



## INHOUR EQUATION BASED ON P1 APPROXIMATION OF NEUTRON TRANSPORT THEORY

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### ABSTRACT

The aim of this work is the analysis and discussion of the *inhour* equation which is derived on the P1 approximation neutron transport theory (P1-*inhour* equation). The classic *inhour* equation (C-*inhour* equation) is based on the neutron diffusion theory which is widely applied in nuclear reactor analysis. The P1-*inhour* equation is compared with the C-*inhour* equation and the differences are discussed.

**Keywords:** inhour equation, P1 Neutron transport, neutron diffusion equation, point reactor kinetics equations, reactivity.

### INTRODUCTION

When reactivity is expressed in terms of the inverse hour or *inhour* unit, it is defined as the reactivity which will make the stable reactor period to 1 hour (c.f., Glasstone & Sesonske, 1981). The widely known *inhour* equation (C-*inhour*) to six delayed groups is given by (Duderstadt & Hamilton, 1976):

$$\rho_0 = \frac{sl}{sl+1} + \frac{1}{sl+1} \sum_{i=1}^6 \frac{s\beta_i}{s+\lambda_i} \quad (1)$$

In the above equation, the values of  $s_j$  are obtained as the roots of the seventh-order polynomial. Here  $l$  is the neutron mean lifetime in the reactor,  $\beta_i$  are the fraction of all fission neutrons (both prompt and delayed) emitted per fission that appear from the  $i$ th precursor group, and  $\lambda_i$  is the decay constant of the  $i$ th precursor, and  $\beta_i$  is the fraction of the  $i$ th precursor fission group of delayed neutrons. Previous works on the inhour equation have been developed (van Dam, 1996, Aboanber & El Mhrawy, 2008).

The work of van Dam (1996) can be considered as an extension of the *inhour* equation that was applied to the core and the reflector with a two-point reactor kinetic model. The conventional  $n$ th group inhour equation is represented in a polynomial form with a degree on  $n+1$  by Ratemi & Eshabo (1998), where the coefficients have a linear dependence on the inserted reactivity. Aboanber & El Mhrawy (2008) derived a mathematical framework to characterize a new version of the two-point inhour equation to determine its roots for reflected reactors. The *inhour* equation is of practical common use as it can be seen in the comprehensive literature (e.g., Hosseini & Vosoughi, 2010; Vyawahare & Nataraj, 2013; Dulla *et al.*, 2014; Dall'Osso, 2015; Sanchez *et al.*, 2017).

In general, the inhour equation is based on the point reactor kinetics equations (Duderstadt & Hamilton, 1976):

$$\frac{dP}{dt} = \frac{\rho_0 - \beta}{\Lambda} P(t) + \sum_{i=1}^m \lambda_i C_i(t) \quad (2)$$

$$\frac{dC_i}{dt} = \frac{\beta_i}{\Lambda} P(t) - \sum_{i=1}^m \lambda_i C_i(t) \quad (3)$$

The power of the reactor is defined as:

$$P(t) = w_f \nu \Sigma_f n(t) \quad (4)$$

where  $w_f$  is the energy released per fission event,  $\nu$  is the neutron velocity, and  $\Sigma_f$  is the fission cross section. It is important to note that  $n(t)$  should be interpreted as the total number of neutrons in the reactor, which implies that  $C_i(t) = w_f \nu \Sigma_f c_i(t)$ , where  $c_i(t)$  should be interpreted as the number of delayed neutron precursors in the reactor. And  $\beta = \sum_i \beta_i$  (Eq. 2) is the total fraction of fission neutrons which are delayed. In one way or another, the *inhour* equation is obtained under certain assumptions from the point equation of the reactor, which in turn is obtained on the diffusion equation:

$$\begin{aligned} & \frac{1}{\nu} \frac{\partial \phi}{\partial t} - \nabla \cdot D \nabla \phi + \Sigma_a \phi(\mathbf{r}, t) \\ & = (1 - \beta) \nu \Sigma_f \phi(\mathbf{r}, t) + \sum_{i=1}^m \lambda_i \hat{C}_i(\mathbf{r}, t) \end{aligned} \quad (5)$$

Here  $\phi$  is the neutron flux,  $\Sigma_a$  is the macroscopic absorption cross section,  $\nu$  is the number of neutrons per fission. The diffusion coefficient is given by

$$D = \frac{1}{3\Sigma_{tr}} \quad (6)$$



where  $\Sigma_{tr}$  is the macroscopic transport cross section.

In this work, the analysis of the *inhour* equation that is derived on the P1 approximation of neutron transport theory is presented.

### POINT EQUATIONS FROM THE P1 APPROXIMATION

The point reactor kinetics equations from the P1 approximation of transport theory is presented by Espinosa-Paredes & Suescún-Díaz (2020). Below are the fundamental aspects to deduce the point model. The one-speed diffusion equation considering that the angular flux is linearly anisotropic, given by Duderstadt and Hamilton (1976):

$$\frac{1}{\nu} \frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{J} + \Sigma_a(\mathbf{r})\phi(\mathbf{r}, t) = S(\mathbf{r}, t) \quad (7)$$

$$\frac{1}{\nu} \frac{\partial \mathbf{J}}{\partial t} + \frac{1}{3} \nabla \phi(\mathbf{r}, t) + \Sigma_{tr}(\mathbf{r}) \mathbf{J} = 0 \quad (8)$$

Now, dividing Eq. (8) by  $\Sigma_{tr}(\mathbf{r})$ , leads to:

$$\frac{1}{\nu \Sigma_{tr}(\mathbf{r})} \frac{\partial \mathbf{J}}{\partial t} + \mathbf{J}(\mathbf{r}, t) = -D \nabla \phi \quad (9)$$

where the coefficient diffusion  $D$  is given by Eq. (6). The term  $(\nu \Sigma_{tr})^{-1}$  has units of time, therefore it can correspond to a relaxation time  $\tau$ . Then, this equation can be rewritten as:

$$\tau(\mathbf{r}) \frac{\partial \mathbf{J}}{\partial t} + \mathbf{J}(\mathbf{r}, t) = -D(\mathbf{r}) \nabla \phi \quad (10)$$

It can be considered that the medium in which the neutrons are diffusing is uniform or homogeneous so that  $D$ ,  $\Sigma_a$  and  $\tau$  do not depend on position. Now, we apply the operator  $\nabla \cdot$  in Eq. (10):

$$\left( \tau \frac{\partial}{\partial t} + 1 \right) \nabla \cdot \mathbf{J} = -D \nabla^2 \phi \quad (11)$$

The term  $\nabla \cdot \mathbf{J}$  is obtained from Eq. (7):

$$\nabla \cdot \mathbf{J} = S(\mathbf{r}, t) - \Sigma_a \phi(\mathbf{r}, t) - \frac{1}{\nu} \frac{\partial \phi}{\partial t} \quad (12)$$

Substituting Eq. (12) into Eq. (11), it leads to:

$$\begin{aligned} \frac{\tau}{\nu} \frac{\partial^2 \phi}{\partial t^2} + \left( \Sigma_a \tau + \frac{1}{\nu} \right) \frac{\partial \phi}{\partial t} + \Sigma_a \phi(\mathbf{r}, t) \\ = D \nabla^2 \phi + S(\mathbf{r}, t) + \tau \frac{\partial S}{\partial t} \end{aligned} \quad (13)$$

where

$$S(\mathbf{r}, t) = (1 - \beta) \nu \Sigma_f \phi(\mathbf{r}, t) + \sum_{i=1}^m \lambda_i \hat{C}_i(\mathbf{r}, t) \quad (14)$$

Eq. (13) represents that the neutron phenomena production is not instantaneous, because they are described by a hyperbolic equation.

Applying the definitions of nuclear reactor analysis (Glasstone and Sesonske, 1981): diffusion area  $L^2 = D(\Sigma_a)^{-1}$ ; the one-group non-leakage probability  $P_{NL} = (1 + L^2 B_g^2)^{-1}$ ; reactivity  $\rho = (k - 1)k^{-1}$ ; prompt-neutron lifetime  $l = P_{NL}(\nu \Sigma_a)^{-1}$ ; Mean neutron generation time  $\Lambda = lk^{-1}$ , a version of the point reactor kinetics equations from the P1 approximation of the transport equations is obtained:

$$\begin{aligned} \tau \frac{d^2 n(t)}{dt^2} + (\tau \nu \Sigma_a + 1) \frac{dn(t)}{dt} \\ = \frac{1}{\Lambda} (\rho - \beta) n(t) + \sum_{i=1}^m \lambda_i c_i(t) + \tau \frac{ds(t)}{dt} \end{aligned} \quad (15)$$

where

$$s(t) = \frac{(1 - \beta)}{\Lambda} n(t) + \sum_{i=1}^m \lambda_i c_i(t) \quad (16)$$

$$\frac{dc_i}{dt} = \frac{\beta}{\Lambda} n(t) - \lambda_i c_i(t), \quad i = 1, 2, \dots, m \quad (17)$$

these two equations can be combined with Eq. (15) to obtain:

$$\begin{aligned} \tau \frac{d^2 n}{dt^2} + \tau \left( \frac{1}{l} - \frac{1 - \beta}{\Lambda} \right) \frac{dn(t)}{dt} + \frac{dn(t)}{dt} \\ = \frac{\rho - \beta}{\Lambda} n(t) + \sum_{i=1}^m \left( \lambda_i c_i(t) + \tau \frac{dc_i}{dt} \right) \end{aligned} \quad (18)$$

To get the first term on the left side, it is considered that the non-leakage probability is one.

### P1-INHOURLY EQUATION

We apply the exponential type solution:

$$n(t) = n_0 e^{\omega t} \quad (19)$$

$$c_i(t) = c_{i0} e^{\omega t}, \quad i = 1, 2, \dots, m \quad (20)$$



where  $\omega$  is a parameter to be determined, which has inverse time dimensions,  $n_0$  and  $c_{i0}$  are the neutron density and the concentration of group- $i$  delayed neutron precursors, respectively at  $t=0$ . It is considered that before this moment the nuclear reactor was in a steady state with a value of  $\rho=0$ , and at  $t=0$  the reactivity undergoes a sudden change, which causes the neutron density and the concentration of precursors to begin to vary over time.

With the use of the solution given by Eq. (20) in the equation of precursors given by Eq. (17), it can be shown that:

$$c_i(t) = \frac{\beta_i n_0}{\Lambda(\omega + \lambda_i)} e^{\omega t}, \quad i = 1, 2, \dots, m \quad (21)$$

The next step is to apply the proposed solution given by Eq. (19) in the P1 equation of neutron density given by Eq. (18):

$$\begin{aligned} &\tau n_0 \omega^2 e^{\omega t} + \tau \left( \frac{1-\beta}{l} - \frac{1-\beta}{\Lambda} \right) n_0 \omega e^{\omega t} + n_0 \omega e^{\omega t} \\ &= \frac{\rho - \beta}{\Lambda} e^{\omega t} + \sum_{i=1}^m \left( \lambda_i c_i(t) + \tau \frac{\beta_i n_0}{\Lambda(\omega + \lambda_i)} \omega e^{\omega t} \right) \end{aligned} \quad (22)$$

where

$$\frac{dn}{dt} = n_0 \omega e^{\omega t} \quad (23)$$

$$\frac{d^2 n}{dt^2} = n_0 \omega^2 e^{\omega t} \quad (24)$$

$$\frac{dc_i}{dt} = \frac{\beta_i n_0}{\Lambda(\omega + \lambda_i)} \omega e^{\omega t} \quad (25)$$

Operating and rearranging terms, we obtain the P1-inhour equation

$$\rho = \Lambda \omega + \tau \left[ \Lambda \omega^2 + \left( \frac{\Lambda}{l} - 1 \right) \omega \right] + \sum_{i=1}^m \left( \frac{\beta_i \omega + \tau \beta_i \omega^2}{\omega + \lambda_i} \right) \quad (26)$$

when  $\tau \rightarrow 0$  we recovered the typical inhour equation (C-inhour equation). This equation is the reactivity equation that will provide the roots of P1 point reactor kinetics equations, which relates the parameter  $\omega$  with nuclear parameters such as  $\Lambda$ ,  $\beta$ ,  $\lambda$  and relaxation time  $\tau$ . For the last term, its value depends on the neutron diffusion coefficient.

**RESULTS AND DISCUSSIONS**

Eq. (26) represents a polynomial as a function of  $\omega$  :

$$\begin{aligned} P_{m+2} &= (\rho - \Lambda \omega) \prod_{k=1}^m (\omega + \lambda_k) \\ &- \tau \left[ \Lambda \omega^2 + \left( \frac{\Lambda}{l} - 1 \right) \omega \right] \prod_{k=1}^m (\omega + \lambda_k) \\ &- (\omega + \tau \omega^2) \sum_{j=1}^m \beta_j \prod_{\substack{k=1 \\ k \neq j}}^m (\omega + \lambda_k) \end{aligned} \quad (27)$$

To analyse the P1-inhour equation we rewrite Eq. (26), as follows:

$$\begin{aligned} &= \omega^2 + \left( \frac{\Lambda}{l} + \dots \right) \omega + \dots \\ &+ \sum_{i=1}^m \frac{\beta_i \omega + \tau \beta_i \omega^2}{\omega + \lambda_i} \end{aligned} \quad (28)$$

The analysis will be carried out considering six groups of neutron precursors, i.e.,  $m = 6$ . The nuclear parameters are:  $\lambda_i$ : 0. 0127, 0. 0317, 0. 155, 0. 311, 1. 4, 3. 87; and  $\beta_i$ : 0. 000266, 0. 001491, 0. 001316 , 0. 002849, 0. 000896, 0. 000182;  $\beta = 0.0002s$ . The behaviour of  $g(\omega) = 0$  is depicted in Figures 1 to 4, where the intersections with the  $\omega$ -axis can be observed.

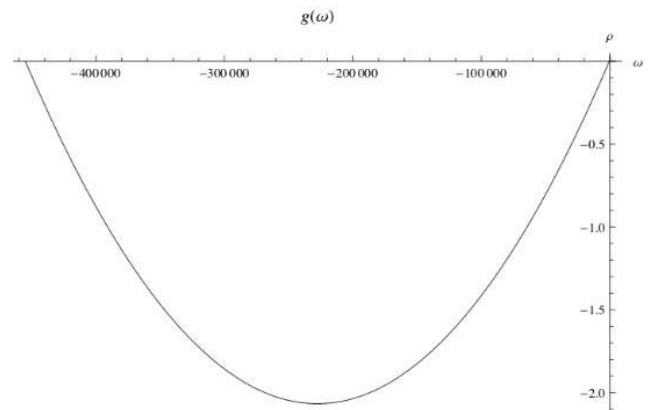


Figure-1. Reactivity with [ 55500, 1500].

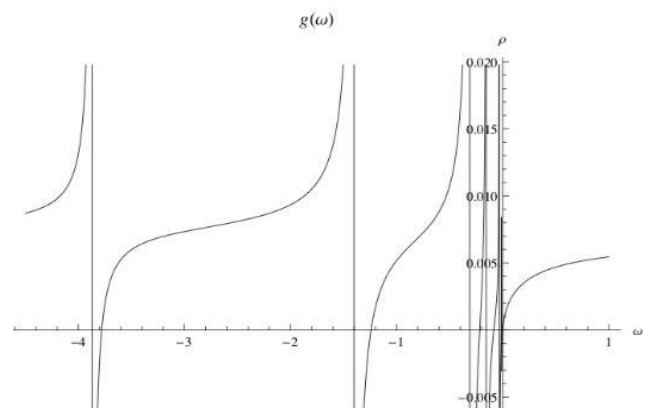


Figure-2. Reactivity with [ 4.5, 1].

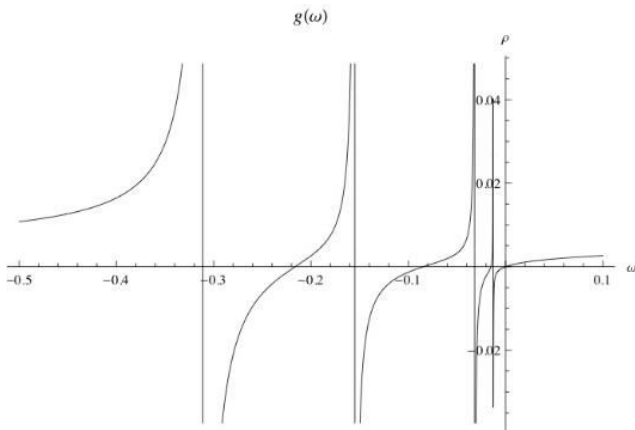


Figure-3. Reactivity with [ 0.5, 0.1].

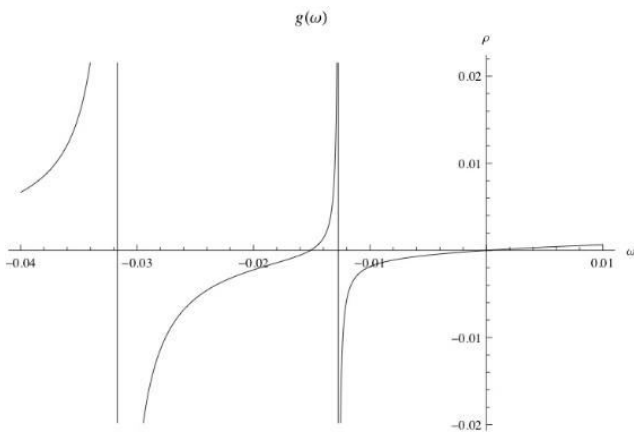


Figure-4. Reactivity with [ 0.04, 0.01].

These Figures show the distribution of the characteristic values of Eq. (28) which in turn are the roots  $\omega$  of the characteristic polynomial [Eq (27)]. It has eight roots which are distributed in the interval of [-55, 000, 1, 500] as follows: Two roots are to the left of the most negative value of  $\lambda$ , that is,  $\gamma < \omega_6 < \omega_7$ ; five roots are among the negative values of  $\lambda$  of the form  $-\lambda_6 < \omega_i < -\lambda_1$ ; and the first root is to the right of the least negative  $\lambda$  value. Then, for any group of  $m$ th neutron precursors, the behavior is illustrated in Figure-5.

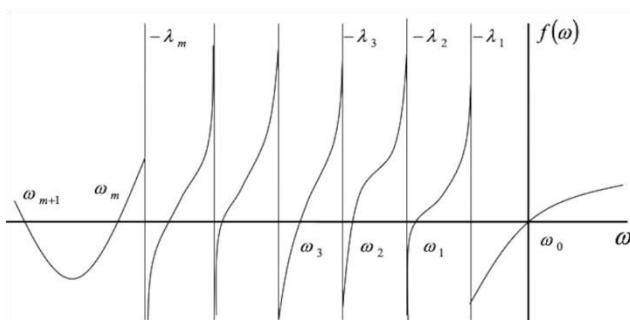


Figure-5. Behavior of Eq. (28) for  $m$  groups of delayed neutron precursors.

Eq. (28), has a unique behavior around each  $\omega = \lambda_i$ , where the term  $1/(\omega + \lambda_i)$  of Eq. (26) and the equivalent equation (28) may not be defined and changes from  $+\infty$  for  $\omega$  less than  $-\lambda_i$  to  $-\infty$  for  $\omega$  greater than  $\lambda_{i+1}$ , as shown in Figure-5. In this figure, it can be observed that for values of  $\rho > 0$  there will be a positive root and all other roots will be negative for  $\rho < 0$ . For large  $|\omega|$  values the summation term in Eq. (28) is negligible, and the  $\rho$  values asymptotic approximate a curve of the form:

$$\rho = \tau\Lambda\omega^2 + \left( \tau\frac{\Lambda}{l} + \tau\beta - \tau + \Lambda \right)\omega + \beta \quad (29)$$

which is a parabola, which in the range of [-43, 5] is a straight line, as shown in Figure-6, where you can see the intersections with the axis-  $\omega$ .

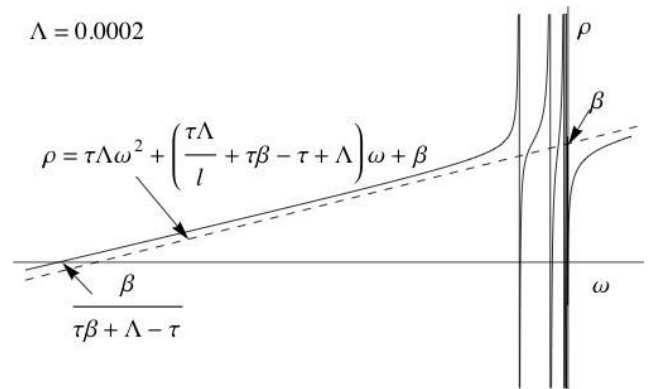
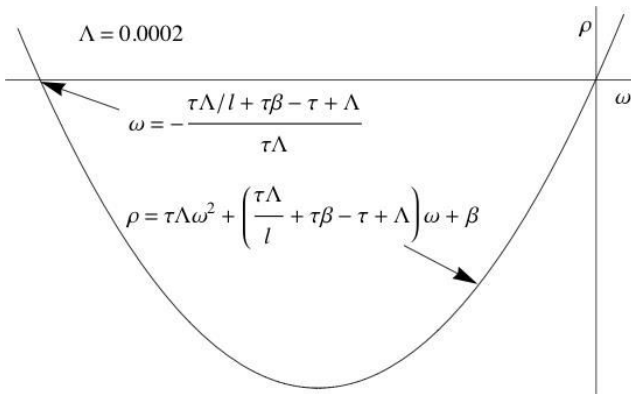


Figure-6. Zoom of the behavior of the Eq. (28)

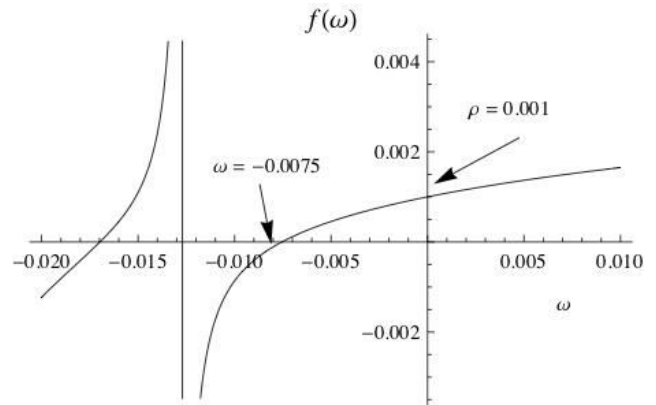
Figure-7 shows the parabola in a greater range, where the intersection with the axis-  $\omega$  are identified.

To obtain the values of  $\omega_i$  for different values of the reactivity  $\rho$ , Eq. (28) is equaled to zero and is considered as a function of the form

$$f(\omega) = \tau\Lambda\omega^2 + \left( \tau\frac{\Lambda}{l} + \tau\beta - \tau + \Lambda \right)\omega + \beta - \sum_{i=1}^m \frac{\lambda_i\beta_i\omega + \tau\lambda_i\beta_i\omega}{\omega + \lambda_i} - \rho \quad (30)$$

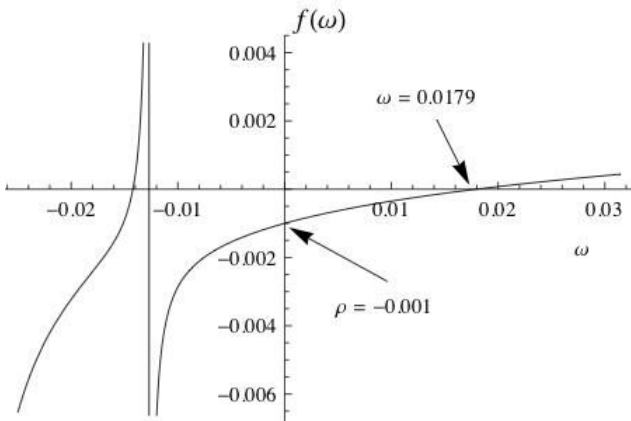


**Figure-7.** Description of the behavior of figure 1 with Eq. (28)



**Figure-9.** Eq. (30) behavior for  $\rho = 0.001$

The behavior of this equation is presented in Figures 8 and 9. In these figures, the reactivity  $\rho$  represents the ordinate to the origin. However, it can be observed in Figure-8 that it is directly intersected the axis-  $f(\omega)$  at a value of  $\rho < 0$ , i.e., if  $\rho < 0$  intersected the axis-  $f(\omega)$  in the negative part and the axis-  $\omega$  in the positive part once and the rest in the negative part.



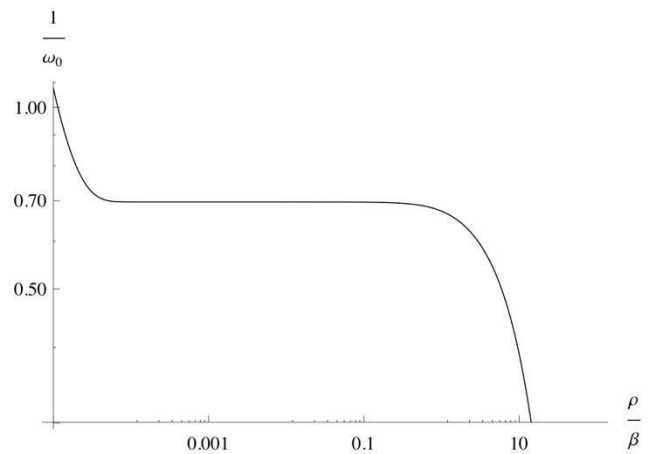
**Figure-8.** Eq. (30) behavior for  $\rho = -0.001$

It can be observed in Figure-9 that all the  $\omega$  values are negative for positive  $\rho$  values, and the intersection with the axis-  $f(\omega)$  is in the positive part.

Table-1 presents the values of  $\omega_i$  for different reactivity values, where it can be observed that all the values are real.

The least negative root represents the most stable period of the reactor (Figure-10):

$$T = \frac{1}{\omega_0} \tag{31}$$



**Figure-10.** Period behavior of a nuclear system.



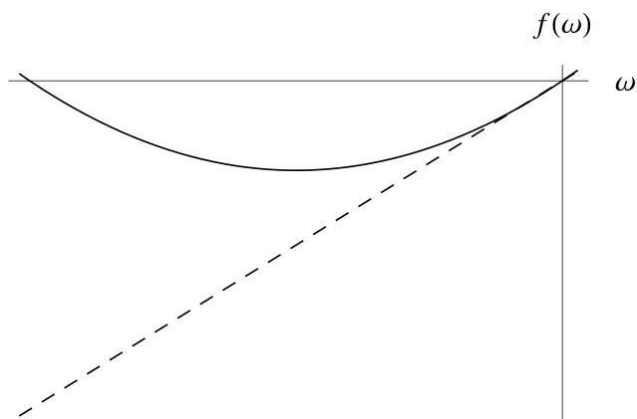
**Table-1.** Values of  $\rho_i$  for different reactivity values.

	- 0.003	- 0.002	- 0.001	0.000	0.001	0.002	0.003
0	- 0.0111	- 0.0102	- 0.0075	0.000	0.0170	0.0508	0.1102
1	- 0.0216	- 0.0194	- 0.0169	- 0.0151	- 0.0142	- 0.0138	- 0.0135
2	- 0.1121	- 0.1042	- 0.0942	- 0.0821	- 0.0692	- 0.0582	- 0.0506
3	- 0.2315	- 0.2257	- 0.2196	- 0.2132	- 0.2070	- 0.2010	- 0.1955
4	- 1.2468	- 1.2271	- 1.2024	- 1.1704	- 1.1286	- 1.0734	- 1.0015
5	- 3.6336	- 3.5491	- 3.4138	- 3.2138	- 2.9605	- 2.6854	- 2.4199
6	- 5.1108	- 4.7731	- 4.4958	- 4.2967	- 4.1702	- 4.0930	- 4.0443
7	- 54 498	- 54 499	- 54 499	- 54 500	- 54 500	- 54 501	- 54 501

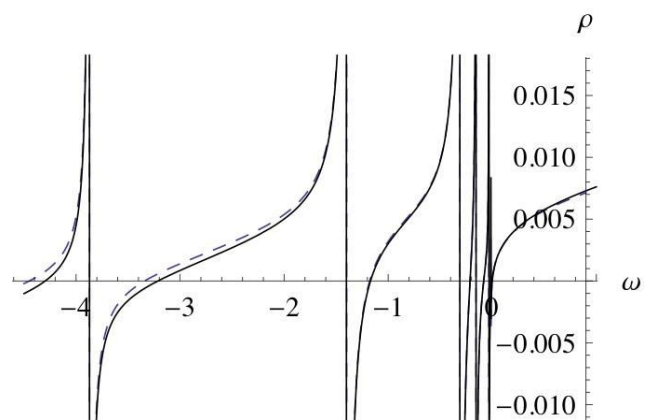
There are also notable differences in the graphs of the reactivity equations corresponding to each theory, Figures 11 and 12 show a comparison between them with the values  $\beta = 0.00$  and  $\beta = 2 \cdot 10^{-5}$ . In Figure-12, it can be observed that for large values of  $\beta$  the graph of the *C-inhour* equation tends asymptotically in a range close to zero to a line, and the *P1-inhour* equation tends to a parabola. Figure-12 shows the zooming of this behaviour in a range around zero.

**CONCLUSIONS**

This work, based on the considerations that were made in the classical theory, deduces the roots of the *P1-inhour* equation, which are the values that relate the nuclear characteristics of the system with the analytical solution of the system equation formed by Eqs. (17) and (18).



**Figure-11.** Behavior comparison of the asymptotes of the *inhour* equation corresponding to each theory in the range  $(-55500, 1500)$ .



**Figure-12.** Behavior comparison of the asymptotes of the *inhour* equation corresponding to each theory in the range  $(-4.5, 0.9)$ . The axes intersect at  $(0.9, 0)$ .

The *P1-inhour* equation allows us to observe the effects generated by the relaxation time in the neutron processes that take place inside a nuclear reactor.

A comparison is made between both theories. The solution presented in this work is called the *P1-inhour*, while the solution of the classical point neutron kinetic equations *C-inhour*. The main differences between each theory are shown for large values of  $\beta$ , where the graph of *C-inhour* equation tends asymptotically in a range close to zero to a line, and the *P1-inhour* equation tends to a parabola.

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