A (Mostly) Coherent talk on the Heisenberg Group

(Coherent States and their Applications in Physics)

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Introduction

Introduction

In the 1920's, Schrödinger was interested in finding quantum states that were dynamically analogous to their classical counterparts, especially a harmonic oscillator. This would require:

- A 'localized' quantum state (wavepacket) with minimal uncertainty
- The wave packet must oscillate at the frequency of the harmonic oscillator
- The wave packet must not spread out in time ($\partial_t \left\{ \Delta x^2 \Delta p^2 \right\} = 0$)
- The relative size of the fluctuations must vanish in the classical limit

It is well known that the ground state of the QHO, being a gaussian, is a minimum uncertainty wavepacket!

Proof.

Consider a quantum particle of mass m in a harmonic potential whose hamiltonian is given by

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

Definine the creation and annihilation operators $a^\dagger$ and a by

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{i}{m\omega}p\right)$$
$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}p\right)$$

which factors the Hamiltonian as

$$\mathcal{H} = h\omega(a^\dagger a + 1/2)$$

Proof.

Rewrite for x^2 and p^2:

$$x^2 = \frac{\hbar}{2m\omega}\left(a + a^\dagger\right)^2$$
$$p^2 = -\frac{m\omega\hbar}{2}\left(a - a^\dagger\right)^2.$$

Critically,

$$\langle 0|(a \pm a^\dagger)(a \pm a^\dagger)|0\rangle = \pm\langle 0|aa^\dagger|0\rangle = \pm 1$$

and so (since $\langle x\rangle_0 = \langle p\rangle_0 = 0$),

$$\langle(\Delta x)^2\rangle_0\,\langle(\Delta p)^2\rangle_0 = -\frac{\hbar^2}{4}[1(-1)] = \frac{\hbar^2}{4}$$

which is minimal!

Proof.

But are all eigenstates minimal?

$$\langle n|(a \pm a^\dagger)(a \pm a^\dagger)|n\rangle = \pm \langle n|aa^\dagger + a \dagger a|n\rangle = \pm(2n+1)$$

which leads to

$$\langle(\Delta x)^2\rangle_0 \langle(\Delta p)^2\rangle_0 = \frac{\hbar^2}{4}(2n+1)^2.$$

Only minimal for the ground state! □

What was different? Crucial step was that

$$a\,|0\rangle = 0 \implies \langle 0|a^\dagger a|0\rangle = 0$$

so if we postulate that $|0\rangle$ is an eigenvector of the annihilation operator a with eigenvalue 0, we are motivated to define the states $|z\rangle$ such that

$$a\,|z\rangle = z\,|z\rangle\,, \quad z \in \mathbb{C}.$$

Therefore

$$\langle z|(a \pm a^\dagger)|z\rangle = (z \pm \bar{z})$$
$$\langle z(a \pm a^\dagger)(a \pm a^\dagger)|z\rangle = (z \pm \bar{z})^2 \pm 1$$

and so

$$\langle(\Delta x)^2\rangle_z \, \langle(\Delta p)^2\rangle_z = \frac{\hbar^2}{4}$$

and we have a class of minimal uncertainty states called *coherent states*.

Coherent states are often expressed in the energy ($|n\rangle$) basis:

$$|z\rangle = \sum_n c_n |n\rangle = \sum_n |n\rangle \langle n|z\rangle .$$

We can construct any $|n\rangle$ state through successive creation operators on the ground state

$$|n\rangle = \frac{\left(a^\dagger\right)^n}{\sqrt{n!}} |0\rangle$$

and thus

$$\begin{aligned} |z\rangle &= \langle 0|z\rangle \sum_n \frac{z^n}{\sqrt{n!}} |n\rangle = e^{-\frac{1}{2}|z|^2} \sum_n \frac{z^n}{\sqrt{n!}} |n\rangle \\ &= e^{-\frac{1}{2}|z|^2 + za^\dagger} |0\rangle \end{aligned}$$

after solving for the normalization constant.

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after solving for the normalization constant.

Time Evolution

However, what we really need is states that remain minimal *for all* t. Clearly $|z\rangle$ states aren't stationary, but note that

$$|z,t\rangle = U(t,0)\,|z,0\rangle = e^{-i\mathcal{H}t/\hbar}e^{-\frac{1}{2}|z(0)|^2}\sum_n \frac{(z(0))^n}{\sqrt{n!}}\,|n\rangle$$
$$= e^{-\frac{1}{2}|z(0)|^2}\sum_n \frac{(z(0))^n}{\sqrt{n!}}\left[e^{-i\omega(n+1/2)t}\frac{\left(a^\dagger\right)^n}{\sqrt{n!}}\,|0\rangle\right]$$
$$= \exp\left\{-\frac{1}{2}|z(0)|^2 - \frac{i}{2}\omega t + z(0)e^{-i\omega t}a^\dagger\right\}|0\rangle\,.$$

Comparing from before, we have that

$$|z,t\rangle = e^{-\frac{i}{2}\omega t}\,|e^{-i\omega t}z(0)\rangle = |z(t)\rangle\,.$$

In other words - *a coherent state remains a coherent state under time evolution*!

We can rewrite this condition as

$$z(t) = e^{-i\omega t} z(0) \implies \frac{d}{dt} z(t) = -i\omega z(t)$$

which in components gives

$$\frac{d}{dt}\Re\{z\} = \omega \Im\{z\}$$
$$\frac{d}{dt}\Im\{z\} = -\omega \Re\{z\}.$$

Then recognizing that

$$\bar{x}(t) = \langle x(t) \rangle_z = \sqrt{\frac{\hbar}{2m\omega}} 2\Re\{z\}$$
$$\bar{p}(t) = \langle p(t) \rangle_z = i\sqrt{\frac{m\hbar\omega}{2}} (-2i)\Im\{z\},$$

we arrive at

$$\bar{p}(t) = m\frac{d}{dt}\bar{x}(t), \qquad \frac{d}{dt}\bar{p}(t) = -m\omega^2 \bar{x}(t)$$

Orthogonality and Completeness

One final property of coherent states is that they are *overcomplete*.

Proof.

Let $z \neq w$ be two distinct coherent states. Then

$$\begin{aligned} \langle z|w\rangle &= \sum_n \langle z|n\rangle \langle n|w\rangle \\ &= e^{-\frac{1}{2}\left(|z|^2+|w|^2\right)} \sum_n \frac{(\bar{z}w)^n}{n!} \\ &= \exp\left\{-\frac{1}{2}\left(|z|^2+|w|^2\right)+\bar{z}w\right\} \end{aligned}$$

and so

$$|\langle z|w\rangle|^2 = e^{-|z-w|^2} \neq 0 \quad \text{for } z \neq w$$

which means that the set of vectors $|z\rangle$ is an *overcomplete* basis! □

However, we can still define a completeness relation:

$$\int d^2z \, |z\rangle \langle z| = \int_{\mathbb{C}} d^2z \, e^{-|z|^2} \sum_{m,n} \frac{(\bar{z})^n z^m}{\sqrt{n!m!}} \, |m\rangle \langle n|$$

where the measure d^2z corresponds to all complex numbers $z \in \mathbb{C}$. Using polar coordinates one can show that

$$\int d^2 ze^{-|z|^2} (\bar{z})^n z^m = \pi n! \delta_{m,n}$$

and so

$$\int \frac{d^2z}{\pi} \, |z\rangle \langle z| = 1.$$

This ends up being the *most* important property of coherent states for many generalizations!

But what does this say about each coherent state $|z\rangle$? Using the completeness relation,

$$|z\rangle = \int_{\mathbb{C}} d^2z' \ |z'\rangle \langle z'|z\rangle = \int_{\mathbb{C}} d^2z' \, e^{-\frac{1}{2}|z-z'|^2} \, |z'\rangle$$

which we can interpret as a coherent state being equivalent (up to a phase) to a *weighted average over all coherent states* with the weights coming from a Gaussian distribution centered at z.

This means that if the oscillator is in the state $|z\rangle$, there is a non-zero probability that the oscillator is *also* in a different state $|w\rangle$ with the probability decreasing as $|z-w|$ increases.

Will return to this in quantum optics.

Summary

- We can construct 'classical' solutions of the quantum harmonic oscillator, called *coherent states*, from the eigenstates of the annihilation operator a.
- Each coherent state has minimal uncertainty for all t.
- The expectation values satisfy the classical equations of motion, satisfying Ehrenfest's Theorem.
- The family of coherent states are *overcomplete*

Formalism and the Heisenberg Group

Canonical or Standard Coherent States (CCS)

The most important properties of CCS are as follows:

P1: The states $|z\rangle$ saturate the Uncertainty Relation

$$\langle \Delta x \rangle_z \langle \Delta p \rangle_z = \frac{\hbar}{2}.$$

P2: The states $|z\rangle$ are eigenvectors of the annihilation operator, with eigenvalue z

$$a\,|z\rangle = z\,|z\rangle\,, \quad z \in \mathbb{C}.$$

Canonical or Standard Coherent States (CCS)

P3: The coherent states $\{|z\rangle\}$ constitute an overcomplete family of basis vectors in the Hilbert space of the H.O., which is encoded by

$$I = \frac{1}{\pi} \int_{\mathbb{C}} d\Re(z) d\Im(z) \, |z\rangle \langle z| \, .$$

P4: The family of states $|z\rangle$ is generated by an exponential operator acting on the ground state $|0\rangle$ of the H.O.

$$|z\rangle = e^{za^\dagger - \bar{z}a} \, |0\rangle \, .$$

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P4: The family of states $|z\rangle$ is the orbit of the ground state $|0\rangle$ under the action of the Weyl-Heisenberg group

$$|z\rangle = e^{za^\dagger - \bar{z}a} \, |0\rangle \, .$$

Heisenberg Algebra

The Heisenberg Algebra $\mathfrak{h}$ is motivated by the canonical commutation relations

$$[Q_i, Q_j] = [P_i, P_j] = 0, \quad [Q_i, P_j] = i\hbar\delta_{ij}.$$

However, it can be generalized to any dimension n:

Definition (Heisenberg Algebra)

The Heisenberg Lie algebra $\mathfrak{h}_\mathfrak{n}$ is a real $2n+1$ dimensional real Lie algebra with basis elements

$$\{P_1, \ldots, P_n, Q_1, \ldots, Q_n, S\}$$

and Lie bracket defined by

$$[P_i, P_j] = [Q_i, Q_j] = [P_i, S] = [Q_i, S] = [S, S] = 0,$$

$$[Q_i, P_j] = S\delta_{ij}$$

It turns out that $\mathfrak{h}$ is isomorphic to a Lie algebra of *strictly* upper triangular matrices. For $n = 1$, $pP + qQ + sS \in \mathfrak{h}$ for $p, q, s \in \mathbb{R}$ can be identified with

$$\begin{pmatrix} 0 & p & s \\ 0 & 0 & q \\ 0 & 0 & 0 \end{pmatrix}$$

and the Lie bracket is just the matrix commutator:

$$\left[\begin{pmatrix} 0 & p & s \\ 0 & 0 & q \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & p' & s' \\ 0 & 0 & q' \\ 0 & 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 & pq' - qp' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which can clearly be identified with the structure constants given before.

This can be generalized for higher n in the following form:

$$P_i = \begin{pmatrix} 0 & \mathbf{p} & 0 \\ 0 & 0_n & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0_n & \mathbf{q} \\ 0 & 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & s \\ 0 & 0_n & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

which satisfy the canonical commutation relations for a given n due to the properties of the cononical basis vectors e_i.

Some Properties of the Heisenberg Algebra

$\mathfrak{h}$ is isomorphic to striclty upper-triangular real matrices.

- However, the algebra can be generalized by allowing q, p, s to be elements of any commutative ring C
- Of special interest are the *extra special groups* in which the ring is of prime order p and the center is the cyclic group $\mathbb{Z}_p$
  - Quantum information, computing, error correction

The center (ideal) of $\mathfrak{h}$ is just S:

$$Z(\mathfrak{h}) = S = \begin{pmatrix} 0 & 0 & s \\ 0 & 0_n & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The only non-zero commutator is $[Q_i, P_i] = S \ (i\hbar)$ and S commutes with all other elements

- Thus $\mathfrak{h}$ is a *nilpotent Lie algebra* and therefore automatically a *solvable Lie algebra*. This is often called 'almost' abelian.

Heisenberg Group

WLOG, restrict to $n = 1$. Then exponentiating elements of $\mathfrak{h}$ gives (using Baker-Campbell-Hausdorff formula):

$$e^{pP+qQ+sS} = e^{pP+qQ+\frac{1}{2}pq[Q,P]+(s+\frac{1}{2}pq)S} = e^{pP}e^{qQ}e^{(s+\frac{1}{2}pq)S}$$

or equivalently in the matrix representation,

$$\exp\begin{pmatrix}0 & p & s\\ 0 & 0 & q\\ 0 & 0 & 0\end{pmatrix} = I + \begin{pmatrix}0 & p & s\\ 0 & 0 & q\\ 0 & 0 & 0\end{pmatrix} + \frac{1}{2!}\begin{pmatrix}0 & 0 & pq\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} + 0$$
$$= \begin{pmatrix}1 & p & s+\frac{1}{2}pq\\ 0 & 1 & q\\ 0 & 0 & 1\end{pmatrix}$$

as $(\cdot)^k = 0$ for $k \geq 3$. Therefore H is represented by $(n+2)\times(n+2)$ upper triangular matrices!

Properties of Heisenberg Group

H is a Lie group of dimension $2n+1$ that is isomorphic to $(n+2)\times(n+2)$ real upper triangular matrices, and is:

- Simply Connected
- Non-compact
- Non-Abelian
- Nilpotent.

H is a semidirect product:

$$H \cong \mathbb{R} \rtimes \mathbb{R}^2$$

where $\mathbb{R}^2$ corresponds to (P, Q) and $\mathbb{R}$ corresponds to S. This is the result of a *central extension* of $\mathbb{R}^2$ by $Z(H)$

H_{2n+1} is a subgroup of the affine group $\mathrm{Aff}(2n)$

.

Representations of the Heisenberg Group

While not the focus of this talk, there are many useful irreducible representations of the Heisenberg Group:

- Schrödinger Representation $(\Pi, L^2(\mathbb{R}))$

$$\Pi(Q)\psi(q) = q\psi(q), \qquad \Pi(P)\psi(q) = -i\hbar\nabla_q\psi(q)$$

- Momentum Representation (Fourier transform of S.R., unitarily equivalent)

$$\Pi(Q)\psi(p) = i\hbar\nabla_p\psi(p), \qquad \Pi(P)\psi_p(p) = p\psi(p)$$

- Fock-Bargmann (Theta) Representation

Yet this brings up a troubling question - when doing calculations involving the CC relations, why do we not have to specify which representation of the operators that we are using?

The remarkable answer is the Stone-von Neumann Theorem:

Theorem (Stone-von Neumann)

All irreducible representations of the Heisenberg group H_{2n+1} on a Hilbert space satisfying

$$\Pi(S) = i\hbar\mathbb{1}$$

are unitarily equivalent.

Once you fix a particular S ($i\hbar$ in our case), there is only one irreps up to unitary transformations!

This tells us that the Heisenberg Group is far more important than just the 'symmetries' of some physical system. It *encodes* the structure of quantum mechanics!

Note: This theorem breakes down for infinite degrees of freedom, as is the case in field theory. Extra difficulties arise in that case.

What's the Connection to Coherent States?

Before we wrote a generic element of $\mathfrak{h}$ as

$$pP + qQ + sS \in \mathfrak{h}.$$

In quantum mechanics, we choose the following convenient convention: We say that $\mathfrak{h}$ is spanned by the basis $\{iQ, iP, iI\}$ and a generic element can be written as

$$is + ipQ - iqP = is + i(pQ - qP) \in \mathfrak{h}, \quad s, q, p \in \mathbb{R}.$$

But we can switch to a different basis and define the complex number z such that

$$z = \frac{q + ip}{\sqrt{2}}, \quad Q = \frac{a + ia^\dagger}{\sqrt{2}}, \quad P = \frac{a - ia^\dagger}{\sqrt{2}i}$$

which gives

$$e^{is + i(pQ - qP)} = e^{is} e^{za^\dagger - \bar{z}a} \equiv e^{is} D(z)$$

where $D(z)$ is a unitary operator called the *Displacement Operator*.

The Displacement Operator

The operator $D(z)$ is (up to a phase factor!) a unitary representation of translations in the complex plane.

Proof.

Clearly $D(z)$ is unitary, as

$$D(z)D^\dagger(z) = e^{za^\dagger - \bar{z}a}e^{\bar{z}^a - za^\dagger} = I$$

from BCH as $[za^\dagger - \bar{z}a, \bar{z}a - za^\dagger] = -[za^\dagger - za, za^\dagger - za] = 0$.
Thus $D^\dagger(z) = D(-z)$. For two states $\alpha \neq \beta$,

$$\begin{aligned} D(\alpha)D(\beta) &= e^{\alpha a^\dagger - \bar{\alpha}a}e^{\beta a^\dagger - \bar{\beta}a} = e^{(\alpha+\beta)a^\dagger - \overline{(\alpha+\beta)}a}e^{\frac{1}{2}(\alpha\bar{\beta} - \bar{\alpha}\beta)} \\ &= e^{2i\Im(\alpha\beta)}D(\alpha+\beta). \end{aligned}$$

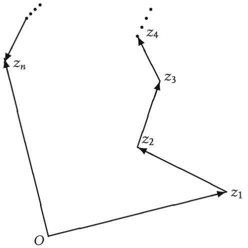
□

(Quick Aside)

More generally, we can write

$$D(z_n)D(z_{n-1})\dots D(z_1) = e^{i\delta} D(z_1 + z_2 + \dots + z_n)$$

where the phase $\delta = 2\sum_{j<k} \Im(z_j z_k)$ Interestingly, this phase factor has a definite topological meaning: it is equal to the oriented area of the polygon outlined by the closed path in $\mathbb{C}$ with vertices z_i as shown below



This represents a discrete version of Stokes' Theorem!

Back to Coherent States!

Using the displacement operator and definitions for z, Q, P in terms of $z, a, a^\dagger$, we see that (with the help of BCH)

$$e^{-\frac{i}{2}qp}e^{ipQ}e^{-iqP} = e^{-\frac{1}{2}|z|^2}e^{za^\dagger}e^{-\bar{z}a} = e^{za^\dagger - \bar{z}a} = D(z)$$

Now clearly

$$a\left|0\right\rangle = 0 \implies e^{-\bar{z}a}\left|0\right\rangle = \left|0\right\rangle + \sum_{n=1}^{\infty}\frac{(-\bar{z}a)^n}{n!}\left|0\right\rangle = \left|0\right\rangle.$$

Combining all of this gives

$$D(z)\left|0\right\rangle = e^{-\frac{1}{2}|z|^2}e^{za^\dagger}e^{\bar{z}a}\left|0\right\rangle = e^{-\frac{1}{2}|z|^2 + za^\dagger} = \left|z\right\rangle$$

and so we finally have our correspondence between coherent states and the Heisenberg Group:

$$\boxed{\left|z\right\rangle = D(z)\left|0\right\rangle}$$

We have now shown the following:

P4: The family of states $|z\rangle$ is the orbit of the ground state $|0\rangle$ under the action of the Weyl-Heisenberg group

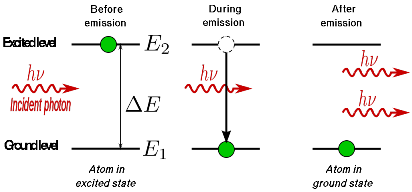
$$|z\rangle = e^{za^\dagger - \bar{z}a} |0\rangle .$$

Applications

Quantum Optics

The first and largest application of coherent states is to quantum optics. Physicist Roy Glauber introduced the term and modern use of the states in the 1960's when he presented the first fully quantum mechanical description of coherence in the electromagnetic field.

Coherent states are needed for understanding even ‘simple’ optical systems like lasers. The usual explanation of the pileup of stimulated emission of 2 photons in a resonant cavity:



$$E_2 - E_1 = \Delta E = h\nu$$

is **incorrect**! The outgoing ‘photons’ are actually the field superposition of no interaction (1 photon) and a stimulated emission (2 photons). After many such interactions and superpositions, we get *coherent light*, which is exactly described by coherent states. Not described by Fock numbers!

Probability Distribution in $|n\rangle$ Representation

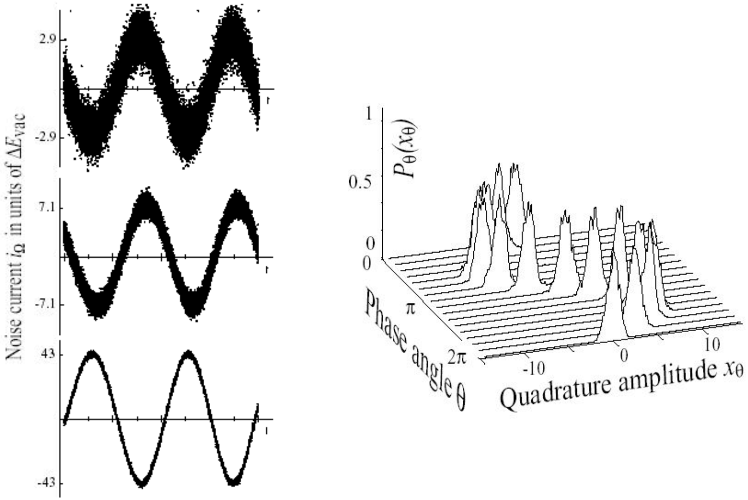
To see this: What's the probability of measuring a particular n (photon #) in a given coherent state $|z\rangle$?

$$P(n) = |\langle n|z\rangle|^2 = \left|\frac{z^n}{\sqrt{n!}}e^{-|z|^2/2}\right|^2 = \frac{|z|^{2n}e^{-|z|^2}}{n!}$$

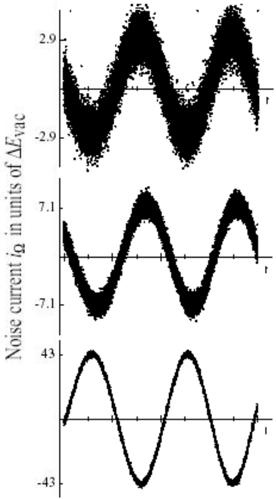
which is a Poisson distribution with $\lambda = |z|^2$. This means that given a *fixed* z, the most probable n is given by floor$\left\{|z|^2\right\}$. From this we find that

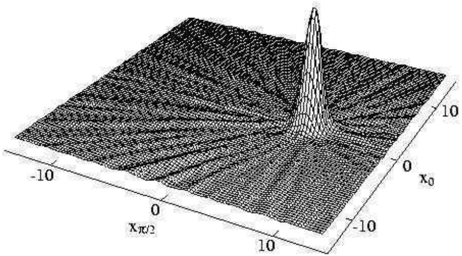
$$\langle H\rangle \approx \hbar\omega|z|^2, \quad \text{for } |z| > 1.$$

The Poisson distribution is a necessary and sufficient condition that all detections are statistically independent. Compare this to a single particle Fock state $|1\rangle$: Once a particle is detected, zero probability of detecting a different one.

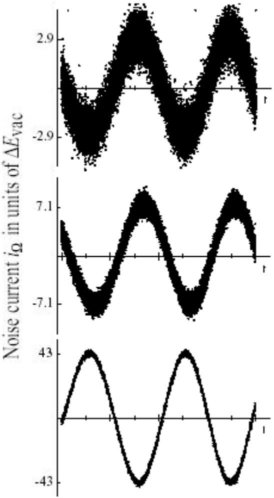


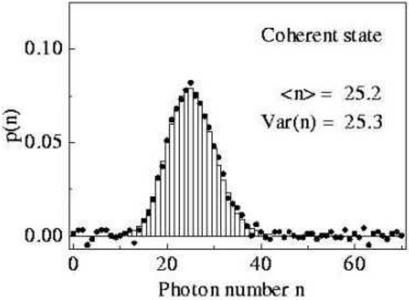
Measurement of the electric field of a laser at different amplitudes, and the oscillating wave packet of the middle coherent state.



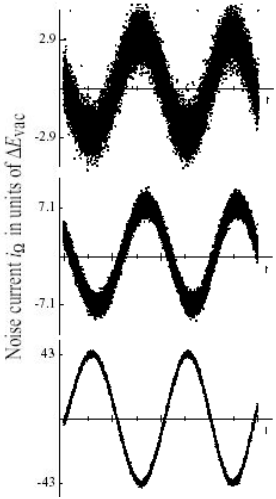


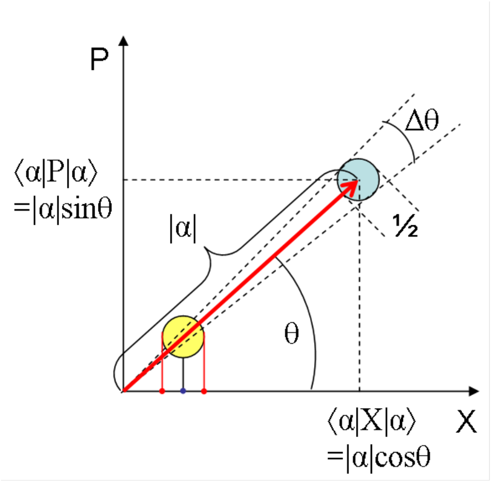
Wigner function at a particular phase of the middle coherent state. Ripples are from experimental errors.



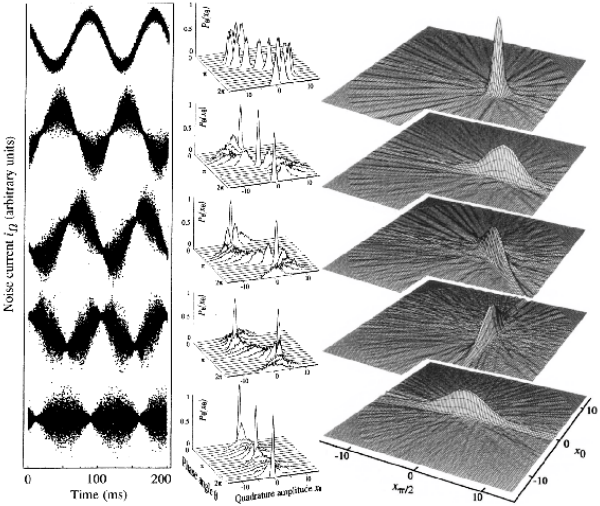


Probability density in $|n\rangle$-basis for bottom coherent state. Bars are data, dots from theory.





Phase space plot of a coherent state. Red dots trace out boundaries of quantum noise in left figure.



If we soften the restriction that $\langle \Delta x \rangle = \langle \Delta p \rangle$, then we can create *squeezed states*.

Wavelets and Signal Processing

What about orbits of the Heisenberg group on states that *are not* the ground state? Define a new class of (non-canonical) coherent states $|\alpha\rangle$ as

$$|\alpha\rangle \equiv D(z)\,|\psi\rangle\,.$$

While most properties of CCS have remained, we have lost one of the main motivations of coherent states:

$$a\,|\alpha\rangle \neq \alpha\,|\alpha\rangle\,.$$

However, crucially (although not proved here) the completeness relation is still satisfied:

$$c_\alpha|\alpha|^2 = \int_{\mathbb{C}} |\,\langle\alpha|D(z)|\alpha\rangle\,|^2 \frac{d^2z}{\pi} \implies I = \int_{\mathbb{C}} |\alpha\rangle\,\langle\alpha|\,\frac{d^2z}{c_\alpha\pi}.$$

The completeness relation allows us to to describe any $|\phi\rangle$ in the Hilbert space as

$$|\phi\rangle = \int_{\mathbb{C}} |\alpha\rangle \langle\alpha|\phi\rangle \frac{d^2 z}{c_\psi \pi}$$

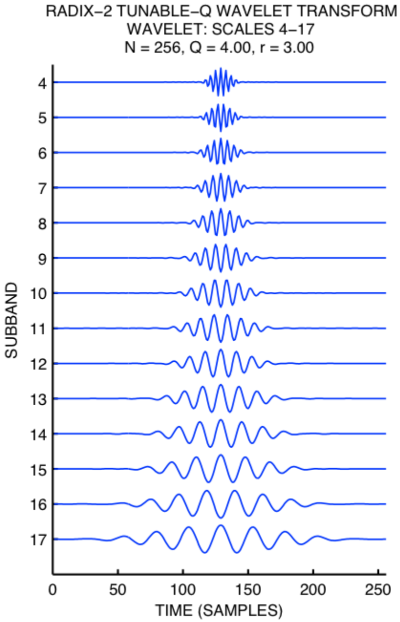
In position (Schrödinger) representation, the states $|\alpha\rangle$ is given by

$$\psi_\alpha(x) = e^{-\frac{i}{2}qp} e^{ipx} \psi(x-q)$$

called the *window*, *Gaboret*, or *wavelet*, and the projection $\langle\alpha|\phi\rangle$ is given by

$$\langle\alpha|\phi\rangle = \int_{-\infty}^{\infty} e^{\frac{i}{2}qp} e^{-ipx} \overline{\psi(x-q)} \phi(x) dx \equiv G_\phi(q,p).$$

This is called the *Gabor transform* or *windowed Fourier transform*. Due to the redundant basis, it represents a 'signal' in both the time-frequency or spatial-frequency domain.


RADIX–2 TUNABLE–Q WAVELET TRANSFORM
WAVELET: SCALES 4–17
N = 256, Q = 4.00, r = 3.00
SUBBAND
4
5
6
7
8
9
10
11
12
13
14
15
16
17
0
50
100
150
200
250
TIME (SAMPLES)

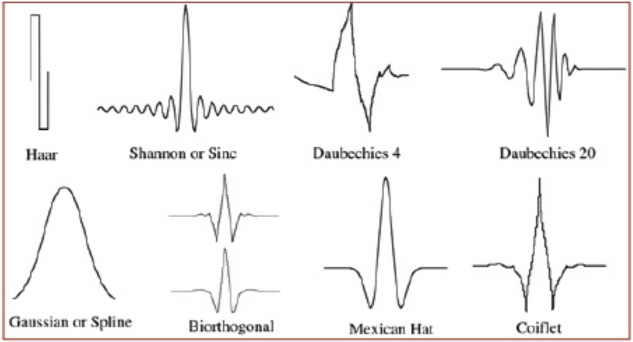
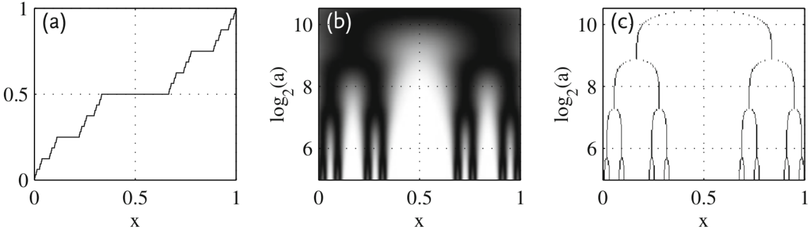


Figure 8

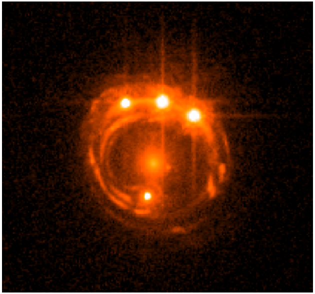
Examples of types of wavelets

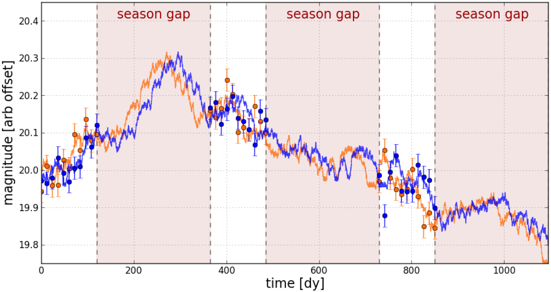


One example: Discontinuity detection on the Devil's Staircase.

Very useful in computer science, astronomy, and cosmology!

- Image compression and denoising
- Deblending
- Point spread function extraction
- Galaxy clustering and correlation functions
- Time-varying fluxes (strong-lens time delays)




season gap
season gap
season gap
magnitude [arb offset]
20.4
20.3
20.2
20.1
20.0
19.9
19.8
0
200
400
600
800
1000
time [dy]

More!

Coherent states is a *huge* topic with far too many applications to cover here. A few more examples:

- Feynmann Path Integration
- Geometric Quantization
- Quantum Information
- Probability and Bayesian Measure
- Quantum Hall Effect
- Partition function statistical mechanics
- Bose-Einstein Condensates
- Superfluidity and Superconductivity

Summary

By looking for the most ‘classical’-like states of the quantum harmonic oscillator, we are motivated to define coherent states as eigenvectors of the annihilation operator:

$$a\,|z\rangle = z\,|z\rangle\,.$$

We found that coherent states are the orbit of the Heisenberg ground state acting on the vacuum, and that $\mathfrak{h}$ encodes the structure of quantum mechanics uniquel (from Stone-von Neumann).

Coherent states constitute an *overcomplete basis* but still satisfies the completeness relation.

CS have numerous applications from theoretical tools in quantization and quantum optics to practical/experimental tools in signal processing.

References

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