# Artin formalism and Euler systems

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*(Based on joint works with David Loeffler and Victor Rotger.)*

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# Motivation

- $K$ number field, $\varrho : G_K \longrightarrow \mathrm{Aut}(V) \simeq \mathrm{GL}_n(\bar{\mathbb{Q}}_p)$ geometric Galois representation.
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- Bloch–Kato conjecture:

$$\mathrm{ord}_{s=0} L(\varrho^*(1), s) = \dim H^1_f(K, \varrho) - \dim H^0(K, \varrho).$$

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  (relation with other well-known problems in number theory, like the Birch and Swinnerton-Dyer conjecture).
- Iwasawa main conjecture: relation between the $p$-adic $L$-function and a Selmer group ("equality between an algebraic and an analytic $p$-adic $L$-function").

# The Bloch–Kato conjecture: examples

- $V = \mathbb{Q}_p(1)$. Consider the different terms:
  - $\mathrm{ord}_{s=0}\, L(V^*(1), s) = \mathrm{ord}_{s=0}\, \zeta_K(s) = r_1 + r_2 - 1$.
  - $\dim H^1_f(K, V) = \dim_{\mathbb{Q}_p}(\mathcal{O}_K^\times \otimes \mathbb{Q}_p) = r_1 + r_2 - 1$ [Dirichlet's unit theorem].
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  - $\dim H^0(K, V) = 0$.
- $V = V_p(E)$.
  - $\mathrm{ord}_{s=0} L(\varrho^*(1), s) = \mathrm{ord}_{s=1} L(E/K, s)$.
  - The image under Kummer's aplication lies in the Bloch–Kato Selmer group

    $$E(K) \otimes \mathbb{Q}_p \hookrightarrow H^1_f(K, V),$$

    with equality if and only if the $p$-part of Sha is finite.
  - $\dim H^0(K, \mathbb{Q}_p(1)) = 0$.

    We recover the Birch and Swinnerton-Dyer conjecture.

# Euler systems

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- $T \subset V$ stable lattice under the Galois action.
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## Definition

*An Euler system for $(T, \Sigma)$ is a collection $c = (c_m)_{m \geq 1}$, with $c_m \in H^1(\mathbb{Q}(\mu_m), T)$, and such that for $m \geq 1$ and $\ell$ prime*

$$N_{\mathbb{Q}(\mu_{m\ell})/\mathbb{Q}(\mu_m)}(c_{m\ell}) = \begin{cases} c_m & \textit{if } \ell \in \Sigma \textit{ or } \ell \mid m \\ P_\ell(V^*(1), \sigma_\ell^{-1}) \cdot c_m & \textit{elsewhere,} \end{cases}$$

*with $\sigma_\ell$ the image of $Frob_\ell$ in $Gal(\mathbb{Q}(\mu_m)/\mathbb{Q})$.*

**Main application:** bound Selmer groups.

# The easiest case: cyclotomic units

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- For $L/K$ finite, the corestriction map corresponds to the norm.
- Fix an embedding $\iota : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Set $\zeta_m = \iota^{-1}(e^{2\pi i/m})$.
- Let $u_m = 1 - \zeta_m$. Define

$$v_m = \begin{cases} u_m & \text{if } p \mid m \\ N_{\mathbb{Q}(\mu_{mp})/\mathbb{Q}(\mu_m)}(u_{pm}) & \text{if } p \nmid m. \end{cases}$$

- The classes $\kappa_p(v_m)$ form an Euler system for $(\mathbb{Z}_p(1), \{p\})$.

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4. **Heegner points.** Anticyclotomic classes coming from a geometric construction.

In most of the cases, tools are based on the manipulation of modular units over modular curves. The case of diagonal cycles just uses geometric cycles.

# New trends

(A) The case of totally real fields.

- BSD over totally real fields?
- Kato's techniques do not generalize: lack of modular units.
- Works of Barrera–Cauchi–Molina–Rotger. Use the geometry of diagonal cycles and use deformation arguments with weight one modular forms.
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(C) Euler systems for unitary groups (Loeffler–Skinner–Zerbes).

(D) Anticyclotomic Euler systems (general theory of Jetchev–Nekovar–Skinner, constructions of Graham–Shah and Alonso–Castellà–R.).

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- Triple $(f, g, h)$ of modular forms of weights $(k, \ell, m)$, with $2t = k - \ell - m \geq 0$. $f$ cuspidal. Rankin–Selberg $p$-adic $L$-functions interpolate $p$-adically

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- Reciprocity law connects that quantity with different Euler systems:
  - **Beilinson–Kato classes** when both $g$ and $h$ Eisenstein.
  - **Beilinson–Flach classes** when $h$ is Eisenstein and $g$ is cuspidal.
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- The geometry of those Euler systems is very different. Connections among them?

# Geometry of Euler systems

- **Beilinson–Kato classes.** Two modular units $u_1,\ u_2 \in H^1(Y, \mathbb{Z}_p(1))$:

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- **Diagonal cycles.** Trivial class $\Delta \in H^0(Y, \mathbb{Z}_p)$. Consider the inclusion $Y \hookrightarrow Y^3$ and get

$$\Delta \in H^0(Y, \mathbb{Z}_p) \hookrightarrow H^4(Y^3, \mathbb{Z}_p(2)).$$

# Objectives

We discuss two different approaches to study that connection.

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Possible applications:

- Non-vanishing results?
- Better comprehension of certain settings (BF classes over totally real fields?).

# Artin formalism

- $V = V_1 \oplus V_2$. The Artin formalism asserts that

$$L(V, s) = L(V_1, s) \cdot L(V_2, s).$$

Example: $V_E$ attached to $E_2(\psi, \tau)$. Then, $V = \mathbb{Z}_p(\psi) \oplus \mathbb{Z}_p(\tau)(1)$ and

$$L(E_2(\psi, \tau), s) = L(\psi, s) \cdot L(\tau, s-1).$$

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- It appears in many settings. Study of the Birch and Swinnerton-Dyer conjecture:

$$L(E/K, s) = L(E, s) \cdot L(E^D, s).$$

- Slogan: look at analogue for the corresponding algebraic structures.

# Artin formalism modulo $p$

- Modulo $p$ versions. What happens if $E_2(\psi,\tau) \equiv f$ modulo $p$. Subtler point.
- Algebraicity results at the level of $L$-functions. Need to normalize by suitable periods to have algebraic values (Shimura).
- Representations $V_E$ and $V_f$ agree up to semisimplication. But $V_f$ modulo $p$ is not a direct sum of two characters.

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- Representations $V_E$ and $V_f$ agree up to semisimplication. But $V_f$ modulo $p$ is not a direct sum of two characters.
- $p$-adic $L$-functions? It depends on the situation.
  - In some cases the interpolation regions match and it is straightforward.
  - In others, much more harder! Gross, Dasgupta...
- Euler systems are the geometric realization of $p$-adic $L$-functions. What does the Artin formalism mean at that level?

# Eisenstein series

Eisenstein series $f = E_{r+2}(\psi, \tau)$, where $\psi$ and $\tau$ Dirichlet characters

$$E_{r+2}(\psi, \tau) = (*) + \sum_{n \geq 1} q^n \Big( \sum_{n = d_1 d_2} \psi(d_1) \tau(d_2) d_2^{r+1} \Big).$$

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$$\alpha = \psi(p), \quad \beta = p^{r+1}\tau(p).$$

May consider either $f_\alpha$ or $f_\beta$. The former gives rise to family of Eisenstein series. Not interesting for us.

If $f = E_{r+2}(\psi, \tau)$ is a $p$-decent Eisenstein series, there are exactly three isomorphism classes of continuous Galois representations $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(L)$ which are unramified at primes $\ell \nmid Np$ and satisfy $\mathrm{tr}\,\rho(\mathrm{Frob}_\ell) = a_\ell(f)$. These are as follows:

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1. The semisimple representation $\psi \oplus \tau\epsilon^{r+1}$.
2. Exactly one non-split representation having $\tau\epsilon^{r+1}$ as a subrepresentation. This representation splits locally at $\ell$ for every $\ell \neq p$, and is crystalline at $p$.
3. Exactly one non-split representation having $\psi$ as a subrepresentation. This representation splits locally at $\ell$ for every $\ell \neq p$, but does not split at $p$, and is not crystalline (or even de Rham).

# The open and closed modular curve

Take $f \equiv E_2(\psi, 1)$ modulo $p^t$.

- $T_Y(f)^*$ maximal quotient of $H^1_{\text{et}}(\overline{Y}, \mathbb{Z}_p(1))$ where the (adjoint) action is via the Hecke eigensystem attached to $f$. Define similarly $T_X(f)^*$.

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- Let $I$ be the Eisenstein ideal. Chain

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- Most of our constructions will happen in the open modular curve.

# The case of Beilinson–Kato

- Beilinson–Kato class in

$$\kappa_f \in H^1(\mathbb{Q}, T_Y(f)^*(1)).$$

When $f \equiv E_2(\psi, 1)$, may project $T_Y(f)^* \otimes \mathbb{Z}/p^t \to \mathbb{Z}/p^t(\psi)$ and get a class

$$\kappa_{f,1} \in H^1(\mathbb{Q}, \mathbb{Z}/p^t(\psi)(1)).$$

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- Canonical class $c_\psi \in H^1(\mathbb{Q}, \mathbb{Z}/p^t(\psi)(1))$. This is the circular unit attached to $\psi$ (weighted combination of cyclotomic units).
- Explicit comparison between both of them:
  - Factorization formula of $p$-adic $L$-functions (Greenberg–Vatsal, Fukaya–Kato).
  - Comparison of Perrin-Riou maps.
  - Local to global statement (Gras).

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- Interpretation on the realm of Sharifi's conjectures, as the cup product of two circular units.
  - Mazur–Wiles isomorphism.
  - Use of Fukaya–Kato results relating this projection with evaluation at infinity.
  - Transition map between $H^2(\mathbb{Q}, \mathbb{Z}/p^t(2))$ and $H^1(\mathbb{Q}, \mathbb{Z}/p^t(2))$. It involves $L_p'(\bar{\psi}, -1)$ (note that $L_p'(\bar{\psi}, -1)$ is a multiple of $p$).

# Galois representations of critical Eisenstein series

Weight two $f_\beta = E_2^{\mathrm{crit}}(\psi, \tau)$.

- Define $V(f_\beta)^*$ as the maximal quotient of $H^1_{\mathrm{et}}(\overline{Y}, \mathbb{Z}_p(1))$ where the action is via the Hecke eigensystem attached to $f_\beta$.
  - 2-dimensional vector space, de Rham representation of $G_{\mathbb{Q}}$.
  - Fits into an exact sequence

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- Define $V^c(f_\beta)^*$ as the analogous space with compactly-supported rather than non-compactly-supported cohomology.

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- Define $V^c(f_\beta)^*$ as the analogous space with compactly-supported rather than non-compactly-supported cohomology.

$$0 \to \mathbb{Q}_p(\psi) \to V^c(f_\beta)^* \to \mathbb{Q}_p(\tau)(1) \to 0.$$

- Both sequences are non-split (classes (2) and (3) with the previous notations).

# The eigencurve at critical Eisenstein points

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- Differences with other cases of critical slope (recent work of Benois and Büyükboduk).
- May consider families of representations $V(\mathbf{f})^*$ and $V^c(\mathbf{f})^*$.
- Let $X$ be a uniformizer at the Eisenstein point. Then,

$$V(\mathbf{f})^* \supset V^c(\mathbf{f})^* \supset XV(\mathbf{f})^* \supset XV^c(\mathbf{f})^* \supset \dots$$

# Beilinson–Flach elements

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- 3-variable $p$-adic $L$-function $L_p^{\mathbf{f}}(\mathbf{f},\mathbf{g})$ interpolating $L(f_k, g_\ell, 1+j)$ along critical region $(\ell+1 \leq j \leq k)$.
- Reciprocity law

$$ \mathrm{Col}_{\eta_{\mathbf{f}}\otimes\omega_{\mathbf{g}}}(\mathrm{loc}_p({}_d\kappa(\mathbf{f},\mathbf{g}))) = C_d(\mathbf{f},\mathbf{g},\mathbf{j}) \cdot L_p^{\mathbf{f}}(\mathbf{f},\mathbf{g}). $$

# Beilinson–Flach vs Beilinson–Kato

- Three-variable BF class: two weight variables (corresponding to two Coleman families) and a cyclotomic variable. Class

$$\kappa(\mathbf{f},\mathbf{g}) \in H^1(\mathbb{Q}, \frac{1}{X} V^c(\mathbf{f},\mathbf{g})).$$

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- Slogan: the class has a simple pole at the critical Eisensteint point
- Take projection to the quotient $\frac{\frac{1}{X} V^c(\mathbf{f},\mathbf{g})}{V(\mathbf{f},\mathbf{g})}$.
- Hence, we can lift it to a class

$$_d\kappa(f_\beta,\mathbf{g}) \in H^1(\mathbb{Q}, \mathbb{Q}_p(\psi) \otimes V(\mathbf{g}) \otimes \mathcal{H}_\Gamma(-\mathbf{j})).$$

# Vanishing of the class

Why does the previous projection of $\kappa(\mathbf{f},\mathbf{g})$ vanish?

- Local properties of Beilinson–Flach elements. Both $V(\mathbf{f})$ and $V(\mathbf{g})$ admit local filtrations

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- Bloch–Kato known in this case. The space where the projected class lives is one-dimensional and spanned by the Kato class.
- The Kato class does not lie in the plus subspace: its projection to the minus quotient is the $p$-adic $L$-function.
- We conclude that the projection of the BF class must be zero.

# Beilinson–Flach vs Beilinson–Kato

**Theorem (Loeffler-R)**

*We have*

$$ {}_d\hat{\kappa}(f_\beta, \mathbf{g}) = \frac{\Big(C \cdot C_d(f_\beta, \mathbf{g}, \mathbf{j}) \log^{[r+1]} \cdot L_p(\mathbf{g} \otimes \tau, \mathbf{j} - 1 - r)\Big)}{L_p(Ad\ \mathbf{g})} \cdot \kappa(\mathbf{g} \times \psi) $$

*for some nonzero constant* $C \in L^\times$.

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Key ingredients.

1. Behaviour of cohomology classes (leading term argument).
2. Artin formalism for $L$-series.
3. Eichler–Shimura isomorphisms at critical Eisenstein points.
4. Bloch–Kato conjecture.

# Beilinson–Flach vs Beilinson–Kato

Other instances:

- Diagonal cycles degenerate to Beilinson–Flach elements.
  - Works in a similar way.
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- Heegner cycles degenerate to elliptic units.
  - Anticyclotomic analogue.
  - Heenger points satisfy a rather strong local condition.
- Beilinson–Kato classes degenerate to circular units (subtler).
  - We plan to explore this in the future.
  - No local conditions.

# Artin formalism and Euler systems

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02/02/2022

35 è SEMINARI DE TEORIA DE NOMBRES DE BARCELONA (STNB)