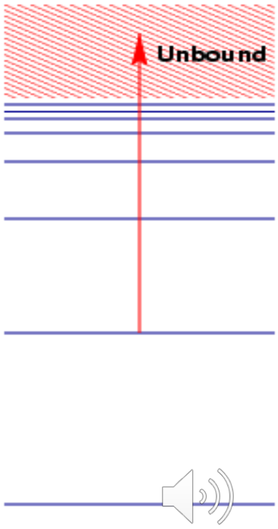
Bound-free transitions (Saha equation)

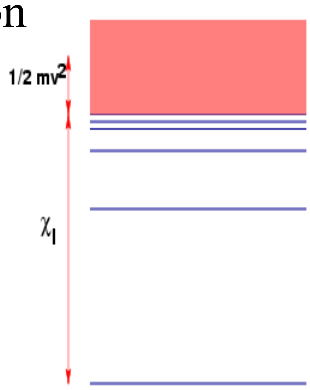
- Bound-free transitions occur between atomic state and an unbound state
- Free electron can have a range of kinetic energies => bound-free transitions produce continuous opacity (not just at lines)
- A minimum photon energy is needed to ionize an atom from a given level, e.g., need $\lambda \leq 91.2$ nm to ionize hydrogen from the n=1 level.



The Saha equation

- Gives the distribution (relative number densities) of atoms in successive stages of ionization. Simplest case: a neutral atom and its first stage of ionization.

- Energy difference between ground state of atom, and free electron having velocity υ, is:
- $$\Delta E = \chi_I + \frac{1}{2} m_e v^2$$
- where χ_I is the ionization potential.



Boltzmann

- The Boltzmann law suggests:

$$\frac{dN_0^+(v)}{N_0} = \frac{g}{g_0}\exp\left[-\frac{(\chi_I + 1/2m_e v^2)}{kT}\right]dv$$

where:
  - $dN_0^+(\upsilon)$ is the number of ions in the ground state with the free electron having velocity between υ and $\upsilon+d\upsilon$
  - N_0 is number of atoms in ground level
  - g_0 is the statistical weight of the atom in the ground state
  - g is the product of the statistical weight of the ion in its ground state g_0^+, and the differential statistical weight of the electron g_e. i.e., $g = g_0^+ g_e$

Statistical weight of free electron

- Uncertainty principle tells us phase space is quantised into cells with volume h^3
- For the electron, with two spin states,

$$g_e = \frac{2\, dx_1 dx_2 dx_3 dp_1 dp_2 dp_3}{h^3}$$

- The volume $dx_1\, dx_2\, dx_3$ contains one electron, so $dx_1\, dx_2\, dx_3 = 1/N_e$, where N_e is the electron density.
- Since the electrons have an isotropic velocity distribution,

$$dp_1 dp_2 dp_3 = 4\pi p^2 dp = 4\pi m_e^3 v^2 dv$$

- which gives,

$$\frac{dN_0^+(v)}{N_0} = \frac{g_0^+}{g_0}\frac{8\pi m_e^3}{N_e h^3}\exp\left[-\frac{(\chi_I + 1/2m_e v^2)}{kT}\right]v^2 dv$$

Eliminating velocity

- We don't care about the electron velocity. Integrating over all possible υ gives,

$$\frac{N_0^+ N_e}{N_0} = \frac{2g_0^+}{g_0}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\frac{\chi_I}{kT}}$$

- where we use the integral

$$\int_0^\infty e^{-x^2} x^2 dx = \frac{\sqrt{\pi}}{4} \qquad x = \sqrt{\frac{m_e}{2kT}}\,v$$

Finally… the Saha equation

- For the ground state, Boltzmann's law gives,

$$\frac{N_0}{N} = \frac{g_0}{U(T)} \quad \text{and} \quad \frac{N_0^+}{N^+} = \frac{g_0^+}{U^+(T)}$$

- Substituting these gives us Saha's equation,

$$\frac{N^+ N_e}{N} = \frac{2U^+(T)}{U(T)}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\frac{\chi_I}{kT}}$$

- where N and N^+ are the number densities of neutral and once-ionized atoms, and U and U^+ are the corresponding partition functions.
- Saha's equation for any two neighbouring states of ionization is just the same, replace N by N^j, N^+ by N^{j+1}, etc.

Ionization of hydrogen - I

- Define the degree of ionization x by

$$x = \frac{N^+}{N + N^+}$$

- For a neutral gas $x = 0$, for a fully ionized gas $x = 1$. Left hand side of Saha equation is then,

$$\frac{N^+ N_e}{N} = \frac{x}{1-x} N_e$$

- Next, eliminate N_e by writing it in terms of the gas pressure.
- If $N_H = N + N^+$ is the total number of hydrogen nuclei, then can write the pressure of the electrons as:

$$P_e = N_e kT = (N_H + N_e)kT \frac{N_e}{N_H + N_e} = P_{gas} \frac{N_e}{N_H + N_e}$$

Ionization of hydrogen - II

- Each ionized atom gives one electron, so for pure hydrogen $N_e = N^+$ and

$$P_e = \frac{x}{1+x} P_{gas}$$

- The Saha equation can then be written,

$$\frac{x^2}{1-x^2} = \frac{1}{P_{gas}} \frac{2U^+(T)}{U(T)} \left(\frac{2\pi m_e}{h^2}\right)^{3/2} (kT)^{5/2} e^{-\frac{\chi_I}{kT}}$$

- a quadratic equation for the degree of ionization. To apply, we need
  - P_{gas} and T. Ionization increases with the temperature (collisions become more violent) and decreases with increasing pressure at fixed T (more recombinations).
  - The partition functions. In practice, can take $U = 2$ (the ground state value) and $U^+ = 1$.
- Even a small abundance of other elements can provide lots of electrons if the ionization potential is low. So the pure hydrogen case is of limited applicability.

Bound-free absorption cross-section

- Bound-free absorption provides an important source of continuum opacity. For a hydrogen-like atom in a level with principal quantum number n, with ionization potential χ_n, the bound-free absorption cross-section σ_{bf} is given by

$$\sigma_{bf} = 0 \quad \text{for} \quad \nu < \frac{\chi_n}{h}$$

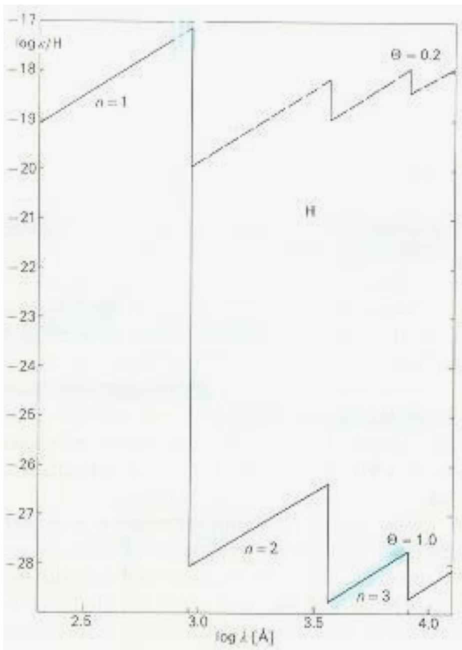
$$\sigma_{bf} \propto \frac{g(\nu,n,l)}{n^5 \nu^3} \quad \text{otherwise}$$

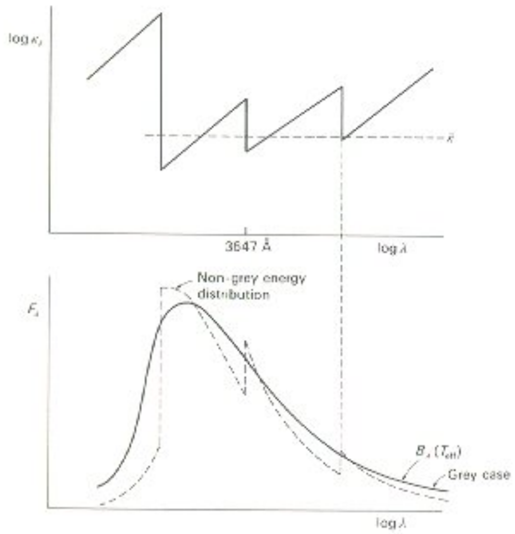
- Here g is the bound-free Gaunt factor (not degeneracy), a quantum mechanical correction factor to the simple scaling
- Properties:
  - Absorption cross-section has sharp rises, absorption edges, at the frequency where the atom in a given level can be ionized
  - At frequencies higher than the edge: $\sigma_{bf} \propto \nu^{-3}$
  - The Gaunt factor is close to unity near the edge

The hydrogen absorption coefficient κ per hydrogen atom is shown as a function of wavelength for two temperatures 5040 and 25200 K, where $\theta = 5040/T$

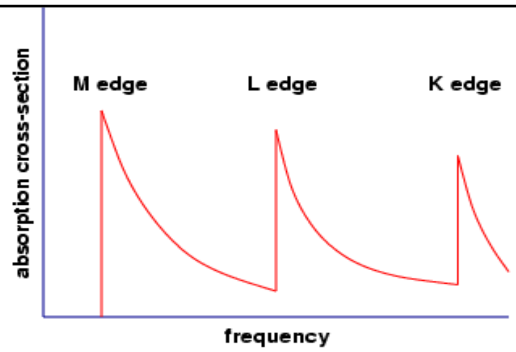
Higher temperatures lead to higher values of κ in the visual spectral region (Paschen continuum, absorption from the level $n = 3$).

The value of κ at the Lyman limit is ~ the cross section of the lowest orbital (0.5×10^{-8} cm) in the hydrogen atom.





Effect of wavelength-dependence of hydrogen absorption coefficient on the observed energy distribution of the star



An atom with many electrons will be characterized by a series of ionization edges as it loses electrons from successive shells.

- Heavy elements, either in the gas phase or in grains, have many inner-shell electrons. They provide large opacity to soft X-rays (below 1 keV).
- Hard X-rays (10 keV or more) see only the ν^{-3} tail (becoming closer to $\nu^{-3.5}$ at high ν). Very hard to absorb these.
- Seeing the absorption at low energies → measurement of the column density towards an X-ray source.

Example: absorption towards an Active Galactic nucleus

The *intrinsic* X-ray spectra of Active galaxies are often taken to be power laws. Superimposed on that we have,

- Absorption at low energy (here modelled as oxygen edges).
- Instrumental features than have not been calibrated quite right (a gold edge).
- Emission from fluorescent iron near the black hole.

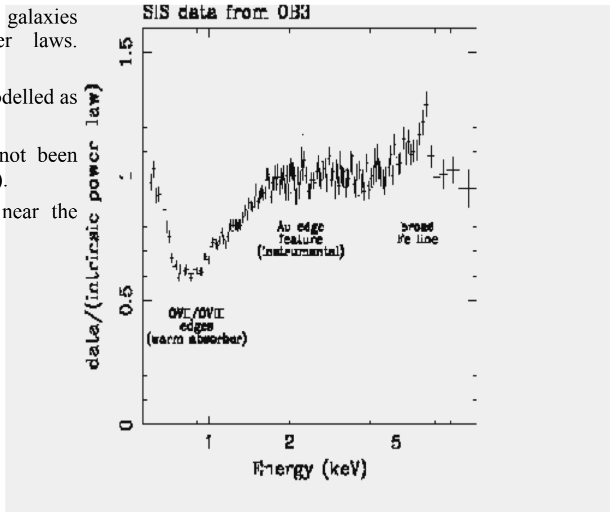


Figure 2.2: Ratio of the full band SIS0 spectrum of MCG–6-30-15 obtained during OB3 to the best fitting intrinsic power-law. The intrinsic continuum is defined by fitting a power-law to the 3–4 keV range (since there is negligibleX-ray reprocessing over this range). Galactic absorption is included with a column density of N_H = 4×10^{20} cm^{-2}

Lecture 9 revision quiz

- Sanity-check integral with respect to v:

$$\frac{dN_0^+(v)}{N_0} = \frac{g_0^+}{g_0}\frac{8\pi m_e^3}{N_e h^3}\exp\left[-\frac{(\chi_I + 1/2 m_e v^2)}{kT}\right] v^2 dv$$

- Plot the degree of ionization of hydrogen as a function of log(P_{gas}) at a fixed T=10^4 K.
- In the spectrum of an early-type star, why is there an abrupt change in flux with wavelength across hydrogen ionization boundaries?
- Do you expect the emergent intensity to be greater at higher or lower frequencies than the ionization threshold frequency? Why?

Lecture 9 revision quiz

- Starting from the Boltzmann excitation equation, fill in the steps to derive the Saha equation.
- Starting from the definition of the ionization fraction *x*, derive the equations for electron pressure and the quadratic form of the Saha equation.