Math 461 - October 3, 2022 - Quiz 12

Name:______________________________

1. State the Tietze Extension Theorem. Use the letters X, and A, and f to denote the objects of the theorem.

2. Suppose that X is the set of all sequences of real numbers $x = (x_1, x_2, x_3, \ldots)$ with the metric $d(x, y) = \sup_{i\geq 1} |x_i - y_i|$. Let A be the subspace of X which consists of all convergent sequences. Define $f\colon A \to \mathbb{R}$ by $f(x) = \lim_{n\to\infty} x_n$.

If we wish to extend the definition of limit to nonconvergent sequences, in such a way that this new, "extended limit for nonconvergent sequences" agrees with the usual limit on convergent sequences and is a continuous function from X to $\mathbb{R}$, is it possible to do so? Why or why not?

3. We studied the proof of the Tietze extension theorem using the example $X = [-9, 9]$, $A = [-9, -1] \cup [1, 9]$, and $f(x) = x$ for $x \in A$. Taking $M = 9$ as the bound for $|f(x)|$ on A, the proof defines:

$$A_1 = \{x \in A \mid f(x) \geq M/3\} \qquad B_1 = \{x \in A \mid f(x) \leq -M/3\}$$

and then defines $g_1\colon X \to [-M/3, M/3]$ by

$$g_1(x) = \begin{cases} M/3, & x \in A_1 \\ -M/3 & x \in B_1 \end{cases}$$

and $g_1(x)$ is defined for $x \in X - (A_1 \cup B_1)$ by Lemma 4.6. Write down the sets A_1 and B_1 explicitly and graph the function $g_1\colon X \to \mathbb{R}$.