GROWTH OF ENTIRE $\mathcal{A}$-SUBHARMONIC FUNCTIONS

TERO KILPELÄINEN AND XIAO ZHONG

Dedicated to John L. Lewis on the occasion of his 59th birthday

ABSTRACT. We prove an estimate of the growth of a nonnegative $\mathcal{A}$-subharmonic function in $\mathbf{R}^n$ in terms of the Wolff potential of its Riesz measure. Our estimate can be viewed as a counterpart to Nevanlinna's first fundamental theorem for subharmonic functions in the nonlinear setting. As a consequence, we prove that a nonnegative $\mathcal{A}$-subharmonic function has the same order as the Wolff potential of its Riesz measure.

1. INTRODUCTION

If u is a nonnegative subharmonic function in $\mathbf{R}^n$, then Nevanlinna's first fundamental theorem tells us that

$$T(r,u) = N(r,u) + u(0)\,;$$

here $T(r,u)$ is the average of u on the sphere $\partial B(0,r)$ and

$$N(r,u) = d_n \int_0^r \frac{\mu(B(0,t))}{r^{n-2}}\,\frac{dt}{t}\,,$$

where $d_n = \max(1, n-2)$ and $\mu = \Delta u$ is the Riesz measure of u. Moreover, $T(r,u)$ and $\max_{B(x,r)} u$ have comparable growth. We refer to [HK, Section 3.9] for more thorough discussion.

In this paper we extend this result in the nonlinear setting. That is, we estimate the growth of nonnegative $\mathcal{A}$-subharmonic functions in $\mathbf{R}^n$ in terms of potentials of their Riesz measures

$$\mu = \operatorname{div} \mathcal{A}(x, \nabla u)\,,$$

where the operator $\operatorname{div} \mathcal{A}(x, \nabla u)$ is similar to the weighted p-Laplacian; see Section 2 below for precise assumptions.

Our first result gives a double sided estimate on the maximal growth of a nonnegative $\mathcal{A}$-subharmonic function u in terms of the (weighted) Wolff potential of its Riesz measure μ,

$$\mathbf{W}^{\mu}_{p,w}(x,r) = \int_0^r \left(t^p \frac{\mu(B(x,t))}{w(B(x,t))} \right)^{1/(p-1)} \frac{dt}{t}. \tag{1.1}$$

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In the unweighted case, where w is Lebesgue measure the Wolff potential takes the form

$$\mathbf{W}^{\mu}_{p,1}(x,r) = \mathrm{const} \int_0^r \left(\frac{\mu(B(x,t))}{t^{n-p}} \right)^{1/(p-1)} \frac{dt}{t}\,,$$

which with an appropriate choice of the constant reduces to $N(r,u)$ if $p=2$.

1.2. Theorem. *Let u be a nonnegative $\mathcal{A}$-subharmonic function in $\mathbf{R}^n$ and $\mu = \operatorname{div}\mathcal{A}(x,\nabla u)$ its Riesz measure. Then there is a constant $\delta = \delta(n,p,\lambda,\Lambda,c_w) \geq 1$ such that for all $r>0$*

$$u(0) + c_1 \mathbf{W}^{\mu}_{p,w}(0,r/2) \leq M(r) \leq 2u(0) + c_2 \mathbf{W}^{\mu}_{p,w}(0,\delta r)$$

where

$$M(r) = \sup_{B(0,r)} u$$

and c_1, c_2 are positive constants depending only on n,p,λ,Λ and constants associated with weight w.

Observe that by the maximum principle

$$M(r) = \sup_{B(0,r)} u = \max_{\partial B(0,r)} u\,.$$

The proof of Theorem 1.2 is based on the pointwise potential estimate for the $\mathcal{A}$-superharmonic functions [KM2] and on a method by Eremenko and Lewis [EL].

1.3. Corollary. *Let u be a nonnegative $\mathcal{A}$-subharmonic function in $\mathbf{R}^n$ and μ its Riesz measure. Then u is bounded in $\mathbf{R}^n$ if and only if*

$$\mathbf{W}^{\mu}_{p,w}(0,\infty) < \infty\,.$$

As in [HK, Definition 4.1], we define the *order* $\bar{\nu}$ and the *lower order* ν of a positive increasing function $S(r)$ by

$$\bar{\nu} = \overline{\lim_{r\to\infty}} \frac{\log S(r)}{\log r}\,, \quad \nu = \underline{\lim_{r\to\infty}} \frac{\log S(r)}{\log r}.$$

If u is a nonnegative $\mathcal{A}$-subharmonic function in $\mathbf{R}^n$, unbounded above, we define the order and the lower order of u be that of $M(r) = \sup_{B(0,r)} u$.

1.4. Corollary. *Let u be a nonnegative $\mathcal{A}$-subharmonic function in $\mathbf{R}^n$ and μ its Riesz measure. Then the order of u and the order of $\mathbf{W}^{\mu}_{p,w}(0,r)$ coincide. The same holds for lower orders.*

Corollaries 1.3 and 1.4 generalize classical results for the Laplacian [HK, Theorem 3.20 and 4.4]. Our results seem to be new even for linear uniformly elliptic equations in divergence form.

2. Preliminaries

In this section, we recall necessary definitions and required preliminary results.

We shall work with weighted setup. A function $w \in L^1_{\text{loc}}(\mathbf{R}^n)$, $w > 0$ a.e., is called a *weight*; also the associated measure is denoted by w, that is,

$$w(E) = \int_E w\,dx$$

for all measurable $E \subset \mathbf{R}^n$. In what follows we shall always assume that w is *p-admissible* in the sense of [HKM], i.e. the following four properties hold:

I *Doubling*: there is a constant $C_I \geq 1$ such that

$$w(B(x, 2r)) \leq C_I w(B(x, r))$$

for all balls $B(x, r) \subset \mathbf{R}^n$.

II *Uniqueness of the gradient*: If Ω is an open set in $\mathbf{R}^n$ and $(\varphi_j) \subset C^\infty(\Omega)$ is a sequence of functions such that as $j \to \infty$,

$$\int_\Omega |\varphi_j|^p\,dw \to 0 \quad \text{and} \quad \int_\Omega |\nabla\varphi_j - v|^p\,dw \to 0,$$

where $v \in L^p(\Omega; w)$, then $v = 0$ a.e. in Ω.

III *Sobolev inequality*: There are constants $\kappa > 1$ and $C_{III} > 0$ such that

$$\left(\frac{1}{w(B)} \int_B |\varphi|^{\kappa p}\,dw\right)^{1/\kappa p} \leq C_{III} r \left(\frac{1}{w(B)} \int_B |\nabla\varphi|^p\,dw\right)^{1/p}$$

for all balls $B = B(x, r) \subset \mathbf{R}^n$ and for all $\varphi \in C_0^\infty(B)$.

IV *Poincaré inequality*: There is a constant $C_{IV} > 0$ such that

$$\int_B |\varphi - \varphi_B|^p\,dw \leq C_{IV} r^p \int_B |\nabla\varphi|^p\,dw$$

for all balls $B = B(x, r) \subset \mathbf{R}^n$ and for all bounded $\varphi \in C^\infty(B)$, where

$$\varphi_B = \frac{1}{w(B)} \int_B \varphi\,dw.$$

In what follows we shall indicate the dependence on the above constants C_I, κ, C_{III}, and C_{IV} by c_w.

The above properties of the weight form a sufficient framework for a theory of quasilinear PDEs. This was first proven by Fabes, Kenig, and Serapioni in [FKS] and further exploited in [HKM]. Nowadays it is known that the uniqueness of the gradient and the Sobolev inequality can be deduced from the other two properties; the first was proven by Semmes (see [HeK]) and the second can be found e.g. in [HaK].

Examples of p-admissible weights are the constant weight $w = 1$, Muckenhoupt's A_p-weights, and certain powers of the Jacobians of quasiconformal mappings [HKM, ch. 15].

Throughout, we let $1 < p < \infty$ be a fixed number and Ω an open set in $\mathbf{R}^n$. The *weighted Sobolev space* $H^{1,p}(\Omega; w)$ is the completion of the set

$$\{\varphi \in C^\infty(\Omega) : ||\varphi||_{1,p,w} < \infty\}$$

with respect to the norm

$$||\varphi||_{1,p,w} = \left(\int_\Omega |\varphi|^p \, dw\right)^{1/p} + \left(\int_\Omega |\nabla\varphi|^p \, dw\right)^{1/p},$$

and $H^{1,p}_{\mathrm{loc}}(\Omega; w)$ the corresponding local space. The closure of $C_0^\infty(\Omega)$ in $H^{1,p}(\Omega; w)$ is denoted by $H_0^{1,p}(\Omega; w)$. For the basic properties of weighted Sobolev spaces we refer to [HKM, ch. 1].

$\mathcal{A}$-subharmonic functions. To define our operator we assume that $\mathcal{A} : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is a mapping satisfying the following properties:

$$\begin{aligned} &\text{the mapping } x \mapsto \mathcal{A}(x,\xi) \text{ is measurable for all } \xi \in \mathbf{R}^n, \text{ and} \\ &\text{the mapping } \xi \mapsto \mathcal{A}(x,\xi) \text{ is continuous for a.e. } x \in \mathbf{R}^n; \end{aligned} \tag{2.1}$$

There are constants $0 < \lambda \le \Lambda \le \infty$ such that for a.e. $x \in \mathbf{R}^n$ and for all $\xi \in \mathbf{R}^n$

$$|\mathcal{A}(x,\xi)| \le \Lambda w(x)|\xi|^{p-1}, \tag{2.2}$$

$$(\mathcal{A}(x,\xi) - \mathcal{A}(x,\zeta)) \cdot (\xi - \zeta) \ge \lambda w(x)(|\xi| + |\zeta|)^{p-2}|\xi - \zeta|^2, \tag{2.3}$$

whenever $\xi, \zeta \in \mathbf{R}^n$, and

$$\mathcal{A}(x, t\xi) = t|t|^{p-2}\mathcal{A}(x,\xi) \tag{2.4}$$

for all $t \in \mathbf{R} \setminus \{0\}$. A basic example of $\mathcal{A}$ satisfying the assumptions (2.1)-(2.4) is the weighted p-Laplacian, $\mathcal{A}(x,\xi) = w(x)|\xi|^{p-2}\xi$; in the unweighted case, where $w = 1$ this reduces to the p-Laplacian and, further, to the classical Laplacian if $p = 2$.

The above properties enable us to define a differential operator as follows. Assume that v is measurable function such that $|v|^{p-1}$ is locally integrable in Ω with respect to w-measure. Then $-\operatorname{div}\mathcal{A}(x, v(x))$ can be defined in the distributional sense:

$$-\operatorname{div}\mathcal{A}(x,v)(\varphi) = \int_\Omega \mathcal{A}\big(x, v(x)\big) \cdot \nabla\varphi \, dx\,, \quad \varphi \in C_0^\infty(\Omega)\,.$$

Then continuous function $u \in H^{1,p}_{\mathrm{loc}}(\Omega; w)$ is called $\mathcal{A}$-*harmonic* in Ω, if

$$-\mathrm{div}\mathcal{A}(x, \nabla u) = 0 \ \text{ in } \Omega\,.$$

The $\mathcal{A}$-subharmonic functions are defined via the comparison principle: an upper semicontinuous function $u : \Omega \to \mathbf{R} \cup \{-\infty\}$ is $\mathcal{A}$-*subharmonic* in Ω if it is not identically $-\infty$ and if for all open $D \subset\subset \Omega$ and $h \in C(\bar{D})$, $\mathcal{A}$-harmonic in D, the condition $h \ge u$ on ∂D implies $h \ge u$ in D. Further, a function v is $\mathcal{A}$-*superharmonic* in Ω if $-v$ is $\mathcal{A}$-subharmonic in Ω.

It is well known that in the case of the Laplacian, i.e. $\mathcal{A}(x,\xi)=\xi$, this definition is one of the equivalent characterizations of subharmonic functions, often defined via a submean value property, see [HK]. For a thorough discussion of $\mathcal{A}$-subharmonic functions see [HKM].

Truncations of $\mathcal{A}$-subharmonic functions belong locally to the Sobolev space $H^{1,p}_{\mathrm{loc}}(\Omega;w)$ which leads to the definition of the *weak gradient*

$$Du=\lim_{k\to\infty}\nabla\max(u,-k)\,.$$

This weak gradient is measurable and $|Du|^{p-1}$ is locally w-integrable. Hence the operator $\operatorname{div}\mathcal{A}(x,Du)$ is well defined. It can be shown that it is a nonnegative distribution whence represented by a Radon measure μ. We call this Radon measure

$$\mu=\operatorname{div}\mathcal{A}(x,Du)$$

the Riesz measure of an $\mathcal{A}$-subharmonic function u. For these properties the reader is referred to [HKM, Ch. 7], [KM1], and [M].

A fundamental property of $\mathcal{A}$-harmonic functions is the Harnack inequality.

2.5. Harnack's inequality. *Let h be a nonnegative $\mathcal{A}$-harmonic function in $B(x_0,r)$. Then*

$$\sup_{B(x_0,\tau r)}h\le c(1-\tau)^{-\beta}\inf_{B(x_0,\tau r)}h\,,$$

where $c=c(n,p,\lambda,\Lambda,c_w)$ and $\beta=\beta(n,p,\lambda,\Lambda,\,c_w)$ are positive constants.

We refer to [HKM, 6.2] for a proof of (2.5) when $\tau=1/2$. The general case follows by iteration.

For the proof of Theorem 1.2 we need the following pointwise estimate for the $\mathcal{A}$-superharmonic functions established in [KM2]; see also [M, Theorem 3.1] and [MZ].

2.6. Theorem. *Let u be a nonnegative $\mathcal{A}$-superharmonic function in $B(x_0,r)$ and*

$$\mu=-\operatorname{div}\mathcal{A}(x,\nabla u).$$

Then

$$c_3\mathbf{W}^{\mu}_{p,w}(x_0,r/2)\le u(x_0)\le c_4\inf_{B(x_0,r/2)}u+c_5\mathbf{W}^{\mu}_{p,w}(x_0,r),$$

where c_3,c_4 and c_5 are positive constants depending only on $n,p,\lambda,\Lambda,$ and c_w, and $\mathbf{W}^{\mu}_{p,w}(x_0,r)$ is the Wolff potential of μ, defined as in (1.1).

3. Proof of Theorem 1.2

For the proof of Theorem 1.2, we need the following lemma, whose proof is similar to that of [EL, Lemma 1].

3.1. Lemma. *Let u be a nonnegative $\mathcal{A}$-subharmonic function in $\mathbf{R}^n$ and μ its Riesz measure. Then there is $\theta=\theta(n,p,\lambda,\Lambda,c_w)$, $0<\theta<1$, such that if $M(tr)\le\theta M(r)$ for some $0<t\le 1$ and $r>0$, then*

$$M(r)\le c_6\left(r^p\,\frac{\mu(B(0,2r))}{w(B(0,2r))}\right)^{1/(p-1)},$$

where $c_6 = c_6(t, n, p, \lambda, \Lambda, c_w) > 0$, and $M(r)$ is defined as in Theorem 1.2.

Proof. We treat two cases: $p \geq 2$ and $1 < p < 2$ separately. First suppose that $p \geq 2$. Let h be the $\mathcal{A}$-harmonic function in $B(0, 2r)$ with boundary value u, that is,

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla h) = 0 & \text{in } B(0, 2r) \\ h - u \in H_0^{1,p}(B(0, 2r); w). \end{cases} \tag{3.2}$$

The existence of h follows from the theory of monotone coercive operators, see [HKM, Corollary 17.3]. We note that $u \in H^{1,p}_{\mathrm{loc}}(\mathbf{R}^n; w)$, since u is $\mathcal{A}$-subharmonic in $\mathbf{R}^n$ and bounded from below [HKM, Corollary 7.20]. By the comparison principle [HKM, 3.18], $0 \leq u \leq h$ in $B(0, 2r)$. From the Harnack inequality 2.5, we have

$$\inf_{B(0,r)} h \geq c_7 \sup_{B(0,r)} h \geq c_7 \sup_{B(0,r)} u = c_7 M(r)\,,$$

where $c_7 = c_7(n, p, \lambda, \Lambda, c_w) \leq 1$. Thus, for all $x \in B(0, tr)$,

$$h(x) - u(x) \geq \inf_{B(0,r)} h - \sup_{B(0,tr)} u \geq c_7 M(r) - \theta M(r) \geq \frac{c_7}{2} M(r), \tag{3.3}$$

by the assumption $M(tr) \leq \theta M(r)$ and choosing $\theta = c_7/2$. The function

$$\varphi = \min_{B(0,2r)} (h - u, \frac{c_7}{2} M(r))$$

is nonnegative in $B(0, 2r)$ and belongs to $H_0^{1,p}(B(0, 2r); w)$. By (3.3),

$$\varphi = \frac{c_7}{2} M(r) \quad \text{on } B(0, tr). \tag{3.4}$$

Let L be the set of points where $\nabla\varphi$ exists and is nonzero. Using φ as a test-function in the equations of u and h, we deduce that

$$\begin{aligned} \lambda \int_{B(0,2r)} |\nabla\varphi|^p \, dw &\leq \lambda \int_L (|\nabla h| + |\nabla u|)^{p-2} |\nabla h - \nabla u|^2 \, dw \\ &\leq \int_{B(0,2r)} (\mathcal{A}(x, \nabla h) - \mathcal{A}(x, \nabla u)) \cdot \nabla\varphi \, dx \\ &= -\int_{B(0,2r)} \mathcal{A}(x, \nabla u) \cdot \nabla\varphi \, dx = \int_{B(0,2r)} \varphi \, d\mu \\ &\leq \frac{c_7}{2} M(r) \mu(B(0, 2r)). \end{aligned} \tag{3.5}$$

Here we employed the fact $p \geq 2$ and assumption (2.3). On the other hand, by (3.4), Hölder's inequality and the Sobolev inequality (III),

$$\begin{aligned} \Big(\frac{c_7}{2} M(r)\Big)^p w(B(0, tr)) &\leq \int_{B(0,2r)} \varphi^p \, dw \\ &\leq w(B(0, 2r))^{1-1/\kappa} \left(\int_{B(0,2r)} \varphi^{\kappa p} \, dw \right)^{1/\kappa} \\ &\leq C_{III}^p (2r)^p \int_{B(0,2r)} |\nabla\varphi|^p \, dw \end{aligned} \tag{3.6}$$

We conclude the proof in this case by combining (3.5), (3.6) and

$$w(B(0,2r)) \le c_8 w(B(0,tr)), \tag{3.7}$$

where $c_8 = c_8(t, C_I) \ge 1$, which easily follows from the doubling property I of w.

To prove the lemma for $1 < p < 2$, let $H = H(\cdot, s)$ be the $\mathcal{A}$-harmonic function in $B(0,s)$, $r < s < 2r$, with $H - u \in H_0^{1,p}(B(0,s); w)$. If $r \le s' < s$, then from the Harnack's inequality (2.5) we find as above that

$$\inf_{B(0,s')} H \ge c(s,s') \sup_{B(0,s')} H \ge c(s,s') M(s'),$$

where

$$c(s,s') = \frac{1}{c}\left(\frac{s-s'}{s}\right)^{\beta}, \tag{3.8}$$

and c, β are constants in (2.5). Thus, for all $x \in B(0,tr)$,

$$H(x,s) - u(x) \ge \inf_{B(0,s')} H - \sup_{B(0,tr)} u \ge c(s,s')M(s') - \theta M(r) \ge \frac{c(s,s')}{2} M(s'),$$

if we assume that

$$c(s,s')M(s') \ge 2\theta M(r). \tag{3.9}$$

Let

$$\varphi = \min_{B(0,s)} (H - u, \frac{c(s,s')}{2} M(s')),$$

and L be the set of points in $B(0,s)$ where $\nabla\varphi$ exists and is nonzero. We note that

$$\varphi = \frac{c(s,s')}{2} M(s') \quad \text{ on } B(0,tr)$$

Using φ as a test-function in the eqautions of H and u, we deduce as in (3.5) that

$$\begin{aligned} I_1 &= \lambda \int_L (|\nabla H| + |\nabla u|)^{p-2} |\nabla H - \nabla u|^2 \, dw \\ &\le \frac{c(s,s')}{2} M(s') \mu(B(0,s)) \le \frac{c(s,s')}{2} M(s') \mu(B(0,2r)). \end{aligned} \tag{3.10}$$

By Hölder's inequality, we have

$$\begin{aligned} \int_{B(0,s)} |\nabla\varphi|^p \, dw &\le \left(\int_L (|\nabla H| + |\nabla u|)^{p-2} |\nabla H - \nabla u|^2 \, dw \right)^{p/2} \times \\ &\times \left(\int_{B(0,s)} (|\nabla H| + |\nabla u|)^p \, dw \right)^{(2-p)/2}. \end{aligned} \tag{3.11}$$

Let

$$I_2 = \int_{B(0,s)} (|\nabla H| + |\nabla u|)^p \, dw.$$

We estimate I_2 as as follows. First, by the quasiminimizing property of $\mathcal{A}$-harmonic functions [HKM, 3.15]

$$\int_{B(0,s)} |\nabla H|^p \, dw \le \left(\frac{\Lambda}{\lambda}\right)^p \int_{B(0,s)} |\nabla u|^p \, dw.$$

Secondly, by the well-known Caccioppoli inequality (see [HKM, 3.27]), we have for $s < s'' \le 2r$

$$\begin{aligned}\int_{B(0,s)} |\nabla u|^p \, dw &\le \left(\frac{\Lambda}{\lambda}\right)^p \frac{(4p)^p}{(s''-s)^p} \int_{B(0,s'')} |u|^p \, dw \\ &\le \left(\frac{\Lambda}{\lambda}\right)^p \frac{(4p)^p}{(s''-s)^p} w(B(0,2r))M(s'')^p,\end{aligned}$$

These together lead us to the estimate

$$I_2 \le \frac{c}{(s''-s)^p} w(B(0,2r))M(s'')^p, \tag{3.12}$$

where $c = c(p, \lambda, \Lambda) > 0$. Combining (3.10)–(3.12), we obtain that

$$\begin{aligned}\int_{B(0,s)} |\nabla \varphi|^p \, dw \le c(s''-s)^{-p(2-p)/2} \left(\frac{c(s,s')}{2} M(s')\mu(B(0,2r))\right)^{p/2} \times \\ \times w(B(0,2r)^{(2-p)/2} M(s'')^{p(2-p)/2}.\end{aligned} \tag{3.13}$$

On the other hand, as in (3.6) and (3.7), we deduce that

$$\frac{1}{c_8}\left(\frac{c(s,s')}{2} M(s')\right)^p w(B(0,2r)) \le C_{III}^p s^p \int_{B(0,s)} |\nabla \varphi|^p \, dw,$$

which, together with (3.13), gives

$$\begin{aligned}M(s')^{p/2} \le cs^p c(s,s')^{-p/2}(s''-s)^{-p(2-p)/2} \times \\ \times w(B(0,2r))^{-p/2}\mu(B(0,2r))^{p/2} M(s'')^{p(2-p)/2},\end{aligned} \tag{3.14}$$

where $c = c(t, p, \lambda, \Lambda, c_w) > 0$. This can be rewritten, after some juggling, as

$$\Psi(s') \le k(s, s', s'')(\Psi(s''))^{2-p}\,, \tag{3.15}$$

where

$$\Psi(\tau) = M(\tau)^{p/2} \left(r^p \frac{\mu(B(0,2r))}{w(B(0,2r))}\right)^{-p/2(p-1)} \tag{3.16}$$

and

$$k(s, s', s'') = cs^p c(s,s')^{-p/2}(s''-s)^{-p(2-p)/2} r^{-p^2/2}.$$

Estimate (3.15) holds for all $r \le s' < s < s'' \le 2r$ if (3.9) is satisfied.

Now, let

$$s_j = 2r(1 - 2^{-j}), j = 1, 2, \dots,$$

and put $s' = s_j, s'' = s_{j+1}$ and $s = (s_j + s_{j+1})/2$. We note that (3.9) is always true for $j = 1$, that is, (3.9) is true for $s' = s_1, s = (s_1 + s_2)/2$, if we choose

$$\theta = \frac{1}{2}c((s_1 + s_2)/2, s_1) = \frac{1}{2c5^\beta}.$$

Let θ be chosen in this way. We prove that the lemma is true for such a θ. Now we have two possibilities:

i) (3.9) is true for all $j = 1, 2, \dots$. Then in this case $M(r) = 0$, which easily follows from (3.9) and the fact $M(2r) < \infty$. The lemma is trivial.

ii) (3.9) is not true for some j. Let j_0 be the smallest number for which it fails. Then $j_0 > 1$, since (3.9) is satisfied for $j = 1$ by our choice of θ. This means (3.9) is true for all $j = 1, 2, \dots, j_0 - 1$, but

$$c((s_{j_0} + s_{j_0+1})/2, s_{j_0})M(s_{j_0}) < 2\theta M(r). \tag{3.17}$$

Consequently (3.15) is true for $s' = s_j$, $s'' = s_{j+1}$ and $s = (s_j + s_{j+1})/2$ for all $j = 1, 2, \dots, j_0 - 1$, and

$$k(s, s', s'') \le c2^{\gamma j}$$

for some $\gamma = \gamma(n, p, \lambda, \Lambda, c_w) \ge 1$. Using this inequality in (3.15) and iterating we obtain

$$\Psi(s_1) \le c2^\gamma \Psi(s_2)^{2-p} \le \cdots \le (c2^\gamma)^\beta \Psi(s_{j_0})^{(2-p)^{j_0-1}}, \tag{3.18}$$

where

$$\beta = \sum_{j=1}^{\infty} j(2-p)^{j-1} < \infty.$$

Taking account of the fact that $1 < p < 2$, we deduce from (3.17) and (3.18) by an easy calculation that

$$\Psi(s_1) \le c,$$

where $c > 0$ depends only on $t, n, p, \lambda, \Lambda, c_w$, not on j_0. This concludes the proof of the lemma. □

Now we are ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2. We first prove the left hand inequality in Theorem 1.2. Let u be a nonnegative $\mathcal{A}$-subharmonic function in $\mathbf{R}^n$ and μ its Riesz measure. Since $v = M(r) - u$ is a nonnegative $\mathcal{A}$-superharmonic function in $B(0, r)$ and

$$-\mathrm{div}\mathcal{A}(x, \nabla v) = \mu$$

The left hand inequality in Theorem 2.6 gives

$$c_3 \mathbf{W}^\mu_{p,w}(0, r/2) \le M(r) - u(0),$$

which prove the left hand inequality in Theorem 1.2.

Next, we prove the right hand inequality in Theorem 1.2. Let

$$\alpha = \frac{2c_4 - 1}{2c_4} < 1,$$

where c_4 is the constant in Theorem 2.6. Let k be the integer such that $\alpha^k < \theta \leq \alpha^{k-1}$, where θ is the constant in Lemma 3.1, and let $t = 2^{-k}$. Now fix $r > 0$. Suppose that there is j, $1 \leq j \leq k$, such that

$$M(2^{-j}r) \geq \alpha M(2^{-j+1}r).$$

Since $M(2^{-j+1}r) - u$ is a nonnegative p-superharmonic function in $B(0, 2^{-j+1}r)$, Theorem 2.6 shows that

$$\begin{aligned} M(2^{-j+1}r) - u(0) &\leq c_4(M(2^{-j+1}r) - M(2^{-j}r)) + c_5 \mathbf{W}^{\mu}_{p,w}(0, 2^{-j+1}r) \\ &\leq c_4(1-\alpha) M(2^{-j+1}r) + c_5 \mathbf{W}^{\mu}_{p,w}(0, r) \\ &= \frac{1}{2} M(2^{-j+1}r) + c_5 \mathbf{W}^{\mu}_{p,w}(0, r), \end{aligned}$$

that is,

$$M(tr) \leq M(2^{-j+1}r) \leq 2u(0) + 2c_5 \mathbf{W}^{\mu}_{p,w}(0, r). \tag{3.19}$$

If for all $j = 1, 2, \ldots, k$,

$$M(2^{-j}r) < \alpha M(2^{-j+1}r),$$

then

$$M(tr) = M(2^{-k}r) < \alpha^k M(r) < \theta M(r).$$

We may now apply Lemma 3.1 to obtain that

$$\begin{aligned} M(tr) \leq M(r) &\leq c_6 \left(r^p \frac{\mu(B(0, 2r))}{w(B(0, 2r))} \right)^{1/(p-1)} \\ &\leq c \mathbf{W}^{\mu}_{p,w}(0, 4r), \end{aligned} \tag{3.20}$$

by the doubling property I of w. Since either (3.19) or (3.20) is true, we arrive at

$$M(tr) \leq 2u(0) + c \mathbf{W}^{\mu}_{p,w}(0, 4r)$$

for all $r > 0$. This is equivalent to the right hand inequality of Theorem 1.2. □

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University of Jyväskylä, Department of Mathematics and Statistics
P.O. Box 35, FIN-40014 Jyväskylä, Finland
E-mail: terok@math.jyu.fi zhong@math.jyu.fi