r to $\xi \equiv \sqrt{rt}$ to obtain

$$h(x, t) = \frac{x}{\pi t^{3/2}} \int_0^\infty e^{-\xi^2} \cos\left(\frac{x}{t^{1/2}}\xi\right) d\xi,$$

and using (A.2.31) we obtain

$$h(x, t) = \frac{x}{2\pi^{1/2}t^{3/2}} e^{-x^2/4t}. \tag{A.2.72}$$

Now we use the convolution theorem (A.2.44) to note that

$$u(x, t) = \int_0^t h(x, \tau)g(t - \tau)d\tau$$

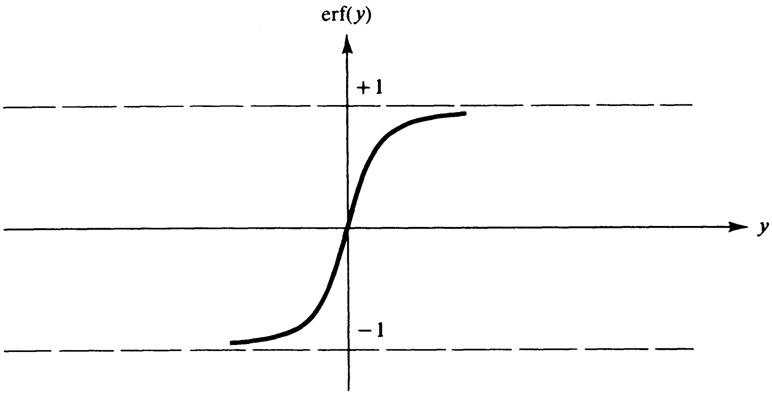

FIGURE A.6. The error function erf(y)

$$= \frac{x}{2\pi^{1/2}} \int_0^t \frac{g(t-\tau)}{\tau^{3/2}} e^{-x^2/4\tau} d\tau. \tag{A.2.73}$$

This result simplifies further if g is a constant g_0. We obtain

$$u(x, t) = \frac{g_0 x}{2\pi^{1/2}} \int_0^t \frac{e^{-x^2/4\tau}}{\tau^{3/4}} d\tau. \tag{A.2.74}$$

Changing the variable of integration from τ to $\eta = x/2\sqrt{\tau}$ gives

$$u(x, t) = \frac{2g_0}{\pi^{1/2}} \int_{x/2t^{1/2}}^{\infty} e^{-\eta^2} d\eta. \tag{A.2.75}$$

The integral in (A.2.75) is a constant times the complementary error function. The error function erf(y) is defined by

$$\text{erf}(y) \equiv \frac{2}{\sqrt{\pi}} \int_0^y e^{-\eta^2} d\eta. \tag{A.2.76}$$

In particular, we see that erf(y) is odd: erf$(-y)$ = erf(y), and erf$(0) = 0$. In the limit $y \to \pm\infty$, we have the asymptotic values erf$(\infty) = 1$ and erf$(-\infty) = 0$. This function is sketched in Figure A.6.

The complementary error function erfc(y) is simply

$$\text{erfc}(y) \equiv 1 - \text{erf}(y) = \frac{2}{\pi^{1/2}} \int_y^{\infty} e^{-\eta^2} d\eta. \tag{A.2.77}$$

Thus, the solution (A.2.75) for our boundary-value problem for $g(t) = g_0 =$ constant is just

$$u(x, t) = g_0 \,\text{erfc}\left(\frac{x}{2t^{1/2}}\right). \tag{A.2.78}$$

A.3 Review of Asymptotic Expansions

In many of our discussions throughout this book we rely on certain "perturbation expansions" to approximate the solution when a dimensionless parameter ϵ is small. Often, we express this perturbation expansion in the form of a power series in ϵ,

$$u(\mathbf{x};\epsilon) = u_0(\mathbf{x}) + \epsilon u_1(\mathbf{x}) + \epsilon^2 u_2(\mathbf{x}) + \ldots, \tag{A.3.1}$$

where u is a scalar that solves a given partial differential equation in terms of the n independent variables $x_1, x_2, \ldots, x_n \equiv \mathbf{x}$. A more precise characterization of the series (A.3.1) is that it is the asymptotic expansion of $u(\mathbf{x};\epsilon)$ with respect to the asymptotic sequence $1, \epsilon, \epsilon^2, \ldots$, in the limit $\epsilon \to 0$. For a comprehensive discussion of asymptotic expansions, the reader is referred to standard texts, such as [15] and [39]. In the remainder of this section, we shall review some of the basic ideas of asymptotic expansions. We shall consider functions of $\mathbf{x}$ and the parameter ϵ where $\mathbf{x}$ ranges over some domain $\mathcal{D}$, and ϵ over the interval $I : 0 \le \epsilon \le \epsilon_0$.

A.3.1 Order Symbols

(i) *The O symbol*

The statement

$$u(\mathbf{x};\epsilon) = O(v(\mathbf{x};\epsilon)) \text{ in } \mathcal{D} \text{ as } \epsilon \to 0 \tag{A.3.2}$$

means that for each point $\mathbf{x}$ in $\mathcal{D}$ there exists a positive $k(\mathbf{x})$ and an interval I : $0 \le \epsilon \le \epsilon_0(\mathbf{x})$, where ϵ_0 depends in general on the choice of $\mathbf{x}$, such that

$$|u| \le k|v| \tag{A.3.3}$$

for every ϵ in I. If (u/v) is defined in $\mathcal{D}$, (A.3.3) implies that $|u/v|$ is bounded above by k.

The statement (A.3.2) is said to be *unifiormly valid* in $\mathcal{D}$ if k is a constant and ϵ_0 does not depend on $\mathbf{x}$. We now illustrate some of the features of this definition with the following examples.

1. Let $\mathbf{x} = (x_1, x_2)$; $\mathcal{D} : x_1^2 + x_2^2 < 1$; $u = (1 - x_1^2 - x_2^2 + \epsilon)^{-1}$; $v = 1$. The statement $u = O(1)$ is correct with $k = (1 - x_1^2 - x_2^2)^{-1}$ and for any $\epsilon_0 < 1$, but this result is *not* uniformly valid because k becomes unbounded as $x_1^2 + x_2^2 \to 1$. However, if we restrict our domain to the interior of the disk $\widetilde{\mathcal{D}} : x^2 + y^2 \le a < 1$, then $u = O(1)$ uniformly in $\widetilde{\mathcal{D}}$ with k equal to the constant $(1-a)^{-1}$.
2. Let $\mathbf{x} = (x_1, x_2)$; $\mathcal{D} : 0 < x_1 < \infty; 0 < x_2 < \infty$; $u = x_1\epsilon^\alpha$; $v = x_2\epsilon^\beta$, where $\alpha =$ constant, $\beta =$ constant, with $\alpha \ge \beta$. We have $u = O(v)$ because we can choose $k(x_1, x_2) = x_1/x_2$, $\epsilon_0 < 1$ to verify that $x_1\epsilon^\alpha \le (x_1/x_2)x_2\epsilon^\beta$ holds as long as x_1 is finite and $x_2 \ne 0$. Thus, the statement that $u = O(v)$ is not uniformly valid in $\mathcal{D}$, but it is uniformly valid in any subdomain $\widetilde{\mathcal{D}} : 0 <$

$x_1 \le X_1 < \infty$; $0 < X_2 \le x_2 < \infty$, where X_1 and X_2 are positive constants, because if x_1 and x_2 are restricted to $\tilde{\mathcal{D}}$, we can pick $k = X_1/X_2 =$ constant.

This example also shows that the O symbol does not necessarily imply that u and v are of the "same order of magnitude"; for example, if $\alpha > \beta$, we see that $u < v$ for any fixed x_1, x_2 in $\mathcal{D}$ if ϵ is sufficiently small. (In fact, $u/v \to 0$ as $\epsilon \to 0$.) In order to describe two functions u and v that are of the "same order of magnitude," we must say that $u = O(v)$ and $v = O(u)$. In this case the limit of $|u/v|$, if it exists, is neither zero nor infinity. The two functions in the first example satisfy this dual requirement, but those in the second do not. In [30] the notation $u = O_s(v)$ for "strictly of the order of" is used to indicate $u = O(v)$ and $v = O(u)$; we shall adopt this notation when needed.

(ii) *The o symbol*

We now define the lowercase o symbol. The statement

$$u(\mathbf{x};\epsilon) = o(v(\mathbf{x};\epsilon)) \text{ in } \mathcal{D} \text{ as } \epsilon \to 0, \tag{A.3.4}$$

for two functions u, v defined in $\mathcal{D}$ and $\epsilon > 0$ means that for each point $\mathbf{x}$ in $\mathcal{D}$ and a given $\delta > 0$ there exists an interval $I : 0 \le \epsilon \le \epsilon_0(\mathbf{x}, \delta)$, which depends in general on the choice of $\mathbf{x}$ and δ, such that

$$|u| \le \delta |v| \tag{A.3.5}$$

for all ϵ in I. Sometimes the notation $u \ll v$ is used to indicate (A.3.4). The inequality (A.3.5) states that $|u|$ becomes arbitrarily small compared to $|v|$, and if (u/v) is defined, (A.3.5) implies that $(u/v) \to 0$ as $\epsilon \to 0$. Note that $u = o(v)$ always implies $u = O(v)$.

Again, we say that (A.3.4) holds uniformly in $\mathcal{D}$ if the right endpoint of the interval I depends only on δ and not on the choice of the point $\mathbf{x}$ in $\mathcal{D}$. The following examples illustrate these ideas.

1. $\mathbf{x} = (x_1, x_2)$; $x_1\epsilon^\alpha = o(x_2\epsilon^\beta)$ in $\mathcal{D} : 0 < x_1 < \infty; 0 < x_2 < \infty$ as $\epsilon \to 0$ for constant α, β as long as $\alpha > \beta$. To prove this we need to show that for any given δ, we can find a neighborhood of $\epsilon = 0$ such that $(x_1/x_2)\epsilon^{\alpha-\beta} \le \delta$. The neighborhood of $\epsilon = 0$ that meets our need is the interval $0 \le \epsilon \le (\delta x_2/x_1)^{1/(\alpha-\beta)}$. This neighborhood shrinks to zero if either $x_2 \to 0$ or $x_1 \to \infty$. Therefore, the statement $x_1\epsilon^\alpha = o(x_2\epsilon^\beta)$ is not uniformly valid in $\mathcal{D}$. However, if we restrict attention to $\tilde{\mathcal{D}} : 0 < x_1 \le X_1 < \infty; 0 < X_2 \le x_2 < \infty$ for positive constants X_1, X_2, then the statement is uniformly valid in $\tilde{\mathcal{D}}$, and the neighborhood of $\epsilon = 0$ that suffices is now $0 \le \epsilon \le (\delta X_2/X_1)^{1/(\alpha-\beta)}$, which depends only on δ.
2. Let $\mathcal{D}$ be the triangular domain $0 < x_1 < \infty; 0 < x_2 < x_1$. For any arbitrarily large positive constant β, we have

$$e^{(x_2-x_1)/\epsilon} = o(\epsilon^\beta) \text{ in } \mathcal{D} \text{ as } \epsilon \to 0. \tag{A.3.6}$$

A function such as $e^{(x_2-x_1)/\epsilon}$ satisfying (A.3.6) is said to be *transcendentally small* as $\epsilon \to 0$. Note that $(x_2 - x_1)/\epsilon < 0$. To prove (A.3.6), it suffices to note that $\lim_{\epsilon\to 0}(\epsilon^{-\beta}e^{-\alpha/\epsilon}) = 0$ for any $\alpha > 0$, $\beta > 0$. Again, (A.3.6) is not uniformly valid in the triangular domain, but it is uniformly valid in the subdomain $\tilde{\mathcal{D}} : 0 < X_1 \leq x_1 < \infty; 0 < x_2 \leq x_1 - X_1$.

A.3.2 *Definition of an Asymptotic Expansion; Generalized Asymptotic Expansion*

Let $\{\phi_n(\epsilon)\}$ with $n = 1, 2, \ldots$ be a sequence of functions of ϵ such that

$$\phi_{n+1}(\epsilon) = o(\phi_n(\epsilon)) \text{ as } \epsilon \to 0 \tag{A.3.7}$$

for each $n = 1, 2, \ldots$. Such a sequence is called an *asymptotic sequence*. Thus, $\{\epsilon^{n-1}\}$ with $n = 1, 2, \ldots$ is an asymptotic sequence; so is the sequence $\log\epsilon$, 1, $\epsilon\log\epsilon$, ϵ, $(\epsilon\log\epsilon)^2$, $\epsilon^2\log\epsilon$, $\epsilon^2, \ldots$.

Let $u(\mathbf{x}; \epsilon)$ be defined for all $\mathbf{x}$ in some domain $\mathcal{D}$ and all ϵ in some neighborhood of $\epsilon = 0$. Let $\{\phi_n(\epsilon)\}$ be a given asymptotic sequence. The series $\sum_{n=1}^{N} \phi_n(\epsilon)u_n(\mathbf{x})$, where the integer N may be finite or infinite, is said to be the *asymptotic expansion* of u with respect to $\{\phi_n\}$ as $\epsilon \to 0$ if for every $M = 1, 2, \ldots, N$,

$$u(\mathbf{x}; \epsilon) - \sum_{n=1}^{M} \phi_n(\epsilon)u_n(\mathbf{x}) = o(\phi_M) \text{ as } \epsilon \to 0. \tag{A.3.8a}$$

A stronger definition, which follows from (A.3.8a), is that

$$u(\mathbf{x}; \epsilon) - \sum_{n=1}^{M} \phi_n(\epsilon)u_n(\mathbf{x}) = O(\phi_{M+1}) \text{ as } \epsilon \to 0 \tag{A.3.8b}$$

for each $M = 1, 2, \ldots, N-1$. The asymptotic expansion is said to hold *uniformly in $\mathcal{D}$* if the order relations in (A.3.8) hold uniformly there.

If $N = \infty$, the following notation is used to indicate an asymptotic expansion

$$u(\mathbf{x}; \epsilon) \sim \sum_{n=1}^{\infty} \phi_n(\epsilon)u_n(\mathbf{x}) \text{ as } \epsilon \to 0. \tag{A.3.8c}$$

We see that once u and the sequence $\{\phi_n\}$ are given, it is a straightforward matter to construct a unique asymptotic expansion by the repeated application of (A.3.8a). Thus,

$$u_1(\mathbf{x}) = \lim_{\epsilon\to 0} \frac{u(\mathbf{x}; \epsilon)}{\phi_1(\epsilon)}, \tag{A.3.9a}$$

$$u_2(\mathbf{x}) = \lim_{\epsilon\to 0} \frac{u(\mathbf{x}; \epsilon) - \phi_1(\epsilon)u_1(\mathbf{x})}{\phi_2(\epsilon)}, \tag{A.3.9b}$$

$$u_m(\mathbf{x}) = \lim_{\epsilon\to 0} \frac{u(\mathbf{x}; \epsilon) - \sum_{n=1}^{m-1} \phi_n(\epsilon)u_n(\mathbf{x})}{\phi_m(\epsilon)}. \tag{A.3.9c}$$

In our study of matched asymptotic expansions and of multiple-scale expansions in Chapter 8, we need more general expansions than those in (A.3.8), where we have a sum of products of functions of $\mathbf{x}$ with functions of ϵ. In particular, we encounter series of functions of both $\mathbf{x}$ and ϵ simultaneously. Accordingly, we say that $\sum_{n=1}^{N} \widetilde{u}_n(\mathbf{x}; \epsilon)$ is the generalized asymptotic expansion of $u(\mathbf{x}; \epsilon)$ as $\epsilon \to 0$ with respect to the asymptotic sequence $\{\phi_n(\epsilon)\}$ if for each $M = 1, 2, \ldots, N$, we have

$$u(\mathbf{x}; \epsilon) - \sum_{n=1}^{M} \widetilde{u}_n(\mathbf{x}; \epsilon) = o(\phi_M) \text{ as } \epsilon \to 0. \tag{A.3.10}$$

A.3.3 Asymptotic Expansion of a Given Function

To illustrate the idea of an asymptotic expansion in its most elementary form, we assume that $u(\mathbf{x}; \epsilon)$ is given explicitly, and we wish to derive its asymptotic expansion with respect to a given sequence $\{\phi_n(\epsilon)\}$. Let $\mathcal{D}$ be the domain $0 \leq x < \infty$; $0 < y < \infty$ and consider the asymptotic expansion of

$$u = \log\left(1 + \frac{\epsilon x}{y} + \epsilon^2 xy\right) \text{ in } \mathcal{D} \text{ as } \epsilon \to 0, \tag{A.3.11}$$

with respect to the sequence $1, \epsilon, \epsilon^2, \ldots$.

Since $z \equiv \epsilon x/y + \epsilon^2 xy$ is small for $\epsilon \ll 1$, as long as x and y are fixed, we calculate the Taylor series of (A.3.11) (with remainder) around $z = 0$:

$$u = \sum_{n=1}^{N} (-1)^{n-1} \frac{z^n}{n} + \frac{(-1)^N z^{N+1}}{(N+1)(1 = \alpha z)^{N+1}}, \tag{A.3.12}$$

where α is a constant, $0 < \alpha < 1$. Now we expand each of the terms z^n in powers of ϵ and retain only terms up to $O(\epsilon^N)$. The result gives the asymptotic expansion of (A.3.11), and our construction guarantees that the condition (A.3.8b) is satisfied. For example, for $N = 4$, we have

$$u = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + R_5(z),$$

where $|R_5| \leq C_5 z^5$ and C_5 is a constant. Thus,

$$\begin{aligned}
u &= \left(\epsilon \frac{x}{y} + \epsilon^2 xy\right) - \frac{1}{2}\left(\epsilon^2 \frac{x^2}{y^2} + 2\epsilon^3 x^2 + \epsilon^4 x^2 y^2\right) \\
&\quad + \frac{1}{3}\left(\frac{\epsilon^3 x^3}{y^3} + 3\epsilon^4 \frac{x^3}{y} + O(\epsilon^5)\right) - \frac{1}{4}\left(\epsilon^4 \frac{x^4}{y^4} + O(\epsilon^5)\right) + O(\epsilon^5) \\
&= \epsilon \frac{x}{y} + \epsilon^2 \left(xy - \frac{x^2}{2y^2}\right) + \epsilon^3 \left(-x^2 + \frac{x^3}{3y^3}\right) \\
&\quad + \epsilon^4 \left(-\frac{x^2 y^2}{2} + \frac{x^3}{y} - \frac{x^4}{4y^4}\right) + O(\epsilon^5).
\end{aligned} \tag{A.3.13}$$

It is clear from the definition of z that (A.3.13) is not uniformly valid in $\mathcal{D}$. However, if we restrict attention to $\tilde{\mathcal{D}} : 0 \leq x \leq X < \infty; 0 < Y_1 \leq y \leq Y_2 < \infty$, where X, Y_1, and Y_2 are arbitrary positive numbers with $Y_1 < Y_2$, then the expansion is uniformly valid. The structure of z also indicates that the sequence 1, ϵ, ϵ^2 is the "natural" one to use for this function.

A.3.4 Asymptotic Expansion of the Root of an Algebraic Equation

A less direct way to define a function $u = U(\mathbf{x}; \epsilon)$ is to have it equal to one of the roots of the algebraic equation

$$R(\mathbf{x}, u; \epsilon) = 0. \tag{A.3.14}$$

Thus, $R(\mathbf{x}, U(\mathbf{x}; \epsilon); \epsilon) \equiv 0$ for all $\epsilon > 0$ and all $\mathbf{x}$ in some domain $\mathcal{D}$. We assume that the limiting value $u_1(\mathbf{x}) \equiv U(\mathbf{x}; 0)$ is known, and we are interested in the asymptotic expansion of the root $U(\mathbf{x}; \epsilon)$ for $\epsilon \to 0$.

The following example illustrates the ideas. Let

$$R(x, y, u; \epsilon) \equiv \epsilon u^2 + 2f(x, y)u - g(x, y) = 0, \tag{A.3.15}$$

where f and g are given functions in $\mathcal{D} : -\infty < x < \infty; -\infty < y < \infty$ and $0 < \epsilon \ll 1$. Since (A.3.15) is quadratic, the solution for the roots is readily obtained in the form

$$U^+(x, y; \epsilon) = \frac{-f(x, y) + [f^2(x, y) + \epsilon g(x, y)]^{1/2}}{\epsilon}, \tag{A.3.16a}$$

$$U^-(x, y; \epsilon) = \frac{-f(x, y) - [f^2(x, y) + \epsilon g(x, y)]^{1/2}}{\epsilon}, \tag{A.3.16b}$$

and for ϵ sufficiently small, the two roots are real.

For a general function R, the exact roots will not be explicitly available, as in (A.3.16). Therefore, we shall ignore these expressions for the time being and illustrate how to go about calculating their asymptotic expansions based only on (A.3.15).

Setting $\epsilon = 0$ in (A.3.15) shows that the limiting value of one root is $u_1^+(x, y) = g(x, y)/2f(x, y)$ whenever $f \neq 0$. Let us restrict attention to the case of $f \neq 0$ and assume an asymptotic expansion for U^+ in the form

$$U^+(x, y; \epsilon) = \frac{1}{2}\frac{g(x, y)}{f(x, y)} + \phi_2(\epsilon)u_2^+(x, y) + \phi_3(\epsilon)u_3^+(x, y) + o(\phi_3), \tag{A.3.17}$$

where $\phi_1 = 1, \phi_2, \phi_3, \ldots$ is an asymptotic sequence yet to be defined. If we substitute (A.3.17) into (A.3.15) and cancel out the terms of $O(1)$, we are left with

$$\epsilon\frac{g^2}{4f^2} + \phi_2(2fu_2^+) + \epsilon\phi_2\frac{g}{f}u_2^+ + \phi_3(2fu_3) = O(\epsilon\phi_3) + O(\epsilon\phi_2^2) + o(\phi_3). \tag{A.3.18}$$

The choice of ϕ_2 and ϕ_3 affects the relative importance of the various terms in the left-hand side of (A.3.18). At any rate, $\epsilon\phi_2 \ll \phi_2$ and $\phi_3 \ll \phi_2$, so the third and fourth terms in the left-hand side of (A.3.18) are negligible in comparison with the second term proportional to ϕ_2. If we assume $\phi_2 \ll \epsilon$, we must set $g = 0$ in order to satisfy (A.3.18) to $O(\epsilon)$, and this is inconsistent in general. On the other hand, if we set $\epsilon \ll \phi_2$, we must either have $f = 0$ (which we have excluded) or $u_2^+ = 0$. Having $u_2^+ = 0$ in (A.3.17) does not provide any new information about the expansion because it is always trivially possible to insert a zero term of some order $\phi_2 : \epsilon \ll \phi_2$. The only nontrivial possibility left is that ϕ_2 be of the same order of magnitude as ϵ, that is,

$$\phi_2 = O_s(\epsilon). \tag{A.3.19}$$

For simplicity, we take $\phi_2 = \epsilon$ and conclude that

$$u_2^+(x, y) = -\frac{g^2(x, y)}{8f^3(x, y)}, \tag{A.3.20}$$

in order to satisfy (A.3.18). This equation now reduces to

$$-\epsilon^2 \frac{g^3}{8f^4} + \phi_3(2fu_3^+) = O(\epsilon\phi_3) + O(\epsilon^3) + o(\phi_3). \tag{A.3.21}$$

The same arguments that we used to determine ϕ_2 now imply that $\phi_3 = O_s(\epsilon^2)$, and we choose $\phi_3 = \epsilon^2$. In this case, we must set

$$u_3^+(x, y) = \frac{g^3(x, y)}{16f^5(x, y)}, \tag{A.3.22}$$

and all the terms that we have neglected in the right-hand side of (A.3.21) are $O(\epsilon^3)$.

Thus, we have determined the asymptotic expansion of U^+ in the form

$$U^+(x, y; \epsilon) = \frac{1}{2}\frac{g}{f} - \epsilon\frac{g^2}{8f^3} + \epsilon^3\frac{g^3}{16f^5} + O(\epsilon^3). \tag{A.3.23}$$

It is reassuring to note that if we expand the radical in (A.3.16a) for $|\epsilon g| < f^2$ and collect terms, we obtain (A.3.23).

The procedure that we have outlined extends to all orders, and we conclude that the sequence $\{\epsilon^{n-1}\}$, $n = 1, 2, \ldots$, is "appropriate" for U^+. Strictly speaking, this sequence is not unique; any sequence satisfying $\phi_n(\epsilon) = O_s(\epsilon^{n-1})$ for each $n = 1, 2, \ldots$ can be used. For example, we may choose the sequence $\{\phi_n(\epsilon)\} = C_{n-1}\epsilon^{n-1}$, $n = 1, 2, \ldots$, for arbitrary nonzero constants C_{n-1}. In this case, the new $u_n^+(x, y)$ that we calculate will differ from those given by (A.3.20), (A.3.22), and the like by the factors $1/C_{n-1}$, but the net result (A.3.23) will be the same. A more elaborate choice has $\psi_2(\epsilon) = \epsilon/(1 + \epsilon)$, $\psi_3(\epsilon) = \epsilon^2/(1 + \epsilon), \ldots$. In this case $\epsilon = \psi_2 + \epsilon\psi_2 = \psi_2 + \psi_3$ and $R = \psi_2 u^2 + \psi_3 u^2 + 2fu - g = 0$. We then compute

$$U^+(x, y; \epsilon) = \frac{1}{2}\frac{g}{f} - \frac{\epsilon}{1+\epsilon}\frac{g^2}{8f^3} + \frac{\epsilon^2}{1+\epsilon}\left[\frac{g^3}{16f^5} - \frac{g^2}{8f^3}\right] + o(\psi_2). \tag{A.3.24}$$

Note that if the expressions for ψ_2 and ψ_3 in (A.3.24) are expanded in powers of ϵ and only terms proportional to 1, ϵ, and ϵ^2 are retained, the expressions in (A.3.23) and (A.3.24) are *asymptotically equivalent* to $O(\epsilon^2)$.

There is no advantage, a priori, in using (A.3.24) as opposed to (A.3.23). For particular choices of f, g, and ϵ, the three-term expansion (A.3.24) may be numerically more accurate than the three-term expansion (A.3.23). For example, if $f = g = 1, \epsilon = 0.1$, we have the exact root $U^+ = 0.4880884817\ldots$, whereas (A.3.23) gives $U^+ = 0.488125\ldots$ and (A.3.24) gives $U^+ = 0.488068\ldots$. In this case (A.3.24) is more accurate than (A.3.23), but in both cases the error is still $O(\epsilon^3)$.

Setting $\epsilon = 0$ in (A.3.15) shows that there is only one root U^+ that is $O(1)$. This root corresponds to the balance to $O(1)$ between the second and third terms in the expression (A.3.15) for R. The balance between the first and second terms to $O(1)$ is the only other possible one if $f \neq 0$. This means that the root U^- must be $O(\epsilon^{-1})$, and we assume that it has an asymptotic expansion of the form

$$U^-(x, y; \epsilon) = \frac{1}{\epsilon}u_1^-(x, y) + \lambda_2(\epsilon)u_2^-(x, y) + \lambda_3(\epsilon)u_3^-(x, y) + o(\lambda_3). \tag{A.3.25a}$$

Proceeding as before, we find that $1/\epsilon, 1, \epsilon, \ldots$ is an appropriate sequence, and we obtain the expansion

$$U^-(x, y; \epsilon) = \frac{1}{\epsilon}(-2f) - \frac{1}{2}\frac{g}{f} + \epsilon\left(\frac{1}{8}\frac{g^2}{f^3}\right) + O(\epsilon^2), \tag{A.3.25b}$$

which also follows from (A.3.16b).

The two expansions (A.3.23) (or (A.3.24)) and (A.3.25b) are uniformly valid in $\mathcal{D}$: $-\infty < x < \infty$; $-\infty < y < \infty$, as long as $f \neq 0$ and g is bounded in $\mathcal{D}$. To illustrate the situation for $f = 0$, g bounded, let us assume that $f(x, y)$ vanishes along some curve $y = h(x)$ but that $f_y(x, h(x)) \neq 0$. Since our expansions fail near $y - h(x) = 0$, let us use a rescaled variable y^* defined by

$$y^* \equiv \frac{y - h(x)}{\alpha(\epsilon)} \tag{A.3.26}$$

instead of y, and retain $x^* = x$ to see what happens to (A.3.15) for various choices of the scale function $\alpha(\epsilon)$. We are interested in $\alpha(\epsilon) \to 0$, as $\epsilon \to 0$ so that holding y^* fixed in this limit implies that $y - h(x) = O(\alpha)$.

We now regard u as a function of x^* and y^* and write (A.3.15) in the form

$$\epsilon u^2 + 2f(x^*, h(x^*) + \alpha y^*)u - g(x^*, h(x^*) + \alpha y^*) = 0. \tag{A.3.27}$$

Assuming that f and g are sufficiently differentiable near $y = h(x)$, we can expand these functions in Taylor series. To leading order, we obtain

$$f(x^*, h(x^*) + \alpha y^*) = f_y(x^*, h(x^*))\alpha y^* + O(\alpha^2), \qquad (A.3.28a)$$

$$g(x^*, h(x^*) + \alpha y^*) = g(x^*, h(x^*)) + O(\alpha), \qquad (A.3.28b)$$

where in the expansion for f we have noted that $f(x^*, h(x^*)) \equiv 0$ for all x^*. The expression for R to leading order then becomes

$$\epsilon u^2 + 2\alpha(\epsilon) f^*(x^*) y^* u - g^*(x^*) = O(\alpha), \qquad (A.3.29)$$

where we have written

$$f_y(x^*, h(x^*)) \equiv f^*(x^*), \quad g(x^*, h(x^*)) \equiv g^*(x^*). \qquad (A.3.30)$$

Unless $g^* \equiv 0$, the only way to achieve a dominant balance in (A.3.29) is to have u large. Suppose that we assume $u = O(\beta^{-1}(\epsilon))$, where $\beta(\epsilon) \ll 1$. We may implement this assumption explicitly by setting

$$u \equiv \frac{u^*}{\beta(\epsilon)}, \qquad (A.3.31)$$

where $u^* = O(1)$. Equation (A.3.29) then becomes

$$\frac{\epsilon}{\beta^2} u^{*2} + \frac{2\alpha}{\beta} f^* y^* u^* - g^* = O(\alpha). \qquad (A.3.32)$$

The choice of α and β for which the most terms in (A.3.32) are in dominant balance corresponds to $\epsilon/\beta^2 = O_s(1), \alpha/\beta = O_s(1)$, or simply $\alpha = \beta = \epsilon^{1/2}$. In this case all three terms are in dominant balance; any other choice would result in a limiting form with fewer terms. This principle for choosing the scale functions α, β is sometimes called the *principle of least degeneracy*. More often, we say that the choice $\alpha = \beta = \epsilon^{1/2}$ leads to the "richest" limiting form of (A.3.32), and this is

$$u^{*2} + 2 f^* y^* u^* - g^* = O(\epsilon^{1/2}) \qquad (A.3.33)$$

to leading order.

The solution of (A.3.33) for u^* gives the leading term of the asymptotic expansion of $U^\pm$ near $y = h(x)$ in the form

$$\epsilon^{1/2} U^\pm = \{-y^* f^*(x^*) \pm [y^{*2} f^{*2}(x^*) + g^*(x^*)]^{1/2}\} + O(\epsilon^{1/2}). \qquad (A.3.34)$$

Although (A.3.33) is somewhat simpler than (A.3.15), it is still a general quadratic and, strictly speaking, not any simpler to solve than (A.3.15). In a less trivial example we would, in general, not be able to solve analytically the limiting expression corresponding to (A.3.33). However, once this solution has been defined—for example, numerically—the calculation of the higher-order terms becomes straightforward. Notice again that when the u^*, x^*, y^* variables are used in (A.3.16), we obtain (A.3.34) in the limit as $\epsilon \to 0$.

A final comment about the expansion that begins in the form (A.3.34) is that this is uniformly valid for *fixed* y^* but fails to be uniform as $|y^*| \to \infty$, that is, if $|y -$

$h(x)| \to \infty$. But, if $|y - h(x)|$ is large, we are in the region where the expansions (A.3.23), (A.3.25b) are valid; these fail if $|y - h(x)|$ is small. Therefore, we have been able to derive two expansions that are uniformly valid in "complementary" subdomains of $\mathcal{D}$ but fail to be uniform in each other's subdomain. This behavior is defined more precisely when we discuss matching of asymptotic expansions in Section 8.2.

Suppose now that $g(x, y)$ is not a bounded function as $|x|$ and/or $|y| \to \infty$. We see that (A.3.23) and (A.3.25b) fail to be uniform in this limit. The source of our difficulty is that we regarded $g(x, y) = O(1)$ in our derivations, and this statement is not uniformly valid in any domain where $|g| \to \infty$. To fix these ideas, let us choose $g(x, y) = x \sin y$. We see that $g = O(x)$ as $|x| \to \infty$. Now, if $|g|$ is large, (A.3.15) implies that both roots must be large also, and this suggests the rescaling of variables

$$\widetilde{x} \equiv \gamma(\epsilon)x, \quad \widetilde{y} \equiv y, \quad \widetilde{u} \equiv \delta(\epsilon)u, \tag{A.3.35}$$

where $\gamma(\epsilon) \ll 1$, $\delta(\epsilon) \ll 1$, and $\widetilde{x}$, $\widetilde{y}$, $\widetilde{u}$ are all $O(1)$. Equation (A.3.15) then becomes

$$\frac{\epsilon}{\delta^2}\widetilde{u}^2 + \frac{1}{\delta}2f\left(\frac{\widetilde{x}}{\gamma}, \widetilde{y}\right)\widetilde{u} - \frac{1}{\gamma}\widetilde{x}\sin\widetilde{y} = 0. \tag{A.3.36a}$$

Assuming that $f(\infty, \widetilde{y})$ exists, we see that setting $\delta = \gamma = O_s(\epsilon)$, or more simply $\delta = \gamma = \epsilon$, results in the richest limiting expression for (A.3.36a) in which all three terms are in dominant balance—that is,

$$\widetilde{u}^2 + 2f(\infty, \widetilde{y})\widetilde{u} - \widetilde{x}\sin\widetilde{y} = o(1). \tag{A.3.36b}$$

Here again, (A.3.36b) is not essentially simpler than (A.3.15) but must be solved in order to compute an expansion that remains valid for $|x|$ large.

A.3.5 Asymptotic Expansion of a Definite Integral

Many applications in Chapters 1–3 involve a linear partial differential equation that can be studied using an integral transform (for example, Laplace transform or Fourier transform). The solution is then defined by an inversion integral, which often cannot be evaluated explicitly. A large body of work concerns the asymptotic expansion of such integrals using various methods such as *stationary phase* and *steepest descents*. We shall not discuss these methods here, and the reader is referred to standard texts (for example, see [8], [15], [39] and [1]). To illustrate some of the ideas, we shall first consider a simple model problem based on an example first proposed by Laplace (see also the Introduction in [15]).

(i) *Expanding the integrand*

Consider the function $f(x, y; \epsilon)$ defined by the real definite integral

$$f(x, y; \epsilon) \equiv \int_0^\infty \frac{e^{-t}}{1 + \epsilon t g(x, y)}\,dt, \tag{A.3.37}$$

where $0 < \epsilon \ll 1$ and $0 < g(x, y)$.

This integral exists and defines $f(x, y; \epsilon)$. Actually, f is a function of ϵg, and by changing variables of integration from t to $s = (1 + \epsilon g t)/\epsilon g$, we can express (A.3.37) as

$$f(\epsilon g) = \frac{e^{1/\epsilon g}}{\epsilon g} E_1\left(\frac{1}{\epsilon g}\right), \tag{A.3.38a}$$

where E_1 is the exponential integral

$$E_1(z) \equiv \int_z^{\infty} \frac{e^{-s}}{s} ds, \quad z > 0. \tag{A.3.38b}$$

The function E_1 cannot be expressed in closed form; numerical values are given in Table 5.1 of [3].

In order to approximate (A.3.37), suppose that we expand the integrand in series form for ϵ small and then integrate this series term by term. In what sense, if any, does the resulting series approximate f? We shall show that this procedure leads to a *divergent series for any fixed* ϵ. Nevertheless, this series is asymptotic as $\epsilon \to 0$, and it provides a useful approximation for f if ϵ is a small number.

We have the *exact* expansion for $1/(1 + \epsilon t g)$ to N terms:

$$\frac{1}{1 + \epsilon t g} = \sum_{n=0}^{N} (-1)^n (\epsilon t g)^n + \frac{(-1)^{N+1}(\epsilon t g)^{N+1}}{1 + \epsilon t g}. \tag{A.3.39}$$

If we now use (A.3.39) in (A.3.37) and integrate the result, we obtain the *exact* expression

$$f = \sum_{n=0}^{N} (-1)^n n! (\epsilon g)^n + R_N(\epsilon g), \tag{A.3.40a}$$

where R_N is the remainder

$$R_N(\epsilon g) = (-1)^{N+1}(\epsilon g)^{N+1} \int_0^{\infty} \frac{e^{-t} t^{N+1} dt}{1 + \epsilon t g}, \tag{A.3.40b}$$

and we have used the following identity for the gamma function of $(n + 1)$, where n is a nonnegative integer:

$$n! = \int_0^{\infty} e^{-t} t^n dt \equiv \Gamma(n + 1). \tag{A.3.40c}$$

We reiterate that for N finite, the expression in (A.3.40a) is exact. If, however, we ignore R_N and let $N \to \infty$, the series in (A.3.40a) diverges for any positive ϵ, as can be seen from the ratio test. We have

$$\left| \frac{(-1)^{n+1}(n + 1)!(\epsilon g)^{n+1}}{(-1)^n n! (\epsilon g)^n} \right| = (n + 1)\epsilon g,$$

and $\lim_{n\to\infty}(n + 1)\epsilon g = \infty$ for any $\epsilon g > 0$.

Nevertheless, it is easy to prove that the series

$$f_N(\epsilon g) \equiv \sum_{n=0}^{N}(-1)^n n!(\epsilon g)^n \qquad \text{(A.3.41)}$$

is the asymptotic expansion of f to N terms (where $N = 0, 1, 2, \ldots$) with respect to the sequence $1, \epsilon, \epsilon^2, \ldots$. To do so, we use (A.3.40) to conclude that

$$\begin{aligned}|f - f_N| &= (\epsilon g)^{N+1}\int_0^\infty \frac{e^{-t}t^{N+1}dt}{1+\epsilon t g} \\ &< (\epsilon g)^{N+1}\int_0^\infty e^{-t}t^{N+1}dt = (N+1)!(\epsilon g)^{N+1}.\end{aligned}$$

Therefore,

$$f - f_N = O(\epsilon^{N+1}) \quad \text{as } \epsilon \to 0,$$

as required in order that f_N be the asymptotic expansion of f. Moreover, if g is bounded in some domain of the xy-plane, then (A.3.41) is uniformly valid there.

The sign of the error R_N is positive if N is odd and negative if N is even. If we denote $E_N \equiv |R_N|$, we see that for any fixed $g = g_0$ and $\epsilon = \epsilon_0$, the error $E_N(\epsilon_0 g_0)$ decreases as N increases up to N equal to some integer $M(\epsilon_0 g_0)$, which depends on the value of $\epsilon_0 g_0$. For $N > M$, E_N increases with increasing N. Thus, for any given $\epsilon_0 g_0$, there is a certain minimum numerical error $E_M(\epsilon g_0)$, which is achieved by retaining M terms in (A.3.41). We cannot improve the accuracy beyond this value, and in fact, retaining more terms beyond the Mth only degrades the accuracy. We can also show that E_M decreases, whereas M increases as $\epsilon_0 g_0$ decreases.

As an illustration, we take $\epsilon = 0.1$, $g_0 = 1$ in (A.3.41) and compare our asymptotic results with the exact value $f(0.1) = 0.9156333394$ obtained from [3]. We find that $M = 9$ and $E_M = 1.7702 \times 10^{-4}$. If we take $\epsilon_0 = 0.08$, $g_0 = 1$, then $f(0.08) = 0.9304409399$. We then find $M = 11$ and $E_M = 1.544 \times 10^{-5}$.

In general, we do not need to have an exact result to determine where an asymptotic expansion begins to diverge. We need only monitor the absolute value of each successive term in the expansion; the optimal cutoff point M occurs when we reach the smallest term in absolute value.

(ii) *Repeated integrations by parts*

An alternative approach for calculating the asymptotic expansion of (A.3.37) is based on the form (A.3.38). Since for $\epsilon \to 0$ we have the argument $\lambda = 1/\epsilon g \to \infty$ in E_1, we consider the general expression

$$E_n(\lambda) \equiv \int_\lambda^\infty \frac{e^{-s}}{s^n}ds, \quad s > 0, \quad n = 1, 2, \ldots, \qquad \text{(A.3.42)}$$

and we are interested in $E_1(\lambda)$.

If we integrate (A.3.42) by parts, we see that

$$E_n(\lambda) = \frac{e^{-\lambda}}{\lambda^n} - nE_{n+1}(\lambda), \quad n = 1, 2, \ldots.$$

This is a recursion relation that can be used to define $E_1(\lambda)$ *exactly* in the form

$$E_1(\lambda) = \frac{1}{\lambda e^{\lambda}} \sum_{n=0}^{N} (-1)^n \frac{n!}{\lambda^n} + (-1)^{N+1} E_{N+1}(\lambda),$$

which is equivalent to (A.3.40).

For a second example of the use of repeated integrations by parts to generate an asymptotic expansion, consider the complementary error function defined in the form (see (A.2.77))

$$\operatorname{erfc}(\lambda) = \frac{2}{\sqrt{\pi}} \int_{\lambda}^{\infty} e^{-\zeta^2} d\zeta. \tag{A.3.43a}$$

Setting $\zeta^2 = \tau$ gives the alternative form

$$\operatorname{erfc}(\lambda) = \frac{1}{\sqrt{\pi}} \int_{\lambda^2}^{\infty} \frac{e^{-\tau} d\tau}{\tau^{1/2}}. \tag{A.3.43b}$$

Now we consider the general expression

$$F_n(\lambda) = \int_{\lambda^2}^{\infty} \frac{e^{-\tau}}{\tau^{\frac{2n+1}{2}}} d\tau, \quad n = 0, 1, 2, \ldots, \tag{A.3.44}$$

and we are interested in $F_0(\lambda)$. Integration by parts gives the recursion relation

$$F_n(\lambda) = \frac{e^{-\lambda^2}}{\lambda^{2n+1}} - \frac{2n+1}{2} F_{n+1}(\lambda), \quad n = 0, 1, \ldots. \tag{A.3.45}$$

Using this recursion relation, we derive the exact result

$$\begin{aligned} F_0(\lambda) &= e^{-\lambda^2} \sum_{n=1}^{N} (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \ldots (2n-3)}{2^{n-1} \lambda^{2n-1}} \\ &+ e^{-\lambda^2} (-1)^N \frac{1 \cdot 3 \cdot 5 \ldots (2N-1)}{2^N} F_N(\lambda), \quad N = 1, 2, \ldots. \end{aligned} \tag{A.3.46}$$

It is easy to show that the remainder term in (A.3.46) is $o(\lambda^{-(2N-1)} e^{-\lambda^2})$, and therefore the series in this equation is the asymptotic expansion of F_0. In particular, we have

$$\operatorname{erfc}(\lambda) \sim \frac{e^{-\lambda^2}}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \ldots (2n-3)}{2^{n-1} \lambda^{2n-1}}. \tag{A.3.47}$$

(iii) *Singular integrals*

In many applications a function of one or more variables defined by a definite integral approaches a critical value. This situation occurs often in the construction of matched asymptotic expansions discussed in Section 8.2. Consider first the simple example, which we revisit in Problem 8.2.4,

$$f(t) = \int_0^t \frac{\sin(t-\tau)}{[1-\sin\tau+\tau\cos\tau]^2}\,d\tau. \tag{A.3.48}$$

We see that $f(t)$ becomes singular as $t \to \pi/2$.

To compute the asymptotic expansion of $f(t)$ as $t \to \pi/2$, let us introduce the new independent variable $s \equiv \pi/2 - t$ and change the variable of integration from τ to $\sigma \equiv \pi/2 - \tau$. Equation (A.3.48) can then be written as

$$\tilde{f}(s) \equiv f\left(\frac{\pi}{2} - s\right) = \cos s \int_s^{\pi/2} \frac{\sin\sigma}{D(\sigma)}\,d\sigma - \sin s \int_s^{\pi/2} \frac{\cos\sigma}{D(\sigma)}\,d\sigma, \tag{A.3.49a}$$

where

$$D(\sigma) \equiv \left[1 - \cos\sigma + \left(\frac{\pi}{2} - \sigma\right)\sin\sigma\right]^2. \tag{A.3.49b}$$

To exhibit the singular behavior of the integrands, we expand these near $\sigma = 0$ to calculate

$$\frac{\sin\sigma}{D(\sigma)} = \frac{4}{\pi^2\sigma} + O(1) \text{ as } \sigma \to 0, \tag{A.3.50a}$$

$$\frac{\cos\sigma}{D(\sigma)} = \frac{4}{\pi^2\sigma^2} + \frac{8}{\pi^3\sigma} + O(1) \text{ as } \sigma \to 0. \tag{A.3.50b}$$

The basic idea now is to subtract the terms in (A.3.50) that become singular as $\sigma \to 0$ from the integrands in (A.3.49) and add these terms back to obtain the identity

$$\begin{aligned}\tilde{f}(s) = {} & \cos \int_s^{\pi/2} \left[\frac{\sin\sigma}{D(\sigma)} - \frac{4}{\pi^2\sigma}\right] d\sigma + \frac{4}{\pi^2}\cos s \int_s^{\pi/2} \frac{d\sigma}{\sigma} \\ & - \sin s \int_s^{\pi/2} \left[\frac{\cos\sigma}{D(\sigma)} - \frac{4}{\pi^2\sigma^2} - \frac{8}{\pi^3\sigma}\right] d\sigma \\ & - \sin s \int_s^{\pi/2} \left(\frac{4}{\pi^2\sigma^2} + \frac{8}{\pi^3\sigma}\right) d\sigma.\end{aligned} \tag{A.3.51}$$

The integrals involving D are now regular as $\sigma \to 0$, and we can evaluate the integrals that become singular as $\sigma \to 0$ to obtain

$$\begin{aligned}\tilde{f}(s) = {} & \cos s \int_s^{\pi/2} F(\sigma)d\sigma + \frac{4}{\pi^2}\left(\log\frac{\pi}{2}\right)\cos s - \frac{4}{\pi^2}(\cos s)\log s \\ & - \sin s \int_s^{\pi/2} G(\sigma)d\sigma + \frac{8}{\pi^3}\sin s - \frac{4}{\pi^2 s}\sin s \\ & - \frac{8}{\pi^3}\left(\log\frac{\pi}{2}\right)\sin s + \frac{8}{\pi^3}(\log s)\sin s,\end{aligned} \tag{A.3.52}$$

where we have introduced the notation

$$F(\sigma) \equiv \frac{\sin\sigma}{D(\sigma)} - \frac{4}{\pi^2\sigma} = O(1) \text{ as } \sigma \to 0, \tag{A.3.53a}$$

$$G(\sigma) \equiv \frac{\cos\sigma}{D(\sigma)} - \frac{4}{\pi^2\sigma^2} - \frac{8}{\pi^3\sigma} = O(1) \text{ as } \sigma \to 0. \tag{A.3.53b}$$

Equation (A.3.52) is still exact, and in a form where the integrals involving F and G are regular as $s \to 0$. Therefore, it is a straightforward exercise in algebra to expand this expression to obtain

$$\tilde{f}(s) = C_1 \log s + C_2 + C_3 s \log s + O(s) \text{ as } s \to 0, \tag{A.3.54a}$$

where $C_1 \equiv -4/\pi^2$, $C_3 \equiv 8/\pi^3$, and

$$C_2 \equiv \int_0^{\pi/2} F(\sigma)d\sigma + \frac{4}{\pi^2}\left(\log\frac{\pi}{2} - 1\right). \tag{A.3.54b}$$

As a second example, consider the potential due to a distribution of sources with strength per unit distance equal to $S(x)$ over the unit interval (see (2.4.15))

$$-4\pi u(x, r) = \int_0^1 \frac{S(\xi)}{[(x-\xi)^2 + r^2]^{1/2}} d\xi. \tag{A.3.55}$$

As $r \to 0$, u becomes singular. Other examples of this type of singular behavior are discussed in Section 2.4.4.

The standard approach is to split the interval of integration into the three intervals $(0, x-\epsilon)$, $(x-\epsilon, x+\epsilon)$, and $(x+\epsilon, 1)$, where $\epsilon(r)$ is small, $0 < \epsilon(r) \ll 1$, in order to isolate the singular contributions to u. We denote by I_1, I_2, and I_3 the contributions arising from the first, second, and third intervals, respectively, and consider I_2 first. The change of variable $\xi = x + \epsilon\sigma$ gives

$$I_2 \equiv \int_{x-\epsilon}^{x+\epsilon} \frac{S(\xi)d\xi}{\sqrt{(x-\xi)^2 + r^2}} = \int_{-1}^{1} \frac{S(x+\epsilon\sigma)}{\sqrt{\sigma^2 + (r/\epsilon)^2}} d\sigma. \tag{A.3.56}$$

We approximate this result for (r/ϵ) fixed not equal to zero by expanding S. The term proportional to $S'(x)$ is odd in σ, so it gives zero contributions, and we obtain

$$\begin{aligned} I_2 &= S(x)\int_{-1}^{1} \frac{d\sigma}{\sqrt{\sigma^2 + (r/\epsilon)^2}} + \frac{\epsilon^2}{2} S''(x) \int_{-1}^{1} \frac{\sigma^2 d\sigma}{\sqrt{\sigma^2 + (2/\epsilon)^2}} + \ldots \\ &= S(x) \log \frac{1 + \sqrt{1 + (r/\epsilon)^2}}{-1 + \sqrt{1 + (r/\epsilon)^2}} + \frac{\epsilon^2}{2} S''(x)\sqrt{1 + (r/\epsilon)^2} + \ldots . \end{aligned}$$

Now if we assume $(r/\epsilon) \to 0$, we obtain

$$I_2 = 2S(x) \log 2\left(\frac{\epsilon}{r}\right) + O(r/\epsilon)^2. \tag{A.3.57}$$

Since the denominators in I_1 and I_3 are not singular, we can expand these for small r to obtain

$$I_1 = \int_0^{x-\epsilon(r)} \frac{S(\xi)}{\sqrt{(x-\xi)^2 + r^2}}\, d\xi$$

$$= \int_0^{x-\epsilon(r)} \frac{S(\xi)}{(x-\xi)} \left[1 - \frac{r^2}{2(x-\xi)^2} + \dots\right] d\xi, \qquad \text{(A.3.58}a\text{)}$$

$$I_2 = \int_{x+\epsilon(r)}^{1} \frac{S(\xi)}{\sqrt{(x-\xi)^2 + r^2}}\, d\xi$$

$$= \int_{x+\epsilon(r)}^{1} \frac{S(\xi)}{(\xi - x)} \left[1 - \frac{r^2}{2(x-\xi)^2} + \dots\right] d\xi \qquad \text{(A.3.58}b\text{)}$$

Integrating these expressions by parts, neglecting small terms, and adding gives

$$I_1 + I_2 = -\,2S(x)\log\epsilon(r) + S(0)\log x + S(1)\log(1-x)$$
$$+ \int_0^1 S'(\xi)\,\mathrm{sgn}(x-\xi)\log|x-\xi|\,d\xi + O\left(\frac{r^2}{\epsilon^2}\right). \qquad \text{(A.3.59)}$$

We note that the $\log\epsilon$ singularities in (A.3.57) and (A.3.59) cancel, and we obtain

$$-4\pi u(x,r) = I_1 + I_2 + I_3 = -2S(x)\log r + S(0)\log 2x$$
$$+ S(1)\log 2(1-x) + \int_0^1 S'(\xi)\,\mathrm{sgn}(x-\xi)\log 2|x-\xi|\,d\xi$$
$$+ O(r^2) \text{ as } r \to 0. \qquad \text{(A.3.60)}$$

This result can be derived much more efficiently by noting that

$$[(x-\xi)^2 + r^2]^{-1/2} = -\frac{\partial}{\partial\xi}\log[x - \xi + \sqrt{(x-\xi)^2 + r^2}] \text{ if } \xi \le x,$$

$$[(x-\xi)^2 + r^2]^{-1/2} = \frac{\partial}{\partial\xi}\log[\xi - x + \sqrt{(x-\xi)^2 + r^2}] \text{ if } x \le \xi.$$

If we now split the integration interval in (A.3.55) into the two subintervals $(0, x)$ and $(x, 1)$, then use the above identities and integrate by parts, we obtain (A.3.60) directly. In general, such an identity is not available, and we have to use the first approach.

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