PRODUCTS OF FRÉCHET SPACES

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Abstract. We give a survey of results and concepts related to certain problems on the Fréchet-Urysohn property in products. This material was presented in a workshop at the 2005 Summer Conference on Topology and its Applications at Denison University.

1. Introduction

Recall that a space X is **Fréchet** (or **Fréchet-Urysohn**) if whenever x is in the closure of a set A, there is a sequence of points a_n in A which converge to x. Although this is a very basic concept, there are still many unsolved problems, including, for example:

(1) (Malychin) Is there in ZFC a separable Fréchet non-metrizable topological group?

(2) (Nogura) For each n in ω, is there in ZFC a compact (or countable) Fréchet space X such that X^n is Fréchet but X^{n+1} is not Fréchet? [For $n = 1$, yes [Sim_1], but unsolved for $n = 2$.]

(3) (Galvin) Is there in ZFC a compact (or countable) X such that X^n is Fréchet for each n in ω, but X^ω is not Fréchet?

The property "Fréchet-Urysohn for finite sets" and its variants, introduced by Reznichenko and Sipacheva[RS], and also studied by Szeptycki and myself[GS_1],[GS_2], are relevant to these questions, as are the α_i properties of Arhangel'skii. Many important examples in the area are based on almost disjoint families of subsets of ω, or on Hausdorff gaps. Some connections to special subsets of the reals, topological games, selection principles, and convergence in function spaces will also be mentioned. Many other questions, some old, some new, will be mentioned; see the last section for a long list of them.

Most of the results presented here can be found in the literature; we will not repeat their proofs, though occasionally we give an outline of the key idea. We do give some arguments for the few new ideas that appear here. All spaces are assumed to be regular and T_1.

2. α_i-properties, a.d. spaces, and bisequentiality

We will essentially restrict our attention to countable spaces, so we first give some motivation for doing so. A prototypical Fréchet non-first-countable space (for κ infinite) is:

Definition 2.1. *The* **sequential fan** S_κ *is the quotient space obtained by identifying the limit points of a topological sum of κ-many convergent sequences.*

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It is in general relatively easy to destroy Fréchetness in products by making the product not even countable tight[1]. E.g., Arhangel'skii[Ar_1] proved that $S_\omega \times S_{\mathfrak{c}}$ is not countably tight, and I showed a few years later that $S^2_{\omega_1}$ is not countably tight. In the mid-1990's, Todorcevic[To_1], and Alster and Pol[AP], showed that, for each $n \in \omega$, there exists spaces of the form X_i, $i \leq n$, such that the product of any n members of $\{C_p(X_i) : i \leq n\}$ is Fréchet , but $\Pi_{i \leq n} C_p(X_i)$ is not countably tight. [$C_p(X)$ is the space of all real-valued continuous functions on X with the topology of pointwise convergence; they did this for $n = 1$, but the methods also work for higher n.]

In this article, we will concentrate on failure of Fréchetness in products, where the product remains countably tight. This is a subtler kind of failure. It also allows us to focus on countable spaces with one non-isolated point–i.e., the theory becomes a theory of certain kinds of filters on ω.

This restriction is not as severe as it may seem. E.g., Malychin[Mal] showed that a finite product of compact countably tight spaces is countably tight. So this restricted theory essentially includes Fréchetness in (finite) products of compact spaces.

The following well-known example shows that it is easy, even in this restricted setting, to destroy Fréchetness in the product:

Example 2.2. *Let S be a non-trivial convergent sequence with its limit. Then $S_\omega \times S$ is not Fréchet .*

Arhangel'skii[Ar_1, Ar_2] introduced the α_i properties as a tool for studying Fréchetness of products.

Definition 2.3. *A point x of X is said to be an α_2-**point** (resp.,α_4-**point**) if whenever we have for each $n \in \omega$ a sequence S_n converging to x, there is a sequence S converging to x such that $S \cap S_n \neq \emptyset$ for all $n \in \omega$(resp., for ∞-many $n \in \omega$). For α_2, it is equivalent to require that $S \cap S_n$ be infinite for every n. An α_3-**point** is defined in the same way as this version of α_2, except that we only require $S \cap S_n$ to be infinite for ∞-many n. And for x to be an α_1-**point**, we require $S_n \backslash S$ to be finite for every n.*[2]

While the definitions of the α_i-properties make sense in any space, we will only consider them within the class of Fréchet -spaces; we say x is an α_i**-FU-point** if x is both a Fréchet -point and an α_i-point. Note that x being an α_2 (resp.,α_4)-FU-point is equivalent to the following "selection" version of the Fréchet property:

$$x \in \overline{A_n} \text{ for all } n \text{ implies: for all } n \text{ (resp., for } \infty\text{-many } n) \ \exists x_n \in A_n \text{ with } x_n \to x.$$

The α_4-FU property has also been called **strongly Fréchet** or **countably bisequential**; α_2 (resp., α_4) is also called the **(weak) diagonal sequence property**. The following is a list of several key facts about the α_i properties.

Theorem 2.4. (a) [Mic] *If S is a non-trivial convergent sequence with its limit, then $X \times S$ is Fréchet iff X is α_4-FU;*
(b) [Ar_2] *If $X \times Y$ is Fréchet , and Y countably compact, then X must be α_4.*
(c) [O][Ar_2] *If X is countably compact and Fréchet , then it is α_4.*

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[1] X is *countably tight* if $x \in \overline{A}$ implies $x \in \overline{C}$ for some countable $C \subset A$

[2] Since we are working in T_2-spaces, it does no harm to confuse a non-trivial sequence with its range, as we have done here.

(d) [Ar$_2$] *If X is α_3 and Y countably compact Fréchet , then $X \times Y$ is Fréchet;*
(e) [No$_2$] *If X is α_i, $i = 1, 2, 3$, then so is X^n for all $n \leq \omega$;*
(f) *It is consistent with ZFC that all countable α_2-FU spaces are α_1*[D]*, and that all countable α_1-FU spaces are first-countable*[DS]*, but there is in ZFC a countable α_2-FU space which is not first-countable*[GN$_1$][Ny$_2$]*;*
(g) [TU$_1$] *Every countable analytic*[3] *α_2-FU space is first-countable;*
(h) *Every Fréchet topological group is α_4*[Ny$_1$]*, but in the Cohen model, there is a countable Fréchet topological group which is not α_3*[S$_1$].

Many examples in the area are of the following simple form:

Definition 2.5. *Let $\mathcal{A}$ be an almost-disjoint (a.d.) family of subsets of ω. The* **ψ-space of $\mathcal{A}$***, denoted $\psi(\mathcal{A})$, is the (locally compact) space $\omega \cup \mathcal{A}$, where points of ω are isolated and for every $A \in \mathcal{A}$, $\{n : n \in A\}$ is a convergent sequence with limit point A. $\psi(A)^*$ denotes the one-point compactification of $\psi(\mathcal{A})$ with compactifying point ∞. We call the subspace $\omega \cup \{\infty\}$ of $\psi(\mathcal{A})^*$ the* **a.d.-space generated by** *$\mathcal{A}$. Note that the nbhds of ∞ in the a.d.-space have the form $\{\infty\} \cup (\omega \setminus (F \cup (\cup \mathcal{F}))$, where $F \in [\omega]^{<\omega}$ and $\mathcal{F} \in [\mathcal{A}]^{<\omega}$.*

A meta-theorem in the area is that, in the class of Fréchet spaces, there is a compact example illustrating a convergence property iff there is one of the form $\psi(\mathcal{A})^*$ iff there is an a.d.-space example. Indeed, I don't know of a relevant counterexample to this.

Since $\psi(\mathcal{A})^*$ is first-countable at every point except ∞, and $\mathcal{A} \cup \{\infty\}$ is the one-point compactification of discrete $\mathcal{A}$, which has most every Fréchet -type property weaker than first-countable that we will consider, to say that the a.d.-space generated by $\mathcal{A}$ has the property is usually equivalent to saying $\psi(\mathcal{A})^*$ has the property (in particular, at the point ∞).

A very important example of this type is due to P. Simon. Recall that an a.d family is **MAD** if it is maximal w.r.t. being pairwise almost disjoint. Simon calls an a.d. family $\mathcal{A}$ of subsets of ω **nowhere-MAD** if for any $X \subset \omega$, if X is not almost covered (i.e., covered mod finite) by some finite subcollection of $\mathcal{A}$, then $\mathcal{A} \restriction X = \{A \cap X : A \in \mathcal{A}\}$ is not a MAD collection of subsets of X.

Proposition 2.6. [Sim$_1$]
(a) *$\psi(\mathcal{A})^*$ is Fréchet iff $\mathcal{A}$ is nowhere-MAD;*
(b) *If $\mathcal{A}_\alpha$, $\alpha \in I$, partitions a MAD $\mathcal{A}$, then $\Pi_{\alpha \in I} \psi(\mathcal{A}_\alpha)^*$ is not Fréchet ;*
(c) *There are nowhere-MAD $\mathcal{A}_0$, $\mathcal{A}_1$ such that $\mathcal{A}_0 \cup \mathcal{A}_1$ is MAD; hence $\psi(\mathcal{A}_0)^*$ and $\psi(\mathcal{A}_1)^*$ are compact Fréchet spaces such that $\psi(\mathcal{A}_0)^* \times \psi(\mathcal{A}_1)^*$ is not Fréchet .*

I used Martin's Axiom to construct a certain countably infinite partition of a MAD family which gave a consistent answer to the question of Galvin mentioned in the introduction, and K. Tamano subsequently used it to get a consistent answer to Nogura's question.

Example 2.7. [G$_2$]*(MA)*
(a) *There is a MAD $\mathcal{A}$ and a partition $\{\mathcal{A}_n\}_{n \in \omega}$ of $\mathcal{A}$ such that $\Pi_{i \leq n} \psi(\mathcal{A}_i)^*$ is Fréchet for each n, but $\Pi_{i \in \omega} \psi(\mathcal{A}_i)^*$ is not Fréchet ;*

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[3] A countable space X with topology τ is **analytic** if $\{\chi_U : U \in \tau\}$ is an analytic subset of 2^ω.

(b) *There exists a almost-disjoint family $\mathcal{A}$ such that, if $X = \psi(\mathcal{A}^*)$, then X^n is Fréchet for each $n < \omega$, but X^ω is not Fréchet .*

Example 2.8. [Ta] *(MA) For each positive integer n, there is a compact Fréchet space X (of the form $\psi(\mathcal{A})^*$) such that X^n is Fréchet but X^{n+1} is not Fréchet .*

Question 1. *Are there ZFC examples X, either compact or countable, satisfying the conditions of Example 2.7 or 2.8 as above?*

Bisequentiality has a rather different flavor from most of the other convergence properties, but is a very relevant to our discussion.

Definition 2.9. *For a filter $\mathcal{F}$ and a collection $\mathcal{G}$ of sets, put $\mathcal{G}\#\mathcal{F}$ iff every member of $\mathcal{F}$ meets every member of $\mathcal{G}$. A space X is said to be* **bisequential** *at $x \in X$ if whenever $\mathcal{F}$ is a filter which clusters at x (i.e, $x \in \overline{F}$ for every $F \in \mathcal{F}$), there is a decreasing sequence G_n of sets converging to x such that $\{G_n\}_{n\in\omega}\#\mathcal{F}$. Equivalently, every ultrafilter which clusters at x contains such a sequence $\{G_n\}_{n\in\omega}$.*

It is easy to see that bisequential spaces are Fréchet . The bisequential notion goes back to Michael and was also studied in depth by Arhangel'skii.

Theorem 2.10. (a) [Mic] *A countable product of bisequential spaces is bisequential;*
(b) [Ar_2] *Bisequential spaces are α_3 and absolutely Fréchet (i.e., every point of X is a Fréchet -point in βX);*
(c) [Ar_2] *If X is bisequential and Y is α_4-FU, then $X \times Y$ is Fréchet ;*
(d) [TU_2] *Every countable analytic α_4-FU space is bisequential;*
(e) [Ar_4] *Bisequential topological groups are metrizable.*

By (b) or (c) above, Simon's examples are not bisequential. But a large class of a.d.-spaces is easily seen to be bisequential. Recall that an a.d. family $\mathcal{A}$ is **$\mathbb{R}$-embeddable**[HH] if there is a one-to-one function $f : \omega \to \mathbb{Q}$ which extends to a continuous one-to-one $\hat{f} : \psi(\mathcal{A}) \to \mathbb{R}$.

Proposition 2.11. *Let $\mathcal{A}$ be an $\mathbb{R}$-embeddable a.d. family. Then $\psi(\mathcal{A})^*$ is bisequential.*

The above result seems to be folklore; for its (easy) proof, use the fact that any ultrafilter on $\mathbb{R}$ converges either to a single point, or to $\pm\infty$. If it's a point x, then bisequentiality is witnessed by a decreasing nbhd base at the point, if necessary minus $f(A)$ for the at most one $A \in \mathcal{A}$ with $x \in \overline{f(A)}$.

It would be good to have a nice characterization of bisequentiality of $\psi(\mathcal{A})^*$. Since bisequential implies α_3 and the property is preserved by closed maps, it follows that $\psi(\mathcal{A})^*$ bisequential implies $\psi(\mathcal{B})^*$ is α_3-FU for every $\mathcal{B} \subset \mathcal{A}$; I don't this consistently reverses, or even if just $\psi(\mathcal{A})^*$ being α_3-FU is consistently equivalent to its bisequentiality. There are however ZFC examples of bisequential $\psi(\mathcal{A})^*$ where $\mathcal{A}$ is not $\mathbb{R}$-embeddable; also, Example 4.3 in these notes is a countable α_2-FU space which is not bisequential.

Theorem 2.10(c) left open the problem of characterizing the spaces whose product with every α_4-FU space is Fréchet . Gerlitz and Nagy[GN_1] solved the problem; more recently, Jordan and Mynard obtained another solution by introducting the following property defined in the same spirit as bisequential. First, for a filter $\mathcal{F}$ on X, call a sequence in X **$\mathcal{F}$-convergent** if it is convergent in the topology on

$X \cup \{\infty\}$, where X is a set of isolated points and nbhds of ∞ have the form $F \cup \{\infty\}$, $F \in \mathcal{F}$. Also, call $\mathcal{F}$ an α_4-**FU filter** if $X \cup \{\infty\}$ is α_4-FU (at ∞) in this topology.

Definition 2.12. [JM] *A space X is called* **productively** α_4-**FU**[4] *at x if whenever $\mathcal{F}$ is an α_4-FU filter which clusters at x, then there is a decreasing sequence G_n of sets converging to x such that $\{G_n\}_{n\in\omega}\#\mathcal{F}$. (Equivalently, since $\mathcal{F}$ is α_4, there is a convergent sequence to x which is also $\mathcal{F}$-convergent.)*

So, productively α_4-FU is the same as bisequentiality with respect to α_4-FU filters.

Theorem 2.13. [JM]

(a) *A space X is productively α_4-FU iff its product with every α_4-FU space is $(\alpha_4$-$)$FU;*
(b) *A finite product of productively α_4-FU spaces is productively α_4-FU;*
(c) *It is consistent that there is a countable productively α_4-FU space which is not α_3.*

It is not known, however, if an example as in (c), or a countable productively α_4-FU space which is not bisequential, exists in ZFC.

Question 2. [JM]

(a) *Is there a ZFC example of a countable productively α_4-FU space which is not bisequential? Not α_3?*
(b) *If X is productively α_4-FU, must X^ω be too? Must it at least be Fréchet ?*

One may analogously define "bisequential w.r.t. α_i filters". If we call such spaces **productively** α_i-**FU**, then the analogue of Theorem 2.13(a) holds with α_4 replaced by α_i, $i = 1, 2, 3$. Simon's example mentioned earlier is productively α_3-FU (by Theorem 2.4(d)), but is not productively α_4-FU. We do not know of an example of a productively α_2-FU space which is not productively α_3-FU.

3. Fréchet -Urysohn for finite sets

Definition 3.1. [RS],[GS$_1$], [GS$_2$] *Let $x \in X$. A collection $\mathcal{P}$ is a* π-**net** *at x, denoted by $x \in \overline{\mathcal{P}}$, if every nbhd of x contains some member of $\mathcal{P}$. X is* **Fréchet-Urysohn for finite sets**($\mathbf{FU}_{fin}$) *at x if whenever $\mathcal{P}$ is a collection of finite sets with $x \in \overline{\mathcal{P}}$, there are $P_n \in \mathcal{P}$ with $P_n \to x$ (by which we mean every nbhd of x contains P_n for all sufficiently large n). For $n \in \omega$,* **Fréchet-Urysohn for n-point sets**($\mathbf{FU}_n$) *is defined analogously, and we say X is* **boundedly** $\mathbf{FU}_{fin}$ *at x iff X is FU_n at x for every n.*

We also consider natural analogues of α_2-FU and α_4-FU for the above stronger FU-type properties. E.g., X is α_2-$\mathbf{FU}_n$ (resp. α_4-$\mathbf{FU}_n$ at x) if whenever $x \in \overline{\mathcal{P}_i}$, $i \in \omega$, where each $\mathcal{P}_i$ consists of n-point sets, there are $P_i \in \mathcal{P}_i$ for each i (resp., for ∞-many i) such that $P_i \to x$.

Note that FU$_1$ and its α_2 and α_4 versions are the same as the standard Fréchet-Urysohn property and its α_2 and α_4 versions. More generally:

Proposition 3.2. [Sip][GS$_1$][GS$_2$] *Let X be a space with one non-isolated point. Then:*

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[4]Jordan and Mynard call this simply "productively Fréchet ."

(a) *X is $FU_n \iff X^n$ is Fréchet*
(b) *X is α_4-$FU_n \iff X^n$ is Fréchet and α_4;*
(c) *X is α_2-$FU_n \iff X^n$ is Fréchet and α_2 (equiv., X^n is Fréchet and X is α_2);*
(d) *X is boundedly $FU_{fin} \iff X^n$ is Fréchet for all $n \in \omega$*

It follows that Example 2.8 gives, under MA, a countable FU_n space which is not FU_{n+1}, for each n, and asking for a ZFC example of an example like 2.8 is equivalent to asking for, in ZFC, a countable or compact FU_n space which is not FU_{n+1}. Also, Example 2.7 gives, under MA, a countable boundedly FU_{fin} space whose countable power is not Fréchet , and we do not know a ZFC example of this either. But it is the case that the countable power of a FU_{fin}-space is Fréchet , even FU_{fin} [GS_2].

The above definition suggests quite a few classes of spaces, and there are some elementary implications between the them, as indicated in the following chart.

$$\mathrm{FU}_{fin} \iff \alpha_2\text{-}\mathrm{FU}_{fin} \iff \alpha_4\text{-}\mathrm{FU}_{fin}$$
$$\Downarrow$$
$$\alpha_2\text{-bddly-}\mathrm{FU}_{fin} \iff \alpha_4\text{-bddly-}\mathrm{FU}_{fin}$$
$$\Downarrow$$
$$\forall n(\alpha_2\text{-}\mathrm{FU}_n) \Rightarrow \text{bddly-}\mathrm{FU}_{fin} = \forall n(\mathrm{FU}_n) \iff \forall n(\alpha_4\text{-}\mathrm{FU}_n)$$
$$\Downarrow \qquad\qquad \Downarrow$$
$$\alpha_2\text{-}\mathrm{FU}_n \qquad \mathrm{FU}_{n+1}$$
$$\Downarrow \qquad \Downarrow$$
$$\alpha_4\text{-}\mathrm{FU}_n$$
$$\Downarrow$$
$$\mathrm{FU}_n$$

The papers [GS_1],[GS_2] of myself and Szepytcki contain a detailed investigation of these properties and relations among them. As indicated in the chart, there are a few equivalences among the suggested classes of spaces, e.g., the α_2 and α_4 versions of FU_{fin} are equivalent to FU_{fin}. But most of the time, there are at least consistent examples separating the properties; in particular, under CH, all of the of the properties not rather easily proven to be equivalent are different even for countable spaces. In many cases however, we don't know if there are ZFC examples; e.g., we don't know of any ZFC example of an (α_2-) FU_2 space which is not boundedly FU_{fin} (resp., FU_{fin}). The most outstanding question in the area is:

Question 3. *Is there, in ZFC, a countable X which is FU_{fin} but not first-countable?*

Reznichenko and Sipcheva[RS] proved:

Proposition 3.3. *If there is a countable X which is FU_{fin} but not first-countable, then there is a countable (equiv., separable) Fréchet topological group which is not first countable (equiv., not metrizable).*

Given the space X, which may be assumed to be of the form $\omega \cup \{\infty\}$, the group can be described simply as the set G of all finite subsets of ω under the symmetric difference operation, and for U open in X, the set $U^* = \{F \in G : F \subset U\}$ is a nbhd of the identity.

Thus, an X answering Question 3 would answer an old question of Malychin, who asked if such groups exist in ZFC. This was Reznichenko and Sipecheva's motivation for introducing FU_{fin}. (The property was noticed earlier by Nyikos[Ny_3] and also Dow and Steprans[DS], but [RS] is the first detailed investigation.) Theorem 2.4(h) indicates Malychin's question is probably not equivalent to Question 3, however.

There are ZFC examples separating boundedly FU_{fin} from FU_{fin}, and α_2-boundedly FU_{fin} from first-countable. Recall that $\psi(\mathcal{A})^*$ is bisequential whenever $\mathcal{A}$ is $\mathbb{R}$-embeddable, and it follows from Theorem 2.10(a) that bisequential spaces are boundedly FU_{fin}.

Example 3.4. (a) [GS_2] *An $\mathbb{R}$-embeddable $\mathcal{A}$ such that $\psi(\mathcal{A})^*$ is not α_2-FU. (Hence $\psi(\mathcal{A})^*$ is boundedly FU_{fin} but not FU_{fin}.)*

(b) *(* [Ny_2]*, essentially; see also* [GS_2]*) If $\mathcal{A} \subset \mathcal{P}(2^{<\omega})$ is the set of branches of a Cantor tree over a λ'-set $X \subset 2^\omega$, then $\psi(\mathcal{A})^*$ is α_2-boundedly FU_{fin} but not first-countable.*

As we'll note in the next section, Nyikos also showed that the space in (b) above is FU_{fin} iff X is what is called a γ-set. But Miller[Mil] showed that there is a model in which every λ'-set is γ, so this will not get a ZFC example separating α_2-boundedly FU_{fin} from FU_{fin} (and none is known).

The only case where an equivalence of these kinds of properties is known to be both consistent with and independent of set theory is given by the following result, which also answered a question of Nogura[No_2].

Theorem 3.5. (a) [Sim_2]*(CH)There is a countable space X which is FU_2 but not α_4-FU_2;*

(b) [To_2] *(OCA[5]) FU_n is equivalent to α_4-FU_n , for $n \geq 2$.*

What Todorcevic actually showed is that $X \times Y$ Fréchet implies $X \times Y$ is α_4, which implies that if X FU_n for $n \geq 2$, then $X^n = X^{n-1} \times X$, being Fréchet , is α_4; thus X is α_4-FU_n.

4. The γ-connection

It has been known for a long time that if there are uncountable γ-sets, then there are countable FU_{fin} spaces which are not first-countable. Here we give a construction which makes the "γ-connection" obvious, and suggests something to look at for possible ZFC examples.

Definition 4.1. *Let $X \subset 2^\omega$ and let $\mathcal{U}$ be a collection of open subsets of X with $X \in \mathcal{U}$. Let $\Gamma(X,\mathcal{U})$ be the set $\mathcal{U}$ with the following topology: X is the only non-isolated point, and for any finite $F \subset X$,*

$$O(F) = \{U \in \mathcal{U} : F \subset U\}$$

is a nbhd of X.

Remark. It's occasionally handy to think of members of $\mathcal{U}$ as open subsets of 2^ω instead of X (see Proposition 4.6(j) and Corollary 4.7 below). This is OK as long as no two members of $\mathcal{U}$ have the same trace on X (which is not a serious restriction for applications).

[5]OCA denotes the Open Coloring Axiom, a consequence of PFA, the Proper Forcing Axiom.

If $X \subset 2^\omega$, let $T(X)$ denote the a.d. space generated by the branches of the Cantor tree over X, i.e., generated by the a.d. family $\{b_x : x \in X\}$, where $b_x = \{x \upharpoonright n : n \in \omega\}$ is the branch of the Cantor tree corresponding to x. This space was studied in detail by Nyikos[Ny$_2$].

Example 4.2. *Let $\mathcal{U}$ be the collection of all complements of basic clopen subsets $[\sigma]$ of the Cantor set 2^ω (i.e., $\sigma \in 2^n$ for some n, and $[\sigma] = \{x \in 2^\omega : x \upharpoonright n = \sigma\}$). Also put $X \in \mathcal{U}$. Then $\Gamma(X,\mathcal{U})$ is homeomorphic to $T(X)$.*

To see this, simply note that for $F \subset X$, $O(F) = \{X\} \cup \{2^\omega \setminus [s] : \sigma \notin \bigcup_{x \in F} b_x\}$, i.e., nbhds of ∞ correspond to complements of finitely many branches over points of X. More generally, the following shows that any a.d. space is of the form $\Gamma(X,\mathcal{U})$:

Proposition 4.3. *Let $\mathcal{A}$ be an a.d. family and let $X = \{\chi_A : A \in \mathcal{A}\} \subset 2^\omega$. For each $n \in \omega$, let $O_n = \{x \in 2^\omega : x(n) = 0\}$, and let $\mathcal{O} = \{X\} \cup \{O_n\}_{n\in\omega}$. Then the a.d. space generated by $\mathcal{A}$ is homeomorphic to $\Gamma(X, \mathcal{O})$.*

Proof. If $F = \{\chi_{A_i} : i < n\}$, then clearly $O(F) = \mathcal{O} \setminus \{O_n : n \in \bigcup_{i<n} A_i\}$. □

Definition 4.4. *A collection $\mathcal{V}$ is called an ω-**cover** of a set or space X (resp., γ-**cover**) if every finite subset of X is contained in some member of $\mathcal{V}$ (resp., every finite subset of X is contained in almost every (mod finite) member of $\mathcal{V}$).*

Definition 4.5. *Let $\mathcal{U}$ be a collection of sets. We call X a $\mathcal{U} - \gamma$-**set** if every ω-cover $\mathcal{V}$ of X, with $\mathcal{V} \subset \mathcal{U}$, has a γ-subcover. If $X \subset \mathbb{R}$, then X is called simply a γ-**set** if X is $\mathcal{U} - \gamma$, where $\mathcal{U}$ =all open sets (equivalently, $\mathcal{U}$ =a countable base closed under finite unions).*

Remark. (a) γ-subsets of the reals have strong measure 0, hence uncountable ones consistently do not exist. However, they do exist in many models (e.g., any subset of $\mathbb{R}$ of cardinality less than $\mathfrak{p}$ is a γ-set, and $\mathfrak{p} = \mathfrak{c}$ implies there is a γ-set of cardinality $\mathfrak{c}$); also, Todorcevic showed that there can exist two γ-sets whose union is not γ (see [GM]).

(b) Nyikos[Ny$_2$] showed that the a.d. space corresponding to the Cantor tree over X is FU$_{fin}$ iff X is a γ-set, and Gerlitz and Nagy[GN$_2$] showed that $C_p(X)$ is (α_2-) FU (and the same proof gets FU$_{fin}$) iff X is a γ-set . The $\Gamma(X,\mathcal{U})$ construction includes the essence of both, and the elementary proof of Proposition 4.6 (especially (a) and (c)) makes the "γ-connection" completely obvious.

Notation: $\mathcal{U}^{\cap fin}$ (resp., $\mathcal{U}^{\cap n}$) is the set of all finite (resp., n-sized) intersections of members of $\mathcal{U}$.

Proposition 4.6. *Consider $\Gamma(X,\mathcal{U})$, where $X \subset 2^\omega$ and $\mathcal{U}$ is a collection of open subsets of X with $X \in \mathcal{U}$. Then:*

(a) *$\forall \mathcal{V} \subset \mathcal{U} (X \in \overline{\mathcal{V}} \iff \mathcal{V}$ is an ω-cover of X);*
(b) *$\forall \mathcal{V} \subset [\mathcal{U}]^{<\omega} (X \in \overline{\mathcal{V}} \iff \{\cap \mathcal{W} : \mathcal{W} \in \mathcal{V}\}$ is an ω-cover of X);*
(c) *For $V_n \in \mathcal{U}$, $n \in \omega$, we have $V_n \to X \iff \{V_n\}_{n\in\omega}$ is a γ-cover of X;*
(d) *For $\mathcal{V}_n \subset \mathcal{U}$, $n \in \omega$, we have $\mathcal{V}_n \to X \iff \{\cap \mathcal{V}_n\}_{n\in\omega}$ is a γ-cover of X;*
(e) *$\Gamma(X,\mathcal{U})$ is FU $\iff$ X is a $\mathcal{U} - \gamma$-set;*
(f) *$\Gamma(X,\mathcal{U})$ is FU$_{fin}$ $\iff$ X is a $\mathcal{U}^{\cap fin} - \gamma$-set ;*
(g) *$\Gamma(X,\mathcal{U})$ is FU$_n$ $\iff$ X is a $\mathcal{U}^{\cap n} - \gamma$-set;*
(h) *$\Gamma(X,\mathcal{U})$ is α_2-FU (resp, α_4-FU) $\iff$ whenever $\mathcal{V}_n \subset \mathcal{U}$ is an ω-cover (equivalently, γ-cover) of X, then for each n (resp., for infinitely many n) there are $V_n \in \mathcal{V}_n$ such that the V_n's are a γ-cover;*

(i) $\Gamma(X,\mathcal{U}))$ *is first-countable iff there are finite* $F_n \subset X$, $n \in \omega$, *and* $X = \bigcup_{n\in\omega} X_n$, *such that, for all* $U \in \mathcal{U}$, $U \supset F_n \iff U \supset X_n$. *In particular,* $\Gamma(X,\mathcal{U})$ *is not first-countable if* $|X| > \omega$, *and for each* $x \in X$ *and finite* $F \subset X$ *with* $x \notin F$, *there is some* $U \in \mathcal{U}$ *with* $x \notin U \supset F$.

(j) *If* $X_0, X_1 \subset 2^\omega$, *then* $\Gamma(X_0 \cup X_1,\mathcal{U})$ *is homeomorphic to the diagonal of* $\Gamma(X_0,\mathcal{U}) \times \Gamma(X_1,\mathcal{U})$.

Proof. (a)-(d) are easy translations from the definitions, using $U \in O(F) \iff U \supset F$. E.g., for (a), $X \in \overline{\mathcal{V}} \iff \forall F \in [X]^{<\omega}[\exists V \in \mathcal{V}(V \in O(F))] \iff \forall F \in [X]^{<\omega}[\exists V \in \mathcal{V}(V \supset F)] \iff \mathcal{V}$ is an ω-cover. For (d), $\mathcal{V}_n \to X \iff \forall F \in [X]^{<\omega}$[for almost all $n \in \omega(\mathcal{V}_n \in O(F))] \iff \forall F \in [X]^{<\omega}$[for almost all $n \in \omega(F \subset \cap\mathcal{V}_n)] \iff \{\cap\mathcal{V}_n\}_{n\in\omega}$ is a γ-cover.

Now (e),(f), (g), and (h) follow straightforwardly from (a)-(d). For (i), if $O(F_n)$, $n \in \omega$, enumerates a countable base, let $X_n = \{x \in X : O(F_n) \subset O(x)\} = \{x \in X : U \supset F_n \to x \in U\}$. And finally for (j), it is easily checked that $U \to (U,U)$ is the desired homeomorphism. □

Corollary 4.7. *Let* $\mathcal{U}$ *be a countable base for* 2^ω *closed under finite unions. Then:*

(a) *If* $X \subset 2^\omega$ *is a* γ*-set, then* $\Gamma(X,\mathcal{U})$ *is a countable* FU_{fin}*-space which is not first-countable;*

(b) *If* X_0 *and* X_1 *are* γ*-sets in* 2^ω *but* $X_0 \cup X_1$ *is not* γ, *then* $\Gamma(X_e,\mathcal{U})$ *is* FU_{fin} *for* $e < 2$, *but* $\Gamma(X_0,\mathcal{U}) \times \Gamma(X_1,\mathcal{U})$ *is not Fréchet .*

Nyikos[Ny_2] showed that the a.d.-space corresponding to the Cantor tree over 2^ω (i.e., the whole Cantor tree) is Fréchet ; so it follows from Proposition 4.6 and previous remarks that 2^ω is $\mathcal{U}-\gamma$, where $\mathcal{U}$ is the collection of complements of basic open subsets of 2^ω. However, we don't know the answer to:

Question 4. *Is there in ZFC an uncountable* $\mathcal{U}-\gamma$*-set* $X \subset 2^\omega$, *where* $\mathcal{U}$ *is a countable collection of open subsets of* 2^ω *satisfying:*

(a) $\mathcal{U}$ *closed under finite intersections;*

(b) *For each* $x \in X$ *and finite* $F \subset X$ *with* $x \notin F$, *there is some* $U \in \mathcal{U}$ *with* $x \notin U \supset F$*?*

If there is such X and $\mathcal{U}$, then by Proposition 4.6, $\Gamma(X,\mathcal{U})$ is a non-first-countable FU_{fin}-space. The following, essentially due to Szeptycki[Sz], eliminates a possibility for a ZFC example.

Proposition 4.8. *Let* $\mathcal{U}$ *be the collection all complements of finite unions from a countable collection* $\mathcal{I}$ *of closed intervals. Then it is consistent (in particular, it holds if* $\mathfrak{b} > \omega_1$ *and there are no real* γ*-sets) that there is no uncountable* $\mathcal{U}-\gamma$*-set* X *satisfying the condition of Question 4(b).*

Proof. Suppose there exists such X and $\mathcal{I}$. Let $\mathcal{I}^0$ be the collection of interiors of members of $\mathcal{I}$. Notice that the condition of Question 4(b) implies that $\mathcal{I}^0$ contains a base at all points of X that are two-sided limits of X; in particular, all but countably many points of X. For each n, let

$$Y_n = \{y \in \mathbb{R} : y \in J \in \mathcal{I}^0 \Rightarrow J \not\subset (y - 1/2^n, y + 1/2^n)\}.$$

Notice that $\overline{Y_n} \subset Y_{n+1}$ and that $\bigcup_{n\in\omega} Y_n$ is the set of all points $y \in \mathbb{R}$ such that $\mathcal{I}^0$ does not contain a base at y. Hence the set

$$X^* = \{x \in \mathbb{R} : \mathcal{I}^0 \text{ contains a base at } x\}$$

is G_δ, hence Polish. If $\mathfrak{b} > \omega_1$, then any uncountable subset of an analytic set has an uncountable subset contained is some compact subset of the analytic set. Applying this to X^* and $X \cap X^*$, there is a compact subset K of X^* such that $K \cap X$ is uncountable. The following claim finishes the proof.

Claim. $K \cap X$ *is a (real)* γ*-set.*

Proof of Claim. Let $\mathcal{O}$ be an ω-cover of $K \cap X$. Let

$$\mathcal{U}(\mathcal{O}) = \{U \in \mathcal{U} : \exists O \in \mathcal{O}(U \cap K \subset O)\}.$$

Clearly it suffices to show that $\mathcal{U}(\mathcal{O})$ contains a γ-subcover of $K \cap X$. Since X is a $\mathcal{U} - \gamma$-set, it suffices then to show that $\mathcal{U}(\mathcal{O})$ is an ω-cover of X.

To this end, suppose $F \subset X$ is finite. There is some $O \in \mathcal{O}$ with $F \cap K \subset O$. Since $\mathcal{I}^0$ contains a base at each point of $K \backslash O$, there is some finite subcollection $\mathcal{J}$ of $\mathcal{I}$ such that $\mathcal{J}$ covers $K \backslash O$ and misses F. Let $U = \backslash \cup \mathcal{J}$. Then $U \in \mathcal{U}(\mathcal{O})$ and $F \subset U$. This completes the proof. □ □

Remark. The same argument works for the real line, and also for ω^ω, where closed intervals are replaced by basic clopen sets $[\sigma]$, $\sigma \in \omega^{<\omega}$. The latter shows that it is consistent that there are no uncountable "weak γ-sets" as defined in [GS$_1$], which answers Question 4 in that paper.

The Laver model for the Borel conjecture satisfies that $\mathfrak{b} > \omega_1$ and there are no γ-sets.

5. Gap spaces

Tamano's examples (see Example 2.8) give examples (under MA) of α_4-FU$_n$ a.d. spaces which are not FU$_{n+1}$, for $n \geq 2$ (for $n = 1$ we have Simon's example in ZFC). What about α_2-FU$_n$ not FU$_{n+1}$? By Theorem 2.4(c)(e), a.d.-spaces or compact spaces won't do for this:

Proposition 5.1. *If X is compact α_2-FU, then X is α_2-FU$_n$ for every n.*

(Remark: Same is true with α_1 or α_3 in place of α_2.)

What works to separate (only consistently for $n \geq 2$) α_2-FU$_n$ from FU$_{n+1}$ turns out to be spaces defined from Hausdorff gaps. An example due to J. Isbell, appearing in [O], produces two countable Fréchet α_2-spaces whose product is not Fréchet. The assumption $2^{\aleph_0} < 2^{\aleph_1}$ is used in [O] in describing Isbell's example, and it is only claimed that the spaces are α_4-FU. But P. Nyikos [Ny$_2$] noticed that the examples are α_2-spaces, and that what is needed to construct the examples is a Hausdorff gap, so they exist in ZFC.

Definition 5.2. *An ω_1-sequence $\{(a_\alpha, b_\alpha) : \alpha < \omega_1\}$ of pairs of infinite subsets of ω is a* **Hausdorff gap** *if*

(a) *$a_\alpha \subset^* a_\beta \subset^* b_\beta \subset^* b_\alpha$ for all $\alpha < \beta < \omega_1$;*
(b) *There is no c such that $a_\alpha \subset^* c \subset^* b_\alpha$ for all $\alpha < \omega_1$.*

(Recall $a \subset^* b$ means $|a \backslash b| < \omega$.)

Given a Hausdorff gap as above, let

$$\mathcal{I}^0 = \{x \subset \omega : |x \cap a_\alpha| < \omega \text{ for all } \alpha < \omega_1\},$$

$$\mathcal{I}^1 = \{x \subset \omega : |x \cap (\omega \backslash b_\alpha)| < \omega \text{ for all } \alpha < \omega_1\}.$$

Then Nyikos's observation is that the spaces $X_e = \omega \cup \{\infty\}$, where neigborhoods of ∞ are complements of members of the ideal $\mathcal{I}^e$, are the same as Isbell's spaces and are Fréchet α_2. We call a space obtained from (either the left or right side of) a Hausdorff gap g in this way a **gap space**. It is not hard to show that the product $X_0 \times X_1$ of the above gap spaces is not Fréchet. (See [O] and [Ny$_2$], or Example 2.4 in [G$_3$].)

The Isbell-Nyikos example in a slightly different form produces, in ZFC, a gap space which is not FU$_2$.

Example 5.3. [GS$_2$] *There is a gap space $X = \omega \cup \{\infty\}$ which is Fréchet α_2 but not FU$_2$.*

The example is obtained by putting together two copies of an arbitrary Hausdorff gap, with one copy the reverse of the other. That is, given a gap $\{(a_\alpha, b_\alpha) : \alpha < \omega_1\}$, if we let $\{(a'_\alpha, b'_\alpha) : \alpha < \omega_1\}$ be the same gap on another copy ω' of ω, then $\{(a_\alpha \cup (\omega' \setminus b'_\alpha), b_\alpha \cup (\omega' \setminus a'_\alpha) : \alpha < \omega_1\}$ yields a non-FU$_2$ gap space. It is not difficult to verify that the set $\{\{n, n'\} : n \in \omega\}$ is a π-net of doubletons with no convergent subsequence.

In the other direction, it is not difficult to construct gap spaces that are FU$_{fin}$ under CH, and they also exist in the Hechler model[GS$_2$]. Hrusak asked if there could be in ZFC a gap space which is FU$_{fin}$. The answer is no: under MA_{ω_1}, no gap space is FU$_2$.

Theorem 5.4. [GS$_2$] *$MA(\omega_1)$ implies that no gap space is FU$_2$.*

The key idea is to show that, given a gap, there are disjoint subsets w and w' of ω such that the restriction of the gap to $w \cup w'$ is essentially like Example 5.3.

Todorcevic showed that under PFA, the only types of gaps that exist are (ω_1, ω_1^*), (ω, ω_2^*), or (ω_2, ω^*) (recall $\mathfrak{c} = \omega_2$ under PFA). Since no compact space is α_3-FU without being boundedly FU$_{fin}$, it follows that under PFA, no gap space or a.d.-space serves to separate α_2-FU$_2$ from boundedly FU$_{fin}$. But this pretty much exhausts the kind of examples we have in this theory! So we ask:

Question 5. *Does PFA imply that every countable α_2-FU$_2$ space is boundedly FU$_{fin}$?*

On the other hand, there are examples in other models. In particular:

Example 5.5. [GS$_2$] *(CH) For each n, there is an α_2-FU$_n$ gap space which is not FU$_{n+1}$.*

The idea is to build by an induction of length ω_1 a gap on $\omega \times (n+1)$, making $\{\{i\} \times (n+1) : i \in \omega\}$ be a π-net of $(n+1)$-sized sets with no convergence sequence, and making sure along the way that potential π-nets of $\leq n$-sized sets will in the end either not remain a π-net or will contain a convergent subsequence.

6. Other connections–games, function spaces, moving off property)

Definition 6.1. *Define the* **convergence game** $G(X,x)^{fin}$ *at $x \in X$ as follows. In the n^{th} round, Player O plays an open nbhd O_n of x, and Player P responds by choosing a finite set $P_n \subset O$. We say O* **wins** *the game if $P_n \to x$, otherwise P wins. Games $G(X,x)^n$ in which P must choose a set of size $\leq n$ are similarly defined. [In* [G$_1$]*, the game $G(X,x)^1$ was studied.]*

Theorem 6.2. [GS$_2$] *Let $*$ be fin or n. A space X is α_2-FU$_*$ at x iff Player P has no winning strategy in the game $G^*(X,x)$.*

It follows that if X is the space of Example 5.3, which is α_2-FU but not FU_2, then P has no winning strategy, i.e., cannot avoid convergence of his chosen points to ∞, if he can only choose a single point on each round, but he does have a winning strategy if he is allowed to choose two points each time. And Example 5.5 gives under CH spaces in which P has no winning strategy in the n-point game but does have one in the $n+1$-point game. We don't know ZFC examples of this for $n \geq 2$.

Since FU_{fin} is equivalent to α_2-FU_{fin}, the theorem also says that FU_{fin} is equivalent to P having no winning strategy in the convergence game where he can choose on each round a finite set of arbitrary size.

A game that has some close connections with the above game essentially replaces finite sets with compact sets. Define the game $G^K(X)$ on a locally compact space X as follows: On the n^{th} round, Player K chooses a compact set K_n, and Player L responds by choosing a compact set L_n disjoint from K_n. We say K **wins** the game if $\{L_n : n \in \omega\}$ is discrete in X. The similarity to $G(X, x)$ is evident when one notes that choosing a compact subset of X is equivalent to choosing a nbhd of the compactifying point ∞ of the one-point compactification of X, and $\{L_n : n \in \omega\}$ discrete is equivalent to $L_n \to \infty$.

The following is a compact set analogue of the FU_{fin} property and its game equivalence. Call a collection $\mathcal{L}$ of compact sets a **moving off collection** if for any compact K, there is $L \in \mathcal{L}$ such that $K \cap L = \emptyset$. (Note that a moving off collection $\mathcal{L}$ is a π-net of compact sets at the compactifying point of the one-point compactification.) We say X has the **Weak Moving Off Property (WMOP)** if for every moving off collection $\mathcal{L}$, there are $L_n \in \mathcal{L}$ such that $\{L_n : n \in \omega\}$ is discrete. If one requires the stronger "$\{L_n : n \in \omega\}$ has a discrete open expansion", this defines the *MOP*. WMOP and MOP are easily seen to be equivalent for normal spaces or locally compact spaces.

The reason this game, and the MOP, became of interest is due to the following result:

Theorem 6.3. [GM] *The following are equivalent for a locally compact space X:*

(a) *$C_k(X)$ is a Baire space;*
(b) *L has no winning strategy in $G^K(X)$;*
(c) *X has the MOP;*
(d) *X has the WMOP.*

Sakai[Sa] recently obtained a result which, for locally compact spaces X, implies that the MOP is also equivalent to the following Fréchet-like property (called κ-Fréchet-Urysohn) of $C_k(X)$: $p \in \overline{U}$, U open, implies there are $x_n \in U$ with $x_n \to p$. Also, Nyikos noticed an interesting connection between the MOP property and Fréchetness of a natural subspace of $C_k(X)$. For X locally compact, let

$$C_0(X) = \{f \in C(X) : f \text{ vanishes at } \infty\}.$$

Theorem 6.4. [Ny_4] *Let X be locally compact, and consider $C_0(X)$ with the compact-open topology. Then*

(a) *$t(C_0(X)) = \omega$ iff every moving off collection of compact subsets of X has a countable moving off subcollection ;*
(b) *$C_0(X)$ is FU_{fin} iff $C_0(X)$ is Fréchet iff X has the MOP.*

Nyikos actually showed part (b) only for Fréchet , but the same argument gets FU_{fin}.

We can use $C_0(X)$ to get other (consistent) examples of countable FU_{fin} not first-countable spaces and groups. Now if X were Lindelöf, $C_0(X)$ would be first-countable; but we have:

Theorem 6.5. *Suppose X is locally compact with a countable π-base, and has the MOP. If X is not Lindelöf, then there is a countable subgroup of $C_0(X)$ which is FU_{fin} but not first countable.*

Proof. For each member B of a countable π-base $\mathcal{B}$, there is a continuous function $f_B : X \to [0,1]$ such that $f_B(x) = 1$ for some $x \in B$ and $f_B(X\backslash B) = \{0\}$. Let F be the collection of these f_B's, together with the constant 0 function c_0. If c_0 had countable character in F, there would be basic open sets $B(c_0, L_n, \epsilon_n) = \{f \in F : \forall x \in L_n(|f(x)| < \epsilon_n)\}$ for $n \in \omega$ that witness this. As X is locally compact but not Lindelöf, there is an open set U with compact closure which is not contained in any L_n. Then for each n, there is some $B_n \in \mathcal{B}$ such that $f_{B_n}(L_n) = \{0\}$ and $1 \in f_{B_n}(U)$. It follows that no set $B(c_0, L_n, \epsilon_n)$ is contained in the neighborhood $B(c_0, \overline{U}, 1/2)$ of c_0, contradiction. So F is not first countable.

Finally, let G be the subgroup generated by F. Then G is not first countable, and since $C_0(X)$ is FU_{fin}, so is G. □ □

Note that since finite sets are compact, it is easy to translate the MOP property of any locally compact non-Lindelöf space X to see that it implies the FU_{fin} property at the compactifying point of its one-point compactification X^*. And if X is also separable, there would be a countable non-first-countable subspace of X^*. So we ask:

Question 6. *Is there in ZFC a separable locally compact non-Lindelöf X which has the MOP? In particular, is there in ZFC an uncountable a.d. $\mathcal{A}$ such that $\psi(\mathcal{A})$ has the MOP?*

For all we know, it could be that the answer is no, yet there is such a space which is FU_{fin} at the compactifying point. There are various consistent examples of a.d. $\mathcal{A}$ with $\psi(\mathcal{A})^*$ FU_{fin} yet $\psi(\mathcal{A})$ does not have the MOP; an a.d. space constructed by Nyikos under $\mathfrak{p} = \mathfrak{b}$ is one example. This space is generated by the a.d. family $\mathcal{A}$ of subsets of $\omega \times \omega$ consisting of the graphs of a $<^*$-increasing $<^*$-unbounded family f_α, $\alpha < \mathfrak{b}$, of increasing functions in ω^ω, together with $\{n\} \times \omega : n \in \omega\}$. In Theorem 5.6 of [Ny$_3$]), Nyikos proves that this space is FU_{fin}. It is easy to check that if x_n is the compactifying point of $\{n\} \times \omega$ in , and $L_n = \{x_n\} \cup (\{n\} \times \omega)$, then the colleciton of all tails of the L_n's form a (countable) moving off collection in $\psi(\mathcal{A})$, no infinite subcollection of which is discrete.

7. Open Problems

Problem 1. [RS] *Is there in ZFC a countable FU_{fin} space which is not first-countable?*

Problem 2. *Is there in ZFC a countable Fréchet topological group which is:*

(a) [Mal] *Not first-countable? [Yes if yes to Problem 1.]*
(b) [S$_2$] *Not FU_2? [There is an example if CH*[Sh]*.]*

Problem 3. (a) [S$_1$] *Is it consistent that every Fréchet topological group is α_3?*

(b) *(Shibakov) Can there be a Fréchet topological group whose product with some compact Hausdorff Fréchet space is not Fréchet ? [The group could not be* α_3*.]*

Problem 4. [GS$_1$][GS$_2$] *Are there in ZFC countable spaces X satisfying:*

(a) FU_n *(resp.,* α_2*-*FU_n*) but not* FU_{n+1} *for* $n \geq 2$*? (Equivalently,* X^n *Fréchet (resp., and* α_2*) but* X^{n+1} *not Fréchet ?) [For* $n = 1$*, yes.]*
(b) FU_2 *(resp.,* α_2*-*FU_2*) but not boundedly* FU_{fin}*? (Equivalently,* X^2 *is Fréchet (resp., and* α_2*) but* X^n *not Fréchet for some n?)*
(c) α_2*-*FU_{fin} *but not* FU_{fin}*?*

Problem 5. *Does* α_2*-*FU_2 *imply boundedly* FU_{fin} *under PFA? [No compact or a.d. spaces can be counterexamples, and if PFA, no gap spaces either.]*

Problem 6. *Is there in ZFC a countable (or compact) space X such that* X^n *is Fréchet for all positive integers n (equivalently, X is boundedly* FU_{fin}*) , but* X^ω *is not Fréchet ? [There is an example under MA.]*

Problem 7. *Is there in ZFC a countable non-bisequential space which is:*

(a) *boundedly* FU_{fin}*?*
(b) [JM] *productively* α_4*-FU?*
(c) [DS] FU_2*?*

Problem 8. *Is there in ZFC a productively* α_4*-FU non-*α_3*-space? [Since bisequential implies* α_3*, a positive answer also implies a positive answer to Problem 7(b).]*

Problem 9. *Is there in ZFC a countable boundedly* FU_{fin} *space which is not productively* α_4*-FU?*

Problem 10. [JM] *If X is productively* α_4*-FU, must* X^ω *be FU? Productively* α_4*-FU?*

Problem 11. [GS$_2$] *Does* X^ω *Fréchet imply X is* α_3*?*

Problem 12. *Let* $\mathcal{A} = \{f_\alpha\}_{\alpha<\mathfrak{b}}$ *be a* $<^*$*-unbounded* $<^*$*-increasing family of non-decreasing functions from* ω *to* ω*. Is* $(\psi(\mathcal{A})^*)^\omega$ *Fréchet always? In particular, can* $(\psi(\mathcal{A})^*)^\omega$ *be Fréchet when* $\psi(\mathcal{A})^*$ *is not* α_3 *(and thus provide a consistent negative answer to the previous problem)? [This* $\psi(\mathcal{A})^*$ *was considered by Nyikos, who noted in* [Ny$_3$] *that it can be made non-*α_3*, hence non-bisequential, if* $\mathfrak{b} = \mathfrak{c}$*; more recently he showed it can also be made bisequential. Jordan and Mynard*[JM] *show that it is always productively* α_4*-FU.]*

Problem 13. *Find a productively* α_2*-FU space which is not productively* α_3*-FU. [Simon's example (Prop. 1.7(c)) is productivlely* α_3*-FU but not productively* α_4*-FU.]*

Problem 14. [DS] *Is there a model of set-theory in which every countable* FU_2 α_1*-space is first-countable, yet there is a countable non-first-countable* α_1*-FU space?*

Problem 15. [Ar$_2$] *Characterize the class of spaces X such that* $X \times Y$ *is Fréchet for every countably compact Fréchet -space Y. [The class includes all productively* α_4*-FU spaces and all* $\alpha_3 - FU$ *spaces.]*

Problem 16. *Suppose $X \times Y$ is Fréchet for every α_3-FU space Y. Must X have a Fréchet countably-compactification? What if X is countable? [This problem is motivated by Theorem 2.4(d). But I don't even know if every productively α_4-FU space has a countably tight countably compactification.]*

Problem 17. *Let P be one of the properties FU_n ($n \geq 2$), boundedly FU_{fin}, or FU_{fin}. Suppose X and Y satisfy P, and $X \times Y$ is Fréchet . Must $X \times Y$ satisfy P?[For FU_n, consistently no: any FU_n not FU_{2n} space X provides a counterexample with $X = Y$. For P= α_4-FU spaces, this was a question of Nogura, which has a negative answer under CH*[Sim$_2$]*, and a positive answer under OCA*[To$_2$]*.]*

Problem 18. *Does there exist in ZFC an uncountable subset X of 2^ω and a collection $\mathcal{U}$ of open sets such that, for each finite $F \subset X$ and $x \in X \backslash F$, there is $U \in \mathcal{U}$ with $F \subset U \subset X \backslash \{x\}$, and :*

(a) *X is a $\mathcal{U}^{\cap fin} - \gamma$ set?*
(b) *X is a $\mathcal{U}^{\cap n}$ but not a $\mathcal{U}^{\cap n+1} - \gamma$ set?*

[Positive answers would give positive answers to Problems 1 and 4a respectively.]

Problem 19. *Is there in ZFC an uncountable a.d. family $\mathcal{A}$ such that:*

(a) *$\psi(\mathcal{A})^*$ is FU_{fin}?*
(b) *$\psi(\mathcal{A})$ has the MOP? [This implies $\psi(\mathcal{A})^*$ is FU_{fin}.]*

Problem 20. *Is there in ZFC a separable locally compact non-Lindelöf space X such that:*

(a) *X^* is FU_{fin} at the compactifying point?*
(b) *X has the MOP?*

[Of course, this is simply a more general version of the previous problem.]

Problem 21. *Is it consistent that $\psi(\mathcal{A})^*$ bisequential is equivalent to*

(a) *$\psi(\mathcal{A})^*$ $\alpha_3 - FU$?*
(b) *$\psi(\mathcal{B})^*$ $\alpha_3 - FU$ for every $\mathcal{B} \subset \mathcal{A}$?*

[The forward directions hold. Nyikos noted that the reverse directions do not hold if there is a point of character $\aleph_1$ in $\beta\omega$.]

Problem 22. *Solve Problems 1,6,7, 8, 9, 10, 11, and the non-parenthetical parts of Problem 4 in the class of a.d. (or compact) spaces.*

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