Module 3

Constitutive Equations

Learning Objectives

- Understand basic stress-strain response of engineering materials.
- Quantify the linear elastic stress-strain response in terms of tensorial quantities and in particular the fourth-order elasticity or stiffness tensor describing Hooke's Law.
- Understand the relation between internal material symmetries and macroscopic anisotropy, as well as the implications on the structure of the stiffness tensor.
- Quantify the response of anisotropic materials to loadings aligned as well as rotated with respect to the material principal axes with emphasis on orthotropic and transversely-isotropic materials.
- Understand the nature of temperature effects as a source of thermal expansion strains.
- Quantify the linear elastic stress and strain tensors from experimental strain-gauge measurements.
- Quantify the linear elastic stress and strain tensors resulting from special material loading conditions.

3.1 Linear elasticity and Hooke's Law

Readings: Reddy 3.4.1 3.4.2
BC 2.6

Consider the stress strain curve $\sigma = f(\epsilon)$ of a linear elastic material subjected to uni-axial stress loading conditions (Figure 3.1).

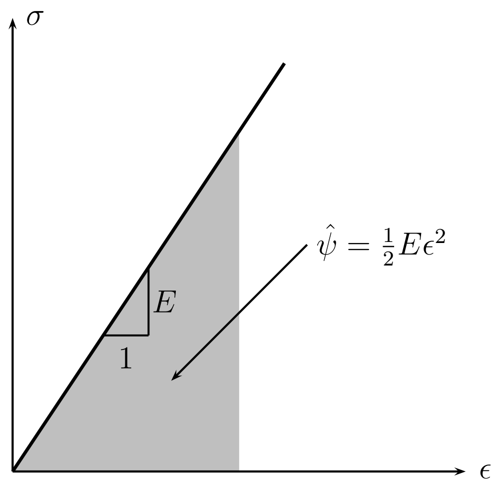


Figure 3.1: Stress-strain curve for a linear elastic material subject to uni-axial stress σ (Note that this is not uni-axial strain due to Poisson effect)

In this expression, E is Young's modulus.

Strain Energy Density

For a given value of the strain ϵ, the *strain energy density (per unit volume)* $\psi = \hat{\psi}(\epsilon)$, is defined as the area under the curve. In this case,

$$\psi(\epsilon) = \frac{1}{2}E\epsilon^2$$

We note, that according to this definition,

$$\sigma = \frac{\partial \hat{\psi}}{\partial \epsilon} = E\epsilon$$

In general, for (possibly non-linear) elastic materials:

$$\sigma_{ij} = \sigma_{ij}(\epsilon) = \frac{\partial \hat{\psi}}{\partial \epsilon_{ij}} \tag{3.1}$$

Generalized Hooke's Law

Defines the most general linear relation among all the components of the stress and strain tensor

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} \tag{3.2}$$

In this expression: C_{ijkl} are the components of the fourth-order *stiffness* tensor of material properties or *Elastic moduli*. The fourth-order stiffness tensor has 81 and 16 components for three-dimensional and two-dimensional problems, respectively. The strain energy density in this case is a quadratic function of the strain:

$$\hat{\psi}(\epsilon) = \frac{1}{2} C_{ijkl} \epsilon_{ij} \epsilon_{kl} \tag{3.3}$$

Concept Question 3.1.1. *Derivation of Hooke's law.*

Derive the Hooke's law from quadratic strain energy function Starting from the quadratic strain energy function and the definition for the stress components given in the notes,

1. derive the Generalized Hooke's law $\sigma_{ij} = C_{ijkl}\epsilon_{kl}$.

3.2 Transformation of basis for the elasticity tensor components

Readings: BC 2.6.2, Reddy 3.4.2

The stiffness tensor can be written in two different orthonormal basis as:

$$\begin{aligned} \mathbf{C} &= C_{ijkl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \\ &= \tilde{C}_{pqrs} \tilde{\mathbf{e}}_p \otimes \tilde{\mathbf{e}}_q \otimes \tilde{\mathbf{e}}_r \otimes \tilde{\mathbf{e}}_s \end{aligned} \tag{3.4}$$

As we've done for first and second order tensors, in order to transform the components from the $\mathbf{e}_i$ to the $\tilde{\mathbf{e}}_j$ basis, we take dot products with the basis vectors $\tilde{\mathbf{e}}_j$ using repeatedly the fact that $(\mathbf{e_m} \otimes \mathbf{e}_n) \cdot \tilde{\mathbf{e}}_k = (\mathbf{e}_n \cdot \tilde{\mathbf{e}}_k)\mathbf{e_m}$ and obtain:

$$\tilde{C}_{ijkl} = C_{pqrs} (\mathbf{e}_p \cdot \tilde{\mathbf{e}}_i)(\mathbf{e}_q \cdot \tilde{\mathbf{e}}_j)(\mathbf{e}_r \cdot \tilde{\mathbf{e}}_k)(\mathbf{e}_s \cdot \tilde{\mathbf{e}}_l) \tag{3.5}$$

3.3 Symmetries of the stiffness tensor

Readings: BC 2.1.1

The stiffness tensor has the following *minor symmetries* which result from the symmetry of the stress and strain tensors:

$$\sigma_{ij} = \sigma_{ji} \;\Rightarrow C_{jikl} = C_{ijkl} \tag{3.6}$$

Proof by (generalizable) example:

$$\text{From Hooke's law we have } \sigma_{21} = C_{21kl}\epsilon_{kl},\ \sigma_{12} = C_{12kl}\epsilon_{kl}$$
$$\text{and from the symmetry of the stress tensor we have } \sigma_{21} = \sigma_{12}$$
$$\Rightarrow \quad \text{Hence } C_{21kl}\epsilon_{kl} = C_{12kl}\epsilon_{kl}$$
$$\text{Also, we have } \big(C_{21kl} - C_{12kl}\big)\epsilon_{kl} = 0 \;\Rightarrow\; \text{Hence } C_{21kl} = C_{12kl}$$

This reduces the number of material constants from $81 = 3 \times 3 \times 3 \times 3 \to 54 = 6 \times 3 \times 3$. In a similar fashion we can make use of the symmetry of the strain tensor

$$\epsilon_{ij} = \epsilon_{ji} \;\Rightarrow C_{ijlk} = C_{ijkl} \tag{3.7}$$

This further reduces the number of material constants to $36 = 6 \times 6$. To further reduce the number of material constants consider equation (10.1), (10.1):

$$\sigma_{ij} = \frac{\partial \hat{\psi}}{\partial \epsilon_{ij}} = C_{ijkl}\epsilon_{kl} \tag{3.8}$$

$$\frac{\partial^2 \hat{\psi}}{\partial \epsilon_{mn} \partial \epsilon_{ij}} = \frac{\partial}{\partial \epsilon_{mn}} \big(C_{ijkl}\epsilon_{kl} \big) \tag{3.9}$$

$$C_{ijkl}\delta_{km}\delta_{ln} = \frac{\partial^2 \hat{\psi}}{\partial \epsilon_{mn} \partial \epsilon_{ij}} \tag{3.10}$$

$$C_{ijmn} = \frac{\partial^2 \hat{\psi}}{\partial \epsilon_{mn} \partial \epsilon_{ij}} \tag{3.11}$$

Assuming equivalence of the mixed partials:

$$C_{ijkl} = \frac{\partial^2 \hat{\psi}}{\partial \epsilon_{kl} \partial \epsilon_{ij}} = \frac{\partial^2 \hat{\psi}}{\partial \epsilon_{ij} \partial \epsilon_{kl}} = C_{klij} \tag{3.12}$$

This further reduces the number of material constants to 21. The most general anisotropic linear elastic material therefore has 21 material constants. We can write the stress-strain relations for a linear elastic material exploiting these symmetries as follows:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ & & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ & & & C_{2323} & C_{2313} & C_{2312} \\ & symm & & & C_{1313} & C_{1312} \\ & & & & & C_{1212} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix} \tag{3.13}$$

3.4 Engineering or Voigt notation

Since the tensor notation is already lost in the matrix notation, we might as well give indices to all the components that make more sense for matrix operation:

$$\begin{bmatrix} \sigma_{1} \\ \sigma_{2} \\ \sigma_{3} \\ \sigma_{4} \\ \sigma_{5} \\ \sigma_{6} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & symm & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{1} \\ \epsilon_{2} \\ \epsilon_{3} \\ \epsilon_{4} \\ \epsilon_{5} \\ \epsilon_{6} \end{bmatrix} \tag{3.14}$$

We have: 1) combined pairs of indices as follows: $()_{11} \rightarrow ()_1, ()_{22} \rightarrow ()_2, ()_{33} \rightarrow ()_3, ()_{23} \rightarrow ()_4, ()_{13} \rightarrow ()_5, ()_{12} \rightarrow ()_6$, and, 2) defined the *engineering shear strains* as the sum of symmetric components, e.g. $\epsilon_4 = 2\epsilon_{23} = \epsilon_{23} + \epsilon_{32}$, etc.

When the material has symmetries within its structure the number of material constants is reduced even further. We now turn to a brief discussion of material symmetries and anisotropy.

3.5 Material Symmetries and anisotropy

Anisotropy refers to the directional dependence of material properties (mechanical or otherwise). It plays an important role in Aerospace Materials due to the wide use of engineered composites.

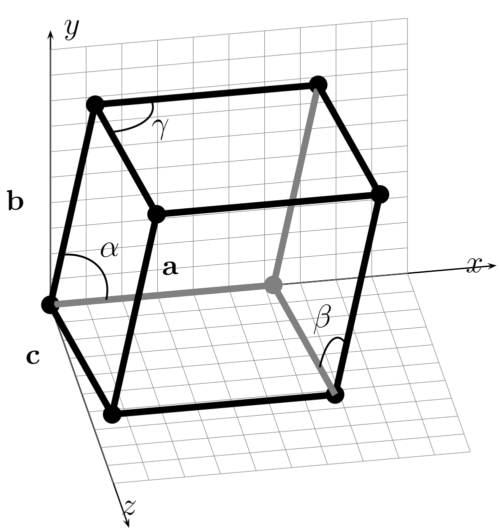
The different types of material anisotropy are determined by the existence of symmetries in the internal structure of the material. The more the internal symmetries, the simpler the structure of the stiffness tensor. Each type of symmetry results in the invariance of the stiffness tensor to a specific *symmetry transformations* (rotations about specific axes and reflections about specific planes). These symmetry transformations can be represented by an orthogonal second order tensor, i.e. $\mathbf{Q} = Q_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, such that $\mathbf{Q}^{-1} = \mathbf{Q}^T$ and:

$$det(Q_{ij}) = \begin{cases} +1 & \text{rotation} \\ -1 & \text{reflection} \end{cases}$$

The invariance of the stiffness tensor under these transformations is expressed as follows:

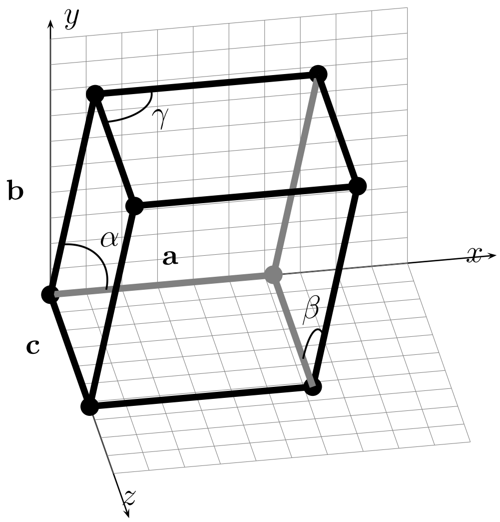
$$C_{ijkl} = Q_{ip}Q_{jq}Q_{kr}Q_{ls}C_{pqrs} \tag{3.15}$$

Let's take a brief look at various **classes of material symmetry**, corresponding **symmetry transformations**, implications on the **anisotropy of the material**, and the **structure of the stiffness tensor**:



Triclinic: no symmetry planes, fully anisotropic.
$\alpha, \beta, \gamma < 90$
Number of independent coefficients: 21
Symmetry transformation: None

$$C = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ & & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ & & & C_{2323} & C_{2313} & C_{2312} \\ & symm & & & C_{1313} & C_{1312} \\ & & & & & C_{1212} \end{bmatrix}$$



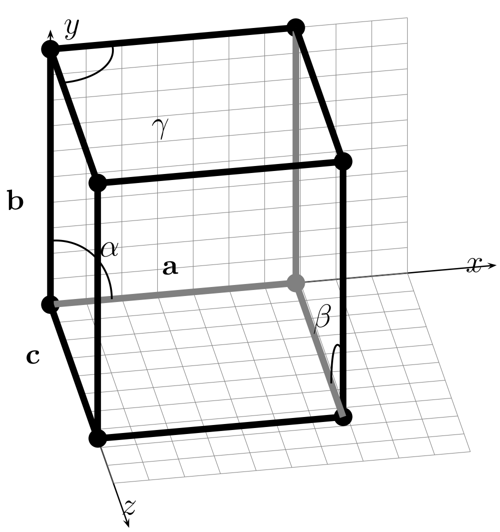
Monoclinic: one symmetry plane (xy). $a \neq b \neq c,\ \beta = \gamma = 90, \alpha < 90$
Number of independent coefficients: 13
Symmetry transformation: reflection about z-axis

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & C_{1112} \\ & C_{2222} & C_{2233} & 0 & 0 & C_{2212} \\ & & C_{3333} & 0 & 0 & C_{3312} \\ & & & C_{2323} & C_{2313} & 0 \\ & symm & & & C_{1313} & 0 \\ & & & & & C_{1212} \end{bmatrix}$$

Concept Question 3.5.1. *Monoclinic symmetry.*
Let's consider a monoclinic material.

1. Derive the structure of the stiffness tensor for such a material and show that the tensor has 13 independent components.



Orthotropic: three mutually orthogonal planes of reflection symmetry. $a \neq b \neq c,\ \alpha = \beta = \gamma = 90$
Number of independent coefficients: 9
Symmetry transformations: reflections about all three orthogonal planes

$$Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ & C_{2222} & C_{2233} & 0 & 0 & 0 \\ & & C_{3333} & 0 & 0 & 0 \\ & & & C_{2323} & 0 & 0 \\ & symm & & & C_{1313} & 0 \\ & & & & & C_{1212} \end{bmatrix}$$

Concept Question 3.5.2. *Orthotropic elastic tensor.*
Consider an orthotropic linear elastic material where **1**, **2** and **3** are the orthotropic axes.

1. Use the symmetry transformations corresponding to this material shown in the notes to derive the structure of the elastic tensor.

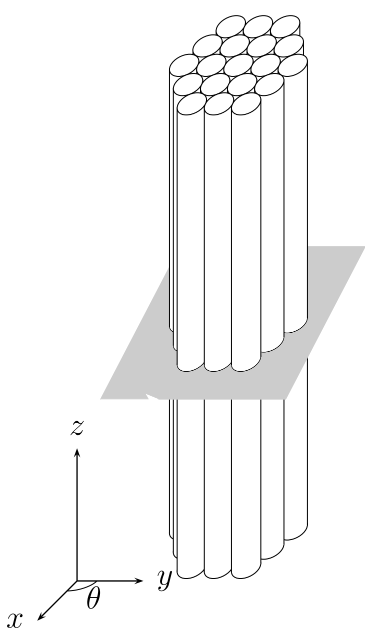
2. In particular, show that the elastic tensor has 9 independent components.

Concept Question 3.5.3. *Orthotropic elasticity in plane stress.*

Let's consider a two-dimensional orthotropic material based on the solution of the previous exercise.

1. Determine (in tensor notation) the constitutive relation $\varepsilon = f(\sigma)$ for two-dimensional orthotropic material in plane stress as a function of the engineering constants (i.e., Young's modulus, shear modulus and Poisson ratio).

2. Deduce the fourth-rank elastic tensor within the constitutive relation $\sigma = f(\varepsilon)$. Express the components of the stress tensor as a function of the components of both, the elastic tensor and the strain tensor.

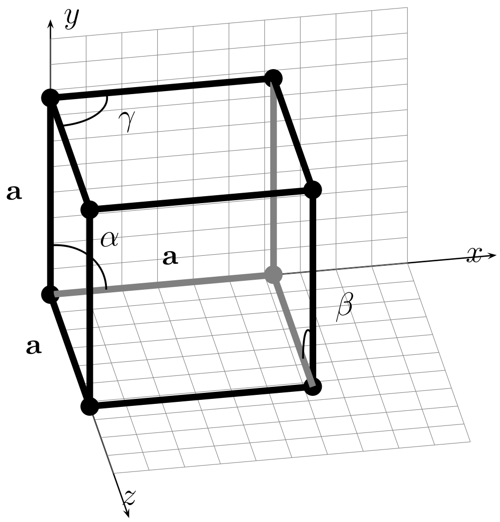


Transversely isotropic: The physical properties are symmetric about an axis that is normal to a plane of isotropy (xy-plane in the figure). Three mutually orthogonal planes of reflection symmetry and axial symmetry with respect to z-axis.
Number of independent coefficients: 5
Symmetry transformations: reflections about all three orthogonal planes plus all rotations about z-axis.

$$Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$Q = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, 0 \le \theta \le 2\pi$$

$$C = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ & C_{1111} & C_{1133} & 0 & 0 & 0 \\ & & C_{3333} & 0 & 0 & 0 \\ & & & C_{2323} & 0 & 0 \\ & symm & & & C_{2323} & 0 \\ & & & & & \frac{1}{2}(C_{1111} - C_{1122}) \end{bmatrix}$$



Cubic: three mutually orthogonal planes of reflection symmetry plus 90° rotation symmetry with respect to those planes. $a = b = c,\ \alpha = \beta = \gamma = 90$
Number of independent coefficients: 3
Symmetry transformations: reflections and 90° rotations about all three orthogonal planes

$$Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} C_{1111} & C_{1122} & C_{1122} & 0 & 0 & 0 \\ & C_{1111} & C_{1122} & 0 & 0 & 0 \\ & & C_{1111} & 0 & 0 & 0 \\ & & & C_{1212} & 0 & 0 \\ & symm & & & C_{1212} & 0 \\ & & & & & C_{1212} \end{bmatrix}$$

3.6 Isotropic linear elastic materials

Most general isotropic 4th order isotropic tensor:

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \big(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \big) \tag{3.16}$$

Replacing in:

$$\sigma_{ij} = C_{ijkl} \epsilon_{kl} \tag{3.17}$$

gives:

$$\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + \mu \big(\epsilon_{ij} + \epsilon_{ji} \big) \tag{3.18}$$

$$\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + \mu \big(\epsilon_{ij} + \epsilon_{ji} \big) \tag{3.19}$$

Examples

$$\sigma_{11} = \lambda \delta_{11} \big(\epsilon_{11} + \epsilon_{22} + \epsilon_{33} \big) + \mu \big(\epsilon_{11} + \epsilon_{11} \big) = \big(\lambda + 2\mu \big) \epsilon_{11} + \lambda \epsilon_{22} + \lambda \epsilon_{33} \tag{3.20}$$

$$\sigma_{12} = 2\mu \epsilon_{12} \tag{3.21}$$

Concept Question 3.6.1. *Isotropic linear elastic tensor.*
Consider an isotropic linear elastic material.

1. Write the three-dimensional elastic/stiffness matrix in Voigt notation.

Compliance matrix for an isotropic elastic material

From experiments one finds:

$$\begin{aligned} \epsilon_{11} &= \frac{1}{E}\Big[\sigma_{11} - \nu\big(\sigma_{22} + \sigma_{33}\big)\Big] \\ \epsilon_{22} &= \frac{1}{E}\Big[\sigma_{22} - \nu\big(\sigma_{11} + \sigma_{33}\big)\Big] \\ \epsilon_{33} &= \frac{1}{E}\Big[\sigma_{33} - \nu\big(\sigma_{11} + \sigma_{22}\big)\Big] \\ 2\epsilon_{23} = \frac{\sigma_{23}}{G},\ \ 2\epsilon_{13} &= \frac{\sigma_{13}}{G},\ \ 2\epsilon_{12} = \frac{\sigma_{12}}{G} \end{aligned} \tag{3.22}$$

In these expressions, E is the Young's Modulus, ν the Poisson's ratio and G the shear modulus. They are referred to as the *engineering constants*, since they are obtained from experiments. The shear modulus G is related to the Young's modulus E and Poisson ratio ν by the expression $G = \frac{E}{2(1+\nu)}$. Equations (3.22) can be written in the following matrix form:

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ & 1 & -\nu & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 2(1+\nu) & 0 & 0 \\ & symm & & & 2(1+\nu) & 0 \\ & & & & & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \tag{3.23}$$

Invert and compare with:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda + 2\mu & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & symm & & & \mu & 0 \\ & & & & & \mu \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix} \tag{3.24}$$

and conclude that:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = G = \frac{E}{2(1+\nu)} \tag{3.25}$$

Concept Question 3.6.2. *Inverted Hooke's law.*
Let's consider a linear elastic material.

1. Verify that the compliance form of Hooke's law, Equation (3.23) can be written in index notation as:
$$\epsilon_{ij} = \frac{1}{E}\Big[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\Big]$$

2. Invert Equation (3.23) (e.g. using Mathematica or by hand) and verify Equation (3.24) using λ and μ given by (3.25).

3. Verify the expression:

$$\sigma_{ij} = \frac{E}{(1+\nu)}\Big[\epsilon_{ij} + \frac{\nu}{(1-2\nu)}\epsilon_{kk}\delta_{ij}\Big]$$

Bulk Modulus

Establishes a relation between the *hydrostatic stress* or pressure: $p = \frac{1}{3}\sigma_{kk}$ and the volumetric strain $\theta = \epsilon_{kk}$.

$$p = K\theta \; ; \; K = \frac{E}{3(1-2\nu)} \tag{3.26}$$

Concept Question 3.6.3. *Bulk modulus derivation.* Let's consider a linear elastic material.

1. Derive the expression for the bulk modulus in Equation (3.26)

Concept Question 3.6.4. *Independent coefficients for linear elastic isotropic materials.*

For a linearly elastic, homogeneous, isotropic material, the constitutive laws involve three parameters: *Young's modulus*, E, *Poisson's ratio*, ν, and the *shear modulus*, G.

1. Write and explain the relation between stress and strain for this kind of material.

2. What is the physical meaning of the coefficients E, ν and G?

3. Are these three coefficients independent of each other? If not, derive the expressions that relate them. Indicate also the relationship with the Lame's constants.

4. Explain why the Poisson's ratio is constrained to the range $\nu \in (-1, 1/2)$. Hint: use the concept of *bulk modulus*.

3.7 Thermoelastic effects

We are going to consider the strains produced by changes of temperature (ϵ^θ). These strains have inherently a dilatational nature (thermal expansion or contraction) and do not cause any shear. Thermal strains are proportional to temperature changes. For isotropic materials:

$$\epsilon_{ij}^\theta = \alpha \Delta\theta \delta_{ij} \tag{3.27}$$

The total strains (ϵ_{ij}) are then due to the (additive) contribution of the *mechanical strains* (ϵ_{ij}^M), i.e., those produced by the stresses and the thermal strains:

$$\begin{aligned} \epsilon_{ij} &= \epsilon_{ij}^M + \epsilon_{ij}^\theta \\ \sigma_{ij} &= C_{ijkl}\epsilon_{kl}^M = C_{ijkl}(\epsilon_{kl} - \epsilon_{kl}^\theta) \text{ or:} \end{aligned}$$

$$\sigma_{ij} = C_{ijkl}(\epsilon_{kl} - \alpha\Delta\theta\delta_{kl}) \tag{3.28}$$

Concept Question 3.7.1. *Thermoelastic constitutive equation.*
Let's consider an isotropic elastic material.

1. Write the relationship between stresses and strains for an isotropic elastic material whose Lamé constants are λ and μ and whose coefficient of thermal expansion is α.

Concept Question 3.7.2. *Thermoelasticity in a fully constrained specimen.* Let's consider a specimen which deformations are fully constrained (see Figure 3.2). The material behavior is considered isotropic linear elastic with E and ν the elastic constants, the Young's modulus and Poisson's ratio, respectively. A temperature gradient $\Delta\theta$ is prescribed on the specimen.

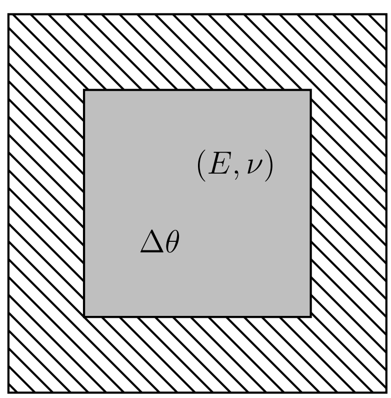


Figure 3.2: Specimen fully constrained.

1. Determine the internal stress state within the specimen.

Material	Mass density $[Mg \cdot m^{-3}]$	Young's Modulus $[GPa]$	Poisson Ratio	Thermal Expansion Coefficient $[10^{-6}K^{-1}$
Tungsten	13.4	410	0.30	5
CFRP	1.5-1.6	70 200	0.20	2
Low alloy steels	7.8	200 - 210	0.30	15
Stainless steel	7.5-7.7	190 - 200	0.30	11
Mild steel	7.8	196	0.30	15
Copper	8.9	124	0.34	16
Titanium	4.5	116	0.30	9
Silicon	2.5-3.2	107	0.22	5
Silica glass	2.6	94	0.16	0.5
Aluminum alloys	2.6-2.9	69-79	0.35	22
GFRP	1.4-2.2	7-45		10
Wood, parallel grain	0.4-0.8	9-16	0.2	40
PMMA	1.2	3.4	0.35-0.4	50
Polycarbonate	1.2 1.3	2.6	0.36	65
Natural Rubbers	0.83-0.91	0.01-0.1	0.49	200
PVC	1.3-1.6	0.003-0.01	0.41	70

Table 3.1: Representative isotropic properties of some materials

3.8 Particular states of stress and strain of interest

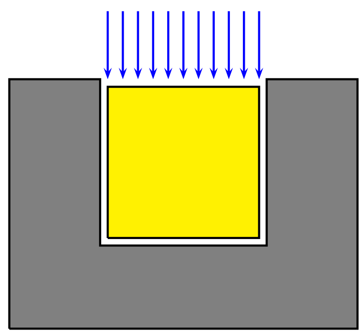
3.8.1 Uniaxial stress

$$\sigma_{11} = \sigma, \text{ all stress components vanish}$$

From (3.23):

$$\epsilon_{11} = \frac{\sigma}{E}, \ \epsilon_{22} = -\frac{\nu}{E}\sigma, \ \epsilon_{33} = -\frac{\nu}{E}\sigma, \ \text{ all shear strain components vanish}$$

3.8.2 Uniaxial strain



$$\epsilon_{11} = \epsilon, \text{ all other strain components vanish}$$

From (3.24):

$$\sigma_{11} = (\lambda + 2\mu)\epsilon_{11} = \frac{(1-\nu)}{(1+\nu)(1-2\nu)}E\epsilon_{11}$$

3.8.3 Plane stress

Consider situations in which:

$$\sigma_{i3} = 0 \tag{3.29}$$

Then:

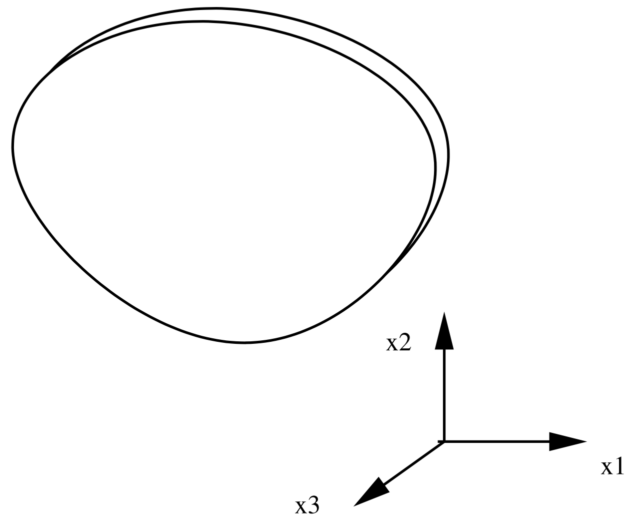
$$\epsilon_{11} = \frac{1}{E}\big(\sigma_{11} - \nu\sigma_{22}\big) \tag{3.30}$$

$$\epsilon_{22} = \frac{1}{E}\big(\sigma_{22} - \nu\sigma_{11}\big) \tag{3.31}$$

$$\epsilon_{33} = \frac{-\nu}{E}\big(\sigma_{11} + \sigma_{22}\big) \neq 0\,!!! \tag{3.32}$$

$$\epsilon_{23} = \epsilon_{13} = 0 \tag{3.33}$$

$$\epsilon_{12} = \frac{\sigma_{12}}{2G} = \frac{(1+\nu)\sigma_{12}}{E} \tag{3.34}$$



In matrix form:

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} \tag{3.35}$$

Inverting gives the:

Relations among stresses and strains for *plane stress*:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix} \tag{3.36}$$

Concept Question 3.8.1. *Plane stress*

Let's consider an isotropic elastic material for a plate in plane stress state.

1. Determine the out-of-plane ϵ_{33} strain component from the measurement of the in-plane normal strains $\epsilon_{11}, \epsilon_{22}$.

3.8.4 Plane strain

In this case we consider situations in which:

$$\epsilon_{i3} = 0 \tag{3.37}$$

Then:

$$\epsilon_{33} = 0 = \frac{1}{E}\Big[\sigma_{33} - \nu\big(\sigma_{11} + \sigma_{22}\big)\Big], \text{ or:} \tag{3.38}$$

$$\sigma_{33} = \nu\big(\sigma_{11} + \sigma_{22}\big) \tag{3.39}$$

$$\begin{aligned}\epsilon_{11} &= \frac{1}{E}\Big\{\sigma_{11} - \nu\big[\sigma_{22} + \nu\big(\sigma_{11} + \sigma_{22}\big)\big]\Big\} \\ &= \frac{1}{E}\Big[\big(1-\nu^2\big)\sigma_{11} - \nu\big(1+\nu\big)\sigma_{22}\Big]\end{aligned} \tag{3.40}$$

$$\epsilon_{22} = \frac{1}{E}\Big[\big(1-\nu^2\big)\sigma_{22} - \nu\big(1+\nu\big)\sigma_{11}\Big] \tag{3.41}$$

In matrix form:

$$\begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ 2\epsilon_{12}\end{bmatrix} = \frac{1}{E}\begin{bmatrix}1-\nu^2 & -\nu(1+\nu) & 0\\ -\nu(1+\nu) & 1-\nu^2 & 0\\ 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} \tag{3.42}$$

Inverting gives the

Relations among stresses and strains for *plane strain*:

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{(1-2\nu)}{2}\end{bmatrix}\begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ 2\epsilon_{12}\end{bmatrix} \tag{3.43}$$

Concept Question 3.8.2. *Plane strain.*
Using Mathematica:

1. Verify equations (3.36) and (3.43)

Concept Question 3.8.3. *Comparison of plane-stress and plane-strain linear isotropic elasticity.*
Let's consider two linear elastic isotropic materials with the same Young's modulus E but different Poisson's ratio, $\nu = 0$ and $\nu = 1/3$. We are interested in comparing the behavior of these two materials for both, plane stress and plane strain models.

1. Express the relation between the stress components and the strain components in the case of both, plane stress and plane strain models.

2. Under which conditions these two materials manifest the same elastic response for each hypothesis, plane strain and plane stress?

3. Derive the equation that relates ϵ_{11} and ϵ_{22} when $\sigma_{22} = 0$ for both, plane strain and plane stress models. For the material having a Poisson's ratio equals to $\nu = 1/3$, for which model (plane stress or plane strain) the deformation ϵ_{22} reaches the greatest value?

4. Let's consider a square specimen of each material, with a length equals to 1 m and the origin of the coordinate system is located at the left bottom corner of the specimen. When a deformation of $\epsilon_{11} = 0.01$ is applied, calculate the displacement u_2 of the point with coordinates $(0.5, 1)$.
