Study On Generalized Nearly P-Sasakian Manifolds

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Abstract—In 1976, 1977, I. Sato [7], [8] dicussed on a structure similar to almost contact structure. In 2011, R. Nivas and A. Bajpai [6] discussed on generalized Lorentzian Para-Sasakian manifolds. Hayden [2] introduced the idea of metric connection with torsion tensor in a Riemannian manifold. Imai [3] studied the properties of semi-symmetric metric connection in a Riemannian manifold. R.S. Mishra and S.N. Pandey [4] discussed on Quarter symmetric metric F-connection. Nirmala S. Agashe and Mangala R. Chafle [5] studied semi-symmetric non-metric connection in a Riemannian manifold. In 2013, Chaubey S.K. and Pandey A.C. [1] studied the properties of semi-symmetric non-metric connection in Sasakian manifolds. In this paper generalized nearly manifolds have been discussed and some of their properties have been established. Semi-symmetric non- metric F-connection in a generalized SP-Sasakian manifold is also discussed.

Keywords—Generalized nearly P-Sasakian manifold, generalized nearly SP-Sasakian manifolds, generalized nearly P-Co-symplectic manifolds, generalized induced connection in a generalized SP-Sasakian manifold.

1. INTRODUCTION

An *n(=2m+1)* dimensional differentiable manifold V_n, on which there are defined a tensor field $F$of type (1, 1), contravariant vector fields T_i, covariant vector fields A_i , where $i = 3,4,5, \ldots\ldots (n-1)$, and a metric tensor g, satisfying for arbitrary vector fields $X, Y, Z, \ldots$

(1.1) $\overline{\overline{X}} = X - \sum_{i=3}^{n-1} A_i(X)T_i, \quad \overline{T_i} = 0, \quad A_i(T_i) = 1, \quad \overline{X} \stackrel{\text{def}}{=} FX, \quad A_i(\overline{X}) = 0,$

rank $F = n - i$

(1.2) $g\,(\overline{X}, \overline{Y}) = g\,(X, Y) - \sum_{i=3}^{n-1} A_i(X)\, A_i(Y),$ where $A_i(X) = g(X, T_i),$

$`F(X, Y) \stackrel{\text{def}}{=} g(\overline{X},\ Y) = `F(Y, X),$

Then V_n is said to be a generalized almost Para-Contact manifold (a generalized almost P-Contact manifold) and the structure $(F,\ T_i,\ A_i,\ g\ )$ is said to be generalized almost Para-Contact structure.

Let D be a Riemannian connection on V_n, then we have

(1.3) (a) $(D_X`F)(\overline{Y},\ Z) + (D_X`F)(Y, \overline{Z}) + \sum_{i=3}^{n-1} A_i(Y)(D_X A_i)(Z) + \sum_{i=3}^{n-1} A_i(Z)(D_X A_i)(Y) = 0$

(b) $(D_X`F)\left(\overline{Y},\ \overline{\overline{Z}}\right) + (D_X`F)\left(\overline{\overline{Y}}, \overline{Z}\right) = 0$

(1.4) (a) $(D_X`F)(\overline{Y},\ \overline{Z}) + (D_X`F)(Y, Z) + \sum_{i=3}^{n-1} A_i(Y)(D_X A_i)(\overline{Z}) + \sum_{i=3}^{n-1} A_i(Z)(D_X A_i)(\overline{Y}) = 0$

(b) $(D_X`F)\left(\overline{\overline{Y}},\ \overline{\overline{Z}}\right) + (D_X`F)(\ \overline{Y},\ \overline{Z}\ ) = 0$

A generalized almost P-Contact manifold is said to be a generalized Para-Sasakian manifold (a generalized P-Sasakian manifold) if

(1.5) (a) $i(D_X F)(Y) + \overline{\overline{X}} \sum_{i=3}^{n-1} A_i(Y) + g(\overline{X}, \overline{Y}) \sum_{i=3}^{n-1} T_i = 0 \Leftrightarrow$

(b) $i(D_X {}`F)(Y,Z) + g(\overline{X}, \overline{Z}) \sum_{i=3}^{n-1} A_i(Y) + g(\overline{X}, \overline{Y}) \sum_{i=3}^{n-1} A_i(Z) = 0 \Leftrightarrow$

(c) $iD_X T_i = \overline{X} + T_i - \sum_{i=3}^{n-1} T_i,$

From which, we get

(1.6) (a) $i(D_X A_i)(\overline{Y}) = g(\overline{X}, \overline{Y}) \Leftrightarrow$

(b) $i(D_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = {}`F(X,Y)$

A generalized almost P-Contact manifold is said to be a generalized Special Para-Sasakian manifold (a generalized SP-Sasakian manifold) if

(1.7) (a) $i(D_X F)(Y) + \overline{X} \sum_{i=3}^{n-1} A_i(Y) + {}`F(X,Y) \sum_{i=3}^{n-1} T_i = 0 \Leftrightarrow$

(b) $i(D_X {}`F)(Y,Z) + {}`F(X,Z) \sum_{i=3}^{n-1} A_i(Y) + {}`F(X,Y) \sum_{i=3}^{n-1} A_i(Z) = 0 \Leftrightarrow$

(c) $iD_X T_i = \overline{\overline{X}} + T_i - \sum_{i=3}^{n-1} T_i$

From which, we get

(1.8) (a) $i(D_X A_i)(\overline{Y}) = {}`F(X,Y) \Leftrightarrow$

(b) $i(D_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = g(\overline{X}, \overline{Y})$

A generalized almost P-Contact manifold is said to be a generalized Para-Co-symplectic manifold (a generalized P-Co-symplectic manifold) if

(1.9) (a) $(D_X F)Y + \sum_{i=3}^{n-1} A_i(Y) \overline{D_X T_i} + \sum_{i=3}^{n-1} (D_X A_i)(\overline{Y}) T_i = 0 \Leftrightarrow$

(b) $(D_X {}`F)(Y,Z) + \sum_{i=3}^{n-1} A_i(Y)(D_X A_i)(\overline{Z}) + \sum_{i=3}^{n-1} A_i(Z)(D_X A_i)(\overline{Y}) = 0$

Therefore a generalized P-Co-symplectic manifold is a generalized P-Sasakian manifold if

(1.10) (a) $i(D_X A_i)(\overline{Y}) = g(\overline{X}, \overline{Y}) \Leftrightarrow$ (b) $i(D_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = {}`F(X,Y) \Leftrightarrow$

(c) $iD_X T_i = \overline{X} + T_i - \sum_{i=3}^{n-1} T_i$

Also a generalized P-Co-symplectic manifold is a generalized SP-Sasakian manifold if

(1.11) (a) $i(D_X A_i)(\overline{Y}) = {}`F(X,Y) \Leftrightarrow$

(b) $i(D_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = g(\overline{X}, \overline{Y}) \Leftrightarrow$ (c) $iD_X T_i = \overline{\overline{X}} + T_i - \sum_{i=3}^{n-1} T_i$

Nijenhuis tensor in a generalized almost P-Contact manifold is given by

(1.12) $`N(X,Y,Z) = (D_{\overline{X}} {}`F)(Y,Z) - (D_{\overline{Y}} {}`F)(X,Z) - (D_X {}`F)(Y,\overline{Z}) + (D_Y {}`F)(X,\overline{Z})$

Where $`N(X,Y,Z) \overset{\text{def}}{=} g(N(X,Y),Z)$

2. GENERALIZED NEARLY PARA-SASAKIAN MANIFOLDS

A generalized almost P-contact manifold is said to be a generalized nearly Para-Sasakian manifold (a generalized nearly P-Sasakian manifold) if

(2.1) $i(D_X`F)(Y,Z) + \sum_{i=3}^{n-1} A_i(Y)\, g(\overline{X},\overline{Z}) + \sum_{i=3}^{n-1} A_i(Z)\, g(\overline{X},\overline{Y})$

$= i(D_Y`F)(Z,X) + \sum_{i=3}^{n-1} A_i(Z)\, g(\overline{X},\overline{Y}) + \sum_{i=3}^{n-1} A_i(X)\, g(\overline{Y},\overline{Z})$

$= i(D_Z`F)(X,Y) + \sum_{i=3}^{n-1} A_i(X)\, g(\overline{Y},\overline{Z}) + \sum_{i=3}^{n-1} A_i(Y)\, g(\overline{X},\overline{Z})$

From which, we get

(2.2) (a) $i(D_X F)Y - i(D_Y F)X + \sum_{i=3}^{n-1} A_i(Y)\overline{\overline{X}} - \sum_{i=3}^{n-1} A_i(X)\overline{\overline{Y}} = 0 \Leftrightarrow$

(b) $i(D_X`F)(Y,Z) - i(D_Y`F)(X,Z) + \sum_{i=3}^{n-1} A_i(Y)\, g(\overline{X},\overline{Z}) - \sum_{i=3}^{n-1} A_i(X)\, g(\overline{Y},\overline{Z}) = 0$

These equations can be written as

(2.3) (a) $i(D_X F)\overline{Y} - i(D_{\overline{Y}} F)X - \sum_{i=3}^{n-1} A_i(X)\overline{Y} = 0 \Leftrightarrow$

(b) $i(D_X`F)(\overline{Y},Z) - i(D_{\overline{Y}}`F)(Z,X) - \sum_{i=3}^{n-1} A_i(X)\, `F(Y,Z) = 0$

(2.4) (a) $i(D_X F)\overline{\overline{Y}} - i(D_{\overline{\overline{Y}}} F)X - \sum_{i=3}^{n-1} A_i(X)\overline{\overline{Y}} = 0 \Leftrightarrow$

(b) $i(D_X`F)\left(\overline{\overline{Y}},Z\right) - i\left(D_{\overline{\overline{Y}}}`F\right)(Z,X) - \sum_{i=3}^{n-1} A_i(X) g(\overline{Y},\overline{Z}) = 0$

(2.5)(a) $i(D_X F)Y - i(D_Y F)X + \sum_{i=3}^{n-1} A_i(Y)\{\overline{D_X T_i} + (D_{T_i} F)X\} - \sum_{i=3}^{n-1} A_i(X)\overline{\overline{Y}} = 0 \Leftrightarrow$

(b) $i(D_X`F)(Y,Z) - i(D_Y`F)(X,Z) + \sum_{i=3}^{n-1} A_i(Y)\{(D_X A_i)(\overline{Z}) + (D_{T_i}`F)(Z,X)\} - \sum_{i=3}^{n-1} A_i(X) g(\overline{Y},\overline{Z}) = 0$

Barring X, Y, Z in (1.12) and using equations (2.1), (1.3) (b), we get $`N(\overline{X},\overline{Y},\overline{Z}) = 0$,which implies that a generalized nearly P-Sasakian manifold is completely integrable.

3. GENERALIZED NEARLY SPECIAL PARA-SASAKIAN MANIFOLDS

A generalized almost P-contact manifold is said to be a generalized nearly Special Para-Sasakian manifold (a generalized nearly SP-Sasakian manifold) if

(3.1) $i(D_X`F)(Y,Z) + \sum_{i=3}^{n-1} A_i(Y)\, `F(Z,X) + \sum_{i=3}^{n-1} A_i(Z)\, `F(X,Y)$

$= i(D_Y`F)(Z,X) + \sum_{i=3}^{n-1} A_i(Z)\, `F(X,Y) + \sum_{i=3}^{n-1} A_i(X)\, `F(Y,Z)$

$= i(D_Z`F)(X,Y) + \sum_{i=3}^{n-1} A_i(X)\, `F(Y,Z) + \sum_{i=3}^{n-1} A_i(Y)\, `F(Z,X)$

From which, we get

(3.2) (a) $i(D_X F)Y - i(D_Y F)X + \sum_{i=3}^{n-1} A_i(Y)\overline{X} - \sum_{i=3}^{n-1} A_i(X)\overline{Y} = 0 \Leftrightarrow$

(b) $i(D_X`F)(Y,Z) - i(D_Y`F)(X,Z) + \sum_{i=3}^{n-1} A_i(Y)\, `F(Z,X) - \sum_{i=3}^{n-1} A_i(X)\, `F(Y,Z) = 0$

This gives

(3.3) (a) $i(D_X F)\overline{Y} - i(D_{\overline{Y}} F)X - \sum_{i=3}^{n-1} A_i(X)\overline{\overline{Y}} = 0 \Leftrightarrow$

(b) $i(D_X `F)(\overline{Y}, Z) - i(D_{\overline{Y}} `F)(Z, X) - \sum_{i=3}^{n-1} A_i(X) g(\overline{Y}, \overline{Z}) = 0$

(3.4) (a) $i(D_X F)\overline{\overline{Y}} - i(D_{\overline{\overline{Y}}} F)X - \sum_{i=3}^{n-1} A_i(X)\overline{Y} = 0 \Leftrightarrow$

(b) $i(D_X `F)\left(\overline{\overline{Y}}, Z\right) - i\left(D_{\overline{\overline{Y}}} `F\right)(Z, X) - \sum_{i=3}^{n-1} A_i(X)\, `F(Y, Z) = 0$

(3.5) (a) $i(D_X F)Y - i(D_Y F)X + \sum_{i=3}^{n-1} A_i(Y)\{\overline{D_X T_i} + (D_{T_i} F)X\} - \sum_{i=3}^{n-1} A_i(X)\}\overline{Y} = 0 \Leftrightarrow$

(b) $i(D_X `F)(Y, Z) - i(D_Y `F)(X, Z) + \sum_{i=3}^{n-1} A_i(Y)\{(D_X A_i)(\overline{Z}) + (D_{T_i} `F)(Z, X)\} - \sum_{i=3}^{n-1} A_i(X)\, `F(Y, Z) = 0$

Barring X, Y, Z in (1.12) and using equations (3.1), (1.3) (b), we get $`N(\overline{X}, \overline{Y}, \overline{Z}) = 0$,which implies that a generalized nearly SP-Sasakian manifold is completely integrable.

4. GENERALIZED NEARLY PARA-CO-SYMPLECTIC MANIFOLDS

A generalized almost P-Contact manifold will be called a generalized nearly Para-Co-symplectic manifold (a generalized nearly P-Co-symplectic manifold) if

(4.1) $(D_X `F)(Y, Z) + \sum_{i=3}^{n-1} A_i(Y)(D_X A_i)(\overline{Z}) + \sum_{i=3}^{n-1} A_i(Z)(D_X A_i)(\overline{Y})$

$= (D_Y `F)(Z, X) + \sum_{i=3}^{n-1} A_i(Z)(D_Y A_i)(\overline{X}) + \sum_{i=3}^{n-1} A_i(X)(D_Y A_i)(\overline{Z})$

$= (D_Z `F)(X, Y) + \sum_{i=3}^{n-1} A_i(X)(D_Z A_i)(\overline{Y}) + \sum_{i=3}^{n-1} A_i(Y)(D_Z A_i)(\overline{X})$

Therefore, a generalized nearly P-Sasakian manifold is a generalized nearly P-Co-symplectic manifold, in which

(4.2) (a) $i(D_X A_i)(\overline{Y}) = g(\overline{X}, \overline{Y}) \Leftrightarrow$

(b) $i(D_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = `F(X, Y) \Leftrightarrow$ (c) $iD_X T_i = \overline{X} + T_i - \sum_{i=3}^{n-1} T_i$

Also a generalized nearly SP-Sasakian manifold is a generalized nearly P-Co-symplectic manifold, in which

(4.3) (a) $i(D_X A_i)(\overline{Y}) = `F(X, Y) \Leftrightarrow$

(b) $i(D_X A_i)(Y) - A_i(Y) + \sum_{i=3}^{n-1} A_i(Y) = g(\overline{X}, \overline{Y}) \Leftrightarrow$ (c) $iD_X T_i = \overline{\overline{X}} + T_i - \sum_{i=3}^{n-1} T_i$

5. GENERALIZED CONNECTION IN A GENERALIZED SP-SASAKIAN MANIFOLDS

Let V_{2m-1} be submanifold of V_{2m+1} and let $c : V_{2m-1} \to V_{2m+1}$ be the inclusion map such that

$$d \in V_{2m-1} \to cd \in V_{2m+1},$$

Where c induces a linear transformation (Jacobian map) $J : T'_{2m-1} \to T'_{2m+1}$.

T'_{2m-1} is a tangent space to V_{2m-1} at point d and T'_{2m+1} is a tangent space to V_{2m+1} at point cd such that

$$\hat{X} \text{ in } V_{2m-1} \text{ at } d \to J\hat{X} \text{ in } V_{2m+1} \text{ at } cd$$

Let $\tilde{g}$ be the induced Lorentzian metric in V_{2m-1} . Then we have

(5.1) (a) $\tilde{g}(\hat{X}, \hat{Y}) \stackrel{\text{def}}{=} g(J\hat{X}, J\hat{Y})$

We now suppose that a generalized semi-symmetric non-metric F-connection B in a generalized SP-Sasakian manifold is given by

(5.2) $iB_X Y = iD_X Y - \sum_{i=3}^{n-1} A_i(Y)X + \sum_{i=3}^{n-1} g(X,Y)T_i - 2\sum_{i=3}^{n-1} A_i(X)Y,$

Where X and Y are arbitrary vector fields of V_{2m+1}. If

(5.3) $T_i = Jt_i + \rho_i M + \sigma_i N$, where $i = 3,4,5, \ldots\ldots (n-1)$.

Where t_i , $i = 3,4,5, \ldots\ldots (n-1)$ are C^∞ vector fields in V_{2m-1} and M and N are unit normal vectors to V_{2m-1}.

Denoting by $\grave{D}$ the connection induced on the submanifold from D, we have Gauss equation

(5.4) $D_{JX} J\hat{Y} = J(\grave{D}_X \hat{Y}) + h(\hat{X},\hat{Y})M + k(\hat{X},\hat{Y})N$

Where h and k are symmetric bilinear functions in V_{2m-1}. Similarly we have

(5.5) $B_{JX} J\hat{Y} = J(\grave{B}_X \hat{Y}) + m(\hat{X},\hat{Y})M + n(\hat{X},\hat{Y})N$,

Where $\grave{B}$ is the connection induced on the submanifold from B and m and n are symmetric bilinear functions in V_{2m-1}

Inconsequence of (5.2), we have

(5.6) $iB_{JX} J\hat{Y} = iD_{JX} J\hat{Y} - \sum_{i=3}^{n-1} A_i(J\hat{Y})J\hat{X} + \sum_{i=3}^{n-1} g(J\hat{X},J\hat{Y})T_i - 2\sum_{i=3}^{n-1} A_i(J\hat{X})J\hat{Y}$

Using (5.4), (5.5) and (5.6), we get

(5.7) $iJ(\grave{B}_X \hat{Y}) + im(\hat{X},\hat{Y})M + in(\hat{X},\hat{Y})N = iJ(\grave{D}_X \hat{Y}) + ih(\hat{X},\hat{Y})M + ik(\hat{X},\hat{Y})N - \sum_{i=3}^{n-1} A_i(J\hat{Y})J\hat{X} +$

$$\sum_{i=3}^{n-1} g(J\hat{X},J\hat{Y})T_i - 2\sum_{i=3}^{n-1} A_i(J\hat{X})J\hat{Y}$$

Using (5.3), we obtain

(5.8) $iJ(\grave{B}_X \hat{Y}) + im(\hat{X},\hat{Y})M + in(\hat{X},\hat{Y})N = iJ(\grave{D}_X \hat{Y}) + ih(\hat{X},\hat{Y})M + ik(\hat{X},\hat{Y})N - \sum_{i=3}^{n-1} a_i(\hat{Y})J\hat{X} +$
$\sum_{i=3}^{n-1} \tilde{g}(\hat{X},\hat{Y})(Jt_i + \rho_i M + \sigma_i N) - 2\sum_{i=3}^{n-1} a_i(\hat{X})J\hat{Y}$

Where $\tilde{g}(\hat{Y},t_i) \stackrel{\text{def}}{=} a_i(\hat{Y})$, where $i = 3,4,5, \ldots\ldots (n-1)$.

This gives

(5.9) $i\grave{B}_X \hat{Y} = i\grave{D}_X \hat{Y} - \sum_{i=3}^{n-1} a_i(\hat{Y})\hat{X} + \sum_{i=3}^{n-1} \tilde{g}(\hat{X},\hat{Y})t_i - 2\sum_{i=3}^{n-1} a_i(\hat{X})\hat{Y}$

Iff

(5.10) (a) $im(\hat{X},\hat{Y}) = ih(\hat{X},\hat{Y}) + \sum_{i=3}^{n-1} \rho_i \tilde{g}(\hat{X},\hat{Y})$

(b) $in(\hat{X},\hat{Y}) = ik(\hat{X},\hat{Y}) + \sum_{i=3}^{n-1} \sigma_i \tilde{g}(\hat{X},\hat{Y})$

Thus we have

Theorem 5.1 The connection induced on a submanifold of a generalised SP-Sasakian manifold with a generalized semi-symmetric non-metric F-connection with respect to unit normal vectors M and N is also semi-symmetric non-metric F-connection iff (5.10) holds.

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