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On the Elimination of Malitz Quantifiers over Archimedian Real Closed Fields

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Rapp [5] proves that the theory of the field $\mathbb{R}$ of reals in the language with Magidor-Malitz quantifiers $Q_1^1, Q_1^2, \ldots$ is decidable.

The quantifier Q_α^n is interpreted in a structure M by:

$M \models Q_\alpha^n x_1, \ldots, x_n \varphi$ iff there exists a set $B \subseteq M$, $\operatorname{card}(B) \geqq \omega_\alpha$, such that for all distinct $b_1, \ldots, b_n \in B$: $M \models \varphi[b_1, \ldots, b_n]$; such a set B is called *homogeneous* for φ. Using Tarski's classical result on real closed fields [6] Rapp's theorem is shown by proving that the quantifiers $Q_1^1, Q_1^2, \ldots$ are effectively eliminable over $\mathbb{R}$.

Bürger [1] addresses the question whether Rapp's methods can be applied to the class of all uncountable archimedian real closed fields (ARCF). Up to isomorphism this is the class of real closed subfields of $\mathbb{R}$. Bürger shows that the uniform eliminability of Q_1^n, for $n \geqq 3$, is independent of ZFC. In particular, the continuum hypothesis CH implies the existence of a counterexample to the eliminability of Q_1^3. (Note that by results of Cowles [2], Vinner [7], and Goltz [3], the quantifiers Q_1^1 and Q_1^2 are always effectively eliminable over uncountable ARCFs.)

In view of these results it seems natural to consider models of $\neg$CH, and study the uniform eliminability of $Q_\alpha^3, Q_\alpha^4, \ldots$ where $2^\omega = \operatorname{card}(\mathbb{R}) = \omega_\alpha$. Here we assume that whenever we talk about the quantifier Q_α^n, only fields of cardinality $\geqq \omega_\alpha$ are considered. We show that again eliminability is independent of ZFC + $\neg$CH:

Theorem 1. *Assume* ZFC + CH, *and let* $\kappa = \omega_\alpha > \omega_1$ *be regular. Let* $V[G]$ *be a generic extension of* V *by a* σ*-product of* κ *Sacks (perfect set) forcings. Then* $V[G] \models 2^\omega = \omega_\alpha$, *and, in* $V[G]$, *the quantifiers* Q_α^n, $n \geqq 3$, *are uniformly eliminable over ARCF.*

Fact 2. *Assume Martin's Axiom MA and* $2^\omega = \omega_\alpha > \omega_1$. *Then the quantifiers* Q_α^n, $n \geqq 3$, *are not uniformly eliminable over ARCF.*

The fact follows from the observation that Bürger's proof [1] of non-eliminability under CH uses CH only to the extent that the union of $< 2^\omega$ meager sets is meager. But this is also a consequence of MA (see Jech [4, p. 55]).

Before we prove the theorem we have to prepare material on the reduction of the quantifier-elimination of the Q_α^n, $n \geqq 1$, to a set theoretic property, and on product Sacks forcing.

Rapp [5] defines formulae ψ_k^n, for $k, n \geqq 1$:

$$\psi_k^n(x_1, \ldots, x_n) \equiv x_1 < \ldots < x_n \\ \rightarrow \bigwedge_{1 \leqq i < j < n} ((x_{i+1} - x_i) < (x_{j+1} - x_j)^k \vee (x_{j+1} - x_j) < (x_{i+1} - x_i)^k).$$

Fact 3 (Rapp [5]). *Let $\omega_\alpha \leqq 2^\omega$. Then ARCF admits effective elimination of the quantifier Q_α^n iff every field R in ARCF (of cardinality $\geqq \omega_\alpha$) satisfies $R \models Q_\alpha^n \psi_k^n$ for all $k \geqq 1$.*

Rapp obtained large homogeneous sets for ψ_k^n in $\mathbb{R}$ by modifying the Cantor discontinuum. Motivated by Rapp's construction we shall define *thin* systems of intervals.

Let $B = {}^{<\omega}2$ be the binary tree of all finite $0-1$-sequences; we use $r, s, t, \ldots$ to denote elements of B. The tree-ordering on B is given by $r \subseteq s$ iff r is an initial segment of s. $r^\frown 0$ and $r^\frown 1$ denote the continuations of r by 0, 1 respectively. We well-order B as follows: let $r \in {}^m 2$, $s \in 2^n$; then $r \lesssim s$ iff $m < n$ or $(m = n \wedge \exists i < m$ $(r \upharpoonright i = s \upharpoonright i \wedge r(i) = 0 \wedge s(i) = 1))$.

By an *interval* we understand a pair $I = \langle a, b \rangle$ of reals such that $a < b$. The intended meaning of I is the closed interval $[a, b]$ in $\mathbb{R}$, but the identification with pairs of reals makes the notion absolute.

If $I = \langle a, b \rangle$, $J = \langle a', b' \rangle$ are intervals, let $d(I, J) = \min\{|x - x'| \mid a \leqq x \leqq b, a' \leqq x' \leqq b'\}$ be their *distance*, $e(I, J) := \max(b, b') - \min(a, a')$ their *extent*, and $l(I) := b - a$ the *length* of I.

A sequence $\underline{I} = \langle I_r \mid r \in B \rangle$ is called a *filtration* if each I_r is an interval and $r \subseteq s \rightarrow I_r \supseteq I_s$. The *fusion* of $\underline{I}$ is defined as $\bigcap_{n<\omega} \bigcup_{r \in {}^n 2} I_r$; it is a closed set of reals. A filtration $\underline{I} = \langle I_r \mid r \in B \rangle$ is *k-thin*, $k < \omega$, provided

(i) $r \in {}^n 2 \rightarrow l(I_r) < 2^{-n}$;

(ii) if there are $s \supseteq r^\frown 0$, $t \supseteq r^\frown 1$ such that $I_s \cap I_t = \emptyset$ then $I_{r^\frown 0} \cap I_{r^\frown 1} = \emptyset$.

(iii) if $r \lesssim s$ and $I_{r^\frown 0} \cap I_{r^\frown 1} = \emptyset$ then $e(I_{s^\frown 0}, I_{s^\frown 1}) < d(I_{r^\frown 0}, I_{r^\frown 1})^k$.

Note that a k-thin filtration in V is also a k-thin filtration in any generic extension of the universe.

Lemma 4. *Let $\underline{I} = \langle I_r \mid r \in B \rangle$ be k-thin, and let $D = \bigcap_{n<\omega} \bigcup_{r \in {}^n 2} I_r$ be the fusion of $\underline{I}$. Then D is homogeneous for ψ_k^n, for all $n < \omega$.*

Proof. Let $x_1 < x_2 \leqq x_3 < x_4$ be elements of D. We have to show that $(x_2 - x_1) < (x_4 - x_3)^k$ or $(x_4 - x_3) < (x_2 - x_1)^k$. Let $r_1, r_2, r_3, r_4 \in B$ such that $x_1 \in I_{r_1}$, $x_2 \in I_{r_2}$, $x_3 \in I_{r_3}$, $x_4 \in I_{r_4}$ and $I_{r_1} \cap I_{r_2} = \emptyset$, $I_{r_3} \cap I_{r_4} = \emptyset$. Let $s = r_1 \cap r_2$, $t = r_3 \cap r_4$. By the thinness of $\underline{I}$, $I_{s^\frown 0} \cap I_{s^\frown 1} = \emptyset$ and $I_{t^\frown 0} \cap I_{t^\frown 1} = \emptyset$. Without loss of generality assume that $x_1 \in I_{s^\frown 0}$, $x_2 \in I_{s^\frown 1}$, $x_3 \in I_{t^\frown 0}$, $x_4 \in I_{t^\frown 1}$.

If $s = t$, we would get $x_3 < x_2$ *or* $x_4 < x_1$, contradiction.

Assume $s \lesssim t$. Then $x_4 - x_3 \leqq e(I_{t^\frown 0}, I_{t^\frown 1}) < d(I_{s^\frown 0}, I_{s^\frown 1})^k \leqq (x_2 - x_1)^k$. The case $t \lesssim s$ is treated analogously. QED

We now recall *Sacks forcing* $P, \leqq$ with perfect binary trees (see Jech [4, p. 15]). $p \subseteq B$ is a *perfect tree* if $p \neq \emptyset$ and $\forall s \in p\, \exists t \in p\, (t \supseteq s$ and $t^\frown 0 \in p, t^\frown 1 \in p)$. Order the set $P := \{p \subseteq B \mid p \text{ is perfect}\}$ by inclusion: $p \leqq q$ iff $p \subseteq q$. For $p \in P$ define $\operatorname{stem}(p) :=$ the unique $s \in p$ such that $\forall t \in p\, (t \subseteq s$ or $s \subseteq t)$ and $s^\frown 0, s^\frown 1 \in p$.

Define

$$p^{\langle 0 \rangle} := \{s \in p \mid s \subseteq \operatorname{stem}(p) \vee \operatorname{stem}(p)^\frown 0 \subseteq s\},$$

and

$$p^{\langle 1 \rangle} := \{s \in p \mid s \subseteq \operatorname{stem}(p) \vee \operatorname{stem}(p)^\frown 1 \subseteq s\}.$$

In general, define p^r, for $r \in B$ recursively: $p^{\emptyset} := p$; $p^{r^\frown 0} := (p^r)^{\langle 0 \rangle}$ and $p^{r^\frown 1} := (p^r)^{\langle 1 \rangle}$. So we obtain the subtree p^r of p by branching through p according to the $0-1$-pattern of r. We say that $s \in p$ is an $(n+1)$-st *branching point* of p, if $s = \operatorname{stem}(p^r)$ for some $r \in {}^n2$. Set $p \leqq_n q$ iff $p \leqq q$ and every n-th branching point of q is an n-th branching point of p $(n \geqq 1)$. Set $p \leqq_0 q$ iff $p \leqq q$. Note that $p \leqq_n q$ iff $p^r \leqq q^r$ for all $r \in {}^n2$. A *fusion sequence* in P is an ω-sequence $p_0 \geqq_0 p_1 \geqq_1 p_2 \geqq_2 p_3 \ldots$.

Fact 5 (see [4, p. 16]). *If $\langle p_n \rangle$ is a fusion sequence then the fusion $p = \bigcap_{n<\omega} p_n$ of $\langle p_n \rangle$ is a perfect tree.*

Let $p \in P$, $r \in {}^n2$, and $q \leqq p^r$. Then the *r-amalgamation* of q into p is defined as $\tilde{p} = (p \backslash p^r) \cup q$. Then $\tilde{p} \leqq_n p$, and $\tilde{p}^r = q$.

In Theorem 1 we force with the σ-product P^* of κ Sacks forcings P, where $\kappa = \omega_\alpha$ is a fixed uncountable regular cardinal (see [4, p. 31]).

Let us assume that the continuum hypothesis CH holds. Let P^κ be the κ-fold cartesian product of P. If $p = \langle p(i) \mid i < \kappa \rangle \in P^\kappa$, set $\operatorname{supp}(p) = \{i < \kappa \mid p(i) \neq B\}$; note that the full binary tree B is the weakest condition in the Sacks forcing P.

Set $P^* = \{p \in P^\kappa \mid \operatorname{card} \operatorname{supp}(p) < \omega_1\}$. Order P^* by coordinatewise inclusion $p \leqq q$ iff $\forall i < \kappa\; p(i) \leqq q(i)$.

Fact 6 [4, p. 31]). *Let G be P^*-generic over V. Then cardinals and cofinalities are absolute between V and $V[G]$, and $V[G] \models 2^\omega = \omega_\alpha$.*

We shall now generalize the ideas of fusion and amalgamation from P to P^*. If we want to "branch" through a condition in P^* according to some $r \in {}^n2$, we need a bookkeeping function b which picks out the coordinates $i < \kappa$, where we perform the branching.

So let us fix a function $b : \omega \to \kappa$, which later will be chosen conveniently. For $n < \omega$ and $i < \kappa$ define $n * i \in \omega$ by recursion on n: $0 * i := 0$ for all $i < \kappa$; $(n+1) * i := n * i$, if $b(n) \neq i$, and $(n+1) * i := (n * i) + 1$, if $b(n) = i$. For $r \in B$ and $i < \kappa$ define $r * i \in B$ by: $\emptyset * i := \emptyset$ for all $i < \kappa$; for $r \in {}^n2$ let $(r^\frown 0) * i := (r^\frown 1) * i := r * i$, if $b(n) \neq i$, and $(r^\frown 0) * i := (r * i)^\frown 0$, $(r^\frown 1) * i := (r * i)^\frown 1$, if $b(n) = i$.

Then define $p^r := \langle p^r(i) \mid i < \kappa \rangle \in P^*$ for $p = \langle p(i) \mid i < \kappa \rangle \in P^*$ and $r \in B$ by: $p^r(i) = (p(i))^{r * i}$.

Set $p \leqq_n q$ for $p, q \in P^*$ if for all $i < \kappa$ $p(i) \leqq_{n * i} q(i)$.

Again, $p \leqq_n q$ for $p, q \in P^*$ iff $p^r \leqq q^r$ for all $r \in {}^n2$.

Let $p \in P^*$, $r \in {}^n2$, $q \leqq p^r$. Then the *r-amalgamation* of q into p is defined as $\tilde{p} = \langle \tilde{p}(i) \mid i < \kappa \rangle$, where $\tilde{p}(i)$ is the $r * i$-amalgamation of $q(i)$ into $p(i)$. $\tilde{p} \leqq_n p$ and $\tilde{p}^r = q$.

In defining a fusion sequence in P^*, we have to stipulate that b takes care of every relevant index i infinitely often:

Let $p_0 \geqq_0 p_1 \geqq_1 p_2 \geqq_2 \ldots$. Let $S = \bigcup_{n<\omega} \operatorname{supp}(p)$. Then $\langle p_n \rangle$ is called a *fusion sequence*, if $S \subseteq \operatorname{range}(b|(\omega \backslash k))$, for all $k<\omega$.

Define the *fusion* $q = \langle q(i) \mid i<\kappa \rangle$ of $\langle p_n \rangle$ by $q(i) = \bigcap_{n<\omega} p_n(i)$. If $\langle p_n \rangle$ is a fusion sequence, then $\langle p_n(i) \rangle_{n<\omega}$ is basically a fusion sequence for all $i<\kappa$, and so the fusion q of $\langle p_n \rangle$ is a condition in P^*; $q \leqq p_n$, for all n.

We now prove our theorem: Assume ZFC + CH and let $\kappa = \omega_\alpha > \omega_1$ be regular. Construct P^*, $\leqq$ as above, and let G be P^*-generic over V. Then $V[G] \models 2^\omega = \omega_\alpha$.

Claim. *If $a \in V[G]$ is a real and $k<\omega$ then there exists a k-thin filtration $\underline{I} = \langle I_s \mid s \in B \rangle$ in the ground model V, such that $V[G] \models$ "a is in the fusion of $\underline{I}$."*

Before we prove the Claim, let us show that it yields a proof of the theorem: Work in $V[G]$. Let $R \subseteq \mathbb{R}$ be an ARCF of cardinality ω_α. Let $k \geqq 1$. For every $a \in R$ choose a k-thin filtration $\underline{I}^a \in V$ such that a is in the fusion of $\underline{I}^a$. Since ω_α is regular $>\omega_1$, there exists $D \subseteq R$, $\operatorname{card}(D) = \omega_\alpha$ and a k-thin filtration $\underline{I} \in V$, such that $\underline{I}^a = \underline{I}$ for $a \in D$. Then $D \subseteq$ fusion of $\underline{I}$, and by Lemma 4, D is a homogeneous set for ψ_k^n for $n<\omega$. By Rapp's Fact 1, the theorem is proved.

Now to prove the Claim it suffices to show the following inside the ground model:

Lemma 7. *Let $p \Vdash_{P^*} \dot{a} \in \mathbb{R}$, and $k \geqq 1$. Then there exists $p' \leqq p$ and a k-thin system $\underline{I}$ such that $p' \Vdash_{P^*}$ "$\dot{a}$ is in the fusion of $\check{\underline{I}}$."*

Proof. We shall construct a fusion sequence $p \geqq p_0 \geqq_0 p_1 \geqq_1 p_2 \geqq_2 p_3 \ldots$ in P^* over a suitable bookkeeping function b. It should be clear that such a b can actually be defined without difficulties during the subsequent construction, so we can proceed as if b were fixed in advance. Together with $\langle p_n \rangle$ we construct a k-thin system $\underline{I} = \langle I_r \mid r \in B \rangle$, such that for $r \in {}^n2$: $p_n^r \Vdash \dot{a} \in \check{I}_r$. If we let p' be the fusion of $\langle p_n \rangle$ then $(p')^r \leqq p_n^r$ for $r \in {}^n2$, and thus $p' \Vdash \dot{a} \in \bigcup_{r \in {}^n2} \check{I}_r$, for $n \in \omega$, hence $p' \Vdash$ "$\dot{a}$ is in the fusion of $\check{\underline{I}}$."

Let me now describe the recursive construction. Choose $p_0 \leqq p$ and an interval $I_\emptyset$ of length <1, such that $p_0 \Vdash \dot{a} \in \check{I}_\emptyset$. Assume that $p_0, \ldots, p_n$ and $\langle I_r \mid r \in {}^{\leqq n}2 \rangle$ are suitably defined. Let $r(1), \ldots, r(2^n)$ be the enumeration of ${}^n2$ in the well-order $\gtrsim$. Set $q_0 := p_n$. We shall define recursively for $i = 1, \ldots, 2^n$ conditions $q_i \leqq_n p_n$ and intervals $I_{r(i)^\frown 0}$, $I_{r(i)^\frown 1}$ such that $q_i^{r(i)^\frown 0} \Vdash \dot{a} \in \check{I}_{r(i)^\frown 0}$, $q_i^{r(i)^\frown 1} \Vdash \dot{a} \in \check{I}_{r(i)^\frown 1}$.

So fix i and assume the construction is suitably carried out for $j<i$. Set $r := r(i)$. Let $d := \min\{d(I_{s^\frown 0}, I_{s^\frown 1}) \mid I_{s^\frown 0} \cap I_{s^\frown 1} = \emptyset \text{ and } s \gtrsim r\}$. $q_{i-1}^r \Vdash \dot{a} \in \check{I}_r$. There are $q' \leqq q_{i-1}^r$ and an interval I_r' such that $q' \Vdash \dot{a} \in \check{I}_r' \subseteq \check{I}_r$ and the length $l(I_r') < \min\{2^{-(n+1)}, d^k\}$.

Case 1. I_r' splits over q', i.e., there are a condition $\tilde{q} \leqq q'$ and disjoint intervals $I_{r^\frown 0}$, $I_{r^\frown 1} \subseteq I_r'$, $I_{r^\frown 0} \cap I_{r^\frown 1} = \emptyset$, such that $\tilde{q}^{\langle 0 \rangle} \Vdash \dot{a} \in \check{I}_{r^\frown 0}$, $\tilde{q}^{\langle 1 \rangle} \Vdash \dot{a} \in \check{I}_{r^\frown 1}$.

Then let q_i be the r-amalgamation of $\tilde{q}$ into q_{i-1} with $I_{r^\frown 0}$, $I_{r^\frown 1}$ as above.

Case 2. Otherwise, then let q_i be the r-amalgamation of q' into q_{i-1}, and set $I_{r^\frown 0} := I_{r^\frown 1} := I_r'$.

When the construction for $i = 1, \ldots, 2^n$ is completed, set $p_{n+1} := q_{2^n}$. $p_{n+1} \leqq_n p_n$, and $p_{n+1}^s \Vdash \dot{a} \in \check{I}_s$ for all $s \in {}^{n+1}2$. This completes the definition of $\langle p_n \rangle$ and $\underline{I} = \langle I_r \mid r \in B \rangle$.

We have to check that $\underset{\sim}{I}$ is a k-thin filtration, i.e., satisfies conditions (i), (ii), and (iii) above. (i) and (iii) are obvious from the construction. For (ii), let $r \in {}^n 2$ and assume there are $s \supseteq r^\frown 0$, $t \supseteq r^\frown 1$ such that $I_s \cap I_t = \emptyset$. Let p' be the fusion of $\langle p_n \rangle$. Then $(p')^s \Vdash \dot{a} \in \check{I}_s$, $(p')^t \Vdash \dot{a} \in \check{I}_t$ witness that in the above construction of $I_{r^\frown 0}, I_{r^\frown 1}, I'_r$ splits over q'. Hence $I_{r^\frown 0}$ and $I_{r^\frown 1}$ are disjoint, as required. This completes the proof of Theorem 1. □

Let us finally comment on the eliminability of the quantifiers Q^n_β over ARCF for regular $\omega_\beta < \omega_\alpha$ within the model $V[G]$ constructed: If $2 \leqq \beta < \alpha$ the same argument as above shows that the quantifiers Q^n_β are uniformly eliminable over ARCF in $V[G]$.

In case $\beta = 1$, however, Bürger's counterexample R to the eliminability of Q^3_1 under CH stays a counterexample in the generic extension.

R is a counterexample because it is a *Lusin set*, i.e., every nowhere dense subset of R is countable. By a standard fusion argument, Lusin sets in V are Lusin in $V[G]$, so R is a counterexample in $V[G]$ to the eliminability of Q^3_1.

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