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"A COMPUTATIONAL APPROACH TO CLASSIFICATION OF ADDITIVE SMOOTH FANO POLYTOPES"

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To the friends and family that were made along the way,
these 6 years of college.

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ABSTRACT

Abstract— Additive smooth Fano varieties show very useful arithmetic properties [Hassett and Tschinkel, 1999]. Although such varieties are hard to construct, a criterion from toric geometry may be used to this effect [Arzhantsev and Romaskevich, 2017]. Their classification has been completed up to dimension 3 through geometric means [Huang and Montero, 2020], but not for higher dimensions, due to clear theoretical limitations. There's another recent criterion to decide whether an additive action on a variety is unique (modulo isomorphism to a normalised action) or not [Dzhunusov, 2022]. Both criteria are combinatorial in essence, and lend themselves to an algorithmic approach to classifying these varieties in the toric case.

We will design and implement these algorithms to classify these varieties for any dimension. We will also present a package that includes these functionalities in Macaulay2, a computational algebraic geometry software system [Grayson et al., 1993].

Keywords— Additive varieties; Toric varieties; Algebraic geometry; Polytopes; Macaulay2.

RESUMEN

***Resumen*—** Las variedades de Fano suaves y aditivas poseen propiedades aritméticas muy deseables [Hassett and Tschinkel, 1999]. Si bien son difíciles de construir, existen criterios para hacerlo usando geometría tórica [Arzhantsev and Romaskevich, 2017]. La clasificación de estas variedades por métodos geométricos ha sido completada hasta dimensión 3 [Huang and Montero, 2020], pero no en dimensiones superiores, debido a las limitaciones de los métodos teóricos. Existe otro criterio reciente que permite decidir si una acción aditiva sobre una variedad es única (módulo isomorfismo con una acción normalizada) o no [Dzhunusov, 2022]. Ambos criterios son, en esencia, combinatorios, y permiten diseñar algoritmos para obtener esta clasificación en el caso tórico.

Diseñaremos e implementaremos estos algoritmos para clasificar estas variedades en dimensión arbitraria. También presentaremos una extensión que incluye estas funcionalidades en el paquete de geometría algebraica computacional Macaulay2 [Grayson et al., 1993].

***Palabras clave*—** Variedades aditivas; Variedades tóricas; Geometría algebraica; Polítopos; Macaulay2.

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INTRODUCTION

Let K be an algebraically closed field, $\mathbb{G}_a = (K, +)$ its additive group, and X an irreducible algebraic variety of dimension n over K. An *additive action* is an effective regular action of the commutative unipotent group $\mathbb{G}_a^n$ on X with an open orbit. We say X is *additive* if it admits an additive action.

Let X be a complete normal variety. We say X is *Gorenstein Fano* is the anticanonical divisor $-K_X$ is Cartier and ample. We say X is *smooth Fano* if it is Gorenstein Fano and smooth.

Toric geometry provides methods to construct algebraic varieties from combinatorial objects, specifically cones and polytopes. There is a correspondence between isomorphism classes of projective (resp., smooth Fano) toric varieties and very ample (resp., smooth Fano) polytopes, and only finitely many such classes of smooth Fano polytopes.

Additive smooth Fano varieties, introduced by [Hassett and Tschinkel, 1999], show very useful arithmetic properties. Additive smooth Fano polytopes have been classified by [Huang and Montero, 2020] up to dimension 3, but not for higher dimensions, due to clear theoretical limitations. Recent results by [Arzhantsev and Romaskevich, 2017], [Dzhunusov, 2022] provide criteria for existence and uniqueness of additive actions on (complete) projective toric varieties, which are combinatorial in essence, and lend themselves to an algorithmic approach to classifying these varieties.

In section 2, we give a brief (but sufficient) introduction to toric geometry, polytope theory, the classification of smooth Fano polytopes, and additive actions on smooth Fano varieties.

In section 3, we use web-scraping on the Graded Ring Database (GRDB) to obtain a database (provided by [Øbro, 2007]) of all smooth Fano polytopes of dimensions 2 to 6. We describe an algorithm to obtain the edges of a polytope that contain a given vertex. Then, we describe algorithms that solve the decision problem of existence and uniqueness of additive actions on (complete) projective toric varieties. We give our results and analyse them qualitatively. Finally, we present a new Macaulay2 package which includes some of these functionalities.

Last, but not least, in section 4, we validate our results by matching them with existing classifications in the literature. We also study the rational cohomology of smooth Fano toric varieties.

SECTION 1
PROBLEM STATEMENT

The asymptotic distribution of rational solutions to polynomial equations is a topic that has been extensively studied in number theory and geometry. Historically, early approaches date back to Diophantus of Alexandria (3th century B. C.) in antiquity. More recently, *and being intentionally vague, in the spirit of painting the big picture*, problems of this class have lent themselves to the same kind of heavy geometric machinery that has been used to prove statements as famous as Fermat's last theorem.

Important developments in the area were pioneered by [Chambert-Loir and Tschinkel, 2002], which led to the discovery of a class of geometric objects with excellent arithmetic properties: additive smooth Fano varieties [Hassett and Tschinkel, 1999]. However, as the very authors observe, these are, in general, very difficult to construct. More recently, [Arzhantsev and Romaskevich, 2017] give a criterion to construct these geometric objects using the methods of toric geometry.

Toric geometry is concerned with the study of toric varieties (in technical terms, algebraic varieties that contain an algebraic torus as a dense open subset), and is a great source of examples, models, and perspectives with which to test many general theories and study objects showing up in other areas of mathematics [Brasselet, 2001]. On the same lines, toric geometry is one of the main bridges joining algebraic geometry and combinatorics. Here, the study of polytopes and their points with integer coordinates plays an especially relevant role. Indeed, there is a correspondence between isomorphism classes of projective (resp., smooth Fano) toric varieties and very ample (resp., smooth Fano) polytopes.

There are only finitely many classes of smooth Fano polytopes, which were obtained explicitly by [Mori and Mukai, 2003] and [Batyrev, 1999] up to dimension 3 and 4, respectively, and later the SFP algorithm was presented by [Øbro, 2007] to do so for any dimension. Using methods from algebraic geometry, all classes of smooth Fano polytopes have been classified as additive and non-additive by [Huang and Montero, 2020] up to dimension 3, but not for higher dimensions, due to clear theoretical limitations.

Another criterion was introduced by [Dzhunusov, 2022] to determine whether an additive action on a complete projective toric variety is unique (modulo isomorphism to a normalised action) or not. Arzhantsev, Romaskevich and Dzhunusov's criteria are combinatorial in essence, and lend themselves to an algorithmic approach to computationally classify additive smooth Fano polytopes of any dimension. We are not aware of any heuristics that may be used to tell with ease whether a high-dimensional smooth Fano polytope is (uniquely) additive or not, thus the associated decision problems are naturally of interest.

Among mathematical circles, it is commonplace to perform algebraic and geometric calculations in Macaulay2, a "software system devoted to supporting research in algebraic geometry and commutative algebra [...]" [Grayson et al., 1993] written in C/C++ that includes its own homonymous interpreted language. It is possible to propose new packages to be distributed on Macaulay2's official repository. Therefore, implementing the solutions to the present problem in this software system may prove to be useful for the mathematical community at large.

The author and his thesis advisor hope that this work, a posteriori, helps with developing more intuition about the geometry of additive smooth Fano varieties. Nowadays, due to the tiny number of explicit examples, this intuition is scarce.

1.1 Objectives

1.1.1 General objectives

To design and implement algorithms that apply the criteria from [Arzhantsev and Romaskevich, 2017] and [Dzhunusov, 2022] to classify additive smooth Fano toric varieties of any dimension, or up to the highest feasible dimension, thus generalising theoretical classification results obtained for small dimensions. Then, to follow guidelines indicated by [Grayson et al., 1993] to submit a new package with these functionalities to Macaulay2, a computational algebraic geometry software system.

1.1.2 Specific objectives

1. To analyse and synthesise existing relevant literature on toric varieties and Fano varieties, and to identify combinatorial aspects in the calculations needed for their classification.
2. To design algorithms that may be used to classify additive smooth Fano polytopes of any dimension, or up to the highest feasible dimension. This algorithm must be able to systematically replicate known results, and be able to classify new additive smooth Fano toric varieties of dimension 4 or higher.
3. To first implement these algorithms in any general-purpose programming language, and later (at least the algorithm related to [Arzhantsev and Romaskevich, 2017]) according to the standards of Macaulay2, a computational algebraic geometry software system, in order to submit a new package that may eventually be distributed on its official repository.
4. To classify new additive smooth Fano toric varieties of dimensions 4 to 6 using these implementations, as they are difficult to classify by theoretical means. To obtain, as a result, a list of all additive smooth Fano polytopes of dimensions 4 to 6.

SECTION 2
PRELIMINARIES

2.1 A brief introduction to toric geometry

The natural starting point of this thesis is a short review of the basic theory of toric varieties, from the construction in the affine case to the more general definition, with the only prerrequisite being a basic knowledge of algebraic geometry. We follow [Brasselet, 2001].

The study of toric varieties lies in the intersection of combinatorics and algebraic geometry, and is full of explicit back-and-forth interactions between the two. [Brasselet, 2001] quotes [Fulton, 1993]: ”Toric varieties provide a [...] way to see many examples and phenomena in algebraic geometry... [... They] have provided a remarkably fertile testing ground for general theories.“ In particular, this thesis is concerned with a specific type of (projective) toric varieties: those constructed from smooth Fano polytopes.

Let N, M be dual lattices with associated vector spaces $N_{\mathbb{R}}, M_{\mathbb{R}}$ of dimension $n \in \mathbb{Z}^+$. Let $\{e_1, \ldots, e_n\} \subset N_{\mathbb{R}}, \{e_1^*, \ldots, e_n^*\} \subset M_{\mathbb{R}}$ be bases of N, M, respectively.

The construction of an affine toric variety consists of the steps represented in the following diagram by the object which each produces:

$$\sigma \to \tilde{\sigma} \to S_\sigma \to R_\sigma \to X_\sigma.$$

In order, the objects in the diagram are a cone $\sigma \in M_{\mathbb{R}}$, a dual cone $\tilde{\sigma} \in N_{\mathbb{R}}$, a finitely generated monoid S_σ, a finitely generated $\mathbb{C}$-algebra R_σ, and an affine algebraic variety X_σ.

We start by defining the cone and dual cone.

Definition 2.1.1. ***(Polyhedral cone generated by a set, zero cone)*** *Let $k \in \mathbb{Z}^+, A = \{v_1, \ldots, v_k\} \subset N_{\mathbb{R}}$ be a finite set. The **polyhedral cone generated by** A is the set*

$$\sigma := \mathbb{R}_0^+ v_1 + \cdots + \mathbb{R}_0^+ v_k \subset N_{\mathbb{R}}.$$

*If $A = \emptyset$, the **zero cone** (generated by A) is $\sigma := \{0\} \subset N_{\mathbb{R}}$.*

Definition 2.1.2. ***(Lattice, rational, strongly convex cone, dimension)*** *Let $A \subset N_{\mathbb{R}}$, and let $\sigma \subset N_{\mathbb{R}}$ be the polyhedral cone generated by A. Then:*

*i) We say σ is a **lattice** or **rational cone** if $A \subset N$.*

*ii) We say σ is a **strongly convex cone** if $\sigma \cap (-\sigma) = 0$.*

*iii) The **dimension** of σ is*

$$\dim(\sigma) = \min_{\substack{N' \subset N_{\mathbb{R}} \text{ linear subspace} \\ \sigma \subset N'}} \dim(N').$$

Definition 2.1.3. ***(Dual cone associated to a cone)*** *Let $\sigma \subset N_{\mathbb{R}}$ be a polyhedral cone. The* ***dual cone associated to*** *σ is the set*

$$\hat{\sigma} := \{u \in M_{\mathbb{R}} : \quad \forall v \in \sigma, \quad \langle u, v \rangle \geq 0\}.$$

Proposition 2.1.4. ***([Brasselet, 2001], Property 1.1)*** *Let $\sigma \subset N_{\mathbb{R}}$ be a lattice cone. Then, $\hat{\sigma} \subset M_{\mathbb{R}}$ is a lattice cone (with respect to M).*

Henceforth, all cones $\sigma \subset N_{\mathbb{R}}, \hat{\sigma} \subset M_{\mathbb{R}}$ we consider are lattice (polyhedral) cones, and all cones $\sigma \subset N_{\mathbb{R}}$ we consider are also strictly convex.

Example 2.1.5. *The following figure contains three examples of cones and their dual cones. Only cones of maximal dimension are labelled.*

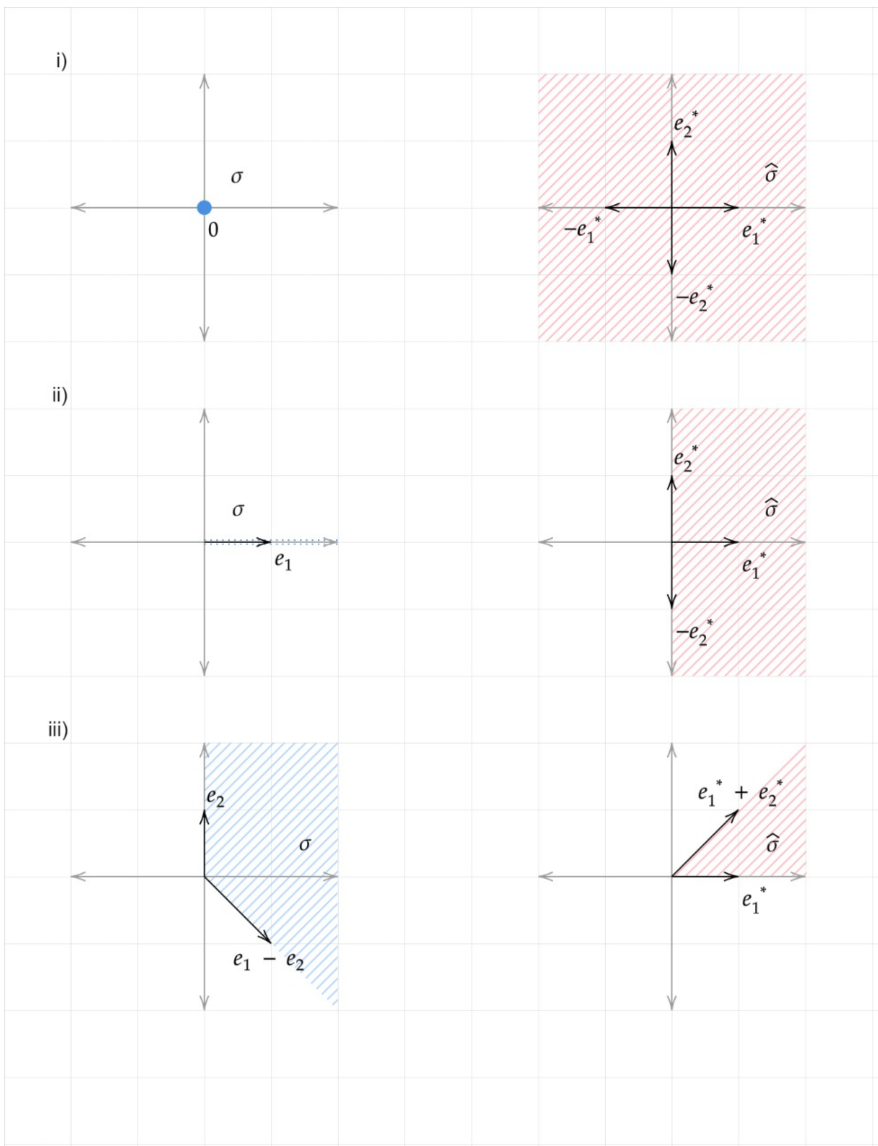


Figure 1: Three examples of cones and their dual cones.

Given a cone, there is a natural way of obtaining subcones, which are an essential part of the theory.

Definition 2.1.6. ***(Face, edge)*** *Let $\sigma \subset N_{\mathbb{R}}$ be a cone. Then:*

i) A ***face*** *of σ is a subset $\tau \subset \sigma$ such that there exists $\lambda \in \hat{\sigma} \cap M$ such that $\tau = \sigma \cap \lambda^{\perp}$. In particular, σ is a face of itself.*

ii) An ***edge*** *is a face of σ of dimension 1.*

Proposition 2.1.7. ***([Brasselet, 2001], Properties 1.2, 1.3, Remark 1.1)*** *Let $\sigma \subset M_{\mathbb{R}}$ be a cone. Then:*

i) Intersections of faces of σ are faces of σ, and faces of faces of σ are faces of σ.

ii) If $\lambda \in \hat{\sigma} \cap M$, and $\tau = \sigma \cap \lambda^{\perp} \subset \sigma$ is a face of σ, then $\tau \subset N_{\mathbb{R}}$ is a cone, $\hat{\tau} = \hat{\sigma} + \mathbb{R}_0^+(-\lambda)$, and $\hat{\sigma} \subset \hat{\sigma}$.

We now discuss about monoids; a specific one is the third object in our sequence of steps.

Definition 2.1.8. ***(Semigroup, monoid)*** *Let $(S, +)$ be a non-empty set S with a binary operation $+ : S^2 \to S$. Then:*

i) S is a ***semigroup*** *if $+$ is associative.*

ii) S is a ***monoid*** *if:*

a) S is a semigroup.

b) $+$ is commutative.

c) There exists a zero element $0 \in S$ (i. e., there exists an element $0 \in S$ such that for all $a \in S$, $a + 0 = 0 + a = a$).

d) Elements of S satisfy the simplification law (i. e., for all $a, b, c \in S$ such that $a+c = b+c$, $a = b$).

Definition 2.1.9. ***(Finitely generated monoid)*** *Let S be a monoid. S is* ***finitely generated*** *if there exist $k \in \mathbb{Z}^+, a_1, \ldots, a_k \in S$ such that $S = \mathbb{Z}_0^+ a_1 + \cdots + \mathbb{Z}_0^+ a_k$.*

Remark 2.1.10. *Let S be a finitely generated monoid. In general, there may exist two sets of generators which may be different, and may even have different cardinalities.*

The following lemma will allow us to associate a finitely generated monoid to a cone.

Lemma 2.1.11. ***([Brasselet, 2001], Lemma 1.3) (Gordon's lemma)*** *Let $\sigma \subset N_{\mathbb{R}}$ be a cone. Then, the monoid $\sigma \cap N$ is finitely generated.*

Definition 2.1.12. ***(Finitely generated monoid associated to a cone)*** *Let $\sigma \subset N_{\mathbb{R}}$ be a cone. The* ***finitely generated monoid associated to*** *σ is*

$$S_{\sigma} := \hat{\sigma} \cap M.$$

Remark 2.1.13. *Let $\sigma \subset N_{\mathbb{R}}$ be a cone. In general, a set of generators of $\hat{\sigma}$ may not generate S_{σ}, as $\hat{\sigma}$ is generated by $\mathbb{R}_0^+$ linear combinations, and S_{σ} by $\mathbb{Z}_0^+$ linear combinations.*

Remark 2.1.14. *Let $\sigma \subset N_{\mathbb{R}}$ be a cone. If $\lambda \in \hat{\sigma} \cap M$, and $\tau = \sigma \cap \lambda^{\perp} \subset \sigma$ is a face of σ, then $\lambda \in S_{\sigma}$.*

The following proposition is a direct consequence of Proposition 2.1.7, and will be used a bit later, when we attempt to give a first definition of a general toric variety,

Proposition 2.1.15. ***([Brasselet, 2001], Proposition 1.1)*** *Let $\sigma \subset N_{\mathbb{R}}$ be a cone, let $\lambda \in S_{\sigma}$, and let $\tau = \sigma \cap \lambda^{\perp} \subset \sigma$ be a face of σ. Then, $S_{\tau} = S_{\sigma} + \mathbb{Z}_0^+(-\lambda)$.*

We now define three classes of polynomials, the first of which will contain an algebra induced by the monoid S_{σ}.

Definition 2.1.16. ***(Laurent polynomial, monomial, monic monomial)***

*i) A **Laurent polynomial** is an element of the ring*

$$\mathbb{C}[z, z^{-1}] := \mathbb{C}[z_1, \dots, z_n, z_1^{-1}, \dots, z_n^{-1}]$$

of polynomials $p : \mathbb{C}^{2n} \to \mathbb{C}$ such that, for all $i \in \{1, \dots, n\}$, $z_1 z_1^{-1} = 1$.

*ii) A **Laurent monomial** is a Laurent polynomial of the form $\lambda z^a := \lambda z_1^{a_1} \cdots z_n^{a_n}, \lambda \in \mathbb{C}^*, a = (a_1, \dots, a_n) \in N$.*

*iii) A **Monic Laurent monomial** is a Laurent monomial such that $\lambda = 1$. Monic Laurent monomials form a multiplicative group, which we will write $\mathbb{C}[z, z^{-1}]_{\text{monic}}$.*

Proposition 2.1.17. *There exists a group isomorphism*

$$\varphi : (M, +) \to (\mathbb{C}[z, z^{-1}]_{\text{monic}}, \cdot)$$
$$a = (a_1, \dots, a_n) \mapsto z^a = z_1^{a_1} \cdots z_n^{a_n}.$$

Definition 2.1.18. ***(Support)*** *Let $p(z, z^{-1}) = \sum_{i=1}^{r} \lambda_i z^{a_i} \in \mathbb{C}[z, z^{-1}], r \in \mathbb{Z}^+, \lambda_i \in \mathbb{C}^*, a_i \in M$ be a Laurent polynomial. The **support** of p is*

$$\text{supp}(p) := \{a_1, \dots, a_r\}.$$

Lemma 2.1.19. ***([Brasselet, 2001], Proposition 2.1)*** *Let $\sigma \subset N_{\mathbb{R}}$ be a cone. Then, the ring $\{p \in \mathbb{C}[z, z^{-1}] : \quad \text{supp}(p) \subset S_{\sigma}\}$ is a finitely generated $\mathbb{C}$-algebra.*

We are now ready to define the coordinate ring of the affine algebraic variety resulting from our construction.

Definition 2.1.20. ***(Finitely generated $\mathbb{C}$-algebra associated to a cone)*** *Let $\sigma \subset N_{\mathbb{R}}$ be a cone. The **finitely generated $\mathbb{C}$-algebra associated to** σ is*

$$R_{\sigma} := \{p \in \mathbb{C}[z, z^{-1}] : \quad \text{supp}(p) \subset S_{\sigma}\}.$$

Reminder 2.1.21. ***(Algebraic geometry)*** *Let R be a (commutative) finitely generated $\mathbb{C}$-algebra. If we choose generators of R, then there exist $k \in \mathbb{Z}^+$ and an ideal $I \subset \mathbb{C}[z] := \mathbb{C}[z_1, \dots, z_k]$ such that $R \cong \mathbb{C}[z]/I$ (as finitely generated $\mathbb{C}$-algebras), and $\text{Spec}(R) \cong \text{Spec}(\mathbb{C}[z]/I) \cong V(I) \subset \mathbb{C}^k$ (as topological spaces with the Zariski topology). If we choose different generators of R, then we may also have different $k' \in \mathbb{Z}^+$ and $I' \subset \mathbb{C}[z] := \mathbb{C}[z_1, \dots, z_{k'}]$, but we nonetheless have $\text{Spec}(R) \cong V(I) \cong V(I')$.*

Definition 2.1.22. ***(Affine toric variety associated to a cone)*** *Let $\sigma \subset N_{\mathbb{R}}$ be a cone. The* ***affine toric variety associated to*** *σ is the affine algebraic variety*

$$X_\sigma := \mathrm{Spec}(R_\sigma).$$

The following (admittedly lengthy) theorem clarifies the steps necessary to realise the above definition in an affine space, and offers a less abstract approach to the latter part of this construction.

Theorem 2.1.23. ***([Brasselet, 2001], Theorem 2.2) (Explicit construction of an affine toric variety)*** *Let $\sigma \subset N_{\mathbb{R}}$ be a cone, and let $A = (a_1, \dots, a_k) \subset S_\sigma, a_i = (a_{i,1}, \dots, a_{i,n}) \in S_\sigma$ be a set of generators of S_σ. For each $i \in \{1, \dots, k\}$, we write $u_i = \varphi(a_i) = z^{a_i} = z_1^{a_{i,1}} \cdots z_n^{a_{i,n}} \in \mathbb{C}[z, z^{-1}]_{monic}$, where φ is as in Proposition 2.1.17.*

We choose generators of R_σ and calculate the ideal $I \subset \mathbb{C}[z] := \mathbb{C}[z_1, \dots, z_k]$ such that $R_\sigma \cong \mathbb{C}[z]/I$. To do this, note that there is a finite number $r \in \mathbb{Z}^+$ of relations between elements of A of the following form, for each $j \in \{1, \dots, r\}$:

$$\mathcal{R}_j : \quad \sum_{i=1}^{k} \alpha_{j,i} a_i = \sum_{i=1}^{k} \beta_{j,i} a_i, \quad \alpha_{j,i}, \beta_{j,i} \in \mathbb{Z}_0^+.$$

Applying φ to both sides, we get a binomial relation in $\mathbb{C}[z, z^{-1}]_{monic}$:

$$\varphi(\mathcal{R}_j) : \quad \prod_{i=1}^{k} u_i^{\alpha_{j,i}} = \prod_{i=1}^{k} u_i^{\beta_{j,i}}.$$

Finally, we have $I = \sum_{j=1}^{r} \mathbb{C}[z](\prod_{i=1}^{k} z_i^{\alpha_{j,i}} - \prod_{i=1}^{k} z_i^{\beta_{j,i}}) \subset \mathbb{C}[z]$, and $X_\sigma = \mathrm{Spec}(R_\sigma) \cong V(I) \subset \mathbb{C}^k$.

Example 2.1.24. ***([Brasselet, 2001], Example 2.2) (Construction of the algebraic torus)*** *The simplest example of the construction above is (as we will see) the namesake of the theory. If $\sigma = \{0\} \subset M_{\mathbb{R}}$, then $\hat{\sigma} = N_{\mathbb{R}}$ (see Figure 1, i).*

As in the figure, $A = (a_1, \dots, a_{2n}) = (e_1^*, \dots, e_n^*, -e_1^*, \dots, -e_n^*) \subset S_\sigma$ is a set of generators of S_σ. For each $i \in \{1, \dots, 2n\}$, we write $u_i = z_i \in \mathbb{C}[z, z^{-1}]_{monic}$, if $i \leq n$, and $u_i = z_{i-n}^{-1} \in \mathbb{C}[z, z^{-1}]_{monic}$, if $i > n$.

We choose generators of R_σ and calculate the ideal $I \subset \mathbb{C}[z] := \mathbb{C}[z_1, \dots, z_{2n}]$ such that $R_\sigma \cong \mathbb{C}[z]/I$. There are n relations between elements of A, for each $j \in \{1, \dots, n\}$:

$$\mathcal{R}_j : \quad a_j + a_{n+j} = 0.$$

Applying φ to both sides, we get a binomial relation in $\mathbb{C}[z, z^{-1}]_{monic}$:

$$\varphi(\mathcal{R}_j) : \quad u_j u_{n+j} = 1.$$

We have $I = \sum_{j=1}^{n} \mathbb{C}[z](z_j z_{n+j} - 1) \subset \mathbb{C}[z]$, and $X_\sigma = \text{Spec}(R_\sigma) \cong \{z = (z_1, \dots, z_{2n}) \in \mathbb{C}^{2n} : \forall j \in \{1, \dots, n\} : \quad z_j z_{n+j} = 1\}$. In particular, if $n = 1$, we have that X_σ is represented by the (complex) equilateral hyperbola. Also note that X_σ is homeomorphic to $(\mathbb{C}^*)^n$.

We may instead choose $A = (a_1, \dots, a_{n+1}) = (e_1^*, \dots, e_n^*, -e_1^* - \dots - e_n^*)$ as the set of generators of S_σ to see that X_σ is also represented by a subset of $\mathbb{C}^{n+1}$.

Definition 2.1.25. ***(Algebraic torus)*** *The **algebraic torus** is the affine algebraic variety $\mathbb{T} = (\mathbb{C}^*)^n$. It is also a multiplicative group, and $\mathbb{T}$ acts on itself by multiplication.*

Remark 2.1.26. ***([Brasselet, 2001], Remark 2.2)*** *The definition of the algebraic torus is justified as clearly $\mathbb{T} \cong (S^1)^n \times \mathbb{R}^+$, where $(S^1)^n$ is the real torus.*

We now investigate more closely the interaction between an affine toric variety and the algebraic torus. The following proposition describes an action which is, in some sense, the multiplicative analogue of the additive actions we will study in this thesis. This multiplicative action is, however, much simpler to understand.

Theorem 2.1.27. ***([Brasselet, 2001], Proposition 2.2, Property 2.1)*** *Let $\sigma \subset N_\mathbb{R}$ be a cone, and let $A = (a_1, \dots, a_k) \subset S_\sigma, a_i = (a_{i,1}, \dots, a_{i,n}) \in S_\sigma$ be a set of generators of S_σ. For any $t = (t_1, \dots, t_n) \in \mathbb{T}, x = (x_1, \dots, x_k) \in \mathbb{C}^k$, and for each $i \in \{1, \dots, k\}$, we write $t^{a_i} = t_1^{a_{i,1}} \cdots t_n^{a_{i,n}} \in \mathbb{C}^*, t \cdot x = (t^{a_1} x_1, \dots, t^{a_k} x_k) \in \mathbb{C}^k$.*

Then, there is an action

$$\mathbb{T} \times X_\sigma \to X_\sigma$$
$$(t, x) = (t_1, \dots, t_n, x_1, \dots, x_k) \mapsto t \cdot x = (t^{a_1} x_1, \dots, t^{a_k} x_k),$$

which extends the action of $\mathbb{T}$ on itself.

The orbit of the point $(1, \dots, 1) \in X_\sigma$ induces an embedding $\mathbb{T} \hookrightarrow X_\sigma$. Thus, X_σ contains a copy of $\mathbb{T}$ as a dense open subset, and $\dim_\mathbb{C}(X_\sigma) = n$.

The second part of this introduction is devoted to the construction of more general toric (algebraic) varieties that may not be realised in affine space. As before, we start with a combinatorial object.

Definition 2.1.28. ***(Fan)*** *A **fan** Δ in $N_\mathbb{R}$ is a finite collection of cones $\sigma_1, \dots, \sigma_r \in N_\mathbb{R}$ such that:*

i) If $\sigma \in \Delta$, and $\tau \subset \sigma$ is a face of σ, then $\tau \in \Delta$.

ii) If $\sigma, \sigma' \in \Delta$, then $\sigma \cap \sigma'$ is a face of both σ and σ'.

Example 2.1.29. *The following figure contains three examples of fans. Only cones of maximal dimension are labelled, faces and intersections of labelled cones of a fan are also cones of the fan.*

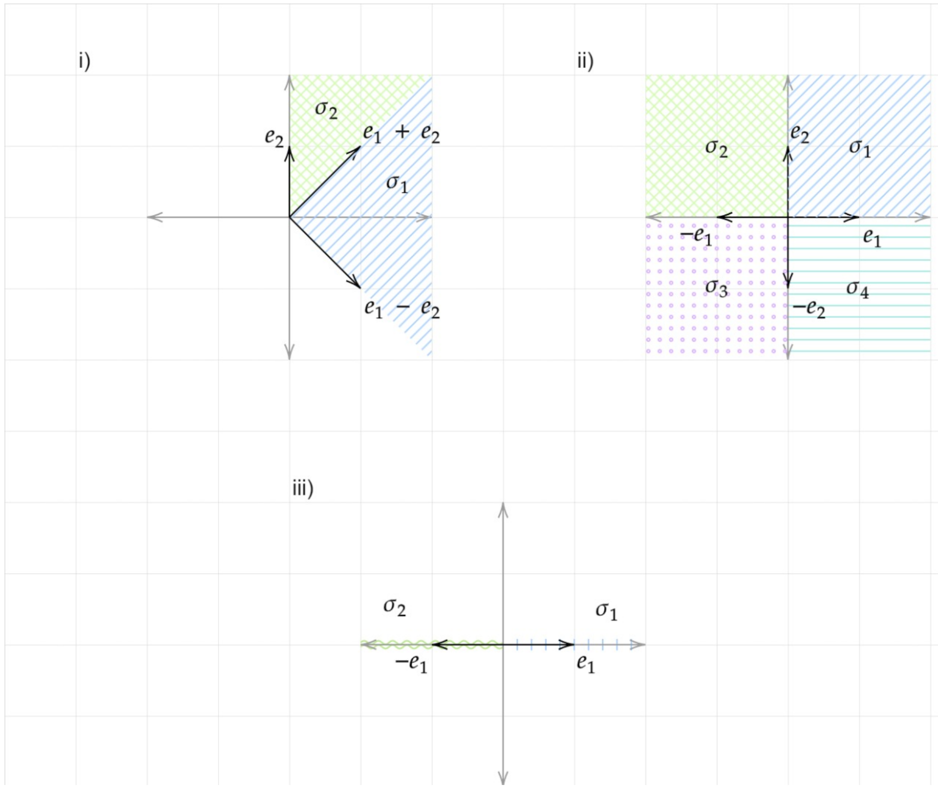


Figure 2: Three examples of fans.

We will define toric varieties from affine charts, and use the following lemma to define changes of charts.

Lemma 2.1.30. ***([Brasselet, 2001], Lemma 3.1) (Explicit construction of the gluing map)*** *Let $\sigma \subset N_{\mathbb{R}}$ be a cone, let $\lambda \in S_\sigma$, let $\tau = \sigma \cap \lambda^\perp \subset \sigma$ be a face of σ, and let $A_\sigma = (a_1, \dots, a_k) \subset S_\sigma, a_i = (a_{i,1}, \dots, a_{i,n}) \in S_\sigma$ be a set of generators of S_σ. Without loss of generality, we can choose $a_k = \lambda$.*

By Proposition 2.1.15, $A_\tau = (a_1, \dots, a_{k+1}) = A_\sigma \cup \{-a_k\}$ is a set of generators of S_τ. We choose generators of R_τ and calculate the ideal $I \subset \mathbb{C}[z] := \mathbb{C}[z_1, \dots, z_{k+1}]$. There's a finite number $r \in \mathbb{Z}^+$ of relations between elements of A_σ given by Theorem 2.1.23, and a new relation:

$$\mathcal{R}_{r+1}: \quad a_k + a_{k+1} = 0$$

We can write $R_\sigma \cong \mathbb{C}[u_1, \dots, u_k]$, where, for each $i \in \{1, \dots, k\}$, u_i is as in Theorem 2.1.23, and think of X_σ as $\mathbb{C}^k$ in the coordinates $(u_1, \dots, u_k)$. We can also write $R_\tau \cong \mathbb{C}[u_1, \dots, u_{k+1}]$, where $u_{k+1} = u_k^{-1}$, and think of X_τ as $\mathbb{C}^{k+1}$ in the coordinates $(u_1, \dots, u_{k+1})$. Thus, the projection

$$\mathbb{C}^{k+1} \to \mathbb{C}^k$$
$$(z_1, \dots, z_{k+1}) \mapsto (z_1, \dots, z_k)$$

induces the isomorphism (of affine algebraic varieties)

$$X_\tau \to \{z = (z_1, \dots, z_k) \in X_\sigma : \quad z_k \neq 0\}.$$

Therefore, if Δ is a fan in $M_\mathbb{R}$, and $\sigma, \sigma' \in \Delta, \tau = \sigma \cap \sigma'$, we can define a gluing map:

$$\psi_{\sigma,\sigma'} : \{z = (z_1, \ldots, z_k) \in X_\sigma : \quad z_k \neq 0\} \to X_\tau \to \{z = (z_1, \ldots, z_{k'}) \in X_{\sigma'} : \quad z_{k'} \neq 0\}.$$

Theorem 2.1.31. ***([Brasselet, 2001], Theorem 3.1)*** *Let Δ be a fan in $M_\mathbb{R}$. Define an equivalence relation on the disjoint union $\coprod_{\sigma\in\Delta} X_\sigma$ by the following:*

Let $x, x' \in \coprod_{\sigma\in\Delta} X_\sigma$.

i) If there exists $\sigma \in \Delta$ such that $x, x' \in X_\sigma$, then $x \sim x'$ if and only if $x = x'$.

ii) If there exist $\sigma, \sigma' \in \Delta, \sigma \neq \sigma'$ such that $x \in X_\sigma, x' \in X_{\sigma'}$, then $x \sim x'$ if and only if $\psi_{\sigma,\sigma'}(x) = x'$, where $\psi_{\sigma,\sigma'}$ is as in Lemma 2.1.30.

Then, $(\coprod_{\sigma\in\Delta} X_\sigma)/\sim$ is a topological space with the topology locally induced by each $X_\sigma, \sigma \in \Delta$, and the union $\cup_{\sigma\in\Delta}(X_\sigma/\sim)$ is an open covering. It is also an algebraic variety (with charts given by binomial relations, and change of charts given by the gluing map).

Definition 2.1.32. ***(First definition of a toric variety)*** *Let Δ be a fan in $M_\mathbb{R}$. The* ***toric variety*** *associated to Δ is the algebraic variety*

$$X_\Delta := \left(\coprod_{\sigma\in\Delta} X_\sigma\right) / \sim .$$

Example 2.1.33. ***([Brasselet, 2001], Example 3.2) (Construction of the complex projective space)*** *An example of the construction above is the complex projective space $\mathbb{P}^2$. We use the following figure:*

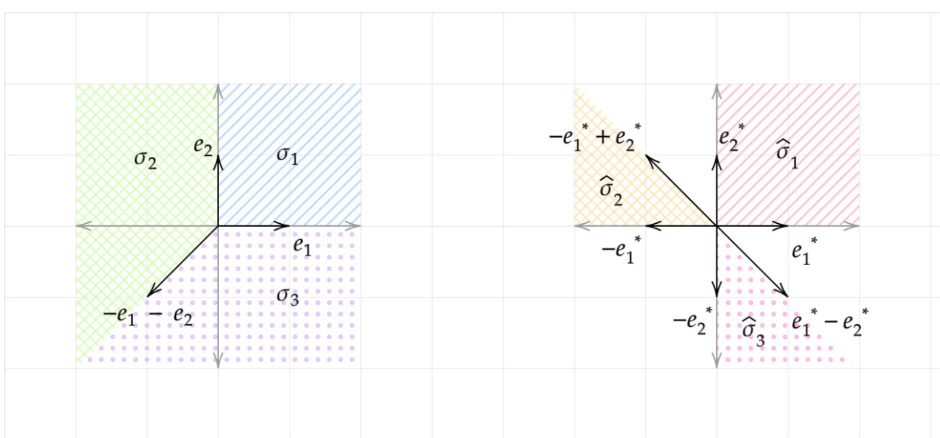


Figure 3: The fan and dual fan of $\mathbb{P}^2$.

Consider the fan and dual fan in Figure 3, in which only cones of maximal dimension are labelled, and their respective sets $A_1 = \{e_1^*, e_2^*\}, A_2 = \{-e_1^* + e_2^*, -e_1^*\}, A_3 = \{-e_2^*, e_1^* - e_2^*\}$ of monoid generators are drawn.

For all $i \in \{1, 2, 3\}$, there are no non-trivial relations between elements of A_i, thus X_1 is $\mathbb{C}^2$ in the coordinates (z_1, z_2), X_2 is $\mathbb{C}^2$ in the coordinates $(z_1^{-1}z_2, z_1^{-1})$, and X_3 is $\mathbb{C}^2$ in the coordinates $(z_2^{-1}, z_1 z_2^{-1})$.

We have gluing maps:

$$\psi_{1,2} : \{(z_1, z_2) \in X_1 : z_1 \neq 0\} \to \{(z_1^{-1}z_2, z_1^{-1}) \in X_2 : z_1^{-1} \neq 0\}$$
$$(z_1, z_2) \mapsto (z_1^{-1}z_2, z_1^{-1}),$$

$$\psi_{2,3} : \{(z_1^{-1}z_2, z_1^{-1}) \in X_2 : z_1^{-1}z_2 \neq 0\} \to \{(z_2^{-1}, z_1z_2^{-1}) \in X_3 : z_1z_2^{-1} \neq 0\}$$
$$(z_1^{-1}z_2, z_1^{-1}) \mapsto (z_2^{-1}, z_1z_2^{-1}),$$

$$\psi_{3,1} : \{(z_2^{-1}, z_1z_2^{-1}) \in X_3 : z_2^{-1} \neq 0\} \to \{(z_1, z_2) \in X_1 : z_2 \neq 0\}$$
$$(z_2^{-1}, z_1z_2^{-1}) \mapsto (z_1, z_2).$$

On the other hand, the classical charts of $\mathbb{P}^2$ *are:*

$$\varphi_1 : U_1 = \{[x_1 : x_2 : x_3] \in \mathbb{P}^2 : \quad x_1 \neq 0\} \to \mathbb{C}^2$$
$$[x_1 : x_2 : x_3] \mapsto (x_2/x_1, x_3/x_1),$$

$$\varphi_2 : U_2 = \{[x_1 : x_2 : x_3] \in \mathbb{P}^2 : \quad x_2 \neq 0\} \to \mathbb{C}^2$$
$$[x_1 : x_2 : x_3] \mapsto (x_1/x_2, x_3/x_2),$$

$$\varphi_3 : U_3 = \{[x_1 : x_2 : x_3] \in \mathbb{P}^2 : \quad x_3 \neq 0\} \to \mathbb{C}^2$$
$$[x_1 : x_2 : x_3] \mapsto (x_1/x_3, x_2/x_3).$$

The change of variables $z_1 = x_2/x_1, z_2 = x_3/x_1$ *induces homeomorphisms:*

$$U_1 \to \varphi_1(U_1) \to X_1,$$
$$U_2 \to \varphi_2(U_2) \to X_2,$$
$$U_3 \to \varphi_3(U_3) \to X_3.$$

Finally, if $\tau_1 = \sigma_1 \cap \sigma_2, \tau_2 = \sigma_2 \cap \sigma_3, \tau_3 = \sigma_3 \cap \sigma_1$*, the gluing maps also induce homeomorphisms compatible with the classical change of charts:*

$$\varphi_1(U_1 \cap U_2) \to X_{\tau_1} \to \varphi_2(U_1 \cap U_2),$$
$$\varphi_2(U_2 \cap U_3) \to X_{\tau_2} \to \varphi_3(U_2 \cap U_3),$$
$$\varphi_3(U_1 \cap U_3) \to X_{\tau_3} \to \varphi_1(U_1 \cap U_3).$$

Thus, both processes yield the same algebraic variety. This process may be generalised to mimic the classical construction of the complex projective space $\mathbb{P}^n$*.*

Proposition 2.1.34. ***([Brasselet, 2001], Proposition 3.1)*** *Let* X *be a toric variety of dimension* n*. Then, there exists an algebraic torus* $\mathbb{T} \cong (\mathbb{C}^*)^n$ *such that* $\mathbb{T} \subset X$ *as a dense open subset.*

We now give some remarks about the geometry of toric varieties.

Definition 2.1.35. ***(Regular cone, fan, complete fan)*** *Let $\sigma \subset \mathbb{N}_{\mathbb{R}}$ be a cone. We say σ is* ***regular*** *if we can choose a set of generators $A \subset N \cap \sigma$ of σ such that A can be extended to a basis of N.*

Let Δ be a fan in $N_{\mathbb{R}}$. We say Δ is ***regular*** *if all cones $\sigma \in \Delta$ are regular. We say Δ is* ***complete*** *if $\cup_{\sigma \in \Delta} \sigma = N_{\mathbb{R}}$.*

Theorem 2.1.36. ***([Brasselet, 2001], Theorem 3.2)*** *Let Δ be a fan in $N_{\mathbb{R}}$. Then:*

i) Δ is a complete fan if and only if X_Δ is a compact algebraic variety.

ii) Δ is a regular fan if and only if X_Δ is a smooth algebraic variety.

The following theorem is a spiritual sequel to Theorem 2.1.27.

Theorem 2.1.37. ***([Brasselet, 2001], Theorem 4.1)*** *Let X be a toric variety. Then, there exists an action of $\mathbb{T}$ which extends the action of $\mathbb{T}$ on itself, induced by the action of $\mathbb{T}$ on each of the affine parts of X.*

To finish this subsection, we provide a second definition of toric varieties, which is less constructive, and may be harder to motivate.

Proposition 2.1.38. ***([Brasselet, 2001], Proposition 4.4)*** *Let X be a toric variety. Then, X is a normal algebraic variety.*

Theorem 2.1.39. ***([Brasselet, 2001], Theorem 4.3) (Second definition of a toric variety)*** *X is a toric variety of dimension n if and only if X is a normal algebraic variety of dimension n such that there exists an algebraic torus $\mathbb{T} \cong (\mathbb{C}^*)^n$ such that $\mathbb{T} \subset X$ as a dense open subset, and there exists an action of $\mathbb{T}$ on X that extends the natural action of $\mathbb{T}$ on itself.*

2.2 Polytopes and their relation to toric geometry

We start this subsection by defining an object central to our discussion. We follow (in order) [Cox et al., 2011], [Ziegler, 2014], and [Brasselet, 2001].

Let M, N be dual lattices with associated vector spaces $M_{\mathbb{R}}, N_{\mathbb{R}}$ of dimension $n \in \mathbb{Z}^+$. Let $\{e_1, \ldots, e_n\} \subset M_{\mathbb{R}}, \{e_1^*, \ldots, e_n^*\} \subset N_{\mathbb{R}}$ be bases of M, N, respectively. Note that we have transposed the lattices N, M defined in the previous subsection. The reason for this will become clear later, when we define the *normal fan* of a polytope.

Definition 2.2.1. ***(Polytope, $\mathcal{V}$-representation)*** *A* ***polytope*** *is a subset $P \subset M_{\mathbb{R}}$ such that $P = Conv(S)$ (i. e., P is the convex hull of S), where $S \subset M_{\mathbb{R}}$ is finite. We say S is a* ***$\mathcal{V}$-representation*** *of P.*

Definition 2.2.2. ***(Dimension, full-dimensional polytope, affine hyperplane, closed half-space, face, supporting affine hyperplane, facet, edge, vertex)*** *Let $P \subset M_{\mathbb{R}}$ be a polytope.*

*i) We say P is of **dimension***

$$\dim(P) = \min_{\substack{M' \subset M_{\mathbb{R}} \text{ affine subspace} \\ P \subset M'}} \dim(M'),$$

*and that P is **full-dimensional** if $\dim(P) = \dim(M_{\mathbb{R}})$.*

*ii) Let $0 \neq u \in N_{\mathbb{R}}, b \in \mathbb{R}$. The **affine hyperplane** $H_{u,b}$ is the subset $\{m \in M_{\mathbb{R}} : \quad \langle m, u \rangle = b\}$, The **closed half-space** $H^+_{u,b}$ is the subset $\{m \in M_{\mathbb{R}} : \quad \langle m, u \rangle \geq b\}$.*

*iii) A **face** of P is a subset $Q \subset P$ such that there exist $0 \neq u \in N_{\mathbb{R}}, b \in \mathbb{R}$ such that $Q = H_{u,b} \cap P$ and $P \subset H^+_{u,b}$. In this situation, we say $H_{u,b}$ is the **supporting affine hyperplane** of Q. By convention, $\emptyset, P$ are also faces of P.*

*iv) A **facet** is a face of P of dimension $\dim(P) - 1$, an **edge** is one of dimension 1, and a **vertex** is one of dimension 0.*

Proposition 2.2.3. *Let $P \subset M_{\mathbb{R}}$ be a polytope given by the $\mathcal{V}$-representation $S \subset M_{\mathbb{R}}$, and let $Q \subset P$ be a face of P. Then, $Q \subset M_{\mathbb{R}}$ is a polytope given by the $\mathcal{V}$-representation $Q = Conv(S \cap Q)$.*

Proposition 2.2.4. ***([Cox et al., 2011], Proposition 2.2.1)*** *Let $P \subset M_{\mathbb{R}}$ be a polytope, and let $Q \subset P$ be a face of P. Then:*

i) The faces $Q' \subset Q$ of Q are exactly the faces $Q'' \subset P$ of P such that $Q'' \subset Q'$.

ii) If $Q \subsetneq P$, then Q is the intersection of the facets $F \subset P$ of P such that $Q \subset F$.

The previous definitions and results suggest an analogy between polytopes and polyhedral cones (see subsection 3.1). Indeed, polytopes (resp., polyhedral cones) are bounded (resp., unbounded) polyhedra.

Proposition 2.2.5. *Let $(H^+_{u_i,b_i})_{1 \leq i \leq s}, 0 \neq u_i \in N_{\mathbb{R}}, b_i \in \mathbb{R}$ be a finite collection of half-spaces. Then, $P = \cap_{i=1}^{s} H^+_{u_i,b_i} \subset M_{\mathbb{R}}$ is a polytope if and only if P bounded.*

The next we do is give another representation of a polytope which will prove immediately useful.

Definition 2.2.6. ***($\mathcal{H}$-representation)*** *Let $P \subset M_{\mathbb{R}}$ be a polytope. If there exists a finite collection of half-spaces $(H^+_{u_i,b_i})_{1 \leq i \leq s}, 0 \neq u_i \in N_{\mathbb{R}}, b_i \in \mathbb{R}$ such that $P = \cap_{i=1}^{s} H^+_{u_i,b_i}$, then we say the collection $(u_i, b_i)_{1 \leq i \leq s} \subset N_{\mathbb{R}} \times \mathbb{R}$ is an **$\mathcal{H}$-representation** of P.*

Proposition 2.2.7. *Let $P \subset M_{\mathbb{R}}$ be a polytope given by the $\mathcal{H}$-representation $(u_i, b_i)_{1 \leq i \leq s} \subset N_{\mathbb{R}} \times \mathbb{R}$, and let $(\lambda_i)_{1 \leq i \leq s} \subset \mathbb{R}^{>0}$. Then, $(\lambda_i u_i, \lambda_i b_i)_{1 \leq i \leq s} \subset N_{\mathbb{R}} \times \mathbb{R}$ is also an $\mathcal{H}$-representation of P.*

Proof. Let $x \in M_{\mathbb{R}}$. Then:

$$x \in P \iff \forall i \in \{1, \ldots, s\}, \quad \langle x, u_i \rangle \geq b_i \iff \forall i \in \{1, \ldots, s\}, \quad \langle x, \lambda_i u_i \rangle \geq \lambda_i b_i.$$

□

The following theorem sums up a characterisation of full-dimensional polytopes that is both surprising and useful, along with a few other facts.

Theorem 2.2.8. ***([Ziegler, 2014], Theorem 2.2.4) (Minkowski-Weyl representation theorem)*** *Let $P \subset M_{\mathbb{R}}$ be a full-dimensional polytope, let V be the set of its vertices, and let $\mathcal{F}$ be the set of its facets. Then:*

i) *$V \subset M_{\mathbb{R}}$ is a $\mathcal{V}$-representation of P. It is also the unique minimal element in the set of $\mathcal{V}$-representations of P partially ordered by inclusion (i. e., if S is a $\mathcal{V}$-representation of P, then $V \subset S$).*

ii) *There exist $\mathcal{H}$-representations of P.*

iii) *For each facet $F \in \mathcal{F}$, there is a unique supporting affine hyperplane H_F of F, corresponding to a closed half-space H_F^+. If we write $H_F = \{m \in M_{\mathbb{R}} : \quad \langle u_F, m\rangle = -a_F\}$, and $H_F^+ = \{m \in M_{\mathbb{R}} : \quad \langle u_F, m\rangle \geq -a_F\}$, then $(u_F, -a_F) \in N_{\mathbb{R}} \times \mathbb{R}$ is unique up to multiplication by $\lambda \in \mathbb{R}^+$.*

iv) *The collection $(u_F, -a_F)_{F\in\mathcal{F}} \subset N_{\mathbb{R}}$ is an $\mathcal{H}$-representation of P (i. e., P = $\cap_{F\in\mathcal{F}} H_F^+$), uniquely determined as per item iii.*

v) *Items i and iv are two equivalent definitions for P, and there exist algorithms to compute each representation given the other.*

The following definitions should be interpreted as shorthand for the *special* representations of a polytope in Theorem 2.2.8. They also justify the presentation of the data on the Graded Ring Database (GRDB) [Brown and Kasprzyk, 2009], which we will heavily use in later subsections.

Definition 2.2.9. ***($\mathcal{V}$, $\mathcal{H}$-polytope, inward-pointing facet normal)*** *A **$\mathcal{V}$-polytope** (resp., **$\mathcal{H}$-polytope**) is a polytope $P \subset M_{\mathbb{R}}$ given by the $\mathcal{V}$-representation (resp., $\mathcal{H}$-representation) in item i (resp., item iv) of Theorem 2.2.8. In this situation, each vector $u_F \in N_{\mathbb{R}}$ is an **inward-pointing facet normal** to its corresponding facet F.*

We now define the *dual* or *polar* polytope $P^\circ \subset N_{\mathbb{R}}$ associated to a polytope $P \subset M_{\mathbb{R}}$ such that $0 \in \text{int}(P)$, and study some of its properties.

Definition 2.2.10. ***([Cox et al., 2011], Exercise 2.2.1) (Dual, polar polytope)*** *Let $P \subset M_{\mathbb{R}}$ be a full-dimensional $\mathcal{H}$-polytope such that $0 \in \text{int}(P)$, and let $\mathcal{F}$ be the set of its facets. The **dual** or **polar polytope** is the polytope $P^\circ \subset N_{\mathbb{R}}$ given by the $\mathcal{V}$-representation $((1/a_F)u_F)_{F\in\mathcal{F}} \subset N_{\mathbb{R}}$.*

Remark 2.2.11. *Let $P \subset M_{\mathbb{R}}$ be a full-dimensional $\mathcal{H}$-polytope such that $0 \in \text{int}(P)$, and let $\mathcal{F}$ be the set of its facets. As $0 \in \text{int}(P)$, for each $F \in \mathcal{F}$ we have $a_F \in \mathbb{R}^+$, thus the collection $((1/a_F)u_F, -1)_{F\in\mathcal{F}}$ is an $\mathcal{H}$-representation of P.*

Proposition 2.2.12. *Let $P \subset M_{\mathbb{R}}$ be a full-dimensional polytope such that $0 \in \text{int}(P)$, let $P^\circ \subset N_{\mathbb{R}}$ be its dual polytope. Then:*

i) $P^\circ = \{u \in N_{\mathbb{R}} : \quad \forall m \in P, \quad \langle u, m\rangle \geq -1\}$.

ii) $(P^\circ)^\circ = P$.

The following important combinatorial structure will presently only allow us to determine the vertices of the dual polytope, but we will revisit it later. We will not define all order-theoretic notions we will use (these may be found in [Ziegler, 2014]), as doing so would not add anything of much relevance to our discussion.

Definition 2.2.13. ***(Face lattice)*** *Let $P \subset M_{\mathbb{R}}$ be a polytope, and let $\mathcal{Q}$ be the set of its faces. The* ***face lattice*** *of P is the poset $(\mathcal{Q}, \subset)$.*

Theorem 2.2.14. ***([Ziegler, 2014], Theorem 2.36)*** *Let $P \subset M_{\mathbb{R}}$ be a polytope. Then, its face lattice is a finite graded lattice of rank $\text{dim}(P) + 1$.*

Theorem 2.2.15. ***([Ziegler, 2014], Theorem 2.55)*** *Let $P \subset M_{\mathbb{R}}$ be a full-dimensional polytope such that $0 \in \text{int}(P)$, and let $P^{\circ} \subset N_{\mathbb{R}}$ be its dual polytope. Then, the face lattices of P and P° are opposite. In other words, there is a bijective correspondence between faces $Q \subset P$ of P and faces $Q' \subset P^{\circ}$ of P° such that $\text{dim}(Q') = n - \text{dim}(Q) - 1$, where, by convention, $\text{dim}(\emptyset) = -1$.*

Corollary 2.2.16. *Let $P \subset M_{\mathbb{R}}$ be a full-dimensional $\mathcal{H}$-polytope such that $0 \in \text{int}(P)$. The dual polytope $P^{\circ} \subset N_{\mathbb{R}}$ is the $\mathcal{V}$-polytope given by the $\mathcal{V}$-representation in Definition 2.2.10.*

Proof. Theorem 2.2.15 implies there is a bijective correspondence between facets of P and vertices of P°. □

We now restrict ourselves lattice polytopes in a similar way to how we restricted ourselves to lattice cones in subsection 3.1.

Definition 2.2.17. ***(Lattice polytope)*** *A* ***lattice polytope*** *is a ($\mathcal{V}$-)polytope $P \subset M_{\mathbb{R}}$ given by a $\mathcal{V}$-representation $V \subset M$.*

It is possible to choose a unique facet representation of a full-dimensional lattice polytope.

Proposition 2.2.18. *Let $P \subset M_{\mathbb{R}}$ be a full-dimensional lattice polytope, and let $\mathcal{F}$ be the set of its facets. Then, for each $F \in \mathcal{F}$, there exists an inward-pointing facet normal $u_F \in N_{\mathbb{R}}$ such that $u_F \in N$, thus there is a unique $\mathcal{H}$-representation $(u_F/\,\text{GCD}(u_F), -a_F/\,\text{GCD}(u_F))_{F\in\mathcal{F}} \subset N_{\mathbb{R}} \times \mathbb{R}$ of P, where $\text{GCD} : \mathbb{Z}^n \to \mathbb{Z}$ gives the greatest common divisor of the n coordinates.*

We will construct projective toric varieties from lattice polytopes that have *enough* lattice points. The following definitions make this notion precise in two different ways:

Definition 2.2.19. ***(Normal, very ample polytope)*** *Let $P \subset M_{\mathbb{R}}$ be a lattice polytope.*

i) P is ***normal*** *if, for all $k, l \in \mathbb{N}$, we have $(kP) \cap M + (lP) \cap M = ((k+l)P) \cap M$, where $+ : (M_{\mathbb{R}})^2 \to M_{\mathbb{R}}$ is the Minkowski sum.*

ii) P is ***very ample*** *if, for each vertex $m \in P$, the semigroup $S_{P,m} = \mathbb{N}(P \cap M - m)$ is saturated.*

Proposition 2.2.20. ***([Cox et al., 2011], Proposition 2.2.18)*** *Let $P \subset M_{\mathbb{R}}$ be a lattice polytope. If P is normal, then it is very ample.*

Corollary 2.2.21. ***([Cox et al., 2011], Corollary 2.2.19)*** *Let $P \subset M_{\mathbb{R}}$ be a full-dimensional lattice polytope, and let $k \in \mathbb{Z}^{+}, k \geq n-1$. Then, $kP \subset M_{\mathbb{R}}$ is a very ample polytope. In particular, if $n = 2$, then P is very ample.*

Throughout, as in subsection 3.1, all cones are strongly convex lattice polyhedral cones.

Theorem 2.2.22. ***([Cox et al., 2011], Theorem 2.3.2, Propositions 2.3.7, 2.3.8, [Brasselet, 2001], Proposition 4.6)*** *Let $P \subset M_{\mathbb{R}}$ be a full-dimensional lattice $\mathcal{V}$-polytope given by the $\mathcal{V}$-representation $(v_i)_{1\leq i\leq r} \subset M_{\mathbb{R}}$. For each $i \in \{1,\ldots,r\}$, define the cones $C_i := \langle P \cap M - v_i\rangle \subset M_{\mathbb{R}}, \sigma_i := \hat{C}_i \subset N_{\mathbb{R}}$ associated to the vertex v_i. Then:*

i) For each $i \in \{1,\ldots,r\}$, $\dim(\sigma_i) = n$.

ii) The fan generated by the collection $(\sigma_i)_{1\leq i\leq r}$ (i. e., the fan that contains $(\sigma_i)_{1\leq i\leq r}$ and all faces and intersections) is a complete fan in $N_{\mathbb{R}}$.

Proposition 2.2.23. *Let $P \subset M_{\mathbb{R}}$ be a full-dimensional lattice $\mathcal{V}$-polytope given by the $\mathcal{V}$-representation $(v_i)_{1\leq i\leq r} \subset M_{\mathbb{R}}$, and let $\mathcal{F}$ be the set of its facets. Then:*

i) Cones in the fan in Theorem 2.2.22 are in bijective correspondence with faces of P.

ii) If $Q \subset P$ is a face of P, the cone

$$\sigma_Q := \{u \in N_{\mathbb{R}} : \quad \forall m \in Q, \forall m' \in P : \quad \langle u, m\rangle \leq \langle u, m'\rangle\}$$

is its corresponding cone in the fan in Theorem 2.2.22. For each $i \in \{1,\ldots,r\}$, if we substitute $Q = \{v_i\}$, then we recover $\sigma_Q = \sigma_i$.

iii) For each face $Q \subset P$ of P,

$$\sigma_Q = \sum_{\substack{F\in\mathcal{F}\\ Q\subset F}} \mathbb{R}_0^+ u_F,$$

where, for each $F \in \mathcal{F}$, $u_F \in N_{\mathbb{R}}$ is an inward-pointing facet normal to F.

Proof. Item i follows from [Cox et al., 2011], Proposition 2.3.7 (as the very author states on page 79). Items ii, iii are the definitions in [Cox et al., 2011], [Brasselet, 2001], respectively. □

Definition 2.2.24. ***(Normal fan, toric variety associated to a polytope)*** *Let $P \subset M_{\mathbb{R}}$ be a full-dimensional lattice polytope. The **normal fan** Δ_P of P is the fan in Theorem 2.2.22. The **toric variety associated to** **P** is the compact toric variety $X_P := X_{\Delta_P}$.*

The following proposition is a technical justification for this procedure, as *a priori* Theorem 2.2.22 is proved in [Cox et al., 2011] for very ample polytopes.

Proposition 2.2.25. *Let $P \subset M_{\mathbb{R}}$ be a full-dimensional lattice polytope, and let $k \in \mathbb{Z}^+$. Then, $\Delta_P = \Delta_{kP}$.*

If the origin is contained in the polytope's interior, the cones of the normal fan fit together beautifully.

Proposition 2.2.26. ***([Brasselet, 2001], Proposition 4.6)*** *Let $P \subset M_{\mathbb{R}}$ be a full-dimensional lattice $\mathcal{V}$-polytope such that $0 \in \mathrm{int}(P)$ given by the $\mathcal{V}$-representation $(v_i)_{1\leq i\leq r} \subset M_{\mathbb{R}}$, and let $\mathcal{F}$ be the set of its facets.*

Then, for each face $Q \subset P$ of P, the cone $\sigma_Q \subset N_{\mathbb{R}}$ covers a face $Q \subset P^\circ$ of P° (i. e., $Q \subset \sigma_Q$). In particular, for each $i \in \{1, \dots, r\}$, the cone $\sigma_i \subset N_{\mathbb{R}}$ covers the facet $F_i := Conv((1/a_F)u_F)_{F \in \mathcal{F}, v_i \in F} \subset P^\circ$ of P° (i. e., $F_i \subset \sigma_i$).

Example 2.2.27. *Figure 4 shows the fan of $\mathbb{P}^2$ (see Figure 3) as the normal fan of a polytope. It also illustrates the behaviour of all cones of maximal dimension involved. The reader is encouraged to verify as many of the definitions and results stated in this subsection as they need to feel comfortable with the situation at hand.*

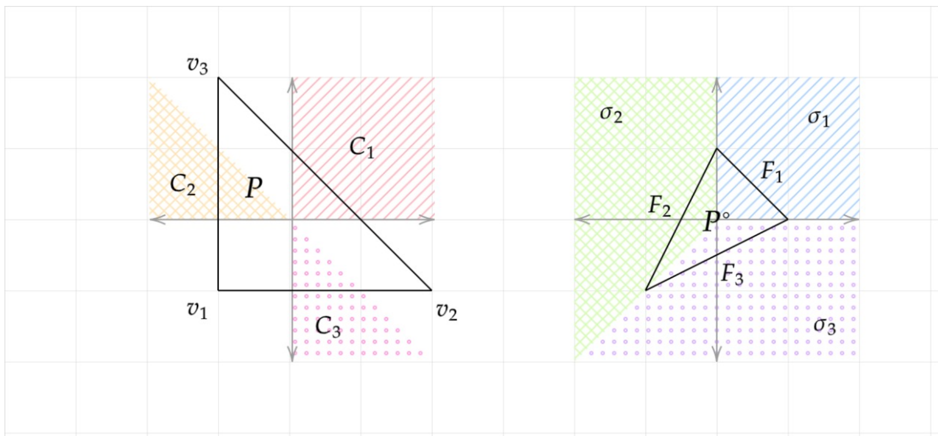


Figure 4: The polytope, dual polytope, and normal fan of $\mathbb{P}^2$.

Full-dimensional lattice polytopes correspond to projective toric varieties, and bijectively to ample Cartier divisors of projective toric varieties.

Theorem 2.2.28. ***([Brasselet, 2001], Theorem 3.2)*** *Let Δ be a fan in $N_{\mathbb{R}}$. Then, X_Δ projective toric variety such that* dim$(X_\Delta) = n$ *if and only if there exists a full-dimensional lattice polytope $P \subset M_{\mathbb{R}}$ such that $X_\Delta = X_P$.*

Theorem 2.2.29. ***([Brasselet, 2001], Theorem 3.2)*** *There is a bijective correspondence between full-dimensional lattice polytopes in $M_{\mathbb{R}}$ and pairs (X, D), where X is a projective toric variety such that* dim$(X) = n$, *and D is an ample Cartier divisor of X.*

We will now impose conditions on the polytope in order for the resulting variety to be non-singular.

Definition 2.2.30. ***(Primitive vector)*** *Let $P \subset M_{\mathbb{R}}$ be a lattice polytope, and let $v \neq v'$ be vertices of P. The* ***primitive vector*** *on the edge $E \subset P$ of P that starts at v and ends at v' is $e := (v' - v)/$* GCD$(v' - v) \in M$, *where* GCD : $\mathbb{Z}^n \to \mathbb{Z}$ *gives the greatest common divisor of the n coordinates.*

Definition 2.2.31. ***(Smooth polytope)*** *Let $P \subset M_{\mathbb{R}}$ be a lattice polytope. We say P is* ***smooth*** *if, for each vertex $v \in P$ of P, the primitive vectors $e \in M$ on the edges $E \subset P$ of P such that $v \in E$ form a subset of a basis of M. In particular, if P is full-dimensional, then, for each vertex $v \in P$ of P, the primitive vectors $e \in M$ form a basis of M.*

Theorem 2.2.32. ***([Cox et al., 2011], Theorem 2.4.3)*** *Let $P \subset M_{\mathbb{R}}$ be a full-dimensional lattice polytope. Then, X_P is a smooth projective toric variety if and only if P is a smooth polytope.*

Proposition 2.2.33. ***([Cox et al., 2011], Proposition 2.4.4)*** *Let $P \subset M_{\mathbb{R}}$ be a full-dimensional lattice polytope. If P is smooth, then P is very ample.*

2.3 Smooth Fano polytopes and their classification

We start this subsection by defining the specific notion of equivalence between lattice polytopes we will use. There is another such notion which is commonly used in other contexts, but coarser, called *combinatorial equivalence* (see [Batyrev, 1999]).

Let M, N be dual lattices with associated vector spaces $M_{\mathbb{R}}, N_{\mathbb{R}}$ of dimension $n \in \mathbb{Z}^{+}$.

Definition 2.3.1. ***(Lattice isomorphic, lattice equivalent polytopes)*** *Let $P, P' \subset M_{\mathbb{R}}$ be lattice polytopes. We say P, P' are **lattice isomorphic** or **lattice equivalent** if there exists an invertible linear map $f : M_{\mathbb{R}} \to M_{\mathbb{R}}$ such that f extends an isomorphism of M (i. e., such that $\det(f) = \pm 1$) and $P = f(P')$.*

Theorem 2.3.2. ***([Batyrev, 1999], Theorem 2.2.4)*** *Let $P, P' \subset M_{\mathbb{R}}$ be full-dimensional lattice polytopes. Then, $X_P, X_{P'}$ are biregularly isomorphic if and only if P, P' are lattice isomorphic.*

We are now ready to introduce the more specific kind of polytopes with which this thesis is concerned.

Definition 2.3.3. ***(Reflexive polytope)*** *Let $P \subset M_{\mathbb{R}}$ be a full-dimensional lattice polytope such that $0 \in \text{int}(P)$. We say P is a **reflexive polytope** if P° is also a lattice polytope.*

Proposition 2.3.4. *Let $P \subset M_{\mathbb{R}}$ be a full-dimensional lattice polytope such that $0 \in P$, and let $\mathcal{F}$ be the set of its facets. Consider the unique $\mathcal{H}$-representation $(u_F / \operatorname{GCD}(u_F), -a_F / \operatorname{GCD}(u_F))_{F \in \mathcal{F}} \subset N_{\mathbb{R}} \times \mathbb{R}$ of P in Proposition 2.2.18. The following are equivalent:*

i) *P is reflexive.*

ii) *P° is reflexive.*

iii) *The collection $(u_F / \operatorname{GCD}(u_F), -1)_{F \in \mathcal{F}} \subset N_{\mathbb{R}} \times \mathbb{R}$ is the unique $\mathcal{H}$-representation of P in Proposition 2.2.18.*

iv) *The collection $(u_F / \operatorname{GCD}(u_F))_{F \in \mathcal{F}} \subset M_{\mathbb{R}}$ is a $\mathcal{V}$-representation of P°.*

Proposition 2.3.5. ***([Cox et al., 2011], Exercise 2.3.5)*** *Let $P \subset M_{\mathbb{R}}$ be a reflexive polytope. Then $\text{int}(P) \cap M = \{0\}$.*

Definition 2.3.6. ***(Smooth Fano polytope)*** *Let $P \subset N_{\mathbb{R}}$ be a lattice polytope such that $0 \in \text{int}(P)$, and let $\mathcal{F}$ be the set of its facets. We say P is **smooth Fano** if, for each facet $F \subset \mathcal{F}$, the vertices $v \in P$ of P such that $v \in F$ form a basis of N.*

Proposition 2.3.7. ***([Cox et al., 2011], Exercise 8.3.6)*** *Let $P \subset N_{\mathbb{R}}$ be a smooth Fano polytope. Then, P and $P^{\circ} \subset M_{\mathbb{R}}$ are reflexive.*

Proposition 2.3.8. ***([Cox et al., 2011], Exercise 8.3.6)*** *Let $P \subset N_{\mathbb{R}}$ be a full-dimensional lattice polytope such that $0 \in \text{int}(P)$. Then, P is smooth Fano if and only if $P^{\circ} \subset M_{\mathbb{R}}$ is reflexive and smooth.*

We mention in passing that there are other intermediate classes of polytopes which are sometimes also called *Fano*, but our definitions seem to align to modern conventions (see [Nill, 2005], Definition 2.3.5).

As we have repeatedly done, we construct certain classes of toric varieties from reflexive and smooth Fano polytopes.

Definition 2.3.9. ***(Gorenstein Fano, smooth Fano variety)*** *Let X be a complete normal algebraic variety. We say X is a **Gorenstein Fano variety** if the anticanonical divisor $-K_X$ is Cartier and ample. In particular, Gorenstein Fano varieties are projective. We say X is a **smooth Fano variety** if it is Gorenstein Fano and smooth.*

Theorem 2.3.10. ***([Cox et al., 2011], Theorem 8.3.4)*** *Let $P \subset M_{\mathbb{R}}$ be a reflexive polytope. Then, X_P is a Gorenstein Fano toric variety. Conversely, if X is a Gorenstein Fano toric variety, then the polytope associated to the anticanonical divisor $-K_X$ is reflexive.*

Corollary 2.3.11. *Let $P \subset N_{\mathbb{R}}$ be a smooth Fano polytope. Then, $X_{P^{\circ}}$ is a smooth Fano toric variety. Conversely, if X is a smooth Fano toric variety, then the polytope associated to the anticanonical divisor $-K_X$ is the dual of a smooth Fano polytope.*

Proposition 2.3.12. ***([Cox et al., 2011])*** *For each fixed dimension $n \in \mathbb{Z}^{+}$, there exist a finite number of isomorphism classes of reflexive (resp., smooth Fano) polytopes.*

By Theorems 2.3.2, 2.3.10, classifying Gorenstein Fano (resp., smooth Fano) toric varieties up to biregular isomorphism is equivalent to classifying reflexive (resp., smooth Fano) polytopes up to lattice isomorphism. This is relatively easy for dimension 2 (see [Cox et al., 2011], [Debarre, 2002]); the resulting classification may be seen in Figures 5, 6, 7. Figure 6 also includes information about the corresponding smooth Fano varieties. The reader is encouraged to verify the relevant definitions and properties for all polytopes, and to identify all smooth Fano polytopes and their duals in Figure 5.

The higher-dimensional classification of reflexive polytopes has been solved by Kreuzer and Skarke (see [Kreuzer and Skarke, 2004]), who implemented a package for a software system with which they found that there are 4319 isomorphism classes of reflexive polytopes of dimension 3, and 473800776 of dimension 4. Quoting [Cox et al., 2011]: "One reason for the interest in these varieties is the relation with mirror symmetry. [... But] since these numbers grow so quickly, most more recent work has focused on subclasses [such as] polytopes giving smooth Fano toric varieties."

On the other hand, there are 18 isomorphism classes of smooth Fano polytopes of dimension 3 (see [Mori and Mukai, 2003]), 124 of dimension 4 (see [Batyrev, 1999]), and $866, 7622$ of dimension $5, 6$, respectively (see [Øbro, 2007]). Similarly, the higher-dimensional classification of smooth Fano polytopes has been solved by Mikkel Øbro (see [Øbro, 2007]), who introduced the SFP algorithm to this effect.

In this thesis, we aim to refine the classification of smooth Fano polytopes to account for the notion of *additiveness* defined in subsection 2.4.

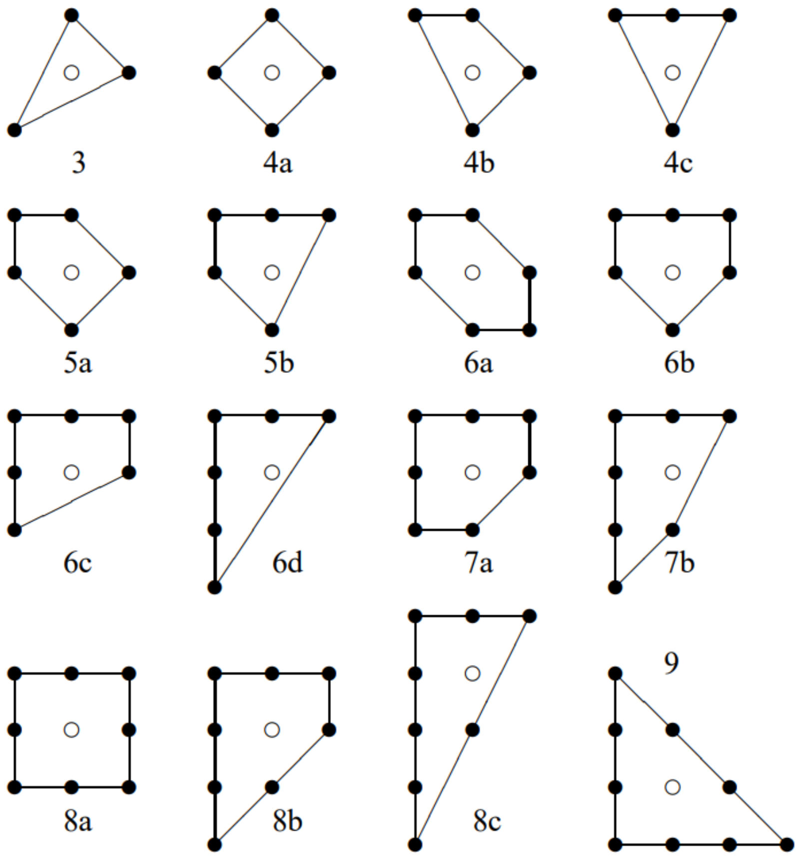


Figure 5: Representatives of the 16 isomorphism classes of reflexive polytopes of dimension 2.

Source: [Cox et al., 2011].

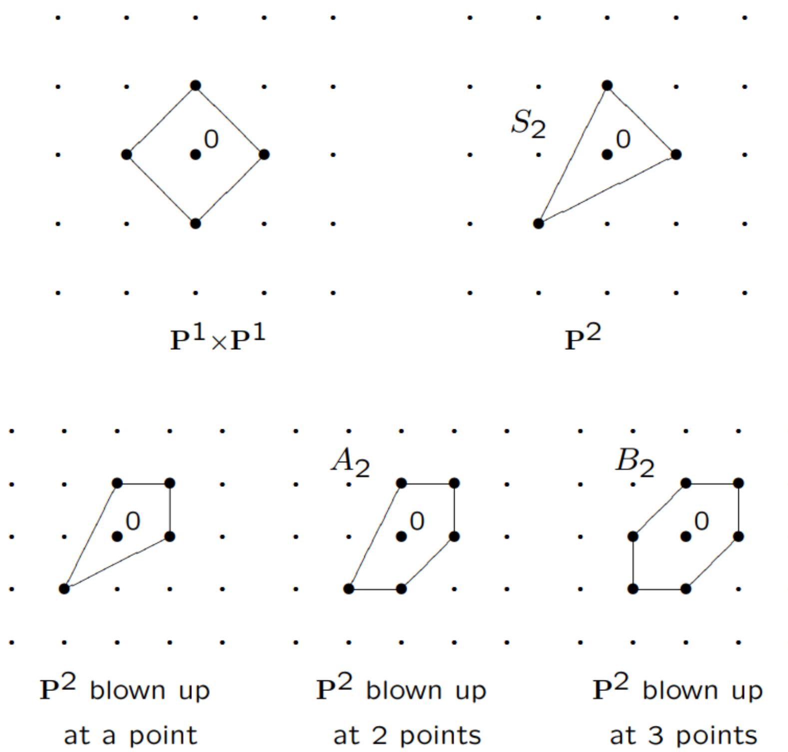


Figure 6: Representatives of the 5 isomorphism classes of smooth Fano polytopes of dimension 2.

Source: [Debarre, 2002].

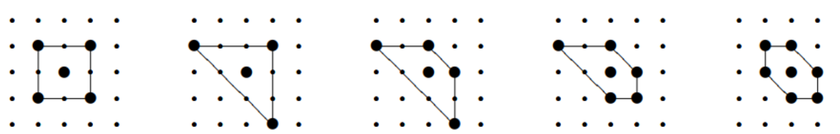


Figure 7: Duals of representatives of the 5 isomorphism classes of smooth Fano polytopes of dimension 2.

Source: [Debarre, 2002].

2.4 Additive actions on smooth Fano toric varieties

As previously suggested (see Theorems 2.1.27, 2.1.37), this thesis is not concerned with the classical *multiplicative* action of the torus on a toric variety, but with less understood *additive* actions of the commutative unipotent group (a power of the additive group underlying the ground field). We now direct our attention to studying these additive actions in general and on smooth Fano toric varieties.

Let M, N be dual lattices with associated vector spaces $M_{\mathbb{R}}, N_{\mathbb{R}}$ of dimension $n \in \mathbb{Z}^+$.

Definition 2.4.1. ***(Additive action on an irreducible algebraic variety, additive variety, uniquely additive variety)*** *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, let X an irreducible algebraic variety of dimension n over $\mathbb{K}$, and let $\mathbb{G}_a = (\mathbb{K}, +)$ be the additive group underlying $\mathbb{K}$. An* ***additive action*** *on X is an (effective, regular) action $\mathbb{G}_a^n \times X \to X$ of the commutative unipotent group $\mathbb{G}_a^n$ on X with an open orbit. We say X is an* ***additive variety*** *(resp., a* ***uniquely additive variety****) if it admits an additive action (resp., if it admits a unique additive action up to isomorphism).*

Definition 2.4.2. ***(Additive polytope)*** *Let $P \subset M_{\mathbb{R}}$ be a full-dimensional lattice polytope. We say P is an* ***additive polytope*** *(resp., a* ***uniquely additive polytope****) if the projective variety X_P associated to P is additive (resp., uniquely additive).*

The original motivation behind the study of additive actions came from problems in arithmetic geometry.

The first systematic study is found in [Hassett and Tschinkel, 1999], wherein a correspondence is established between additive actions on $\mathbb{P}^n$ and local $(n+1)$-dimensional commutative associative algebras with unit. This correspondence has been exploited to classify additive actions on many different kinds of algebraic varieties (the introduction of [Arzhantsev and Romaskevich, 2017] includes a brief survey).

Furthermore, work on Manin's conjecture yielded two papers by Chambert-Loir, Tschinkel on asymptotic results about the distribution of rational points of bounded height on certain equivariant compactifications (resp., embeddings) of the vector group [Chambert-Loir and Tschinkel, 2002] (resp., [Chambert-Loir and Tschinkel, 2012]). Incidentially, Manin's conjecture for smooth projective toric varieties was proved in [Batyrev and Tschinkel, 1995] using other techniques.

Example 2.4.3. ***([Hassett and Tschinkel, 1999], Proposition 3.2)*** *There are two distinct additive actions on $\mathbb{P}^2$ given, for all $a = (a_1, a_2) \in \mathbb{G}_a^2$ and $x = [x_1 : x_2 : x_3] \in \mathbb{P}^2$, by:*

$$\tau(a)(x) = \begin{bmatrix} 1 & 0 & a_2 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \rho(a)(x) = \begin{bmatrix} 1 & a_1 & a_2 + \frac{1}{2}a_1^2 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

We now state a result from [Arzhantsev and Romaskevich, 2017] which we will use to determine the existence of additive actions on smooth Fano varieties using only combinatorial methods.

Definition 2.4.4. ***(Polytope inscribed in a rectangle)*** *Let $P \subset M_{\mathbb{R}}$ be a very ample polytope, and let $\mathcal{F}$ be the set of its facets. We say P is* ***inscribed in a rectangle*** *if there exists a vertex $v_0 \in P$ of P such that:*

i) The primitive vectors on the edges $E_i \subset P$ of P starting at v_0 form a basis $e_1, \dots, e_n \in M$ of M.

ii) For all $F \in \mathcal{F}$ and $i \in \{1, \dots, n\}$, if $v_0 \notin F$, then $\langle -u_F, e_i \rangle \geq 0$.

Theorem 2.4.5. ***([Arzhantsev and Romaskevich, 2017], Theorem 5.2)*** *Let $P \subset M_{\mathbb{R}}$ be a very ample polytope. Then, the projective variety X_P associated to P is additive if and only if P is inscribed in a rectangle.*

Let $P \subset N_{\mathbb{R}}$ be a smooth Fano polytope. By Proposition 2.3.8, $P^\circ \subset M_{\mathbb{R}}$ is smooth. In particular, every vertex of P° satisfies condition i of Definition 2.4.4. P° is also very ample by Proposition 2.2.33 Therefore, Theorem 2.4.5 translates nicely to the following:

Corollary 2.4.6. *Let $P \subset N_{\mathbb{R}}$ be a smooth Fano polytope, and let $\mathcal{F}$ be the set of facets of $P^\circ \subset M_{\mathbb{R}}$. Then, the smooth Fano variety X_{P° associated to P° is additive if and only if there exists a vertex $v_0 \in P^\circ$ such that, if $e_1, \ldots, e_n$ are the primitive vectors on the edges $E_i \subset P^\circ$ of P° starting at v_0, then for all $F \in \mathcal{F}$ and $i \in \{1, \ldots, n\}$, if $v_0 \notin F$, then $\langle -u_F, e_i \rangle \geq 0$.*

Example 2.4.7. ***([Huang and Montero, 2020])*** *The dual of the smooth Fano polytope II_{33} (see Figure 8) is given by the $\mathcal{V}$-representation*

$$\{(-1,-1,-1),(-1,-1,3),(-1,2,-1),(-1,2,0),(2,-1,-1),(2,-1,0)\}$$

*and **is** additive, as it is inscribed in a rectangle. It is easy to verify visually that the inward-pointing facet normals of facets not containing v_0 lie in the negative octant of the basis given by the primitive vectors on the edges starting at v_0.*

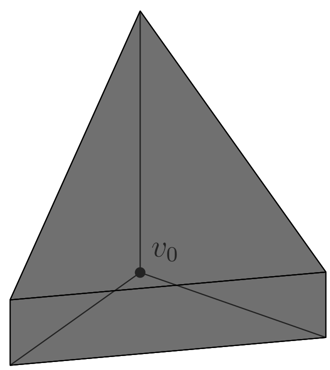


Figure 8: The smooth Fano polytope II_{33}.
Source: [Huang and Montero, 2020].

Example 2.4.8. ***([Huang and Montero, 2020])*** *The dual of the smooth Fano polytope III_{25} (see Figure 9) is given by the $\mathcal{V}$-representation*

$$\{(-1,2,-1),(-1,0,-1),(-1,2,0),(-1,0,2),(2,-1,-1),(0,-1,-1),$$
$$(2,-1,0),(0,-1,2)\}$$

*and **is not** additive, as it is not inscribed in a rectangle. This is also easy to verify visually, although a bit lengthier.*

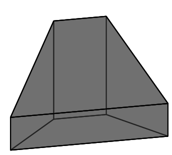


Figure 9: The smooth Fano polytope III_{25}.
Source: [Huang and Montero, 2020].

The following result will be immediately useful:

Proposition 2.4.9. ***([Dzhunusov, 2022], Proposition 2)*** *Let $P \subset M_{\mathbb{R}}$ be a full-dimensional lattice $\mathcal{H}$-polytope given by the unique $\mathcal{H}$-representation $(p_i, -a_i)_{1\leq i\leq s} \subset N_{\mathbb{R}} \times \mathbb{R}$ in Proposition 2.2.18. Then, X_P is additive if and only if the primitive vectors $(p_i)_{1\leq i\leq s} \subset N_{\mathbb{R}}$ can be ordered in a way such that the vectors $(p_i)_{1\leq i\leq n}$ form a basis $B \subset N$ of N and the vectors $(p_i)_{n+1\leq i\leq s}$ lie in the negative octant with respect to B.*

Remark 2.4.10. *Let $P \subset M_{\mathbb{R}}$ be a very ample (full-dimensional lattice) polytope. The previous proposition provides a second, albeit less algorithmically efficient, criterion for additiveness on a projective toric variety. It is direct to check that it is equivalent to being inscribed in a rectangle: if $v_0 \in P$ is the vertex in Definition 2.4.4, and $C_0 \subset M_{\mathbb{R}}$ is the cone generated by the primitive vectors $e_1, \ldots, e_n \in M$, then, by Proposition 2.2.23, $\sigma_0 = \hat{C}_0 \subset N_{\mathbb{R}}$ is the cone generated by the inward-pointing facet normals to the facets containing v_0. The generators of σ_0 form a basis $B \subset N$ of N, and the remaining inward-pointing facet normals lie in the negative octant with respect to B.*

Definition 2.4.11. ***(Ordered primitive $\mathcal{H}$-representation)*** *Let $P \subset M_{\mathbb{R}}$ be an additive full-dimensional lattice $\mathcal{H}$-polytope given by the unique $\mathcal{H}$-representation $(p_i, -a_i)_{1\leq i\leq s} \subset N_{\mathbb{R}} \times \mathbb{R}$ in Proposition 2.2.18. An **ordered primitive $\mathcal{H}$-representation** of P is the collection $\mathcal{P} := (p_i, -a_i)_{1\leq i\leq s} \subset N_{\mathbb{R}} \times \mathbb{R}$ ordered as in Proposition 2.4.9.*

We now state the main result from [Dzhunusov, 2022] which we will use to determine the uniqueness of additive actions on smooth Fano varieties using only combinatorial methods.

Definition 2.4.12. ***(Demazure root)*** *Let $P \subset M_{\mathbb{R}}$ be a full-dimensional lattice $\mathcal{H}$-polytope given by the unique $\mathcal{H}$-representation $(u_i, -a_i)_{1\leq i\leq s} \subset N_{\mathbb{R}} \times \mathbb{R}$ in Proposition 2.2.18. For each $i \in \{1, \ldots, s\}$, define the set:*

$$\mathfrak{R}_i := \{x \in M : \quad \langle u_i, x\rangle = -1, \quad (\forall j \in \{1, \ldots, s\} \setminus \{i\} : \quad \langle u_j, x\rangle \geq 0)\}.$$

*A **Demazure root** is an element of the set $\cup_{i=1}^{s} \mathfrak{R}_i \subset M$.*

Remark 2.4.13. *In the previous definition, if $P \subset M_{\mathbb{R}}$ is reflexive, $i \in \{1, \ldots, s\}$, and F_i is the facet associated to u_i, then $\mathfrak{R}_i = \text{rint}(F_i) \cap M$, where $\text{rint}(F_i)$ is the relative interior of F_i (i. e., the interior in the induced topology of the smallest affine subspace of $M_{\mathbb{R}}$ containing F_i).*

Theorem 2.4.14. ***([Dzhunusov, 2022], Theorem 4)*** *Let $P \subset M_{\mathbb{R}}$ be an additive full-dimensional lattice $\mathcal{H}$-polytope given by an ordered primitive $\mathcal{H}$-representation $\mathcal{P} = (p_i, -a_i)_{1\leq i\leq s} \subset N_{\mathbb{R}} \times \mathbb{R}$. If X_P is additive, and $(p_i^*)_{1\leq i\leq s} \subset M_{\mathbb{R}}$ is the dual basis of $(p_i)_{1\leq i\leq s} \subset N_{\mathbb{R}}$ (in the linear algebraic sense), then X_P is uniquely additive if and only if for each $i \in \{1, \ldots, n\}$ one has $\mathfrak{R}_i = \{-p_i^*\}$.*

Similarly to Corollary 2.4.6, let $P \subset N_{\mathbb{R}}$ be an additive smooth Fano polytope. As P° is full-dimensional, Theorem 2.4.14 applies to X_{P°.

Example 2.4.15. *Figure 10 illustrates Remarks 2.4.10 and 2.4.13 with the polytope and dual polytope associated to $Bl_{p,q}(\mathbb{P}^2)$, the blow-up of $\mathbb{P}^2$ at two general points p, q.*

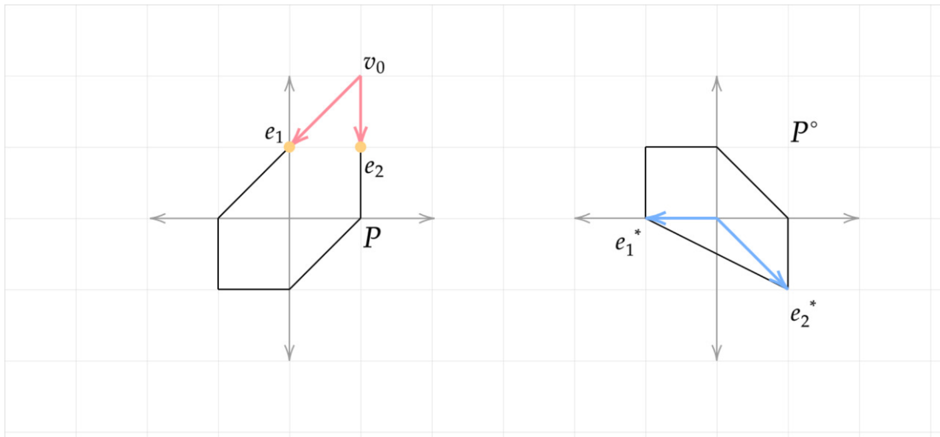


Figure 10: The polytope, dual polytope, and Demazure roots of $Bl_{p,q}(\mathbb{P}^2)$.

Indeed, $v_0 \in P$ is the vertex in Definition 2.4.4, $e_1, e_2 \in M$ are the primitive generators of $C_0 \subset M_\mathbb{R}$, $e_1^*, e_2^* \in N$ are the primitive generators of $\sigma_0 \subset N_\mathbb{R}$ and form a basis $B \subset N$ of N, and the remaining inward-pointing facet normals of P lie in the negative octant with respect to B.

Elements of $\mathfrak{R}_1 \cup \mathfrak{R}_2$ are coloured yellow, and if $F_1 \subset P$ (resp., $F_2 \subset P$) is the facet of P normal to e_1^* (resp., e_2^*), then $\mathfrak{R}_1 = \{-e_1\} = \text{rint}(F_1) \cap M$ (resp., $\mathfrak{R}_2 = \{-e_2\} = \text{rint}(F_2) \cap M$).

We conclude that $Bl_{p,q}(\mathbb{P}^2)$ is additive and uniquely additive.

Example 2.4.16. *Figure 11 illustrates Remarks 2.4.10 and 2.4.13 with the polytope and dual polytope associated to $\mathbb{P}^2$.*

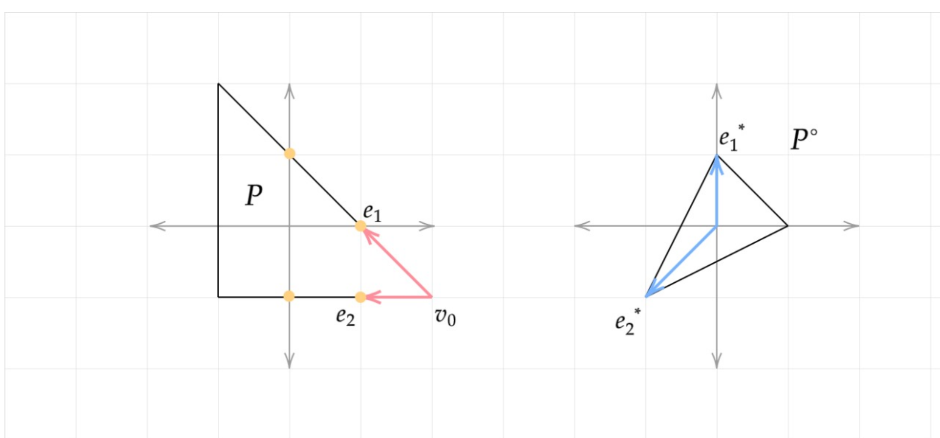


Figure 11: The polytope, dual polytope, and Demazure roots of $\mathbb{P}^2$.

Indeed, $v_0 \in P$ is the vertex in Definition 2.4.4, $e_1, e_2 \in M$ are the primitive generators of $C_0 \subset M_\mathbb{R}$, $e_1^*, e_2^* \in N$ are the primitive generators of $\sigma_0 \subset N_\mathbb{R}$ and form a basis $B \subset N$ of N, and the remaining inward-pointing facet normal of P lies in the negative octant with respect to B.

Elements of $\mathfrak{R}_1 \cup \mathfrak{R}_2$ are coloured yellow, and if $F_1 \subset P$ (resp., $F_2 \subset P$) is the facet of P normal to e_1^* (resp., e_2^*), then $\mathfrak{R}_1 = \text{rint}(F_1) \cap M$ (resp., $\mathfrak{R}_2 = \text{rint}(F_2) \cap M$).

We conclude that $\mathbb{P}^2$ is additive but ***not*** *uniquely additive, as $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are not singletons.*

SECTION 3
PROPOSED SOLUTION

3.1 Web scraping the GRDB

The Graded Ring Database (GRDB) is, in its authors' own words, "a database of graded rings in algebraic geometry, including classifications of toric varieties, [...] and Fano 3-folds and 4-folds" [Brown and Kasprzyk, 2009]. The website provides a front-end to a database written by Gavin Brown and Alexander Kasprzyk, with data computed by various collaborators. We entered a blank search in the Smooth Toric Fano Varieties (STFV) section of the website, which returns polytope data of all 8635 such varieties of dimension less than or equal to 6, spread across 864 pages. This data is provided to the GRDB by Mikkel Øbro, who authored *An algorithm for the classification of smooth Fano polytopes* [Øbro, 2007].

We chose the free tier of Google Colab [Google, 2022] to compute most of our results in (except, of course, those explicitly stated to be related to our Macaulay2 package), as we think it is the ideal environment for general numerical scientific research that is sufficiently undemanding resource-wise (indeed, the maximum amount of RAM we used was around 2.5 GB).

Selenium is an open-source project which "supports automation of all the major browsers in the market through the use of WebDriver, [...] an API and protocol that defines a language-neutral interface for controlling the behaviour of web browsers" [Selenium, 2022]. We installed it in our runtime, and used its Python bindings to scrape data from the website. We previously experimented with (simpler [Amery, 2019]) HTTP libraries Requests [Reitz, 2022] and BeautifulSoup [Richardson, 2022] but failed, as the GRDB website dinamically generates relevant data by requiring the user to press a *More details* button for each polytope (see Figure 12).

The steps in our pipeline were as follows:

i) *Do the web scraping* (see Algorithm 1).

   The function "SleepAndPressMoreDetailsButtons(t)" puts the runtime to sleep for t seconds, and then tells Webdriver to press all *More details* buttons on the current page. It was added as we found the *More details* button of some polytopes took a while to render, thus sometimes causing the script to stop its execution prematurely. As per Algorithm 1, this function is called in a nested sequence of try-except blocks with increasing values of t, in order to prevent an otherwise safer sleeping timer from stalling the execution of the script for unnecessarily long.

   The function "# PolytopesOnPage(i)" returns 5 if i is the last page, or 10 else.

   Each execution of Algorithm 1 typically took us around 50 minutes.

ii) *Perform error correction*.

   In rare occasions (610 out of 8635 polytopes, or around 7.06% of all polytopes, in our first trial), pressing the *More details* button of some polytopes did not actually render any new data. This may be due to a bug in the website or to a limitation of Selenium. We repeated step i of the

Algorithm 1 Web scraping the STFV section of the GRDB website to obtain polytope data of all such varieties of dimension ≤ 6.

Precondition: Selenium is correctly installed in the runtime, and the GRDB website is up online.

Postcondition: “out.txt” is a text file containing all scraped polytope data.

```
Get the webpage of the STFV section of the GRDB website.
f ← Open “out.txt” file in append mode.
# Pages ← 864
for 0 ≤ i < # Pages do
    try
        SleepAndPressMoreDetailsButtons(0)        ▷ Read pipeline step 1 for more details.
    except
        try
            SleepAndPressMoreDetailsButtons(4)
        except
            try
                SleepAndPressMoreDetailsButtons(16)
            except
                SleepAndPressMoreDetailsButtons(64)
            end
        end
    end
    for 0 ≤ j < # PolytopesOnPage(i) do           ▷ Read pipeline step 1 for more details.
        r ← Find row with data for polytope j on page i by its ID.
        Append text on r to f.
    end for
    if i < # Pages - 1 then
        l ← Find link to next page by its CSS selector.
        Click l.
    end if
end for
Close f.
```

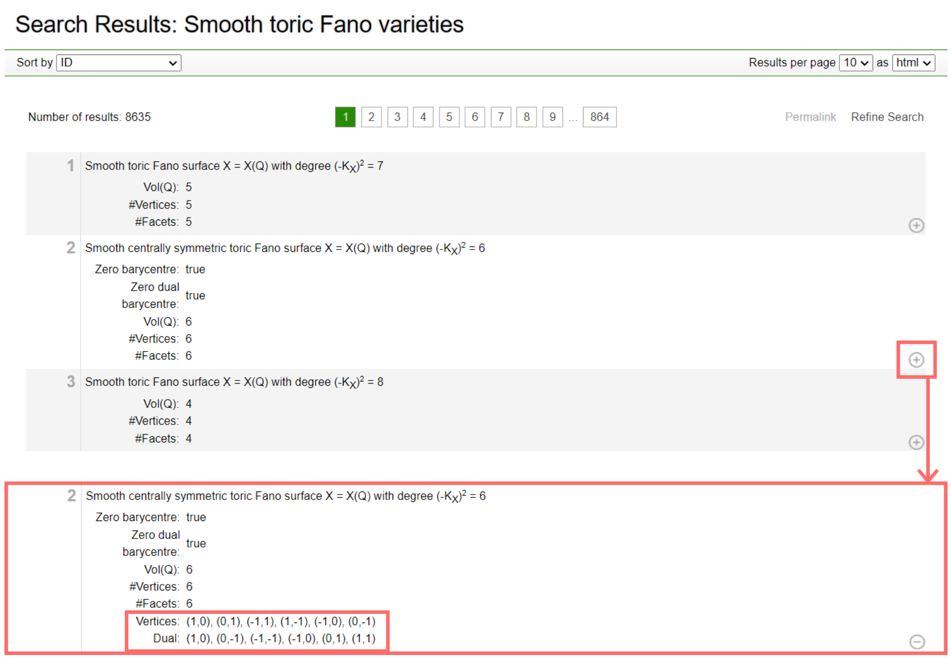


Figure 12: The GRDB website requires the user to press a *More details* button for each polytope.

pipeline twice and ran a simple error correction algorithm (henceforth, SEC algorithm) twice, which consisted of reading two output files of Algorithm 1, identifying the indices of polytopes with missing data on the first output file, appending the corresponding data from the second output file (if it was not also missing), and finally writing a new output file with less errors. The remaining error was corrected manually (see Table 1 for a summary of the error correction process).

Each execution of the SEC algorithm took us less than 1 second.

Task n°	**Task description**	**# Errors before execution**	**# Errors after execution**
1	Run the SEC algorithm	610/8635 (7.06%)	21/8635 (0.24%)
2	Run the SEC algorithm	21/8635 (0.24%)	1/8635 (0.01%)
3	Manual error correction	1/8635 (0.01%)	0

Table 1: List of tasks of the error correction process.

iii) *Parse the data.*

We split the output of the previous steps into a list, parsed the data of each smooth Fano polytope $P \subset N_{\mathbb{R}}$ on GRDB STFV using the following regular expression:

```
(\d+)
Smooth ([^\n]+)?toric Fano (?:[^\n]+) X = X\(Q\) with degree \(-KX\)(\d+) = (\d+)(?:
Zero barycentre: (\w+))?(?:
Zero dual barycentre: (\w+))?
```

```
Vol\(Q\): (\d+)
#Vertices: (\d+)
#Facets: (\d+)
Vertices: ([^\n]+)
Dual: ([^\n]+)
```

The 11 capturing groups obtain the following attributes of the polytope, in order:

1) ID on GRDB STFV.
2) If the polytope is centrally symmetric.
3) Dimension.
4) Degree.
5) If the polytope has zero barycentre.
6) If the polytope has zero dual barycentre.
7) Volume.
8) Number of vertices.
9) Number of facets.
10) List of vertices.
11) List of facets.

Furthermore, the following tests were run for the sake of data integrity:

1) Assert that 11 groups have been captured.
2) Assert that the captured ID on GRDB STFV corresponds to the index on the list of polytopes.
3) Assert that the number of vertices is equal to the length of the list of vertices.
4) Assert that the number of facets is equal to the length of the list of facets.

The number (resp., list) of vertices and the number (resp., list) of facets were then transposed, as we will require $\mathcal{V}$ and $\mathcal{H}$-representations of the polar polytope $P^\circ \subset M_\mathbb{R}$ instead of the smooth Fano polytope $P \subset N_\mathbb{R}$. The resulting parsed list was typecasted into a NumPy array of objects and was saved as a NumPy file.

3.2 Obtaining the edges of a polytope given by both its $\mathcal{V}$ and $\mathcal{H}$-representations

Let N, M be dual lattices with associated vector spaces $N_\mathbb{R}, M_\mathbb{R}$ of dimension $d \in \mathbb{Z}^+$.

Let $P \subset M_\mathbb{R}$ be a full-dimensional polytope.

3.2.1 Motivation

If P is given by its $\mathcal{V}$-representation (resp., $\mathcal{H}$-representation), the problem of obtaining an $\mathcal{H}$-representation (resp., $\mathcal{V}$-representation) is called *facet enumeration* (resp., *vertex enumeration*). By duality results, these problems have the same complexity. It is unknown whether there is an algorithm that does this in the general case in time complexity polynomial on the input size (i. e., on the dimension and number of vertices or facets). [Bremner et al., 1998] Quoting Bremner, Fukuda and Maretta: "Several polynomial algorithms [...] are known under strong assumptions of non-degeneracy, which restrict input polytopes to be simple in the case of vertex enumeration and simplicial in the case of facet enumeration. However, it appears to be extremely hard to determine whether there is a polynomial algorithm in general" [Bremner et al., 1998]. On the other hand, it is known that, for unbounded polyhedra, the problem is NP-Hard [Khachiyan et al., 2008]. In particular, if $\sigma \subset M_{\mathbb{R}}$ is a polyhedral cone, obtaining generators of its dual cone $\hat{\sigma} \subset N_{\mathbb{R}}$ is NP-Hard.

The problem of facet enumeration is clearly equivalent to finding the convex hull of a finite set of points. A worst-case optimal algorithm for doing this has time complexity exponential on d and, for fixed d, polynomial on the number of vertices of P [Chazelle, 1993].

Now, for fixed d, given both $\mathcal{V}$ and $\mathcal{H}$-representations of P, computing its entire face lattice is can be done in time complexity $O(\min\{|I|, |J|\} \cdot \#VF \cdot \#\mathcal{Q})$, where $|I|$ is the number of vertices, $|J|$ is the number of facets, $\#VF$ is the number of vertex-facet incidences, and $\#\mathcal{Q}$ is the total number of faces [Kaibel and Pfetsch, 2002].

One step of Algorithm 4 will require us to find all (primitive) vectors on the edges of P starting at a given vertex of P. By Remark 2.4.10, this may be done by calculating the dual $C_0 \subset M_{\mathbb{R}}$ of $\sigma_0 = \hat{C}_0 \subset N_{\mathbb{R}}$, the cone generated by the inward-pointing facet normals to the facets containing the vertex. Since we also require, in particular, that the cone C_0 be full-dimensional (see Definition 2.4.4), this is likely not computationally hard. However, to satisfy our curiosity, we will study the more general case of any full-dimensional polytope given by both its $\mathcal{V}$ and $\mathcal{H}$-representations and any vertex (i. e., not only lattice polytopes, and not only "smooth" vertices). In this subsection we analyse naïve approaches to this problem for low values of d, and then present an algorithm that solves it in time complexity $O(|I| \cdot \#VF)$ if VF is sparse (e. g., if P has many facets). This algorithm may possibly be generalised to compute more of the face lattice of P, although possibly in sub-optimal time complexity.

Let $I = \{0, \ldots, |I| - 1\}$, $J = \{0, \ldots, |J| - 1\}$, let $P \subset M_{\mathbb{R}}$ a full dimensional lattice polytope given by its vertices $V = (m_i)_{i \in I}$ and primitive inward-pointing facet normals $H = (u_j)_{j \in J}$ to the facets $\mathcal{F} = (F_j)_{j \in J}$ (see Theorem 2.2.8.i and Proposition 2.2.18, respectively). Given a vertex m_0 of P, we want to find vectors on all edges of P starting at m_0. Note that it is enough to find all vertices m of P, $m \neq m_0$, such that m and m_0 form an edge of P.

3.2.2 Naïve approaches

If $d = 2$, all facets are edges, thus it is enough to find all vertices m of P that share a facet with m_0.

If $d = 3$, it can be shown that it is enough to find all vertices m of P such that there exist two facets F, F' such that $m, m_0 \in F \cap F'$. [Henk et al., 1997]

In general, if $d \leq 4$, it can be shown that it is enough to find all vertices m of P that share $d-1$ facets with m_0. [Henk et al., 1997]

However, if $d \geq 5$, there may exist vertices that share many facets and do not form an edge! [Henk et al., 1997]

3.2.3 A correct solution

The author apologises for the somewhat dense formalism that follows: it was created only to simplify writing the resulting algorithm and the complexity analysis. The basic idea is nonetheless already present in [Henk et al., 1997]: two vertices form an edge if and only if they are the only vertices in the facets that contain them.

Definition 3.2.1. ***(Vertices-facets matrix)*** *The* ***vertices-facets matrix*** *associated to P is $VF := (vf_{ij})_{i\in I, j\in J} \in \mathcal{M}_{|I|\times|J|}(\mathbb{Z})$, where $vf_{ij} = 1$ if $m_i \in F_j$, and $vf_{ij} = 0$ else.*

Definition 3.2.2. ***(To share a facet, shared facets vector)*** *Let $i, i' \in I, j \in J$. The vertices m_i and $m_{i'}$* ***share*** *the facet F_j if $m_i \in F_j$ and $m_{i'} \in F_j$. In other words, two vertices share a facet if and only if $vf_{ij}vf_{i'j} = 1$. Let $n < |I|$, and fix $i_0, \ldots, i_n \in I$. The* ***shared facets vector*** *associated to $i_0, \ldots, i_n$ is $S^{i_0,\ldots,i_n} := (s_i^{i_0,\ldots,i_n})_{i\in I} := (VF)(((vf_{i_0 j})_{j\in J})^t \circ \cdots \circ ((vf_{i_n j})_{j\in J})^t) \in \mathcal{M}_{|I|\times 1}(\mathbb{Z})$, where $\circ : (\mathcal{M}_{|I|\times 1}(\mathbb{Z}))^2 \to \mathcal{M}_{|I|\times 1}(\mathbb{Z})$ is the Hadamard or coordinate-wise product.*

Proposition 3.2.3. *Let $n \in \mathbb{Z}_0^+, n < |I|$, and fix $i_0, \ldots, i_n \in I$. For each $i \in I$, the coordinate $s_i^{i_0,\ldots,i_n} \in \mathbb{Z}$ equals the number of facets shared by all the vertices $m_i, m_{i_0}, \ldots, m_{i_n}$ at once.*

Proof.

$$s_i^{i_0,\ldots,i_n} = \sum_{j\in J} vf_{ij}vf_{i_0 j}\cdots vf_{i_n j}.$$

□

From the expansion above also follow two more propositions:

Proposition 3.2.4. *Let $n \in \mathbb{Z}_0^+, n < |I|$, and fix $i, i_0, \ldots, i_n \in I$. For any permutation $\sigma \in S_{n+2}$ we have $s_i^{i_0,\ldots,i_n} = s_{\sigma i}^{\sigma i_0,\ldots,\sigma i_n}$.*

Proposition 3.2.5. *Let $n \in \mathbb{Z}_0^+, n < |I|$, and fix $i_0, \ldots, i_n \in I$. If $k, k' \in \{0, \ldots, n\}$ are such that $k < k'$, then (element-wise) $S^{i_0,\ldots,i_k,\ldots,i_{k'}} \leq S^{i_0,\ldots,i_k}$.*

Lemma 3.2.6. *Fix $i_0, i_1 \in I, i_0 \neq i_1$. There exists an edge E of P from m_{i_0} to m_{i_1} if and only if the following two conditions hold:*

1. $s_{i_1}^{i_0} \geq d - 1$.

2. *For each* $i \in I \setminus \{i_0, i_1\}$ *such that* $s_i^{i_0} \geq d-1$ *we have* $s_i^{i_0,i_1} < s_{i_1}^{i_0}$.

Proof. Fix $i_0, i_1 \in I, i_0 \neq i_1$.

Suppose there exists an edge E of P from m_{i_0} to m_{i_1}. Let $n \in \mathbb{Z}^+, n \leq d$. A non-empty intersection of n hyperplanes of $M_\mathbb{R}$ is of codimension at most n, thus the vertices m_{i_0} and m_{i_1} must lie in the intersection of at least $d-1$ facets [1].

Let $i \in I \setminus \{i_0, i_1\}$. By Propositions 3.2.4 and 3.2.5, we already have $s_i^{i_0,i_1} \leq s_{i_1}^{i_0}$ (if the reader has trouble seeing it, they may just expand as in Proposition 3.2.3). Assume for the sake of contradiction equality on some $i \in I \setminus \{i_0, i_1\}$. Then, by Proposition 3.2.3, m_i lies on the intersection of all facets of P shared by m_{i_0} and m_{i_1} or, equivalently, m_i lies on E. Indeed, to see this, note that, by Proposition 2.2.4, E is the intersection of all facets of P that contain E, and that if a facet F of P is shared by m_{i_0} and m_{i_1} then it is shared by all convex combinations of m_{i_0} and m_{i_1}, thus $E \subset F$. This is a contradiction, as edges contain only two vertices.

Suppose conditions 1 and 2 hold. Assume for the sake of contradiction that there exists a third vertex m_i on the intersection of all facets of P shared by m_{i_0} and m_{i_1}. Then, by Propositions 3.2.3 and 3.2.5, $s_i^{i_0} \geq s_i^{i_0,i_1} = s_{i_1}^{i_0} \geq d-1$, which contradicts condition 2. As a non-empty intersection of facets of P is a face of P, and a face that contain only two vertices is an edge, this implies that there exists an edge E from m_{i_0} to m_{i_1}. □

Example 3.2.7. *The polytope* $P \subset N_\mathbb{R}$ *with ID* $= 4$ *(see Figure 13) on GRDB STFV has vertices*

$$\{m_0 = (1,0), m_1 = (0,1), m_2 = (-1,0), m_3 = (0,-1)\},$$

and primitive inward-pointing facet normals

$$\{u_0 = (1,-1), u_1 = (-1,-1), u_2 = (-1,1), u_3 = (1,1)\}.$$

The vertex-facet inclusions are $m_0 \in F_1, m_0 \in F_2, m_1 \in F_0, m_1 \in F_1, \ldots$*, thus the vertices-facets matrix is:*

$$VF = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Supoose we want to calculate all primitive vectors on the edges of P *starting at* $i_0 = m_1$*. Lemma 3.2.6 gives us a powerful tool appropriate to the task. We firstly calculate* S^1*:*

$$S^1 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

[1] in practice, we will fix only i_0, and then use this necessary condition to prune possible values of i_1

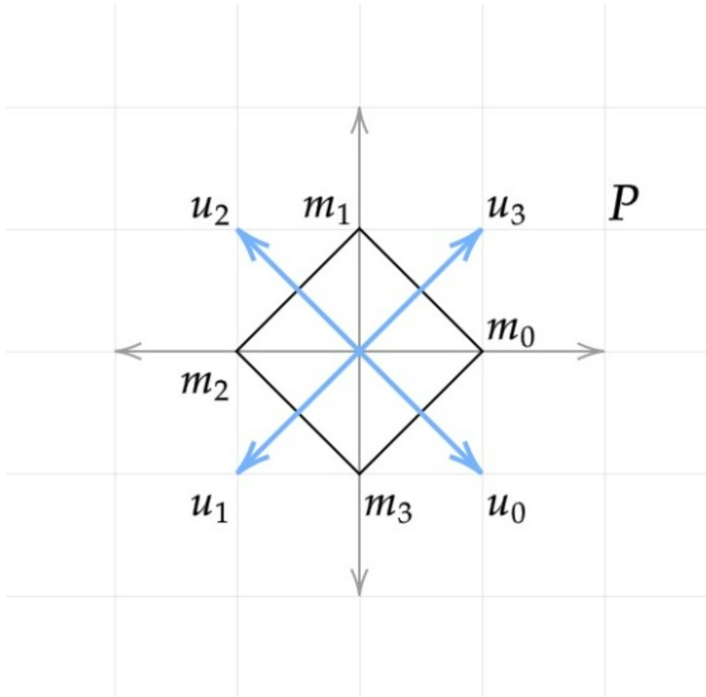


Figure 13: The polytope $P \subset N_{\mathbb{R}}$ with ID $= 4$ on GRDB STFV.

As $s_3^1 = 0 < 1 = d - 1$, there does not exist an edge E of P from m_1 to m_3. We now calculate $S^{1,0}$ and $S^{1,2}$:

$$S^{1,0} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \circ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$S^{1,2} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \circ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Since $s_0^1 = 1 \geq 1 = d - 1$ and $s_2^{1,0} = s_3^{1,0} = 0 < 1 = d - 1$, there exists an edge $(1, -1) = (1, 0) - (0, 1)$ from m_1 to m_0. Similarly, there exists an edge $(-1, -1) = (-1, 0) - (0, 1)$ from m_1 to m_2. These are already the primitive vectors on their respective edges: if they were not, we would have to divide them by the (positive) GCD of their coordinates.

3.2.4 Algorithms

Lemma 3.2.6 gives us Algorithm 2, which is a subroutine of Algorithm 4.

Algorithm 2 Function to find all primitive vectors on the edges of a full-dimensional lattice polytope starting at a given vertex.

Precondition: $V[|I|][d], VF[|I|][|J|], i$ are as in Algorithm 4 ($P \subset M_{\mathbb{R}}$ is a full-dimensional lattice polytope, $V[|I|][d]$ its vertices $VF[|I|][|J|]$ its vertices-facets matrix, and i the index of one of its vertices.)

Postcondition: The function returns a dynamic array containing all primitive vectors on the edges of P starting at $V[i]$.

1: Edges $\leftarrow$ A dynamic array.
2: $S^i \leftarrow VF(VF[i,:])^t$ ▷ See Definition 3.2.2.
3: **for** $0 \leq j < |I|$ **do**
4:     **if** $i \neq j$ && $S^i[j] \geq d-1$ **then**
5:         **if** CheckVertexLemma3.2.6(V, VF, i, S^i, j) **then** ▷ See Algorithm 3.
6:             Edges « $(V[j,:] - V[i,:])/GCD(V[j,:] - V[i,:])$ ▷ See Definition 2.2.30.
7:         **end if**
8:     **end if**
9: **end for**
10: Return Edges.

Algorithm 3 Function to check if a vertex satisfies the condition described in Lemma 3.2.6 or not.

Precondition: $V[|I|][d], VF[|I|][|J|], i, S^i, j$ are as in Algorithms 2 and 4.

Postcondition: The function returns True if $V[j]$ satisfies the condition described in Lemma 3.2.6, and False if not.

1: $S^{i,j} \leftarrow VF((VF[i,:])^t \circ (VF[j,:])^t)$ ▷ See Definition 3.2.2.
2: **for** $0 \leq k < |I|$ **do**
3:     **if** $k \neq i$ && $k \neq j$ && $S^i[k] \geq d-1$ && $S^{i,j}[k] = S^i[j]$ **then**
4:         Return False.
5:     **end if**
6: **end for**
7: Return True.

Theorem 3.2.8. ***(Correctness and time complexity of Algorithms 2 and 3)*** *For fixed d, Algorithms 2 and 3 are correct and have time complexity $O(|I|^2|J|)$, if VF is dense, or $O(|I|\#VF)$, where $\#VF$ is the number of non-zero entries of VF, if VF is sparse.*

Proof. By Lemma 3.2.6, the algorithms are correct.

The most computationally expensive steps are:

- Algorithm 2, line 2, which adds an $O(|I||J|)$ term.
- Algorithm 2, line 3, which implies that Algorithm 3 is executed at most $|I|$ times.

- Algorithm 3, line 1, which adds an $O(|I||J|)$ term.
- Algorithm 3, line 2, which adds an $O(|I|)$ term.

Thus, the time complexity is:

$$O(|I||J|) + |I|(O(|I||J|) + O(|I|)) = O(|I|^2|J|).$$

It is known that, if A is a sparse matrix with a non-zero entries, the matrix-vector multiplication Ax can be performed in time $O(a)$. VF is often sparse (e. g., if P has many facets), and then we have:

$$O(\#VF + |J|) + |I|(O(\#VF + |J|) + O(|I|)) = O(|I|\#VF + |I||J| + |I|^2),$$

where $\#VF$ is the number of non-zero entries of VF. Since $|I|, |J| \leq \#VF$, this reduces to $O(|I|\#VF)$. □

3.3 Existence algorithm

Let $I = \{0, \ldots, |I| - 1\}$, $J = \{0, \ldots, |J| - 1\}, d = \dim(M_\mathbb{R})$, and let $P^\circ \subset M_\mathbb{R}$ be the dual of a smooth Fano polytope given by its vertices $V[|I|][d]$ and primitive inward-pointing facet normals $H[|J|][d]$ (see Theorem 2.2.8.i and Proposition 2.2.18, respectively).

One of the two main problems we aim to solve in this thesis is designing an algorithm that decides if P° is additive or not. We call this decision problem the *existence problem*. To this effect, we have two equivalent criteria, given by Corollary 2.4.6 and Proposition 2.4.9. The latter gives an algorithm with a step of time complexity which is likely exponential on d and $|J|$. Indeed, the algorithm requires us to check all subsets of H of length d. An elementary bound on binomial coefficients, and a conjecture of Batyrev (see [Debarre, 2002]), which states that, if P is smooth Fano, then $d \leq |J| \leq 3d$, yield:

$$\binom{|J|}{d} < \left(\frac{|J|e}{d}\right)^d \leq (3e)^d.$$

Furthermore, for large d and $|J|$, Stirling's approximation yields:

$$\binom{|J|}{d} \sim \sqrt{\frac{|J|}{2\pi d(|J| - d)}} \frac{|J|^{|J|}}{d^d(|J| - d)^{(|J|-d)}}.$$

Since in subsection 3.6 we will also be interested in general very ample polytopes, to which Batyrev's conjecture does not apply, it seems reasonable to implement the algorithm given by Corollary 2.4.6 instead.

We implemented Algorithm 4 (which, in turn, depends on Algorithms 2, 3, and 5) based on results from [Arzhantsev and Romaskevich, 2017] (see Corollary 2.4.6). The following list is a more thorough description of the steps in our procedure:

- *Algorithm 4, line 1:* See Definition 3.2.1.
- *Algorithm 4, lines 2 to 7:* This *for* loop iterates through the columns of VF and fills in each of them at each step. For each $j \in J$, there exists $b_j \in \mathbb{R}$ such that for all $x \in P$ we have $\langle H[j], x\rangle \geq b_j$, as $H[j]$ is an inward-pointing facet normal. Thus, for each $i \in I, j \in J$, we have $\langle H[j], V[i]\rangle \geq b_j$. In particular, in the case of equality, $V[i]$ is in the supporting affine hyperplane of the facet F_j defined by $H[j]$, thus $V[i] \in F_j$. The converse is also clearly true.
- *Algorithm 4, lines 8 to 12:* This *for* loop iterates through the rows of VF and, at each step, checks if the vertex associated to the row satisfies the condition described in Corollary 2.4.6 or not, and returns True if it does. To this effect, we use Algorithm 5.
- *Algorithm 5, line 1:* We calculate the primitive vectors on the edges of P starting at $V[i]$, and store them in the dynamic array Edges. To this effect, we use Algorithms 2 and 3, see subsection 3.2.
- *Algorithm 5, lines 2 to 10:* This *for* loop iterates through the columns of VF and, at each step, if the vertex $V[i]$ is not in the facet F_j defined by $H[j]$, it iterates through the primitive vectors on the edges of P starting at $V[i]$ (i. e., the entries of Edges). In turn, this *for* loop checks if the negation of the inequality in Corollary 2.4.6 is true at each step, and returns False if it is.
- *Algorithm 5, line 11:* Return True if none of the pairs facet-edge in Corollary 2.4.6 satisfy the negation of the inequality therein.
- *Algorithm 4, line 13:* Return False if none of the vertices of P satisfy the condition in Corollary 2.4.6.

Algorithm 4 Algorithm to decide if the dual of a smooth Fano polytope given by both its vertices and primitive inward-pointing facet normals is additive or not.

Precondition: $P^\circ \subset M_\mathbb{R}$ is the dual of a smooth Fano polytope. $V[|I|][d]$ are the vertices of P°. $H[|J|][d]$ are the primitive inward-pointing facet normals of P°.

Postcondition: The algorithm returns True if P° is additive, and False if not.

1: $VF \leftarrow$ A zero array of dimension $|I| \times |J|$. ▷ See Definition 3.2.1.
2: **for** $0 \leq j < |J|$ **do** ▷ Fill in VF.
3: IndicesInFacet $\leftarrow Argmin_{0\leq i<|I|}\{\langle H[j], V[i]\rangle\}$
4: **for** $i \in$ IndicesInFacet **do**
5: $VF[i, j] \leftarrow 1$
6: **end for**
7: **end for**
8: **for** $0 \leq i < |I|$ **do** ▷ Check each vertex.
9: **if** CheckVertexCorollary2.4.6(V, H, VF, i) **then** ▷ See Algorithm 5.
10: Return True.
11: **end if**
12: **end for**
13: Return False.

Algorithm 5 Function to check if a vertex satisfies the condition described in Corollary 2.4.6 or not.

Precondition: $V[|I|][d]$, $H[|J|][d]$, $VF[|I|][|J|]$, i are as in Algorithm 4.
Postcondition: The function returns True if $V[i]$ satisfies the condition described in Corollary 2.4.6, and False if not.

1: Edges $\leftarrow$ FindEdgesFromVertex(V, VF, i) ▷ See Algorithm 2.
2: **for** $0 \leq j < |J|$ **do**
3:     **if** $VF[i][j] == 0$ **then**
4:         **for** $0 \leq k <$ # Edges **do**
5:             **if** $\langle H[j]$, Edges$[k]\rangle > 0$ **then**
6:                 Return False.
7:             **end if**
8:         **end for**
9:     **end if**
10: **end for**
11: Return True.

Theorem 3.3.1. ***(Correctness and time complexity of Algorithms 4 and 5)*** *For fixed d, Algorithms 4 and 5 are correct and have time complexity $O(|I|^3|J|)$, if VF is dense, or $O(|I|^2\#VF)$, where #VF is the number of non-zero entries of VF, if VF is sparse.*

Proof. By the previous analysis, the algorithms are correct.

The most computationally expensive steps are:

- Algorithm 4, lines 2 to 7, which add an $O(|I||J|)$ term.
- Algorithm 4, line 8, which implies that Algorithm 5 is executed at most $|I|$ times.
- Algorithm 5, line 1, which adds an $O(|I|^2|J|)$ or $O(|I|\#VF)$ term (see Theorem 3.2.8).
- Algorithm 5, lines 2 to 10, which add an $O(|J|)$ term (note that # Edges $= d = O(1)$).

Thus, the time complexity is $O(|I|^3|J|)$ or $O(|I|^2\#VF)$. □

Remark 3.3.2. *The procedure above may be slightly modified to generalise it to arbitrary very ample (full-dimensional lattice) polytopes, by using Theorem 2.4.5. In particular, we need just add a few lines before Algorithm 5, line 2, to check if the primitive vectors on the edges of P starting at $V[i]$ form a basis of M (i. e., if # Edges $= d$ and* det(*Edges*) $= \pm 1$*). We indeed use this observation in subsection 3.6 to implement our Macaulay2 package.*

3.4 Uniqueness algorithm

Let $J = \{0, \ldots, |J| - 1\}, d = \dim(M_{\mathbb{R}})$, and let $P^\circ \subset M_{\mathbb{R}}$ be the dual of a smooth Fano polytope given by the primitive inward-pointing facet normals $H[|J|][d]$ (see Proposition 2.2.18).

The second main problem analysed in this thesis is that of deciding if P° is uniquely additive or not. We call this decision problem the *uniqueness problem*. To this effect, we have the criterion given by Theorem 2.4.14, which may be used in conjunction with Algorithm 4, and optimised using Remark 2.4.10, which allows us to obtain a basis $B \subset N$ of N of inward-pointing facet normals such that the remaining lie in the negative octant with respect to B, whilst avoiding the combinatorial explosion described in subsection 3.3. We will, however, implement the algorithm exactly as suggested by [Dzhunusov, 2022], as we will only use it on small instances.

We implemented Algorithm 6 based on results from [Dzhunusov, 2022] (see Theorem 2.4.14). The following list is a more thorough description of the steps in our procedure:

- *Algorithm 6, lines 1 to 2:* PossibleBases is a dynamic array containing all subsets of H of length d. The Python library itertools, which "standardizes a core set of fast, memory efficient tools [that] form an "iterator algebra" making it possible to construct specialized tools succinctly and efficiently in pure Python" [Python, 2022], was very helpful. The for loop iterates through the elements $B[d][d]$ of PossibleBases.
- *Algorithm 6, lines 3 to 7:* If $\det(B)$ is close to ± 1, then $B \subset N$ is a basis of N.

  Sort*(H, B) sorts H in any way such that the first d columns of H are equal to the columns of B, in the same order. SortedH is a dynamic array containing the output of Sort*(H, B).

  After the execution of line 5, $R[d][|J| - d]$ is a dynamic array containing the columns of H not in B. After the execution of line 6, R is changed to the basis B.

  IsNonPositive(R) returns True if all entries of R are non-positive (i. e., $R[i, j] \leq 0$ for all $i \in \{0 \ldots, d - 1\}, j \in \{0, \ldots, |J| - d - 1\}$, or, equivalently, if the columns of R lie in the negative octant with respect to the basis $B \subset N$ of N), and False if not.

  These conditions are stated before on page 2 of [Dzhunusov, 2022] (before the main result), and are also described in Proposition 2.4.9.
- *Algorithm 6, line 8:* $B^* \subset M$ is the linear algebraic dual of B (i. e., the columns of B^* are the dual basis of the basis given by the columns of B).
- *Algorithm 6, lines 9 to 17:* This for loop iterates through indices $i \in \{0, \ldots, d - 1\}$ of columns of B, and for each i defines an integer linear programming problem $\mathcal{P}_i$:

  Let $x = (x_j)_{0 \leq i < d} \in M$. Maximise

$$f(x) = \sum_{\substack{j=0 \\ j \neq i}}^{d-1} \langle H[:, j], x \rangle,$$

subject to

$$\langle H[:,i],x\rangle = -1,$$
$$\langle H[:,j],x\rangle \geq 0, \quad \forall j \in J, j \neq i.$$

If a solution $S_1 \in M$ to $\mathcal{P}_i$ exists, then it is:

1) Equal to $S_0 = -B^*[:,i] \in M$, the negative of the i-th column of B^*, and then $\mathfrak{R}_i = \{S_0\}$ (see Definition 2.4.12).
2) Not equal to $S_0 \in M$, and then $\mathfrak{R}_i \neq \{S_0\}$.

Indeed, the set of feasible solutions to $\mathcal{P}_i$ is exactly $\mathfrak{R}_i$. If $S_0 \notin \mathfrak{R}_i$, we clearly have 2. If $S_0 \in \mathfrak{R}_i$, then note that $f(x) \geq 0$ for all $x \in \mathfrak{R}_i$, with equality if and only if $x = S_0$. Thus, if $S_0 \in \mathfrak{R}_i$ and $S_0 = S_1$, then $\max_{x\in\mathfrak{R}_i} f(x) = f(S_1) = f(S_0) = 0$, $f(x) = 0$ for all $x \in \mathfrak{R}_i$, and we have 1. If $S_0 \in \mathfrak{R}_i$ and $S_0 \neq S_1$, then $\{S_0, S_1\} \subset \mathfrak{R}_i$, and we have 2.

If the set of solutions to $\mathcal{P}_i$ is empty, then $\mathfrak{R}_i$ is empty, and $\mathfrak{R}_i \neq \{S_0\}$.

Since P° is uniquely additive if and only if $\mathfrak{R}_i = \{S_0\}$ for all $i \in \{0,\ldots,d-1\}$, we return False if there exists an $i \in \{0,\ldots,d-1\}$ such that $S_0 \neq S_1$ or the set of solutions of $\mathcal{P}_i$ is empty, and True if not.

For this last part we used PuLP [Mitchell et al., 2022], a powerful and widely used linear programming modeling Python library which also supports mixed and integer linear programming.

Algorithm 6 Algorithm to decide if the dual of an additive smooth Fano polytope given by its primitive inward-pointing facet normals is uniquely additive or not.

Precondition: $\varepsilon > 0$ is a small tolerance. $P^\circ \subset M_\mathbb{R}$ is the dual of an additive smooth Fano polytope. $H[d][|J|]$ are the primitive inward-pointing facet normals of P°. *Note: H is the transpose of its homonym in Algorithm 4!*

Postcondition: The algorithm returns True if P° is uniquely additive, and False if not.

PossibleBases $\leftarrow$ Subsets(H, d) ▷ Read description for more details.
for $B \in$ PossibleBases **do**
  if $||det(B)| - 1| \leq \varepsilon$ **then** ▷ If B is a basis.
    SortedH $\leftarrow$ Sort*(H, B) ▷ Read description for more details.
    $R \leftarrow$ SortedH$[:, d:]$ ▷ Slice vectors in H that are not in B.
    $R \leftarrow B^{-1}R$ ▷ Change of basis.
    if IsNonPositive(R) **then** ▷ Read description for more details.
      $B^* \leftarrow (B^{-1})^t$ ▷ Linear algebraic dual of B.
      for $0 \leq i < d$ **do**
        $S_0 \leftarrow -B^*[:, i]$ ▷ A priori solution.
        Model $\leftarrow$ DefineLPModel(Maximise, $x = (x_j)_{0 \leq j < d}$, $f(x)$, A, b) ▷ Read description for more details.
        $S_1 \leftarrow$ LPSolve(Model)
        if Model.status != “Optimal” or $\|S_1 - S_0\| \geq \varepsilon$ **then**
          Return False.
        end if
      end for
      Return True.
    end if
  end if
end for

3.5 Results

The algorithms in subsections 3.2 to 3.4 were implemented in Google Colab, and later in the Macaulay2 framework. The Python and Macaulay2 code, raw and processed data, plots, etc. may be found in the Github repository: `https://github.com/iZafiro/SFTV`.

3.5.1 Summary

The following table summarises the results of classifying all additive smooth Fano polytopes of dimensions 2 to 6. The raw data was web-scraped from the GRDB website [Brown and Kasprzyk, 2009], and was provided by [Øbro, 2007] (see subsection 3.1). For dimensions 2 to 4, these results will be verified by matching them with the existing literature (see section 4). The column labels indicate, in order:

1. **d**: Dimension.
2. **#NAP**: Number of non-additive polytopes.
3. **#(NU)AP**: Number of (not uniquely) additive polytopes.
4. **#UAP**: Number of uniquely additive polytopes.

d	#NAP	#(NU)AP	#UAP	Total	Note
2	1	2	2	**5**	
3	4	12	2	**18**	#(NU)AP + #UAP = 14 obtained by [Huang and Montero, 2020].
4	45	75	4	**124**	
5	396	466	4	**866**	
6	4194	3420	8	**7622**	
Total	**4640**	**3975**	**20**	**8635**	

Table 2: Summary of the classification of additive smooth Fano polytopes of dimensions 2 to 6.

3.5.2 Data visualisation

We used Seaborn [Waskom, 2022a], a modern Python data visualisation library, to plot histograms of the distribution of the three classes of smooth Fano polytopes (*non-additive*, *(not uniquely) additive*, and *uniquely additive*) of dimensions 2 to 6 with respect to various quantities. The plots show absolute frequency of each class on the y-axis, and *number of facets*, *vertices*, *volume*, *degree*, and boolean values for *centrally symmetric*, *zero barycentre*, and *zero dual barycentre* on the x-axis. We overlayed non-stacked histograms with kernel density estimates (KDEs), "a [method that] represents the data using a continuous probability density curve" [Waskom, 2022b], to make global tendencies easier to spot.

In order to be consistent with previous subsections, the number of facets and vertices of all polytopes have been transposed (i. e., if $P \subset N_{\mathbb{R}}$ is a smooth Fano polytope, its associated data have the number of facets and vertices corresponding to its polar polytope $P^{\circ} \subset M_{\mathbb{R}}$).

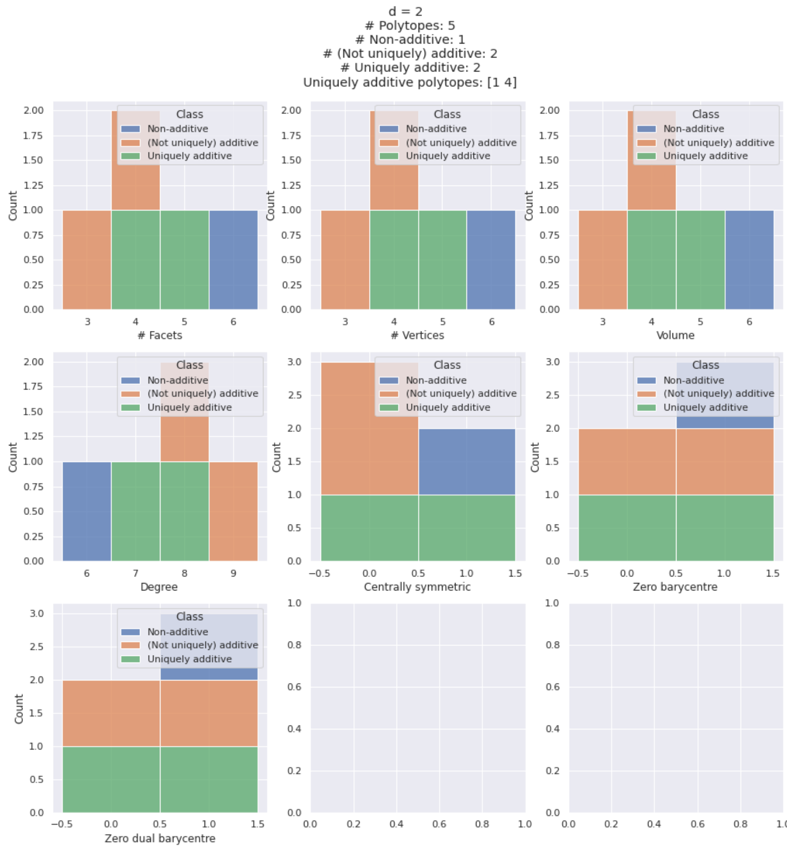


Figure 14: Distribution of the three classes of smooth Fano polytopes of dimension 2 with respect to various quantities.

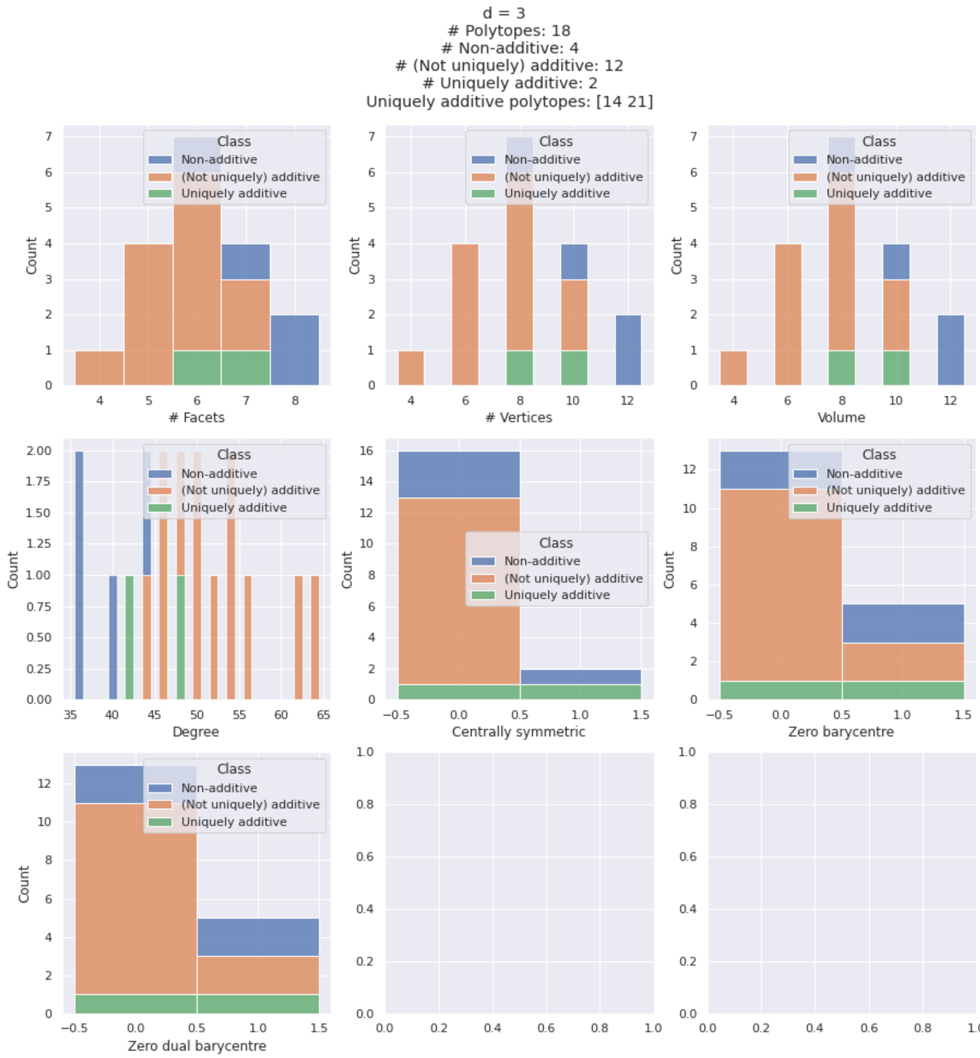


Figure 15: Distribution of the three classes of smooth Fano polytopes of dimension 3 with respect to various quantities.

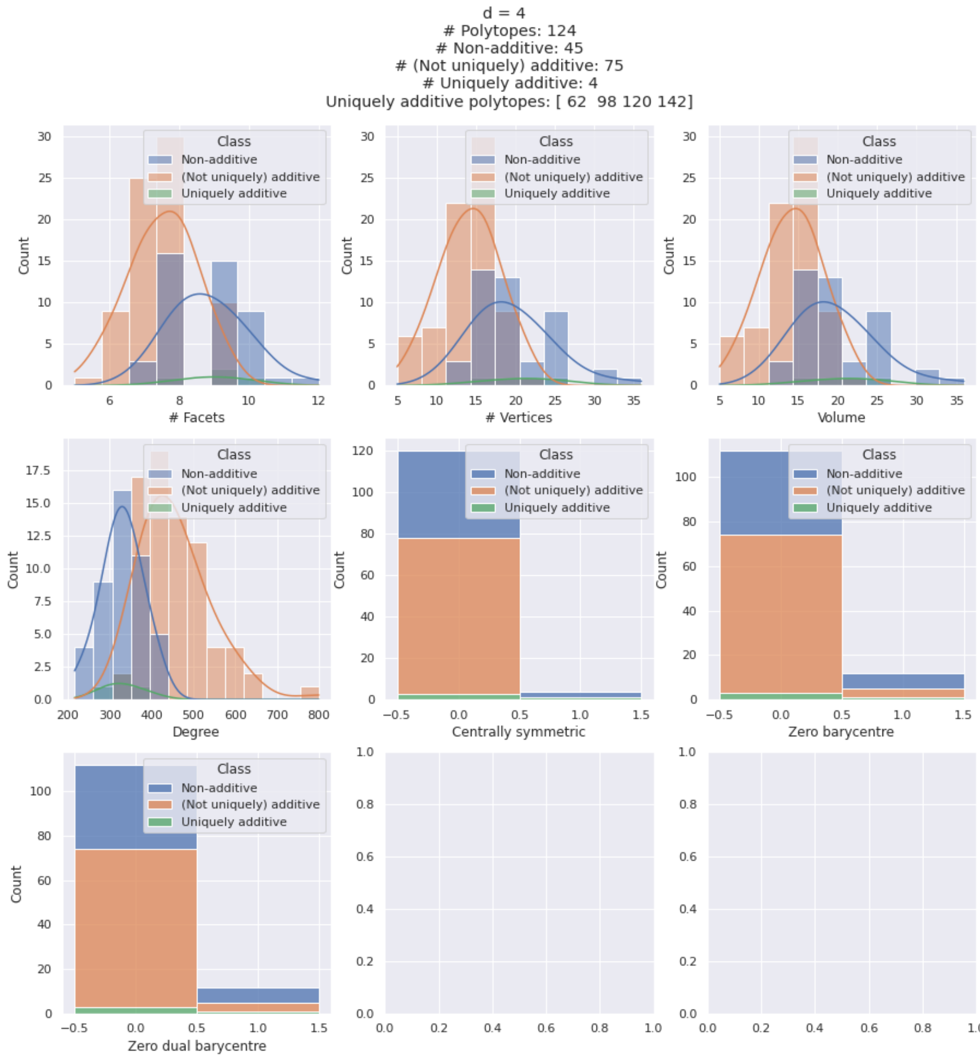


Figure 16: Distribution of the three classes of smooth Fano polytopes of dimension 4 with respect to various quantities.

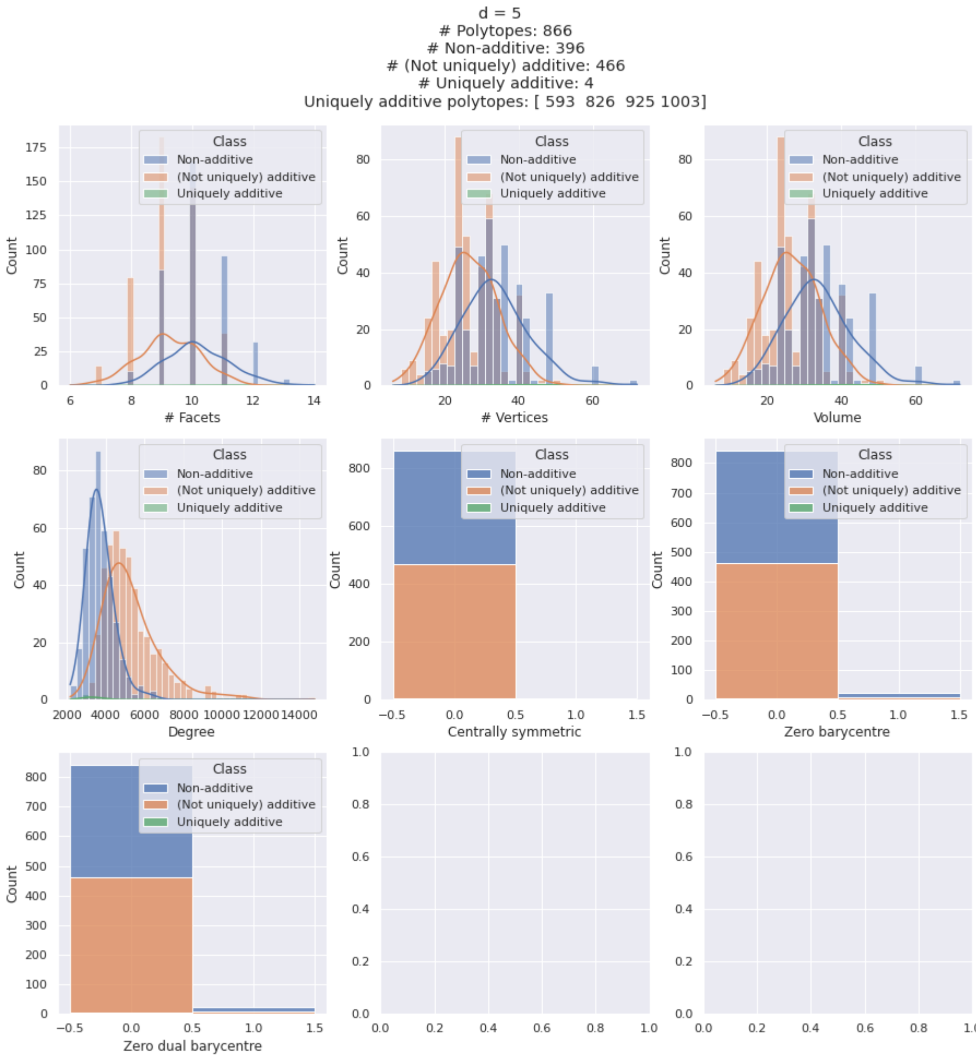


Figure 17: Distribution of the three classes of smooth Fano polytopes of dimension 5 with respect to various quantities.

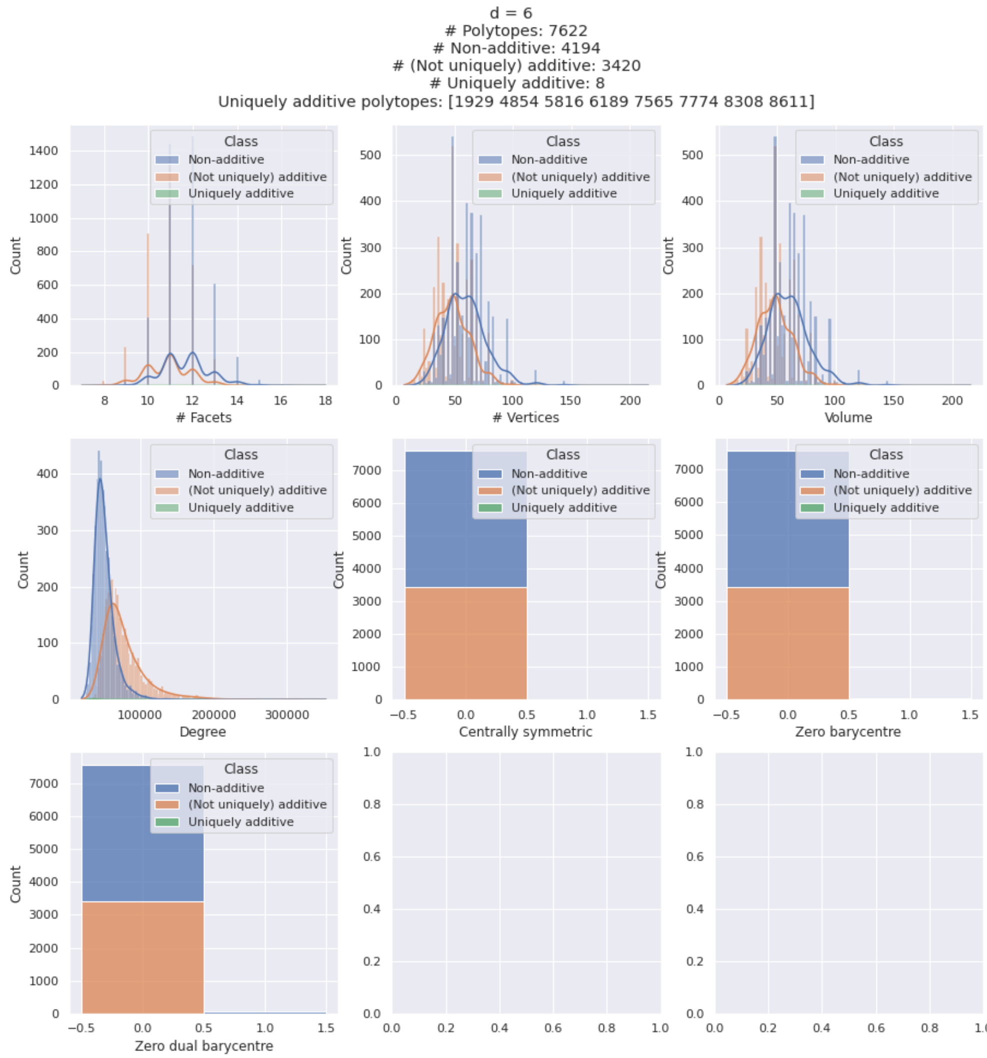


Figure 18: Distribution of the three classes of smooth Fano polytopes of dimension 6 with respect to various quantities.

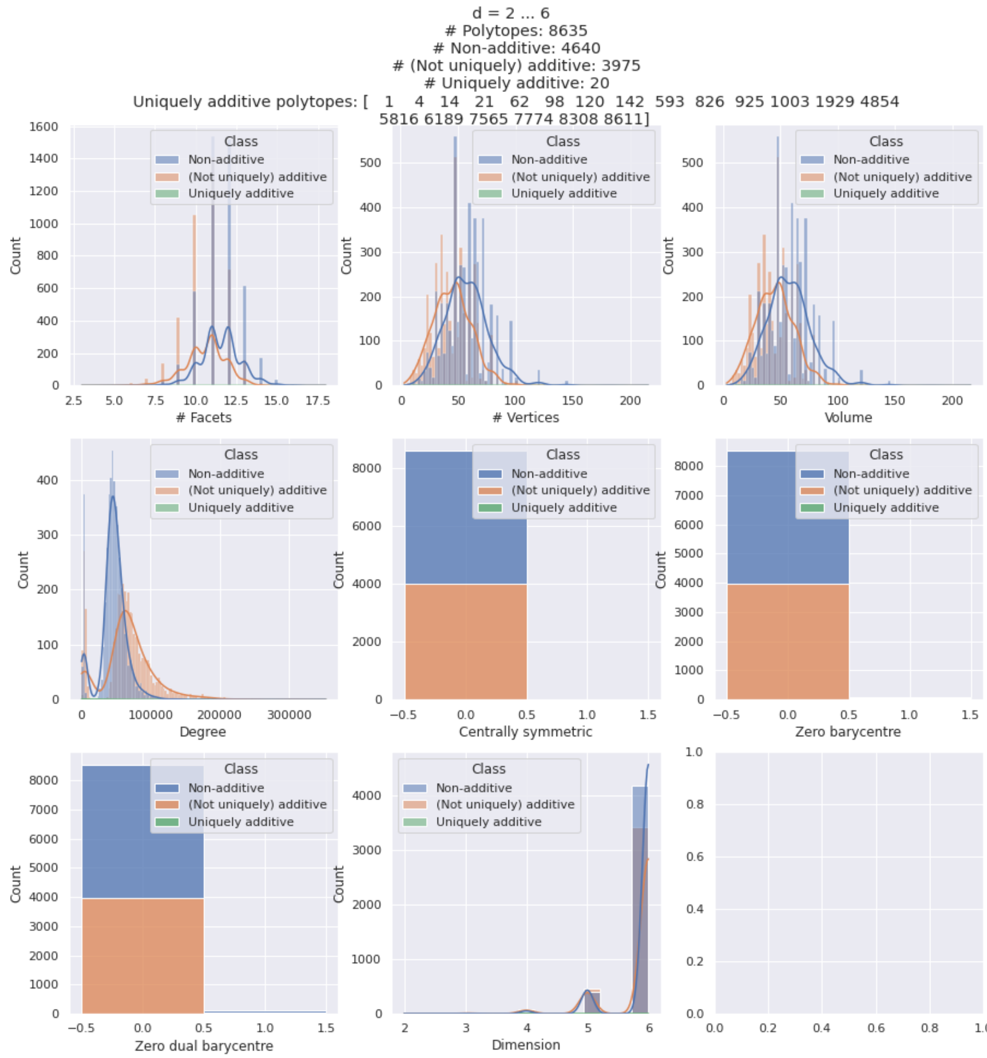


Figure 19: Distribution of the three classes of smooth Fano polytopes of dimensions 2 to 6 with respect to various quantities.

3.5.3 Qualitative analysis

We may infer the following qualitative notions from inspecting the histograms in Figures 14 to 19:

1. For $d = 2$ to 5, there are more additive polytopes than non-additive. For $d = 6$ (and thus also in the final histogram), this trend is reversed.
2. For $d = 2$, **number of facets** and **number of vertices** data are identical. This will be explained by the cohomological analysis in section 4.
3. **Volume** and **number of vertices** data are identical.
4. The frequency distribution of the **number of facets** and **number of vertices** for the non-additive and additive classes appears to be roughly normal, with a mean that increases with d. In all of these histograms, the mean for the non-additive class is higher than the mean for the additive class. This last observation is to be expected, at least in the **number of facets** case, as Demazure roots of polytopes with a higher **number of facets** are solutions to a more restrictive system of linear inequalities, thus additive actions on such polytopes are probably less common.
5. For $d = 3$ or 4 to 6, the frequency distribution of the **degree** for the non-additive and additive classes appears to have a low mean and a heavy tail to the right. This mean also increases with d. As a consequence, the final histogram shows several low-dimensional outliers to the left of the main distribution. In all of these histograms, the mean for the non-additive class is lower than the mean for the additive class, and the distribution for the non-additive class appears to be taller.
6. Most smooth Fano polytopes are not **centrally symmetric** nor have **zero (dual) barycentre**.
7. Although the class of smooth Fano polytopes is much less numerous than the class of reflexive polytopes, they also appear to increase exponentially with d.
8. A conjecture of Batyrev [Debarre, 2002] states that a smooth Fano polytope of dimension d has at most $3d$ vertices, if d is even, or $3d - 1$ vertices, if d is odd. This is clearly verified in the histograms of the **number of facets** for each dimension.
9. Finally, we note that the technique of producing a KDE "has the potential to introduce distortions if the underlying distribution is bounded or not smooth" [Waskom, 2022b]. This phenomenon can be seen in many plots, as the values on the x-axis are not continuous. However, "the quality of the representation also depends on the selection of good smoothing parameters" [Waskom, 2022b]. We experimented a fair bit with these parameters before being satisfied with the plots.

Giving a more convincing mathematical argument for many of these observations is outside the scope of this thesis. This may be difficult in some cases (for example, observations 5 and 8) or trivial in others, and may be the subject of future research. In any case, the latter part of this subsection is evidence for the usefulness of data visualisation in higher mathematics.

3.6 Macaulay2 package

The Macaulay2 package and installation instructions may be found in the Github repository: `https://github.com/iZafiro/SFTV`.

We created AdditiveProjectiveToricVarieties, a new Macaulay2 package with methods for working with additive actions on projective toric varieties, based on algorithms discussed on this thesis. We followed package writing guidelines and standards described in the Macaulay2 website [Grayson et al., 1993]. The package is still a work in progress, so there will be improvements and more testing. In particular, algorithms for calculating Demazure roots, and algorithms on subsections 3.2 and 3.4 of this thesis, are missing. Nevertheless, one of the specific objetives of this thesis was to implement the existence algorithm, and this was done.

The package also includes a pre-processed database which classifies as additive or non-additive smooth Fano toric varieties of dimension one to five as per the database on the NormalToricVarieties package.

The package dependencies are packages NormalToricVarieties and Polyhedra. The exported methods are *isAdditive*, *listAdditiveSmoothFanoToricVarieties*, and *randomAdditiveSmoothFanoToricVariety*. The following is a description of each method:

- *isAdditive:* This method requires either an object P of type Polyhedron, an object X of type NormalToricVariety, or two integers d, n which correspond to a smooth Fano toric variety as per the database on the NormalToricVarieties package. Thus, this method is polymorphic, and has three different implementations:
  1) If the input is an object P of type Polyhedron, we call the main routine.

     We run three guard clauses for input validation: one that checks if P is compact, another that checks if P is full-dimensional, and another that checks if P is a lattice polytope. If P is not very ample, we scale P by $d-1$, so that it is, as the algorithm requires it.

     We implemented the generalised version of the existence algorithm described in Remark 3.3.2. We also wrote two auxiliary methods, *findEdgesFromVertex*, and *checkVertex*, which correspond to the Algorithms 2, and 5, respectively, although *findEdgesFromVertex* was implemented using pre-existing methods in the Polyhedra package. *isAdditive* proved to be very slow for some polytopes of dimension greater than or equal to six, and we suspect this is the reason, as these pre-existing methods may be used to calculate the entire face lattice of the polytope, and may not be optimised for calculating only edges.

     This method returns True, if the associated projective variety is additive, and False if not.
  2) If the input is an object X of type NormalToricVariety, we call a second routine. We run a guard clause for input validation, which checks if X is projective. Then, we calculate the normal fan of X using a method in the NormalToricVarieties package, get the convex hull and polar polytope using methods in the Polyhedra package, and call the main routine with the resulting polytope.
  3) If the input are two integers d, n, we call a third routine. We run various guard clauses for input validation, which check if d, n correspond to valid inputs for the SmoothFanoToricVariety method on the NormalToricVarieties package. If the dimension is equal to

six, we call the second routine with the corresponding smooth Fano toric variety. If the dimension is less than or equal to five, we fetch the return value from the pre-processed database mentioned earlier.

- *listAdditiveSmoothFanoToricVarieties:* This method requires an integer d such that $d \geq 1$ and $d \leq 5$. We run a guard clause for input validation, which checks if d is valid. Then, we return a list containing indices as per the database on the NormalToricVarieties package, and the corresponding additive smooth Fano toric varieties as objects of type NormalToricVariety. This method is fast, as we fetch the data from the pre-processed database. This list is comprehensive: it contains all such varieties of the given dimension d.

- *randomAdditiveSmoothFanoToricVariety*: This method requires an integer d such that $d \geq 1$ and $d \leq 5$. We run a guard clause for input validation, which checks if d is valid. Then, we return a list containing an index as per the database on the NormalToricVarieties package, and the corresponding additive smooth Fano toric variety as an object of type NormalToricVariety. This variety is randomly (uniformly) sampled from the pre-processed database, and is of the given dimension d.

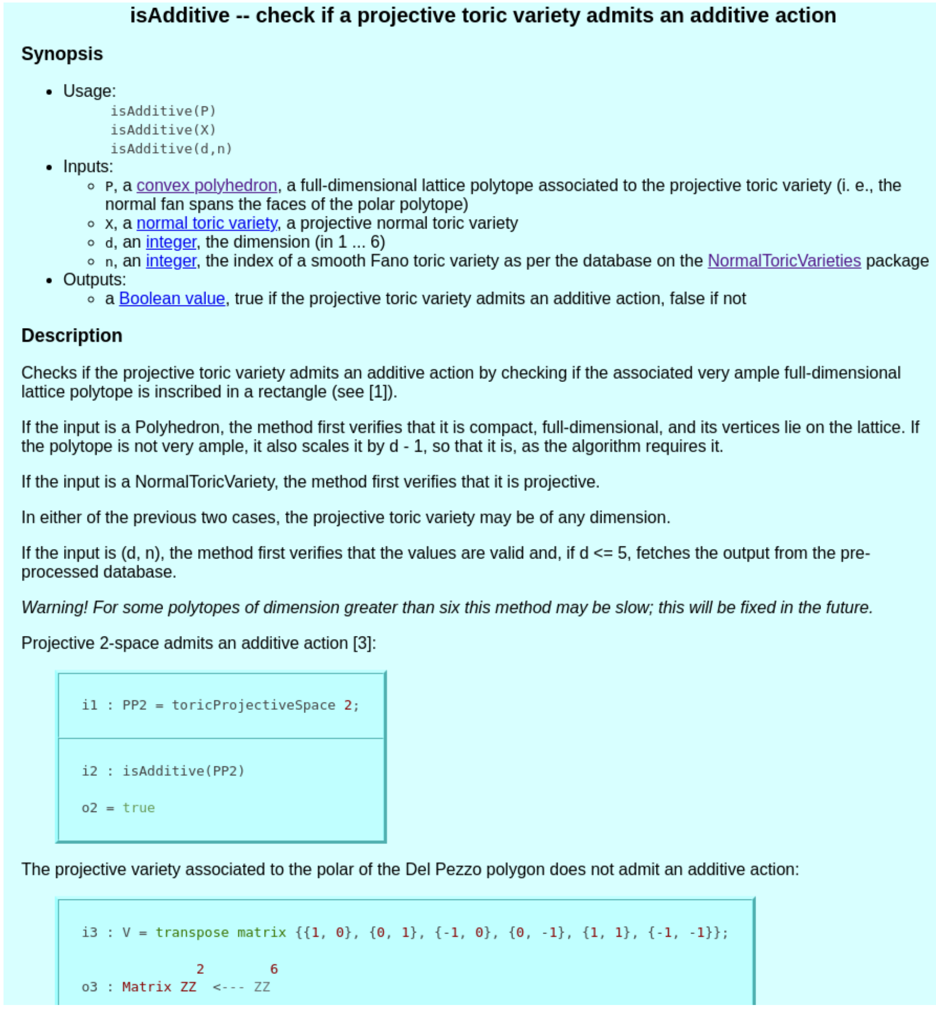
The package also includes documentation and examples for each method, which are automatically built as HTML files upon installation (see Figure 20).

Last, but not least, the package includes eight automatic tests, which may be run at any time. The tests do the following:

- The first six test the three implementations of the *isAdditive* method with smooth Fano toric varieties of dimension 1 to 4, by counting the number of times the method returns true and comparing it with results on subsection 3.5.

- The seventh tests the *listAdditiveSmoothFanoToricVarieties* method similarly, by comparing the lengths of the returned lists.

- The eighth tests the *randomAdditiveSmoothFanoToricVariety* method by calling it once for each $d \in \{1, \ldots, 5\}$, and also calling the *isAdditive* method to assert that the returned variety is additive. This is the only random test.



isAdditive -- check if a projective toric variety admits an additive action

Synopsis

- Usage:
```
isAdditive(P)
isAdditive(X)
isAdditive(d,n)
```
- Inputs:
  - P, a convex polyhedron, a full-dimensional lattice polytope associated to the projective toric variety (i. e., the normal fan spans the faces of the polar polytope)
  - X, a normal toric variety, a projective normal toric variety
  - d, an integer, the dimension (in 1 ... 6)
  - n, an integer, the index of a smooth Fano toric variety as per the database on the NormalToricVarieties package
- Outputs:
  - a Boolean value, true if the projective toric variety admits an additive action, false if not

Description

Checks if the projective toric variety admits an additive action by checking if the associated very ample full-dimensional lattice polytope is inscribed in a rectangle (see [1]).

If the input is a Polyhedron, the method first verifies that it is compact, full-dimensional, and its vertices lie on the lattice. If the polytope is not very ample, it also scales it by d - 1, so that it is, as the algorithm requires.

If the input is a NormalToricVariety, the method first verifies that it is projective.

In either of the previous two cases, the projective toric variety may be of any dimension.

If the input is (d, n), the method first verifies that the values are valid and, if d <= 5, fetches the output from the pre-processed database.

Warning! For some polytopes of dimension greater than six this method may be slow; this will be fixed in the future.

Projective 2-space admits an additive action [3]:

```
i1 : PP2 = toricProjectiveSpace 2;

i2 : isAdditive(PP2)

o2 = true
```

The projective variety associated to the polar of the Del Pezzo polygon does not admit an additive action:

```
i3 : V = transpose matrix {{1, 0}, {0, 1}, {-1, 0}, {0, -1}, {1, 1}, {-1, -1}};

               2        6
o3 : Matrix ZZ  <--- ZZ
```

Figure 20: Automatically built HTML file corresponding to documentation for the method *isAdditive*.

SECTION 4

RESULTS VALIDATION

We will match our results for dimensions 2 to 4 with already existing classifications in the literature. Smooth Fano polytopes are simplicial, therefore we can use results in [Fulton, 1993] to calculate the rational cohomology of their associated varieties in terms of their number of vertices and facets. This will prove immediately useful.

Let N, M be dual lattices with associated vector spaces $N_{\mathbb{R}}, M_{\mathbb{R}}$ of dimension $d \in \mathbb{Z}^+$. Let $\{e_1, \ldots, e_d\} \subset N_{\mathbb{R}}, \{e_1^*, \ldots, e_d^*\} \subset M_{\mathbb{R}}$ be bases of N, M, respectively.

4.1 Cohomology of smooth Fano toric varieties

Let $P \subset N_{\mathbb{R}}$ be a simplicial polytope with $|I|$ vertices and $|J|$ facets. We will use the following simplifying abuse of notation throughout this entire section: we will denote by X_P the projective toric variety associated to the fan which covers the faces of P (i. e., the projective toric variety which we have hitherto referred to as X_{P°).

Definition 4.1.1. ***(Betti number, face number, constant Betti number)*** *Let $i \in \{0, \ldots, 2d\}, p \in \{0, \ldots, d\}$, and let $H^i(X_P, \mathbb{Q})$ be the i-th rational cohomology group of X_P. The i-th* ***Betti number*** *of X_P is $b_i := \dim(H^i(X_P, \mathbb{Q}))$ (in particular, $b_2 = |Pic(X_P)|$, where $Pic(X_P)$ is the Picard group of X_P). We also define $h_p := b_{2p}$.*

The $i-1$-th ***face number*** *f_{i-1} of P is its number of faces of dimension $i-1$ (in particular, $f_0 = |I|, f_{d-1} = |J|$), if $i > 0$, or $f_{-1} = 1$, if $i = 0$.*

Finally, we say a Betti number of X_P is ***constant*** *if it is equal to an expression which does not depend on any face number.*

Proposition 4.1.2. ***(Poincaré duality, Vanishing of odd Betti numbers)*** *By Poincaré duality, for each $p \in \{0, \ldots, \lfloor \frac{d}{2} \rfloor\}$, we have $h_p = h_{d-p}$. Furthermore, the rational cohomology of X_P vanishes in odd dimensions, thus so do odd Betti numbers.*

Proposition 4.1.3. ***(Formula for even Betti numbers)*** *Let $p \in \{0, \ldots, d\}$. We have:*

$$h_p = b_{2p} = \sum_{i=p}^{d} (-1)^{i-p} \binom{i}{p} f_{d-i-1}.$$

In particular:

i) The constant even Betti numbers of X_P are $h_0 = b_0 = \sum_{i=0}^{d} (-1)^i f_{d-i-1} = h_d = b_{2d} = 1$.

ii) $h_1 = b_2 = \sum_{i=1}^{d} (-1)^{i-1} i f_{d-i-1} = h_{d-1} = b_{2d-2} = f_0 - d = |I| - d$.

Proof. (Propositions 4.1.2, 4.1.3.) See [Fulton, 1993], section 5 for the application of Poincaré duality, vanishing of odd Betti numbers, and formula for even Betti numbers. The consequences follow easily. □

4.2 Dimension 2

Let $P \subset N_{\mathbb{R}}$ be a simplicial polytope of dimension 2 with $|I|$ vertices and $|J|$ facets.

Proposition 4.2.1. *The following are true:*

i) The only non-constant Betti number of X_P is $h_1 = b_2 = |Pic(X_P)| = |I| - 2$.

ii) $|I| = |J|$.

Remark 4.2.2. *Item ii of this proposition justifies item 2 of the qualitative analysis in subsection 3.5.3.*

Proof. Item i follows from Proposition 4.1.2, and from expanding Proposition 4.1.3 for $p = 1$. Item ii follows from expanding Proposition 4.1.3 for $p = 0, 2$. □

The following tables contain the matching of additive smooth Fano polytopes of dimension 2 (i. e., our results) with the classification in Debarre's *Fano Varieties and Polytopes* [Debarre, 2002] (see subsection 2.3). This is easily done by inspecting each polytope. Non-additive polytopes are omitted. The rows are ordered as in [Debarre, 2002]. The column labels indicate, in order:

1. **N°**: The position in [Debarre, 2002].
2. **ID**: The ID in the GRDB.
3. **Degree**: The degree $(-K_X)^2$, obtained from the GRDB.
4. **b$_2$**: The second Betti number, calculated using Proposition 4.2.1.
5. $|$**I**$|$, $|$**J**$|$: The number of vertices and facets of the smooth Fano polytope, obtained from the GRDB.
6. **Notation**: The notation in [Debarre, 2002].
7. **Uniquely additive?**: *Yes* if the polytope is uniquely additive, *no* if it is (not uniquely) additive.
8. **Smooth Fano variety** (Table 4): The associated smooth Fano variety, obtained from [Debarre, 2002].

N°	ID	Degree	b$_2$	|I|	|J|	Notation	Uniquely additive?
1	4	8	2	4	4	$\mathbb{P}^1 \times \mathbb{P}^1$	Yes
2	5	9	1	3	3	S_2	No
3	3	8	2	4	4	$Bl_p(\mathbb{P}^2)$	No
4	1	7	3	5	5	A_2	Yes

Table 3: Matching of additive smooth Fano polytopes of dimension 2 with the classification in Debarre's *Fano Varieties and Polytopes* (see [Debarre, 2002]).

N°	ID	Notation	Smooth Fano variety
1	4	$\mathbb{P}^1 \times \mathbb{P}^1$	$\mathbb{P}^1 \times \mathbb{P}^1$
4	1	A_2	$Bl_{p,q}(\mathbb{P}^2)$

Table 4: Matching of uniquely additive smooth Fano polytopes of dimension 2 with the classification in Debarre's *Fano Varieties and Polytopes* (see [Debarre, 2002]).

4.3 Dimension 3

Let $P \subset N_{\mathbb{R}}$ be a simplicial polytope of dimension 3 with $|I|$ vertices and $|J|$ facets.

Proposition 4.3.1. *The non-constant Betti numbers of X_P are $h_1 = b_2 = h_2 = b_4 = |Pic(X_P)| = |I| - 3$.*

Proof. This is an immediate consequence of Propositions 4.1.2, 4.1.3.i, ii. □

The following tables contain the matching of additive smooth Fano polytopes of dimension 3 (i. e., our results) with the classification by [Mori and Mukai, 2003]. This is mostly done by matching the $(-K_X)^3$ and B_2 in [Mori and Mukai, 2003] with the degree obtained from the GRDB and the second Betti number calculated using Proposition 4.3.1. Non-additive polytopes are omitted. The rows are ordered as in [Mori and Mukai, 2003]. The column labels indicate, in order:

1. **N°**: The n° in [Mori and Mukai, 2003].
2. **ID**: The ID in the GRDB.
3. **Degree**: The degree $(-K_X)^3$, obtained from [Mori and Mukai, 2003] and the GRDB.
4. **b$_2$**: The second Betti number, obtained from [Mori and Mukai, 2003] and calculated using Proposition 4.3.1.
5. $|$**I**$|$, $|$**J**$|$: The number of vertices and facets of the smooth Fano polytope, obtained from the GRDB.
6. **Notation**: The notation in [Huang and Montero, 2020].
7. **Uniquely additive?**: *Yes* if the polytope is uniquely additive, *no* if it is (not uniquely) additive.
8. **Smooth Fano variety** (Table 6): The associated smooth Fano variety, obtained from [Mori and Mukai, 2003].

N°	ID	Degree	b$_2$	$\|$I$\|$	$\|$J$\|$	Notation	Uniquely additive?
-	23	64	1	4	4	$\mathbb{P}^3$	No
33	22	54	2	5	6	II_{33}	No
34	19	54	2	5	6	II_{34}	No
35	20	56	2	5	6	II_{35}	No
36	7	62	2	5	6	II_{36}	No
26	16	46	3	6	8	III_{26}	No
27	21	48	3	6	8	III_{27}	Yes
28	17	48	3	6	8	III_{28}	No
29, 30	6	50	3	6	8	III_{29}, III_{30}	No
29, 30	12	50	3	6	8	III_{29}, III_{30}	No
31	11	52	3	6	8	III_{31}	No

10	14	42	4	7	10	IV_{10}	Yes
11	10	44	4	7	10	IV_{11}	No
12	8	46	4	7	10	IV_{12}	No

Table 5: Matching of additive smooth Fano polytopes of dimension 3 with the classification by Mori, Mukai (see [Mori and Mukai, 2003], [Huang and Montero, 2020]).

Nº	**ID**	**Notation**	**Smooth Fano variety**
27	21	III_{27}	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$
10	14	IV_{10}	$\mathbb{P}^1 \times S_7$

Table 6: Matching of uniquely additive smooth Fano polytopes of dimension 3 with the classification by Mori, Mukai (see [Mori and Mukai, 2003], [Huang and Montero, 2020]).

Remark 4.3.2. *The polytope S_7 is the unique (up to isomorphism) Del Pezzo surface of degree 7, which is given by $Bl_{p,q}(\mathbb{P}^2)$, the blow-up of $\mathbb{P}^2$ at two general points p, q.*

Remark 4.3.3. *As previously mentioned, the number of additive smooth Fano toric threefolds appears initially in [Huang and Montero, 2020], who found it through theoretical means.*

4.4 Dimension 4

Let $P \subset N_{\mathbb{R}}$ be a simplicial polytope of dimension 4 with $|I|$ vertices and $|J|$ facets.

Proposition 4.4.1. *The non-constant Betti numbers of X_P are $h_1 = b_2 = h_3 = b_6 = |Pic(X_P)| = |I| - 4, h_2 = b_4 = -2|I| + |J| + 6$.*

Proof. Proposition 4.1.3.ii implies $h_1 = b_2 = f_2 - 2f_1 + 3|I| - 4 = h_3 = b_6 = |I| - 4$. Then,

$$f_2 - 2f_1 + 2|I| = 0. \quad (1)$$

On the other hand, Proposition 4.1.3.i implies $h_0 = b_0 = |J| - f_2 + f_1 - |I| + 1 = h_4 = b_8 = 1$. Then,

$$|J| - f_2 + f_1 - |I| = 0. \quad (2)$$

Adding (1) and (2) gives $f_1 = |I| + |J|$, thus the formula in Proposition 4.1.3 implies $h_2 = b_4 = f_1 - 3|I| + 6 = -2|I| + |J| + 6$. □

The following tables contain the matching of additive smooth Fano polytopes of dimension 4 (i. e., our results) with the classification by [Batyrev, 1999]. This is done by using the matching in [Doi and Yotsutani, 2015], and verifying its correctness by matching the c_1^4, b_2 and b_4 in [Batyrev, 1999] with the degree obtained from the GRDB and the second and fourth Betti numbers calculated using Proposition 4.4.1. Non-additive polytopes are omitted. The rows are ordered as in [Batyrev, 1999]. The column labels indicate, in order:

1. **N°**: The n° in [Batyrev, 1999].
2. **ID**: The ID in the GRDB.
3. **Degree**: The degree $(-K_X)^4$, obtained from [Batyrev, 1999] and the GRDB.
4. $\mathbf{b_2}, \mathbf{b_4}$: The second and fourth Betti numbers, obtained from [Batyrev, 1999] and calculated using Proposition 4.4.1.
5. $|\mathbf{J}|, |\mathbf{I}|$: The number of vertices and facets of the smooth Fano polytope, obtained from the GRDB.
6. **Notation**: The notation in [Batyrev, 1999].
7. **Uniquely additive?**: *Yes* if the polytope is uniquely additive, *no* if it is (not uniquely) additive.
8. **Smooth Fano variety** (Table 8): The associated smooth Fano variety, obtained from [Batyrev, 1999].

N°	ID	Degree	b_2	b_4	$\|J\|$	$\|I\|$	Notation	Uniquely additive?
1	147	625	1	1	5	5	$\mathbb{P}^4$	No
2	25	800	2	2	8	6	B_1	No
3	139	640	2	2	8	6	B_2	No
4	144	544	2	2	8	6	B_3	No
5	145	512	2	2	8	6	B_4	No
6	138	512	2	2	8	6	B_5	No
7	44	594	2	3	9	6	C_1	No
8	141	513	2	3	9	6	C_2	No
9	70	513	2	3	9	6	C_3	No
10	146	486	2	3	9	6	C_4	No
11	24	605	3	3	11	7	E_1	No
12	128	489	3	3	11	7	E_2	No
13	127	431	3	3	11	7	E_3	No
14	30	592	3	4	12	7	D_1	No
15	31	576	3	4	12	7	D_2	No
16	49	560	3	4	12	7	D_3	No
17	35	560	3	4	12	7	D_4	No
18	42	496	3	4	12	7	D_5	No
19	129	496	3	4	12	7	D_6	No
20	97	486	3	4	12	7	D_7	No
21	134	480	3	4	12	7	D_8	No
22	66	464	3	4	12	7	D_9	No
23	132	464	3	4	12	7	D_{10}	No
24	117	459	3	4	12	7	D_{11}	No
25	140	448	3	4	12	7	D_{12}	No
26	143	432	3	4	12	7	D_{13}	No
27	133	432	3	4	12	7	D_{14}	No

28	135	432	3	4	12	7	D_{15}	No
29	68	432	3	4	12	7	D_{16}	No
33	41	529	3	5	13	7	G_1	No
34	40	450	3	5	13	7	G_2	No
35	64	433	3	5	13	7	G_3	No
36	60	417	3	5	13	7	G_4	No
37	69	406	3	5	13	7	G_5	No
38	137	401	3	5	13	7	G_6	No
39	26	558	4	5	15	8	H_1	No
40	45	505	4	5	15	8	H_2	No
41	28	478	4	5	15	8	H_3	No
42	118	447	4	5	15	8	H_4	No
43	123	415	4	5	15	8	H_5	No
46	124	378	4	5	15	8	H_8	No
49	74	480	4	6	16	8	L_1	No
50	75	464	4	6	16	8	L_2	No
51	83	448	4	6	16	8	L_3	No
52	105	432	4	6	16	8	L_4	No
53	95	416	4	6	16	8	L_5	No
54	112	400	4	6	16	8	L_6	No
55	106	384	4	6	16	8	L_7	No
56	142	384	4	6	16	8	L_8	Yes
57	130	384	4	6	16	8	L_9	No
62	33	496	4	6	16	8	I_1	No
63	29	463	4	6	16	8	I_2	No
64	47	442	4	6	16	8	I_3	No
65	38	433	4	6	16	8	I_4	No
67	93	411	4	6	16	8	I_6	No
68	37	400	4	6	16	8	I_7	No
69	115	384	4	6	16	8	I_8	No
70	94	390	4	6	16	8	I_9	No
71	111	389	4	6	16	8	I_{10}	No
72	59	384	4	6	16	8	I_{11}	No
74	126	368	4	6	16	8	I_{13}	No
77	61	385	4	7	17	8	M_1	No
78	50	417	4	7	17	8	M_2	No
79	58	369	4	7	17	8	M_3	No
80	57	369	4	7	17	8	M_4	No
81	110	364	4	7	17	8	M_5	No
84	71	442	5	8	20	9	Q_1	No
85	79	405	5	8	20	9	Q_2	No
86	73	394	5	8	20	9	Q_3	No
87	77	405	5	8	20	9	Q_4	No
88	81	373	5	8	20	9	Q_5	No

89	84	368	5	8	20	9	Q_6	No
90	91	363	5	8	20	9	Q_7	No
91	90	352	5	8	20	9	Q_8	No
93	102	336	5	8	20	9	Q_{10}	No
94	120	336	5	8	20	9	Q_{11}	Yes
105	89	332	5	9	21	9	R_1	No
117	62	307	5	11	23	9	See Remark 4.4.2	Yes
119	98	294	6	11	25	10	$S_2 \times S_2$	Yes

Table 7: Matching of additive smooth Fano polytopes of dimension 4 with Batyrev's classification (see [Batyrev, 1999], [Doi and Yotsutani, 2015]).

N°	**ID**	**Notation**	**Smooth Fano variety**
56	142	L_8	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$
94	120	Q_{11}	$\mathbb{P}^1 \times \mathbb{P}^1 \times S_2$
117	62	See Remark 4.4.2	See Remark 4.4.2
119	98	$S_2 \times S_2$	$S_2 \times S_2$

Table 8: Matching of uniquely additive smooth Fano polytopes of dimension 4 with Batyrev's classification (see [Batyrev, 1999], [Doi and Yotsutani, 2015]).

Remark 4.4.2. ***(Tables 7, 8)*** *Polytope N° 117 is isomorphic to the polytope given by the $\mathcal{V}$-representation*

$$\left\{ \pm e_1, \pm e_2, \pm e_3, \pm e_4, \sum_{i=1}^{4} e_i \right\}.$$

SECTION 5
CONCLUSIONS

At last! We have arrived to this momentuous (and somewhat dreaded) section. We have provided the world with yet another introduction to the beautiful topics of toric geometry, polytope theory, and their intertwinings; albeit one disturbingly lacking in proofs and not particularly original. The author can only hope that it has served its purpose of making this thesis self-contained, well-motivated, and understandable. The reader, on the other hand, may rest assured that it was written with care, and is encouraged to read the references for more comprehensive justifications of each result. Nomenclature may vary from reference to reference, but throughout this text it tries to be consistent and to adhere to modern conventions (in particular, reading the dissertation by Nill [Nill, 2005] is a great way to be persuaded to keep the adjective *smooth* on *smooth Fano polytope*).

We have designed and implemented an algorithm to obtain the edges of a polytope of any dimension which compares favourably to existing algorithms in the literature.

We have provided a short review of the papers from Arzhantsev, Romaskevich, and Dzhunusov [Arzhantsev and Romaskevich, 2017] [Dzhunusov, 2022], and have used their results to design and implement algorithms to systematically classify (complete) projective toric varieties as additive, uniquely additive, or non-additive. We have used these algorithms to classify smooth Fano toric varieties of dimension up to six: this is our main new result, as it was previously only partially done in the cases of surfaces and threefolds [Huang and Montero, 2020].

We have created a new Macaulay2 package with methods for working with additive actions on projective toric varieties, and followed package writing guidelines described in the Macaulay2 website [Grayson et al., 1993] accordingly.

Let N, M be dual lattices with associated vector spaces $N_{\mathbb{R}}, M_{\mathbb{R}}$ of dimension $d \in \mathbb{Z}^+$. Let $\{e_1, \ldots, e_d\} \subset N_{\mathbb{R}}, \{e_1^*, \ldots, e_d^*\} \subset M_{\mathbb{R}}$ be bases of N, M, respectively.

We have identified a small error in the initial classification of smooth Fano toric fourfolds by Batyrev [Batyrev, 1999], which identifies polytope 117 on Table 7 with the polytope given by the $\mathcal{V}$-representation

$$\left\{ \pm e_1, \pm e_2, \pm e_3, \pm e_4, \pm \sum_{i=1}^{4} e_i \right\},$$

which is incorrect, as it contains ten vertices instead of nine, and is not additive. The correct statement is given by Remark 4.4.2.

5.1 Future lines

Possible future lines of work and research include:

- Studying the approach we followed on subsection 3.2.3, and trying to generalise it to produce an algorithm to obtain the face lattice of a polytope, instead of only the edges.

- Classifying sevenfolds and eightfolds, as this may still be computationally feasible, although it will require new polytope data.
- Implementing algorithms for calculating Demazure roots, and algorithms on subsections 3.2 and 3.4, on the new Macaulay2 package. Creating a fork and pull request to the main Macaulay2 repository, submitting the package for community review, writing a companion article, and submitting it to a journal dedicated to computational algebraic geometry.
- Working towards a more rigurous justification for the observations in subsection 3.5.3.

Another very interesting immediate line of research is given by the following problem:

Problem 5.1.1. *Characterise all uniquely additive smooth Fano toric varieties.*

To this effect, it may be worthwhile to consider the following definitions:

Definition 5.1.2. ***($\mathbb{P}^1$, Del Pezzo, pseudo Del Pezzo polytope)***

*i) The $\mathbb{P}^1$ **polytope** is the polytope $[-1, 1] \subset \mathbb{R}$.*

*ii) If n is even, the **Del Pezzo polytope** is the polytope $P \subset N_{\mathbb{R}}$ given by the $\mathcal{V}$-representation:*

$$\left\{ \pm e_1, \ldots, \pm e_n, \pm \sum_{i=1}^{n} e_i \right\}.$$

*iii) If n is even, the **pseudo Del Pezzo polytope** is the polytope $P \subset N_{\mathbb{R}}$ given by the $\mathcal{V}$-representation:*

$$\left\{ \pm e_1, \ldots, \pm e_n, - \sum_{i=1}^{n} e_i \right\}.$$

Definition 5.1.3. ***(Centrally-symmetric vertices, polytope splitting)*** *Let $P \subset N_{\mathbb{R}}$ be a polytope.*

1. *We say two vertices $v, v' \in P$ of P are **centrally-symmetric** if $v = -v'$.*
2. *We say P **splits** into two polytopes $Q, Q' \subset N_{\mathbb{R}}$ if P is lattice isomorphic to the convex hull of $(Q \times \{0\}) \cup (\{0\} \times Q')$.*

The following is a theorem of Casagrande:

Theorem 5.1.4. ***([Casagrande, 2003], Theorem 5, Proposition 7)*** *Let $P \subset N_{\mathbb{R}}$ be a smooth Fano polytope. If P has n linearly independent pairs of centrally-symmetric vertices, then P splits into $\mathbb{P}^1$ polytopes, Del Pezzo polytopes, and pseudo Del Pezzo polytopes. The converse is clearly true.*

Theorem 5.1.4 improves on a weaker theorem of Ewald [Nill, 2005], which consists of similar statement but for centrally-symmetric facets. Theorem 5.1.4 is originally stated in a stronger form, as it also characterises smooth Fano polytopes with less pairs of linearly independent centrally-symmetric vertices.

This is all very good news! We note that the Del Pezzo polytope of dimension 2 is self-dual and does not have any Demazure roots, thus it is not additive (see Figure 21). This is likely true for the Del Pezzo polytope of any even dimension. We also note that our results imply that all uniquely additive smooth Fano polytopes of dimension up to 4 split into $\mathbb{P}^1$ polytopes and pseudo Del Pezzo polytopes. Thus, we may state the following conjecture, which would solve Problem 5.1.1:

Conjecture 5.1.5. *Let $P \subset N_{\mathbb{R}}$ be a smooth Fano polytope. Then, P is uniquely additive if and only if P splits into $\mathbb{P}^1$ polytopes, Del Pezzo polytopes, and pseudo Del Pezzo polytopes.*

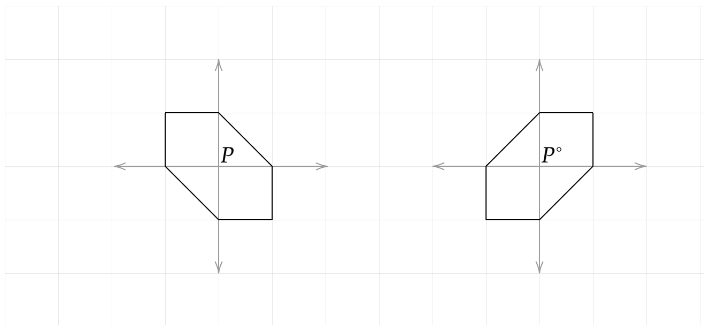


Figure 21: The Del Pezzo polytope of dimension 2 is self-dual.

A simple dimensional analysis lets us verify that Conjecture 5.1.5 is true up to dimension 4 (see Table 2). By considering all possible products of $\mathbb{P}^1$, Del Pezzo, and pseudo Del Pezzo polytopes that result in a polytope of dimeension 5, we also verify that it is true in dimension 5. However, there are only 7 such products that result in a polytope of dimension 6, and 8 uniquely additive smooth Fano toric sixfolds (see Figure 22).

Figure 22: *There is another*, a popular internet meme.
Source: [Lucas, 1980], [BabaSherif, 2019].

Is it possible that a high-dimensional polytope with a Del Pezzo factor can be additive, or does this polytope have a new factor with less vertex symmetry? In the future, we will carefully analyse our results and see.

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