Physics 351 — Friday, April 10, 2015

- Hand in HW11. Pick up HW12 (penultimate!) handout.
- Tanner had some grueling midterms these past two weeks, so he'll catch up on grading HW10 this weekend.
- FYI — intuitive description of precession (mentioned in class on Wednesday — Nate C. and Alex K. found it helpful):

http://positron.hep.upenn.edu/p351/files/0331_george_abell_precession.pdf

- For Monday: read (partly skim) Chapters 1+2 of *Quantum Mechanics and Path Integrals* by Feynman and Hibbs. I wrote up questions for you to answer. This reading is just for fun ("enrichment"). The point is to show you the deep connection between QM and the Lagrangian formulation of CM.
- For Wednesday: read (mostly skim) Taylor's Chapter 12 ("Nonlinear mechanics and chaos"). I haven't written up questions yet, but will do so ASAP (by Saturday!). We'll do maybe one or two Mathematica-based problems from Ch12 — anything additional will be XC.
- Today: keep working through Hamiltonian examples.

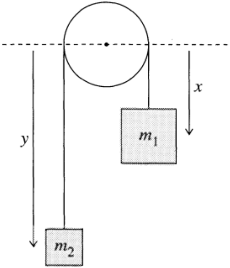
Let's try out Taylor's "procedure" for Hamilton's equations.

This example illustrates the general procedure to be followed in setting up Hamilton's equations for any given system:

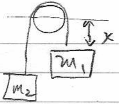
1. Choose suitable generalized coordinates, $q_1, \cdots, q_n$.
2. Write down the kinetic and potential energies, T and U, in terms of the q's and $\dot{q}$'s.
3. Find the generalized momenta $p_1, \cdots, p_n$. (We are now assuming our system is conservative, so U is independent of $\dot{q}_i$ and we can use $p_i = \partial T/\partial \dot{q}_i$. In general, one must use $p_i = \partial \mathcal{L}/\partial \dot{q}_i$.)
4. Solve for the $\dot{q}$'s in terms of the p's and q's.
5. Write down the Hamiltonian $\mathcal{H}$ as a function of the p's and q's. [Provided our coordinates are "natural" (relation between generalized coordinates and underlying Cartesians is independent of time), $\mathcal{H}$ is just the total energy $\mathcal{H} = T + U$, but when in doubt, use $\mathcal{H} = \sum p_i \dot{q}_i - \mathcal{L}$. See Problems 13.11 and 13.12.]
6. Write down Hamilton's equations (13.25).

(We went through this quickly at the end of Wednesday's class.)

Taylor 13.3. Consider the Atwood machine of Figure 13.2, but suppose that the pulley is a uniform disc of mass M and radius R. Using x as your generalized coordinate, write down $\mathcal{L}$, the generalized momentum p, and $\mathcal{H} = p\dot{x} - \mathcal{L}$. Write Hamilton's equations and use them to find $\ddot{x}$.



<u>Taylor 13.3</u>



$$U = (m_2 - m_1) g x$$

$$T = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{\dot{x}}{R}\right)^2$$
$$= \frac{1}{2}\left(m_1 + m_2 + \frac{1}{2}M\right)\dot{x}^2$$

$$\mathcal{L} = T - U = \frac{1}{2}\left(m_1 + m_2 + \frac{M}{2}\right)\dot{x}^2 + (m_1 - m_2) g x$$

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = \left(m_1 + m_2 + \frac{M}{2}\right)\dot{x} \implies \dot{x} = \frac{p_x}{m_1 + m_2 + \frac{M}{2}}$$

$$\mathcal{H} = p\dot{x} - \mathcal{L} = \frac{p_x^2}{m_1 + m_2 + \frac{M}{2}} - \frac{1}{2}\left(m_1 + m_2 + \frac{M}{2}\right)\left(\frac{p_x}{m_1 + m_2 + \frac{M}{2}}\right)^2 + (m_2 - m_1) g x$$

$$\mathcal{H} = \frac{p_x^2}{2\left(m_1 + m_2 + \frac{M}{2}\right)} + (m_2 - m_1) g x$$

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p_x} = \frac{p_x}{m_1 + m_2 + \frac{M}{2}} \qquad \dot{p}_x = -\frac{\partial \mathcal{H}}{\partial x} = (m_1 - m_2) g$$

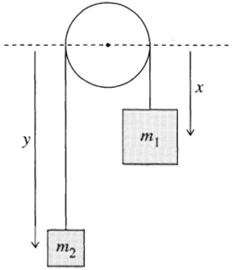
$$\ddot{x} = \frac{\dot{p}_x}{m_1 + m_2 + \frac{M}{2}} = \frac{(m_1 - m_2) g}{m_1 + m_2 + \frac{M}{2}}$$

By the way, if you take the two original Cartesian coordinates to be x, y, and ϕ, then the one generalized coordinate is $q = x$.

$$x = q$$

$$y = \text{const.} - q$$

$$\phi = q/R$$



All of these are time-independent and don't involve the velocities, so the generalized coordinate q is "natural" (or "scleronomous" in Goldstein's language). Goldstein's word for "unnatural" is "rheonomous."

So we found

$$\mathcal{H} = T + U$$

Taylor 13.11. The simple form $\mathcal{H} = T + U$ is true only if your generalized coordinates are "natural" (relation between generalized and underlying Cartesian coordinates is independent of time). If the generalized coordinates are not "natural," you must use

$$\mathcal{H} = \sum p\dot{q} - \mathcal{L}$$

To illustrate: Two children play catch inside a railroad car moving with varying speed V along a straight horizontal track. For generalized coordinates you can use (x, y, z) of the ball relative to a fixed point in the car, but in setting up $\mathcal{H}$ you must use coordinates in an inertial frame. Find $\mathcal{H}$ for the ball and show that it is not equal to $T + U$ (neither as measured in the car, nor as measured in the ground-based frame).

Taylor wrote way back on p. 270 (Eq. 7.91) that $\mathcal{H} = T + U$ if

$$\boldsymbol{r}_\alpha = \boldsymbol{r}_\alpha(q_1, \cdots, q_n)$$

(i.e. there is no "t" and no $\dot{q}_i$ when writing $\boldsymbol{r}_\alpha$ in terms of the q_i's.)

$$T = \frac{1}{2}m\left((\dot{x}+V)^2+\dot{y}^2+\dot{z}^2\right) \qquad U = mgz$$

$$p_x = \frac{\partial T}{\partial \dot{x}} = m(\dot{x}+V) \qquad p_y = m\dot{y} \qquad p_z = m\dot{z}$$

$$\mathcal{H} = \sum p\dot{q} - \mathcal{L} = p_x\left(\frac{p_x}{m}-V\right) + p_y\left(\frac{p_y}{m}\right) + p_z\left(\frac{p_z}{m}\right)$$
$$-\frac{1}{2}m\left(\left(\frac{p_x}{m}\right)^2+\left(\frac{p_y}{m}\right)^2+\left(\frac{p_z}{m}\right)^2\right) + mgz$$

$$\mathcal{H} = \frac{p_x^2}{2m} - p_xV + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + mgz$$

$$(T+U)_{\substack{\text{train}\\\text{car}\\\text{frame}}} = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) + mgz = \frac{1}{2}m\left(\left(\frac{p_x}{m}-V\right)^2+\left(\frac{p_y}{m}\right)^2+\left(\frac{p_z}{m}\right)^2\right) + mgz$$
$$= \frac{\vec{p}^{\,2}}{2m} - p_xV + \frac{1}{2}mV^2 + mgz \neq \mathcal{H}$$

$$(T+U)_{\substack{\text{ground}\\\text{frame}}} = \frac{1}{2}m\left((\dot{x}+V)^2+\dot{y}^2+\dot{z}^2\right) + mgz$$
$$= \frac{1}{2}m\left(\left(\frac{p_x}{m}\right)^2+\left(\frac{p_y}{m}\right)^2+\left(\frac{p_z}{m}\right)^2\right) + mgz = \frac{\vec{p}^{\,2}}{2m} + mgz \neq \mathcal{H}$$

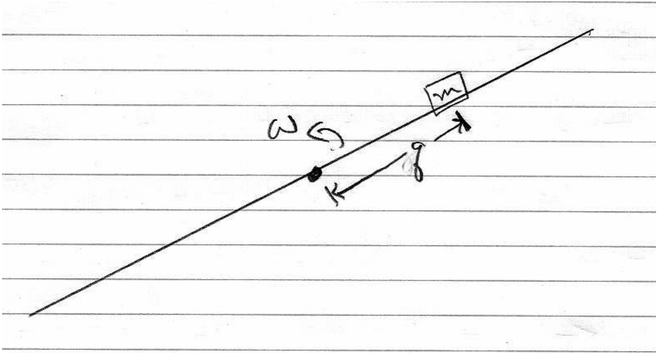
$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p_x} = \frac{p_x}{m} - V \qquad \dot{p}_x = -\frac{\partial \mathcal{H}}{\partial x} = 0 \implies \ddot{x} = -\dot{V}$$

$$\dot{y} = \frac{\partial \mathcal{H}}{\partial p_y} = \frac{p_y}{m} \qquad \dot{p}_y = -\frac{\partial \mathcal{H}}{\partial y} = 0 \implies \ddot{y} = 0$$

$$\dot{z} = \frac{\partial \mathcal{H}}{\partial p_z} = \frac{p_z}{m} \qquad \dot{p}_z = -\frac{\partial \mathcal{H}}{\partial z} = -mg \implies \ddot{z} = -g$$

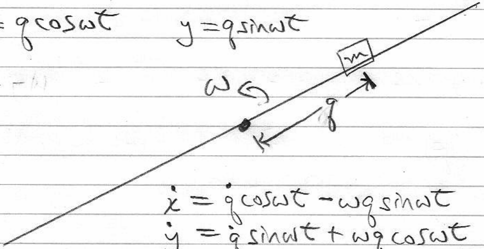
Taylor 13.12. Same as previous problem but use this system:

A bead of mass m is threaded on a frictionless, straight rod, which lies in a horizontal plane and is forced to spin with constant angular velocity ω about a vertical axis through the midpoint of the rod. Find $\mathcal{H}$ for the bead and show that $\mathcal{H} \neq T + U$.



(I suggest this generalized coordinate q.)

$x = q\cos\omega t \qquad y = q\sin\omega t$



$$\dot{x} = \dot{q}\cos\omega t - \omega q\sin\omega t$$
$$\dot{y} = \dot{q}\sin\omega t + \omega q\cos\omega t$$

$$T = \tfrac{1}{2}m\dot{q}^2 + \tfrac{1}{2}m\omega^2 q^2 \quad \rightarrow \mathcal{L} = \tfrac{1}{2}m\dot{q}^2 + \tfrac{1}{2}m\omega^2 q^2$$

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m\dot{q} \Rightarrow \dot{q} = \frac{p}{m}$$

$$\mathcal{H} = p\dot{q} - \mathcal{L} = p\left(\frac{p}{m}\right) - \tfrac{1}{2}m\left(\frac{p}{m}\right)^2 - \tfrac{1}{2}m\omega^2 q^2 = \frac{p^2}{2m} - \tfrac{1}{2}m\omega^2 q^2$$

$$(T+U)_{\substack{\text{relative}\\ \text{to rod}}} = \tfrac{1}{2}m\dot{q}^2 = \frac{p^2}{2m} \neq \mathcal{H}$$

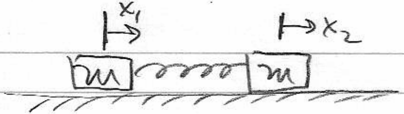
$$(T+U)_{\substack{\text{relative}\\ \text{to}\\ \text{ground}}} = \tfrac{1}{2}m\dot{q}^2 + \tfrac{1}{2}m\omega^2 q^2 \neq \mathcal{H}$$

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m} \qquad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q} = +m\omega^2 q \quad \Rightarrow \ddot{q} = \omega^2 q$$

(q increases exponentially)

Morin 15.28. Two beads of mass m are connected by a spring (with spring constant k and relaxed length ℓ) and are free to move along a frictionless horizontal wire. Let their positions be x_1 and x_2. Find $\mathcal{H}$ in terms of x_1 and x_2 and their conjugate momenta, then write down the four Hamilton's equations.

Morin 15.28



$$T = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) \qquad U = \frac{k}{2}\left(x_1 + l - x_2\right)^2$$

$$p_1 = \frac{\partial \mathcal{L}}{\partial \dot{x}_1} = \frac{\partial T}{\partial \dot{x}_1} = m\dot{x}_1 \qquad p_2 = m\dot{x}_2$$

$$\dot{x}_1 = \frac{p_1}{m} \qquad \dot{x}_2 = \frac{p_2}{m} \longrightarrow \frac{m}{2}(\dot{x}_1^2) = \frac{m}{2}\left(\frac{p_1}{m}\right)^2 = \frac{p_1^2}{2m}$$

$$\mathcal{H} = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{k}{2}\left(x_1 + l - x_2\right)^2$$

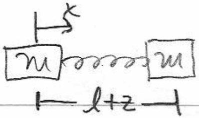
$$\dot{x}_1 = \frac{\partial \mathcal{H}}{\partial p_1} = \frac{p_1}{m} \qquad \dot{x}_2 = \frac{\partial \mathcal{H}}{\partial p_2} = \frac{p_2}{m}$$

$$\dot{p}_1 = -\frac{\partial \mathcal{H}}{\partial x_1} = -k\left(x_1 + l - x_2\right) \qquad \dot{p}_2 = -\frac{\partial \mathcal{H}}{\partial x_2} = +k\left(x_1 + l - x_2\right)$$

$$\Rightarrow \ddot{x}_1 = -\frac{k}{m}\left(x_1 + l - x_2\right) \qquad \ddot{x}_2 = \frac{k}{m}\left(x_1 + l - x_2\right)$$

Morin 15.8. Two beads of mass m are connected by a spring (with spring constant k and relaxed length ℓ) and are free to move along a frictionless horizontal wire. Let the position of the left bead be x, and let z be the stretch of the spring (w.r.t. equilibrium). Find $\mathcal{H}$ in terms of x and z and their conjugate momenta, then write down the four Hamilton's equations.

Morin 15.8



$$\mathcal{U} = \tfrac{1}{2} k z^2 \qquad T = \frac{m}{2}\left(\dot{x}^2 + (\dot{x}+\dot{z})^2\right)$$
$$= \tfrac{1}{2} m\left(2\dot{x}^2 + 2\dot{x}\dot{z} + \dot{z}^2\right)$$
$$= m\dot{x}^2 + m\dot{x}\dot{z} + \tfrac{1}{2} m\dot{z}^2$$

In terms of last problem's coordinates: $\left.\begin{aligned} x_1 &= x \\ x_2 &= x + z + l \end{aligned}\right\} \begin{aligned} &\neq f(t) \checkmark \\ &\neq f(\dot{x}) \\ &\neq f(\dot{z}) \end{aligned}$

$$P_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = 2m\dot{x} + m\dot{z}$$

$$P_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = m\dot{x} + m\dot{z} \longrightarrow \dot{x} + \dot{z} = \frac{P_z}{m}$$

$$P_x - P_z = m\dot{x} \Rightarrow \dot{x} = \frac{P_x - P_z}{m}$$

$$m\dot{z} = P_z - m\dot{x} = P_z - (P_x - P_z) = 2P_z - P_x \Rightarrow \dot{z} = \frac{2P_z - P_x}{m}$$

$$\mathcal{H} = \frac{m}{2}\left(\left(\frac{P_x - P_z}{m}\right)^2 + \left(\frac{P_z}{m}\right)^2\right) + \frac{kz^2}{2} = \frac{1}{2m}\left(P_x^2 - 2P_xP_z + 2P_z^2\right) + \frac{kz^2}{2}$$

$$\mathcal{H} = \frac{P_x^2}{2m} - \frac{P_xP_z}{m} + \frac{P_z^2}{m} + \frac{kz^2}{2}$$

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial P_x} = \frac{P_x}{m} - \frac{P_z}{m} \qquad \dot{z} = \frac{\partial \mathcal{H}}{\partial P_z} = \frac{2P_z}{m} - \frac{P_x}{m}$$

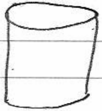
$$\dot{P}_x = 0 \qquad \dot{P}_z = -\frac{\partial \mathcal{H}}{\partial z} = -kz$$

$$\left(\ddot{x} = +\frac{k}{m}z = -\frac{\ddot{z}}{2}\right) \qquad \ddot{z} = \frac{2}{m}(-kz) = -\frac{2k}{m}z$$

What do you expect the general solution to the motion to look like? (It's similar to last night's HW problem 2.)

Taylor 13.13. Consider a particle of mass m constrained to move on a frictionless cylinder or radius R, given by the equation $\rho = R$ in (ρ, ϕ, z) coords. The mass is subject to force $\boldsymbol{F} = -kr\hat{\boldsymbol{r}}$, where k is a positive constant, r is distance from the origin, and $\hat{\boldsymbol{r}}$ points away from the origin. Using z and ϕ as generalized coordinates, find $\mathcal{H}$, write down Hamilton's equations, and describe the motion.

$$\vec{F} = -kr\hat{r} \Rightarrow U = \tfrac{1}{2}kr^2 = \tfrac{1}{2}k(\rho^2+z^2)$$
$$= \tfrac{1}{2}k(R^2+z^2)$$



so might as well write $U = \tfrac{1}{2}kz^2$

$$T = \tfrac{1}{2}m(\dot{z}^2 + R^2\dot{\phi}^2)$$

$$p_z = \frac{\partial T}{\partial \dot{z}} = m\dot{z} \qquad p_\phi = \frac{\partial T}{\partial \dot{\phi}} = mR^2\dot{\phi} \rightarrow \dot{\phi} = \frac{p_\phi}{mR^2}$$

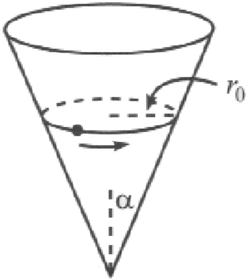
$$\mathcal{H} = T+U = \frac{p_z^2}{2m} + \tfrac{1}{2}mR^2\left(\frac{p_\phi}{mR^2}\right)^2 + \tfrac{1}{2}kz^2 = \frac{p_z^2}{2m} + \frac{p_\phi^2}{2mR^2} + \tfrac{1}{2}kz^2$$

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{mR^2} \qquad \dot{p}_\phi = -\frac{\partial \mathcal{H}}{\partial \phi} = 0$$

$$\dot{z} = \frac{\partial \mathcal{H}}{\partial p_z} = \frac{p_z}{m} \qquad \dot{p}_z = -\frac{\partial \mathcal{H}}{\partial z} = -kz \Rightarrow \ddot{z} = -\frac{k}{m}z$$

Here's a familiar problem from HW6 and from the midterm. Let's work through it it using Hamilton's equations instead.

3. A particle slides on the inside surface of a frictionless cone. The cone is fixed with its tip on the ground and its axis vertical. The half-angle of the cone is α, as shown in the left figure below. Let ρ be the distance from the particle to the axis, and let ϕ be the angle around the cone. (a) Find the EOM for ρ and for ϕ. (One EOM will identify a conserved quantity, which you can plug into the other EOM.) (b) If the particle moves in a circle of radius $\rho = r_0$, what is the frequency ω of this motion? (c) If the particle is then perturbed slightly from this circular motion, what is the frequency Ω of the oscillations about the radius $\rho = r_0$? (d) Under what conditions does $\Omega = \omega$?



$$T = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) \qquad U = mgz$$

$$\rho = z\tan\alpha \Rightarrow z = \rho\cot\alpha \equiv c\rho$$

Generalized coordinates : ρ, ϕ ("natural")

$$\left[z = c\rho \qquad x = \rho\cos\phi \qquad y = \rho\sin\phi\right]$$

$$T = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + c^2\dot{\rho}^2\right) \qquad U = mgc\rho$$

$$\mathcal{H} = T + U = \frac{m}{2}\left((1+c^2)\dot{\rho}^2 + \rho^2\dot{\phi}^2\right) + mgc\rho$$

need to rewrite in terms of P_ρ and P_ϕ

$$P_\phi = \frac{\partial T}{\partial\dot{\phi}} = m\rho^2\dot{\phi} \Rightarrow \dot{\phi} = \frac{P_\phi}{m\rho^2}$$

$$P_\rho = \frac{\partial T}{\partial\dot{\rho}} = m(1+c^2)\dot{\rho} \Rightarrow \dot{\rho} = \frac{P_\rho}{m(1+c^2)}$$

$$\mathcal{H} = \frac{m}{2}\left((1+c^2)\left(\frac{P_\rho}{m(1+c^2)}\right)^2 + \rho^2\left(\frac{P_\phi}{m\rho^2}\right)^2\right) + mgc\rho$$

$$\mathcal{H} = \frac{P_\rho^2}{2m(1+c^2)} + \frac{P_\phi^2}{2m\rho^2} + mgc\rho$$

$$\mathcal{H} = \frac{p_\rho^2}{2m(1+c^2)} + \frac{p_\phi^2}{2m\rho^2} + mgc\rho$$

$p_\phi \equiv$ const. (ϕ is ignorable/ cyclic)

$\longrightarrow$ we have a 1D problem

$$\dot{p}_\rho = \frac{\partial \mathcal{H}}{\partial \rho} = -\frac{p_\phi^2}{m\rho^3} + mgc \quad \Big| \quad \dot{\rho} = -\frac{\partial \mathcal{H}}{\partial p_\rho} = \frac{-p_\rho}{m(1+c^2)}$$

$$\ddot{\rho} = \frac{1}{m(1+c^2)}\left(\frac{p_\phi^2}{m\rho^3} - mgc\right)$$

$$\ddot{\rho} = 0 \Rightarrow mgc = \frac{p_\phi^2}{mr_0^3} \Rightarrow \boxed{r_0^3 = \frac{p_\phi^2}{m^2gc}} = \frac{(mr_0^2\omega_0)^2}{m^2gc} \Rightarrow \omega = \sqrt{\frac{gc}{r_0}}$$

Consider small oscillations of ρ about r_0.

$$\ddot{\rho} = \frac{1}{m(1+c^2)}\left(\frac{P_\phi^2}{m\rho^3} - mgc\right) \equiv f(\rho)$$

$$f(r_0+\varepsilon) = f(r_0) + \varepsilon f'(r_0) + \cancel{\mathcal{O}(\varepsilon^2)}$$

$$f(r_0) = \frac{1}{m(1+c^2)}\left(\frac{P_\phi^2}{mr_0^3} - mgc\right) = 0$$

$$f'(\rho) = \frac{1}{m(1+c^2)}\left(\frac{P_\phi^2}{m}\right)\left(\frac{-3}{\rho^4}\right) = -\frac{3P_\phi^2}{m^2(1+c^2)\rho^4}$$

$$f'(r_0) = -\frac{3P_\phi^2}{m^2(1+c^2)r_0^4}$$

$$\rho = r_0 + \varepsilon \Rightarrow \ddot{\varepsilon} = \ddot{\rho}$$

$$\ddot{\varepsilon} = -\frac{3P_\phi^2}{m^2(1+c^2)r_0^4}\,\varepsilon \quad \Rightarrow \Omega^2 = \frac{3P_\phi^2}{m^2(1+c^2)r_0^4} = \frac{3mgc}{mr_0(1+c^2)}$$

$$\Omega^2 = \frac{3g}{r_0}\frac{c}{1+c^2} = \frac{3g}{r_0}\cos\alpha\sin\alpha$$

or writing $P_\phi = m\rho^2\dot{\phi} = mr_0^2\omega_0$

$$\Omega^2 = \frac{3(mr_0^2\omega_0)^2}{m^2(1+c^2)r_0^4} = \frac{3\omega_0^2}{(1+c^2)} = 3\omega_0^2\sin^2\alpha$$

$$\Rightarrow \Omega = \omega_0\sin\alpha\sqrt{3}$$

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