Solutions to Review Questions, Exam 1

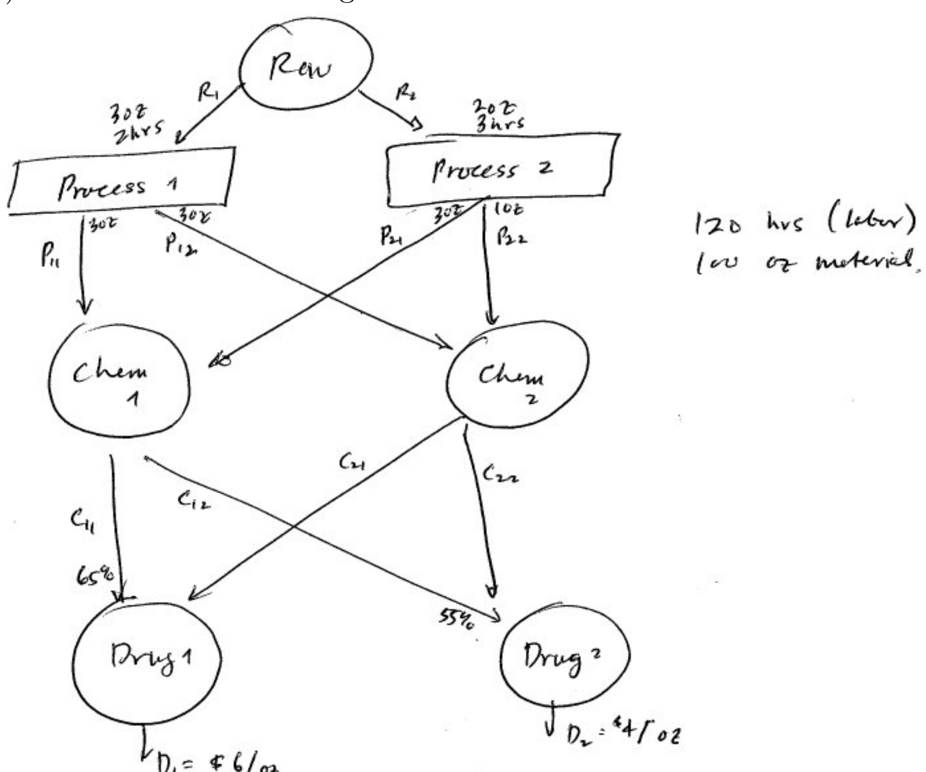
1. What are the four possible outcomes when solving a linear program? Hint: The first is that there is a unique solution to the LP.

   SOLUTION:

   - No solution - The feasible set is empty.
   - A unique solution (either with or without an unbounded feasible set).
   - An unbounded solution - The feasible set is unbounded.
   - An infinite number of solutions - Either by an unbounded set or the isoprofit lines are coincident with a boundary at the optimum.

2. The following are to be sure you understand the process of constructing a linear program:

   (a) Draw a production process diagram and set up the LP for Exercise 6, p. 98 (Sect. 3.9) SOLUTION: See the figure below.



opt: $\max\ 6D_1 + 4D_2$

Total time: $2R_1 + 3R_2 \le 120$

Total oz: $R_1 + R_2 \le 100$

Process 1: $3P_{11} + 3P_{12} = 3R_1$

Process 2: $3P_{21} + P_{22} = 2R_2$

Chem 1, 2: $C_{11} + C_{12} = P_{11} + P_{21}, \quad C_{21} + C_{22} = P_{12} + P_{22}$

Drug 1, 2: $D_1 = C_{11} + C_{21}, \quad D_2 = C_{12} + C_{22}$

$C_{11}/D_1 \ge 0.65, \quad C_{12}/D_2 \ge 0.55.$ ☒

(b) Exercise 2, 31 Chapter 3 review

SOLUTIONS: Be sure you can do these graphically.

- Solution to exercise 2: The optimal value is 69/7, where we have 36/7 chocolate cake and 66/7 vanilla (yes, the "divisibility" assumption is violated here).
- Solution to exercise 31: The LP is unbounded (no solution).

(c) Exercise 6, 18 Chapter 3 review (A ton is 2000 lbs)

SOLUTIONS: Be sure you introduce your variables!

6. Let x_1 be the pounds of Alloy 1 used to produce one ton of steel and x_2 be the pounds of Alloy 2. Then the objective function is:

$$\min z = 190/2000x_1 + 200/2000x_2$$

With:

- Carbon constraints:

$$0.03x_1 + 0.04x_2 \geq (0.032)(2000) \qquad 0.03x_1 + 0.04x_2 \leq (0.035)(2000)$$

- Silicon:

$$0.02x_1 + 0.025x_2 \geq (0.018)(2000) \qquad 0.02x_1 + 0.025x_2 \leq (0.025)(2000)$$

- Lastly, nickel:

$$0.01x_1 + 0.015x_2 \geq 18 \qquad 0.01x_1 + 0.015x_2 \leq 24$$

- Tensile strength:

$$\frac{42,000x_1 + 50,000x_2}{2,000} \geq 45,000$$

- Relationship between variables: $x_1 + x_2 = 2000$
- Non-negative: $x_{1,2} \geq 0$.

- An interesting application of "blending"- The solution is in the back of the text.

(d) Exercise 22, Chapter 3 review. Hint: Consider using a triple index on your variables.

SOLUTION:

Let x_{ijk} is the units of product 1, machine i, month j, for sale in month k.

Let y_{ijk} is the units of product 2, machine i, month j, for sale in month k.

In order to simplify things, we note some quantities that are useful:

- Amount of Product 1 for sale in Month 1: $x_{111} + x_{211}$
- Amount of Product 2 for sale in Month 1: $y_{111} + y_{211}$
- Amount of Product 1 for sale in Month 2: $x_{112} + x_{212} + x_{122} + x_{222}$
- Amount of Product 2 for sale in Month 2: $y_{112} + y_{212} + y_{122} + y_{222}$

Then we have the objective function to maximize:
$55(x_{111}+x_{211})+12(x_{112}+x_{212}+x_{122}+x_{222})+65(y_{111}+y_{211})+32(y_{112}+y_{212}+y_{122}+y_{222})$
For constraints, here are Machine 1 hour constraints (for Months 1, 2):

$$4(x_{111}+x_{112})+7(y_{111}+y_{112})\le 500 \qquad 4x_{122}+7y_{122}\le 500$$

Similarly, for Machine 2:

$$3(x_{211}+x_{212})+4(y_{211}+y_{212})\le 500 \qquad 3x_{222}+4y_{122}\le 500$$

Sales constraints (Month 1, then Month 2):

$$x_{111}+x_{211}\le 100 \qquad y_{111}+y_{211}\le 140$$

$$x_{112}+x_{212}+x_{122}+x_{222}\le 190 \qquad y_{112}+y_{212}+y_{122}+y_{222}\le 130$$

Also, all variables are non-negative.

(e) Exercise 47, 53 in Chapter 3 review.
SOLUTIONS: For Exercise 47, see the back of the book.
SOLUTION, Exercise 53: Here is a list of the variables:

- $T_{1,2}$: Number of Type 1 and 2 Turkeys that are purchased
- $D_{1,2}$: Pounds of dark meat used in Cutlet 1, 2 (resp)
- $W_{1,2}$: Pounds of white meat used in Cutlet 1, 2 (resp)

The objective function is to maximize: $4(W_1+D_1)+3(W_2+D_2)-10T_1-8T_2$
Here are the constraints:

- (Cutlet 1 demand) $W_1+D_1\le 50$
- (Cutlet 2 demand) $W_2+D_2\le 30$
- (Don't use more white meat than you have) $W_1+W_2\le 5T_1+3T_2$
- (Don't use more dark meat than you have) $D_1+D_2\le 2T_1+3T_2$
- (70% white meat) $W_1/(W_1+D_1)\ge 0.7$
- (60% white meat) $W_2/((W_2+D_2)\ge 0.6$
- $T_{1,2}$, $D_{1,2}$, and $W_{1,2}$ are all non-negative.

3. Convert the following LP to one in standard form. Write the result in matrix-vector form, giving $\mathbf{x}$, $\mathbf{c}$, A, $\mathbf{b}$ (from our formulation).

$$\begin{aligned} \min z = \;& 3x-4y+2z \\ \text{st}\;\; & 2x-4y\ge 4 \\ & x+z\ge -5 \\ & y+z\le 1 \\ & x+y+z=3 \end{aligned}$$

with $x\ge 0, y$ is URS, $z\ge 0$.
SOLUTUION: Let $\mathbf{x}=[x,y^+,y^-,z,e_1,s_1,s_2]^T$. Then

$$\mathbf{c}=[3,-4,4,2,0,0,0]^T \qquad A=\begin{bmatrix} 2 & -4 & 4 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 1 \\ 1 & 1 & -1 & 1 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{b}=\begin{bmatrix} 4 \\ 5 \\ 1 \\ 3 \end{bmatrix}$$

4. Consider again the "Wyndoor" company example we looked at in class:

$$\begin{array}{rl} \min z = & 3x_1 + 5x_2 \\ \text{st} & x_1 \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 18 \end{array}$$

with x_1, x_2 both non-negative.

(a) Rewrite so that it is in standard form.
SOLUTION:
Define the extra variables x_3, x_4, x_5.
Using the extra variables in order, the constraints become:

$$\begin{array}{cccccc} x_1+ & & x_3 & & & = 4 \\ & 2x_2+ & & x_4 & & = 12 \\ 3x_1+ & 2x_2+ & & & x_5 & = 18 \end{array}$$

And from this, it is easy to read off the coefficient matrix A.

(b) Let s_1, s_2, s_3 be the extra variables introduced in the last answer. Is the following a basic solution? Is it a basic feasible solution?

$$x_1 = 0, x_2 = 6, s_1 = 4, s_2 = 0, s_3 = 6$$

Which variables are BV, and which are NBV?
SOLUTION: The matrix A has rank 3. If the solution has $n - m = 5 - 3 = 2$ zeros (and it is a solution), then it is a basic solution: Yes, this is a basic solution. It is also a basic *feasible* solution since every entry of the basic solution is non-negative. The variables x_2, x_3 and x_5 are the basic variables (BV) and the variables x_1 and x_4 are NBV.

(c) Find the basic feasible solution obtained by taking s_1, s_3 as the non-basic variables.
In this case, we can row reduce the augmented matrix (remove columns 3 and 5 from the original):

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 2 & 1 & 12 \\ 3 & 2 & 0 & 18 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

In this case, we have the (full) solution:

$$x_1 = 4, \quad x_2 = 3, \quad x_3 = 0, \quad x_4 = 6, \quad x_5 = 0$$

5. Consider Figure 1, with points $A(1,1), B(1,4)$ and $C(6,3)$, $D(4,2)$ and $E(4,3)$.

- Write the point E as a convex combination of points A, B and C.
  SOLUTION: First we'll find the point of intersection between line $\overline{AE}$ and $\overline{BC}$. Call it E'. We found it to be $E'\left(\frac{58}{13}, \frac{43}{13}\right)$ (Sorry about the fractions!).
  By the time we're done, you should have:

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \frac{2}{15}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{13}{15}\left(\frac{4}{13}\begin{bmatrix} 1 \\ 4 \end{bmatrix} + \frac{9}{13}\begin{bmatrix} 6 \\ 3 \end{bmatrix}\right)$$

- Can E be written as a convex combination of A, B and D? If so, construct it.
  SOLUTION: No. The point E is above the convex hull of A, B and D (which is the triangle whose vertices are at A, B, D).
- Can A be written as a *linear* combination of A, B and D? If so, construct it.
  SOLUTION: Obvious typo there- I meant to say E can be written ...
  Using E, we can set up the matrix and solve:

$$\left[\begin{array}{ccc|c} 1 & 1 & 4 & 4 \\ 1 & 4 & 3 & 2 \end{array}\right] \quad \Rightarrow$$

  If the coefficients for the linear combination are c_1, c_2, c_3, we find them to be:

$$\begin{array}{rcl} C_1 & = & \frac{14}{3} - \frac{13}{3}C_3 \\ C_2 & = & -\frac{2}{3} + \frac{1}{3}C_3 \\ C_3 & = & C_3 \end{array}$$

  Therefore, there are an infinite number of ways to make this linear combination (which was expected, since three vectors in $\mathbb{R}^2$ are not linearly independent).

6. Draw the feasible set corresponding to the following inequalities:

$$x_1 + x_2 \leq 6, \quad x_1 - x_2 \leq 2 \quad x_1 \leq 3, \quad x_2 \leq 6$$

with x_1, x_2 non-negative.

   (a) Find the set of extreme points.
   SOLUTION: $(0,0)$, $(0,6)$, $(2,0)$, $(3,3)$, $(3,1)$.

   (b) Write the vector $[1,1]^T$ as a convex combination of the extreme points.
   SOLUTION: Since I get to choose, let's make it easy:

$$\left[\begin{array}{c} 1 \\ 1 \end{array}\right] = \frac{1}{3}\left[\begin{array}{c} 3 \\ 3 \end{array}\right] + \frac{2}{3}\left[\begin{array}{c} 0 \\ 0 \end{array}\right] + 0\left[\begin{array}{c} 0 \\ 6 \end{array}\right] + 0\left[\begin{array}{c} 2 \\ 0 \end{array}\right] + 0\left[\begin{array}{c} 3 \\ 1 \end{array}\right]$$

   Or, a little more complex:

$$\left[\begin{array}{c} 1 \\ 1 \end{array}\right] = \frac{1}{6}\left[\begin{array}{c} 0 \\ 6 \end{array}\right] + \frac{1}{2}\left[\begin{array}{c} 2 \\ 0 \end{array}\right] + \frac{1}{3}\left[\begin{array}{c} 0 \\ 0 \end{array}\right] + 0\left[\begin{array}{c} 3 \\ 3 \end{array}\right] + 0\left[\begin{array}{c} 3 \\ 1 \end{array}\right]$$

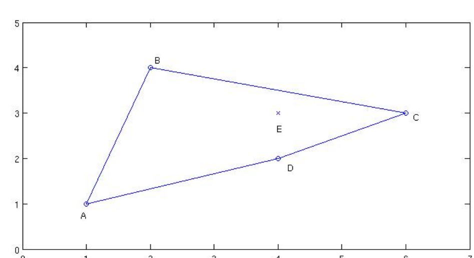


Figure 1: Figure for the convex combinations, Exercise 5.

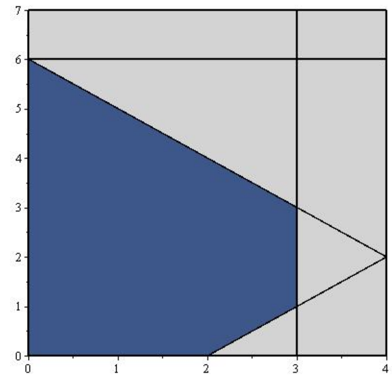


Figure 2: Figure for Question 10

(c) True or False: The extreme points of the region can be found by making exactly two of the constraints binding, then solve.

SOLUTION: If we follow this recipe, we will get extreme points, but we'll also get non-feasible points (for example, the point $(3, 6)$). Therefore, FALSE.

(d) If the objective function is to maximize $2x_1 + x_2$, then (a) how might I change that into a minimization problem, and (b) solve it.

SOLUTION: For part (a), we convert it by minimizing $-z$, or $\min -2x_1 - x_2$. For part (b), solve it graphically to get that the maximum occurs at $(3, 3)$ and the maximum is 9.

7. Consider the unbounded feasible region defined by

$$x_1 - 2x_2 \leq 4, \qquad -x_1 + x_2 \leq 3$$

with x_1, x_2 non-negative. Consider the vector $\mathbf{p} = [5, 2]$.

(a) Show that $\mathbf{p}$ is in the feasible region.

SOLUTION: Substitute the values into the constraints to see that they are both valid.

(b) Set up the system you would solve in order to write $\mathbf{p}$ in the form given in Theorem 2 (provide a specific vector $\mathbf{d}$).

SOLUTION: Directions of unboundedness can have "slopes" between 1/2 and 1, so we could choose $\mathbf{d} = [1, 1]^T$. But then $[5, 2]^T - [1, 1]^T$ is not in the convex hull of the vertices, so we can make $\mathbf{d} = [2, 2]^T$. Therefore,

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \sigma_1 \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \sigma_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \sigma_3 \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

And we could go ahead and solve "by inspection", getting $\sigma_1 = 3/4$, $\sigma_2 = 0$ and $\sigma_3 = 1/4$.

8. Finish the definition: Two basic feasible solutions are said to be **adjacent** if:

SOLUTION: Two basic feasible solutions are adjacent if they share all but one basic variable.

9. Let $\mathbf{d}$ be a direction of unboundedness. Using the *definition*, prove that this means that $r\mathbf{d}$ is also a direction of unboundedness, for any constant $r \geq 0$.

   SOLUTION: We assume an LP in standard form, so our set $S = \{\mathbf{x}|A\mathbf{x} = \mathbf{b}\}$. Then, $\mathbf{d} \neq 0$ is a direction of unboundedness for S if $\mathbf{x} + \lambda\mathbf{d} \in S$ for all $\mathbf{x} \in S$ and $\lambda \geq 0$.

   Therefore, in what is given, we can let $\mathbf{u} = r\mathbf{d}$ and show that $\mathbf{u}$ is a direction of unboundedness:

   Let $\mathbf{x}$ be any point of S and $\lambda \geq 0$. Then:

   $$\mathbf{x} + \lambda\mathbf{u} = \mathbf{x} + (\lambda r)\,\mathbf{d}$$

   which must be in S since $\mathbf{d}$ was a direction of unboundedness.

10. If C is a convex set, then $\mathbf{d} \neq 0$ is a direction of unboundedness for C iff $\mathbf{x} + d \in C$ for all $\mathbf{x} \in C$ (Use the *definition* of unboundedness).

    SOLUTION: We have two directions-

    - $\mathbf{d} \neq 0$ is a direction of unboundedness for C implies $\mathbf{x} + d \in C$ is trivially true, since we can just make $\lambda = 1$.
    - We now show that, if $\mathbf{x}+\mathbf{d} \in C$ for all $\mathbf{x} \in C$, then $\mathbf{d}$ is a direction of unboundedness for C:

      Let $\mathbf{x}_0$ be any point in S, and $\lambda \geq 0$. Then show that $\mathbf{x}_0 + \lambda\mathbf{d} \in S$.

      Let $\lambda = N + \alpha$, where N is a non-negative integer, and $0 \leq \alpha < 1$. Then

      $$\mathbf{x}_0 + \lambda\mathbf{d} = \mathbf{x}_0 + (N + \alpha)\mathbf{d} = (\mathbf{x}_0 + \mathbf{d}) + ((N-1) + \alpha)\mathbf{d}$$

      Now, since $\mathbf{x}_1 = \mathbf{x}_0 + \mathbf{d}$, then $\mathbf{x}_1 \in C$, we can write this as:

      $$\mathbf{x}_1 + \mathbf{d} + ((N-2) + \alpha)\mathbf{d}$$

      and so on. Therefore, we have that $\mathbf{x}_0 + N\mathbf{d} \in S$ Finally, since $\mathbf{x}_0 + N\mathbf{d} + \mathbf{d} \in C$, then because C is convex, so will the vector $\mathbf{x}_0 + N\mathbf{d} + \alpha\mathbf{d} \in C$.

11. For an LP in standard form (see above), prove that the vector $\mathbf{d}$ is a direction of unboundedness iff $A\mathbf{d} = 0$ and $\mathbf{d} \geq 0$.

    Solution:

    - Show that if $A\mathbf{d} = \mathbf{0}$, with $\mathbf{d} \geq 0$, then $\mathbf{d}$ is a direction of unboundedness.

      Note that this means we have to show that $\mathbf{y} = \mathbf{x} + \lambda\mathbf{d} \in S$ for every λ. Let $\mathbf{x}$ be in the feasible set, $\mathbf{x} \in S$ so that $\mathbf{x} \geq 0$. Now,

      $$A\mathbf{y} = A\left(\mathbf{x} + \lambda\mathbf{d}\right) = A\mathbf{x} + \lambda A\mathbf{d} = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

      Wait! We're not done- Check that $\mathbf{y} \geq 0$ (it is since $\lambda, x, d \geq 0$).
    - Now go in the reverse: Suppose that we know that $\mathbf{d}$ is a direction of unboundedness. We show that $A\mathbf{d} = \mathbf{0}$ and $\mathbf{d} \geq 0$.

Let $\mathbf{x}$ be in the feasible set. One path we could take is to suppose that, by way of contradiction, that $A\mathbf{d} \neq \mathbf{0}$. Then

$$A(\mathbf{x} + \lambda\mathbf{d}) = A\mathbf{x} + \lambda A\mathbf{d} = \mathbf{b} + \lambda\mathbf{k} \neq \mathbf{b}$$

But then $\mathbf{x} + \lambda\mathbf{d}$ is not an element of S (contradiction).

The other part: Is $\mathbf{d} \geq 0$? If not, then at least one coordinate $d_i < 0$. But then it is possible to find λ so that the i^{th} coordinate of $\mathbf{x} + \lambda\mathbf{d}$ is negative (contradiction).

12. Show that the set of optimal solutions to an LP (assume in standard form) is convex.

SOLUTION: Define $S = \left\{\mathbf{x} | A\mathbf{x} = \mathbf{b},\ \mathbf{c}^T\mathbf{x} = L\right\}$ Now, let $\mathbf{y}_1, \mathbf{y}_2 \in S$. We show that all points on the line segment between them is also in S. Let $\mathbf{y}$ be a point between- Then there is a $0 \leq t \leq 1$ so that:

$$\mathbf{y} = t\mathbf{y}_1 + (1-t)\mathbf{y}_2$$

Now, $\mathbf{y}$ is also feasible, since

$$A\mathbf{y} = tA\mathbf{y}_1 + (1-t)A\mathbf{y}_2 = t\mathbf{b} + (1-t)\mathbf{b} = \mathbf{b}$$

And $\mathbf{y}$ will give the same optimal value,

$$\mathbf{c}^T\mathbf{y} = t\mathbf{c}^T\mathbf{y}_1 + (1-t)\mathbf{c}^T\mathbf{y}_2 = tL + (1-t)L = L$$

13. Let a feasible region be defined by the system of inequalities below:

$$\begin{array}{rl} -x_1 + 2x_2 & \leq 6 \\ -x_1 + x_2 & \leq 2 \\ x_2 & \geq 1 \\ x_1, x_2 \geq 0 & \end{array}$$

The point $(4,3)$ is in the feasible region. Find vectors $\mathbf{d}$ and $\mathbf{b}_1, \cdots \mathbf{b}_k$ and constants σ_i so that the Representation Theorem is satisfied (NOTE: Your vector $\mathbf{x}$ from that theorem is more than two dimensional).

SOLUTION: Graphing the region in 2-d, we see that the extreme points are:

$$\mathbf{b}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

And $\mathbf{d}$ can be any vector pointing outwards with a slope between 0 and 1/2. The easiest method to get the representation is to "aim backwards" at an extreme point, but using a vector that will be an allowable $\mathbf{d}$. In this case, we can write:

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Using the matrix A from the LP in standard form (with variables in order: x_1, x_2, s_1, s_2, e_1), we have:

$$A = \begin{bmatrix} -1 & 2 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{bmatrix}$$

From this, it is easy to solve for the remaining dimensions:

$$A\mathbf{x} = \mathbf{b} \quad \Rightarrow \quad \begin{array}{ll} s_1 & = 6 + x_1 - 2x_2 \\ s_2 & = 2 + x_1 - x_2 \\ e_1 & = -1 + x_2 \end{array}$$

And for the vector in the null space, replace $(6, 2, 1)$ by $(0, 0, 0)$. Therefore,

$$\begin{bmatrix} 4 \\ 3 \\ 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

14. Let a feasible region be defined by the system of inequalities below:

$$\begin{array}{rl} -x_1 + x_2 & \leq 2 \\ x_1 - x_2 & \leq 1 \\ x_1 + x_2 & \leq 5 \\ x_1, x_2 \geq 0 & \end{array}$$

The point $(2, 2)$ is in the feasible region. Find vectors $\mathbf{d}$ and $\mathbf{b}_1, \cdots \mathbf{b}_k$ and constants σ_i so that the Representation Theorem is satisfied (NOTE: Your vector $\mathbf{x}$ from that theorem is more than two dimensional).

SOLUTION: The point given is between two extreme points, $[0, 2]^T$ and $[3, 2]^T$. Therefore, in two dimensions we have

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = t \begin{bmatrix} 0 \\ 2 \end{bmatrix} + (1 - t) \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \Rightarrow \quad t = \frac{1}{3}$$

We also get the matrix A in standard form, with column variables (in order): x_1, x_2, s_1, s_2, s_3, and

$$A = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{ll} s_1 & = x_1 - x_2 + 2 \\ s_2 & = -x_1 + x_2 + 1 \\ s_3 & = -x_1 - x_2 + 5 \end{array}$$

From which we get:

$$\begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 0 \\ 2 \\ 0 \\ 3 \\ 3 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} 3 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

15. Suppose that $\lambda_1 \leq \lambda_2 \leq \cdots \lambda_n$, and let $\sigma_1, \cdots, \sigma_n$ be non-negative constants so that $\sum_{i=1}^{n} \sigma_i = 1$. Show that

$$\lambda_1 \leq \sigma_1 \lambda_1 + \sigma_2 \lambda_2 + \cdots \sigma_n \lambda_n \leq \lambda_n$$

SOLUTION:

$$\sigma_1 \lambda_1 + \sigma_2 \lambda_2 + \cdots \sigma_n \lambda_n \leq \sigma_1 \lambda_n + \sigma_2 \lambda_n + \cdots \sigma_n \lambda_n = (\lambda_n) \sum_i \sigma_i = \lambda_n$$

(A similar proof works the other way, too)

16. Show that, if $\mathbf{x}$ is in the convex hull of vectors $\mathbf{b}_1, \cdots \mathbf{b}_k$, then for any constant vector $\mathbf{c}$,

$$\mathbf{c}^T\mathbf{x} \le \max_i \left\{\mathbf{c}^T\mathbf{b}_i\right\}$$

SOLUTION: If $\mathbf{x}$ is in the convex hull of the vectors $\mathbf{b}_i$, then we can write $\mathbf{x}$ as a convex combination of them:

$$\mathbf{x} = \alpha_1\mathbf{b}_1 + \cdots + \alpha_k\mathbf{b}_k$$

Taking the dot product of both sides with $\mathbf{c}$, we have:

$$\mathbf{c}^T\mathbf{x} = \alpha_1(\mathbf{c}^T\mathbf{b}_1) + \cdots \alpha_k(\mathbf{c}^T\mathbf{b}_k)$$

which is the set up to the previous problem with $\lambda_j = \mathbf{c}^T\mathbf{b}_j$. By that exercise, we know that

$$\mathbf{c}^T\mathbf{x} = \alpha_1(\mathbf{c}^T\mathbf{b}_1) + \cdots \alpha_k(\mathbf{c}^T\mathbf{b}_k) \le \max_i \left\{\mathbf{c}^T\mathbf{b}_i\right\}$$

SIDE REMARK: Notice that this is a key element of the fundamental theorem of linear programming- The maximum and minimum of the objective function are attained at extreme points.

17. True or False, and explain: The Simplex Method will always choose a basic feasible solution that is **adjacent** to the current BFS.

SOLUTION: That is true. It is because we will only replace one of the current basic variables with a new variable, therefore, the new BFS will keep all but one of the current set of basic variables.

18. Given the current tableau (with variables labeled above the respective columns), answer the questions below.

x_1	x_2	s_1	s_2	rhs
0	-1	0	2	24
0	$1/3$	1	$-1/3$	1
1	$2/3$	0	$1/3$	4

(a) Is the tableau optimal (and did your answer depend on whether we are maximizing or minimizing)? For the remaining questions, you may assume we are maximizing.
ANSWER: This tableau is not optimal for either. If we were minimizing, we could still pivot using s_2. If we were maximizing, we could still pivot in x_2.

(b) Give the current BFS.
ANSWER: The current BFS is $x_1 = 4, x_2 = 0, s_1 = 1$ and $s_2 = 0$.

(c) Directly from the tableau, can I increase x_2 from 0 to 1 and remain feasible? Can I increase it to 4?
ANSWER: From the ratio test, x_2 can be increased to 3 in the first, and 6 in the second. However, increasing it to 4 would violate the first constraint. Summary: I can increase x_2 from 0 to 1, but not to 4.

(d) If x_2 is increased from 0 to 1, compute the new value of z, x_1, s_1 (assuming s_2 stays zero).
SOLUTION:

$$z = 25 \qquad x_1 = \frac{10}{3} \qquad s_1 = \frac{2}{3}$$

(e) Write the objective function and all variables in terms of the non-basic (or free) variables, and then put them in vector form.

SOLUTION: For the current tableau, $z = 24 + x_2 - 2s_2$, with

$$\begin{aligned} x_1 &= 4 - 2/3x_2 - 1/3s_2 \\ x_2 &= \quad\quad\quad x_2 \\ s_1 &= 1 - 1/3x_2 + 1/3s_2 \\ s_2 &= \quad\quad\quad\quad\quad\quad s_2 \end{aligned} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \frac{x_2}{3} \begin{bmatrix} -2 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \frac{s_2}{3} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$