Strain

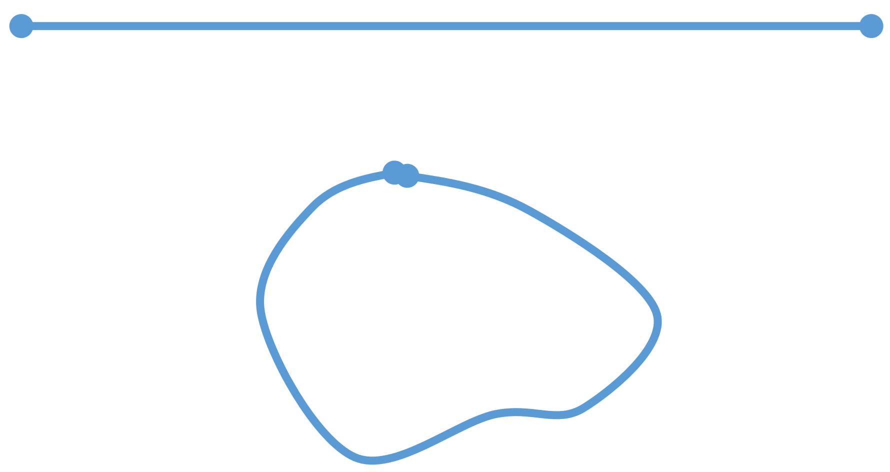
Lecture 3 – deformation, strain

Introduction

- Real bodies are ***deformable***
- Under the action of an external loading, the body transforms from a ***reference configuration*** to a ***current configuration***.
- A "configuration" is a set containing the positions of all particles of the body
- This transformation is called: **deformation**.

A deformation may be caused by:

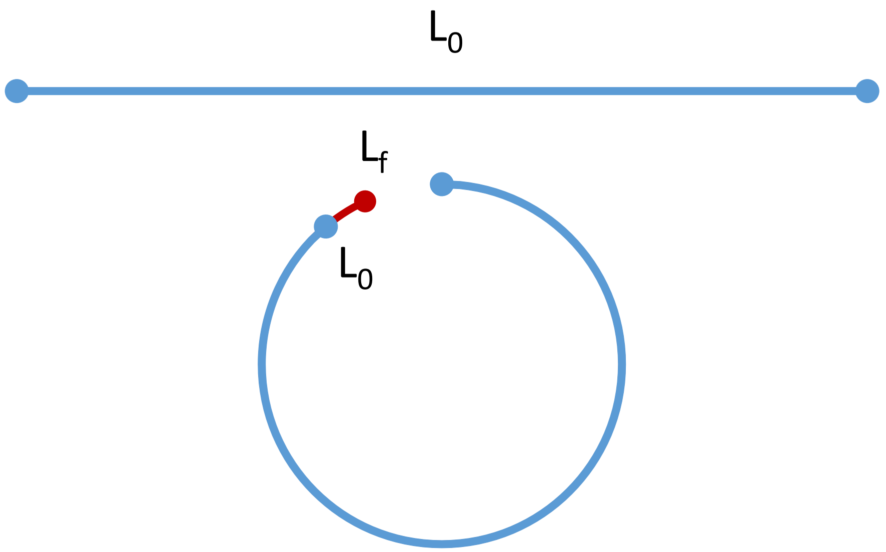
- external loads,
- body forces (such as gravity or electromagnetic forces),
- changes in temperature, moisture content, or chemical reactions, etc.



Deformation of a thin rope

Deformation vs strain

- **Strain** is a measure of deformation representing the displacement between particles in the body relative to a reference length.
- In a continuous body, a deformation field results from a stress field induced by applied forces or is due to changes in the temperature field inside the body.
- The relation between stresses and induced strains is expressed by constitutive equations, e.g., Hooke's law for linear elastic materials.



Deformation of a thin rope

Deformation and strain

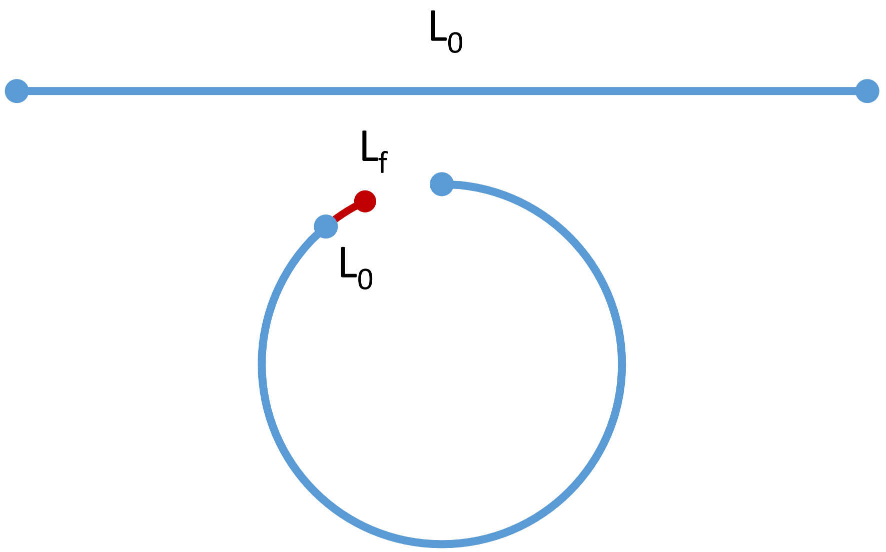
- **Strain** is a measure of deformation representing the displacement between particles in the body relative to a reference length.

- A general deformation of a body can be expressed in the form ***x=F(X)***, where ***X*** is the reference position of the body

- This definition does not distinguish between rigid body motions (translations and rotations) and changes in shape (and size) of the body

$$\boldsymbol{\varepsilon} = \frac{\partial}{\partial \boldsymbol{X}}(\boldsymbol{x} - \boldsymbol{X}) = \boldsymbol{F}' - \boldsymbol{I}$$

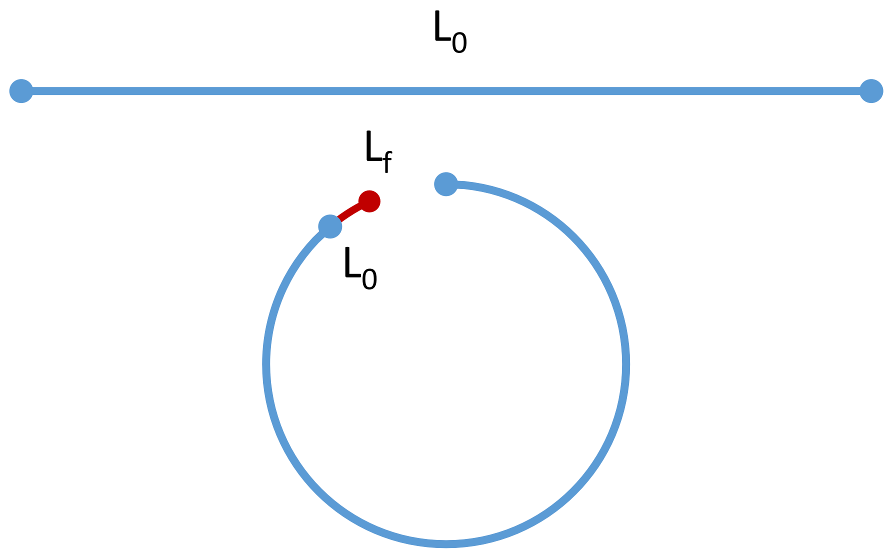
- Strains are dimensionless



Deformation of a thin rope

Strain measures

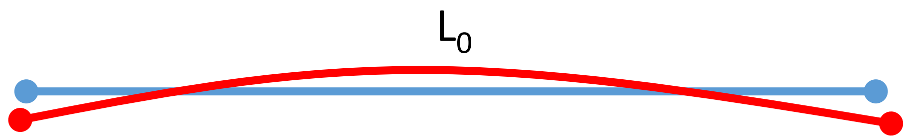
- Depending on the amount of strain, three main theories are for the analysis of deformation:
  - Infinitesimal strain theory
  - Finite strain theory
  - Large-displacement (large rotation) theory



Deformation of a thin rope

Strain measures

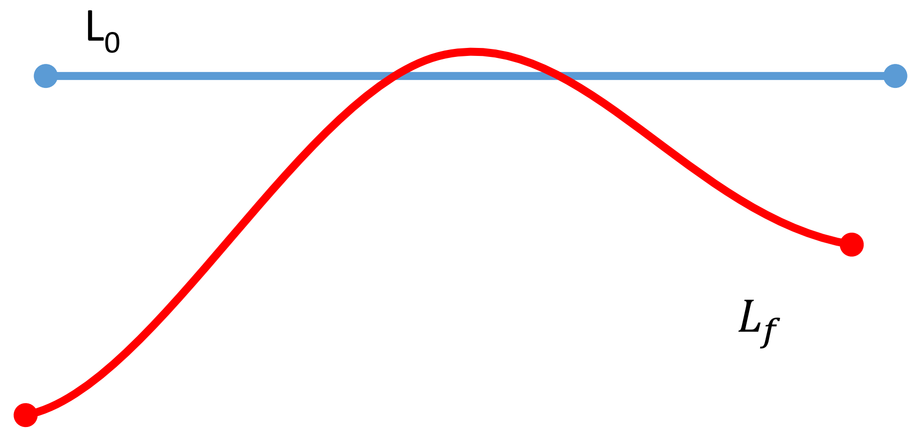
- Infinitesimal strain theory
  - Also called small strain theory, small deformation theory, small displacement theory, or small displacement-gradient theory.
  - **Strains and rotations are both small**. The undeformed and deformed configurations of the body can be assumed identical.
  - The infinitesimal strain theory is used in the analysis of deformations of materials exhibiting elastic behavior, such as materials found in mechanical and civil engineering applications, e.g. concrete and steel.



$$L_f = L_o + \Delta L$$

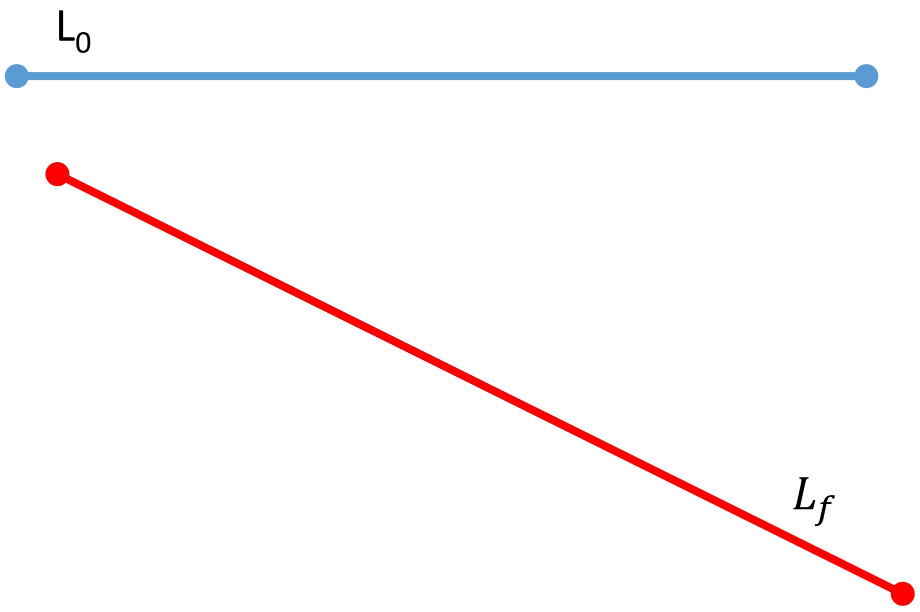
Strain measures

- Finite strain theory
  - also called large strain theory, large deformation theory
  - Deals with deformations in which both **rotations** and **strains** are arbitrarily large. In this case, the undeformed and deformed configurations of the continuum are significantly different and a clear distinction has to be made between them.
  - This is commonly the case with elastomers, plastically-deforming materials and other fluids and biological soft tissue.



Strain measures

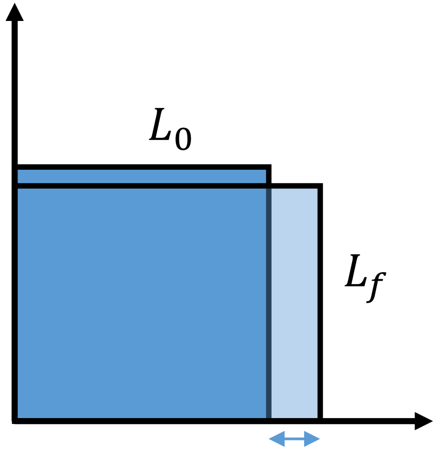
- Large strain theory
    - *Large-displacement* or *large-rotation theory*, which assumes small strains but large rotations and displacements.



Strain measures

- In each theory the definition of strain is different

- **Engineering strain.** The Cauchy strain or engineering strain is expressed as the ratio of total deformation to the initial dimension of the material body in which the forces are being applied.



  - **Engineering normal strain** of a material line element or fiber axially loaded is given by the change in length ΔL per unit of the original length L of the line element or fibers. The normal strain is positive if the material fibers are stretched and negative if they are compressed.

$$\varepsilon = \frac{L_f - L_0}{L_0}$$

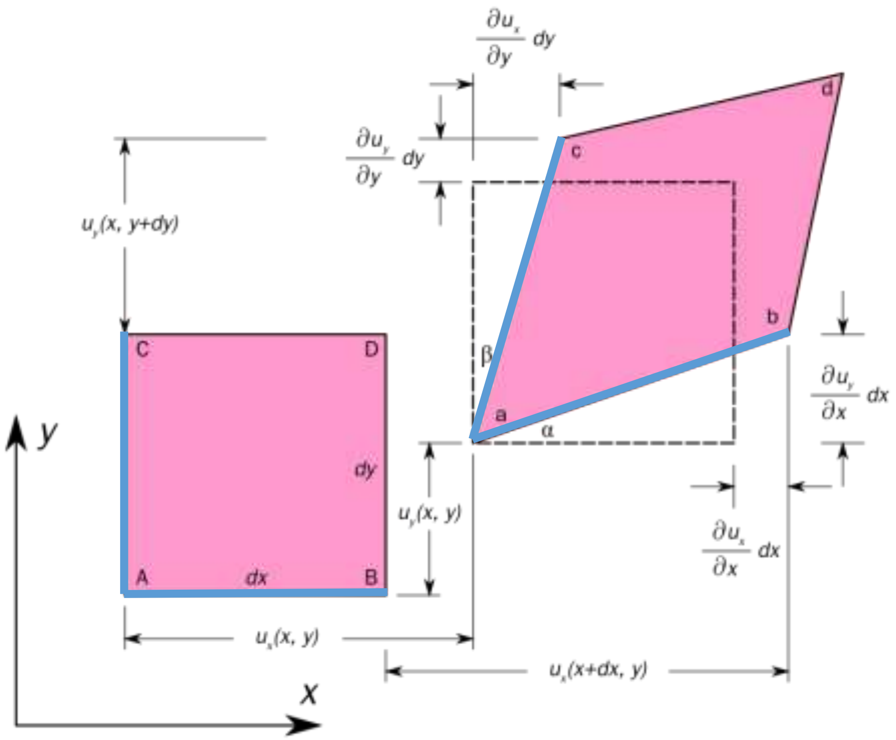
$$L_f \approx dx + \frac{\partial u}{\partial x} dx$$

Strain measures

- In each theory the definition of strain is different

- **Engineering strain.** The Cauchy strain or engineering strain is expressed as the ratio of total deformation to the initial dimension of the material body in which the forces are being applied.

    - **Engineering shear strain** is defined as the change in angle between lines *AC* and *AB*.



$$\gamma_{xy} = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}$$

Strain measures

- Different definitions

- Engineering strain: $\varepsilon_{eng} = \frac{\Delta l}{l_0}$

- True engineering strain $\varepsilon = \int \frac{dl}{l} = ln\left(1 + \frac{\Delta l}{l_0}\right)$

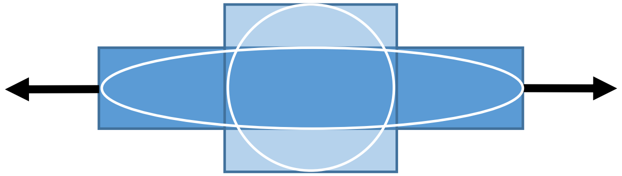
- Green strain $\varepsilon_G = \frac{1}{2}\left(\frac{l^2 - l_0^2}{l_0^2}\right)$

- Almansi strain $\varepsilon_G = \frac{1}{2}\left(\frac{l^2 - l_0^2}{l^2}\right)$

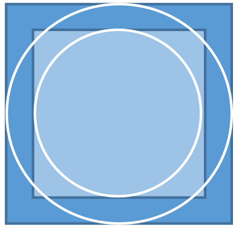
Examples of homogeneous deformation

Straight lines remain straight, parallel lines remains parallel

- uniform extension

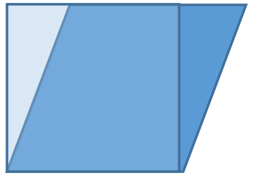


- pure dilation

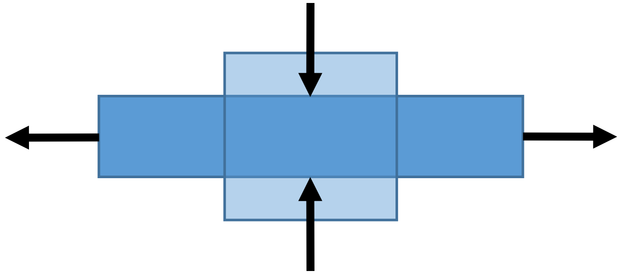


$$\varepsilon = \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{bmatrix}$$

- simple shear



- pure shear



Strain tensor

STRAIN TENSOR (infinitesimal deformations). A tridimensional deformation state is described by the strain tensor ε

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$

Same properties as for the stress tensor: strain invariants

$$I_1 = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

$$I_2 = \varepsilon_{11}\varepsilon_{22} + \varepsilon_{22}\varepsilon_{33} + \varepsilon_{33}\varepsilon_{11} - \varepsilon_{12}^2 - \varepsilon_{23}^2 - \varepsilon_{31}^2$$

$$I_3 = det\left|\varepsilon_{ij}\right|$$

Strain tensor

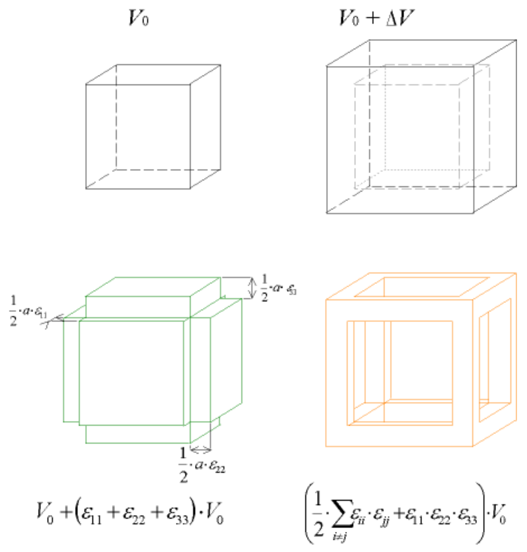
PRINCIPAL STRAIN. At every point in a deformed body there are at least three planes, called principal planes, with normal vectors, called principal directions, where the corresponding strain vector is perpendicular to the plane and where there are no shear strain.

The three strain normal to these principal planes are called **principal strains**.

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix}$$

VOLUMETRIC STRAIN. The *dilatation* (the relative variation of the volume) is the trace of the tensor:

$$\varepsilon_{kk} = \delta = \frac{\Delta V}{V_0} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$



Strain tensor

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$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix}$$

STRAIN DEVIATOR. The infinitesimal strain tensor, similarly to the Cauchy stress tensor, can be expressed as the sum of two other tensors. The strain deviator account for *distortion*.

$$\varepsilon_{ij} = \varepsilon'_{ij} + \frac{1}{3}\varepsilon_{kk}\delta_{ij}$$

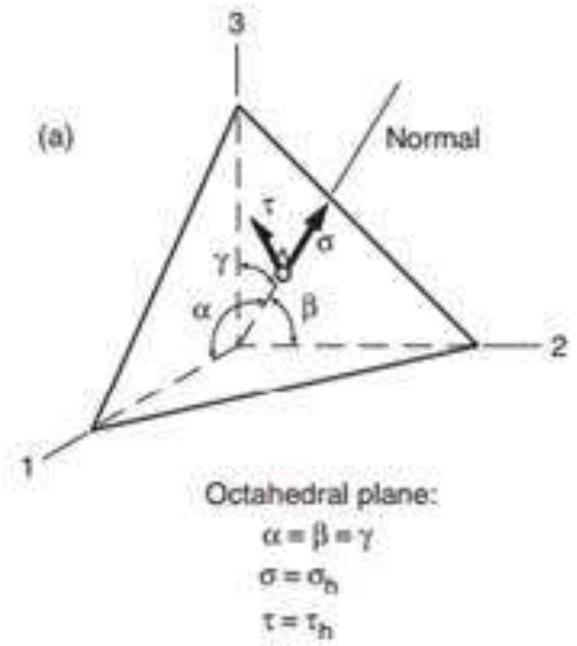
$$\varepsilon'_{ij} =\equiv \begin{bmatrix} \varepsilon_{11} - \delta & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{y} - \delta & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{z} - \delta \end{bmatrix}$$

Strain tensor

OCTAHEDRAL STRAINS. An octahedral plane is one whose normal makes **equal angles with** the three principal directions.

The engineering shear strain on an octahedral plane is called the octahedral shear strain and is given by

$$\gamma_{oct} = \frac{2}{3}\sqrt{(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2}$$



Strain tensor: special cases

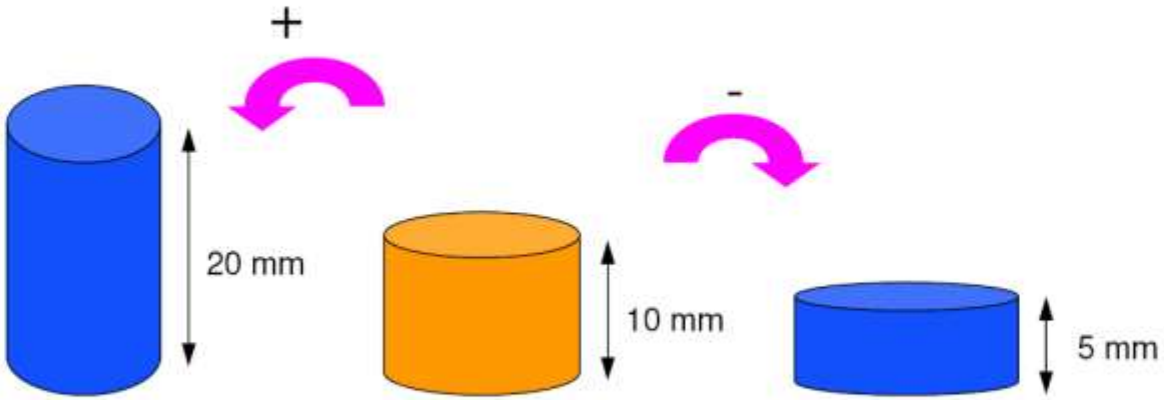
PLANE STRAIN. The plane strain condition is when there are no shear strain in any plane normal to the plane considered and the strain component normal to such plane is zero

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{21} & \varepsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

ANTIPLANE STRAIN. This state of strain is achieved when the displacements in the body are zero in the plane of interest but nonzero in the direction perpendicular to the plane.

$$\varepsilon_{ij} = \begin{bmatrix} 0 & 0 & \varepsilon_{13} \\ 0 & 0 & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & 0 \end{bmatrix}$$

Symmetry and additivity



$$e_1 = \frac{20-10}{10} = 1. \Rightarrow 100\%$$

$$e_2 = \frac{5-10}{10} = -0.5 \Rightarrow -50\%$$

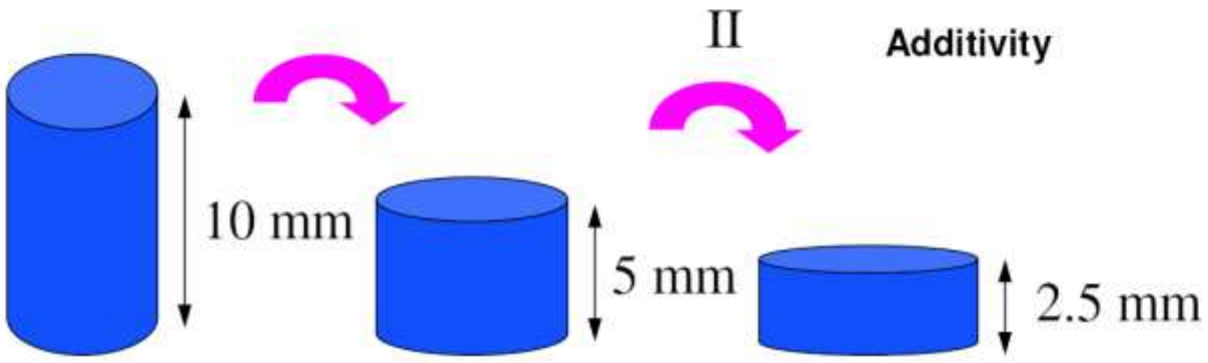
$$\bar{\varepsilon}_1 = Ln\left(\frac{20}{10}\right) = 0.693 \approx 0.7$$

$$\bar{\varepsilon}_2 = Ln\left(\frac{5}{10}\right) = -0.693 \approx -0.7$$

Symmetry and additivity

$$\bar{\varepsilon} = Ln\left(\frac{l}{l_0}\right)$$

$$e = \frac{\Delta l}{l_0}$$



$$e_1 = \frac{5-10}{10} = -0.5 \qquad e_2 = \frac{2.5-5}{5} = -0.5 \qquad \bar{\varepsilon}_1 = Ln\left(\frac{5}{10}\right) = -0.693 \qquad \bar{\varepsilon}_2 = Ln\left(\frac{2.5}{5}\right) = -0$$

$$e_{tot} = \frac{2.5-10}{10} = -0.75 \qquad e_1 + e_2 = -1.0 \qquad \bar{\varepsilon}_{tot} = Ln\left(\frac{2.5}{10}\right) = -1.3863 \qquad \bar{\varepsilon}_1 + \bar{\varepsilon}_2 = -1.3863$$

$$e_1 + e_2 \neq e_{tot} \qquad \bar{\varepsilon}_1 + \bar{\varepsilon}_2 = \bar{\varepsilon}_{tot}$$

STRAIN RATE

Engineering strain rate

$$\dot{\varepsilon}_{eng} = \frac{d\varepsilon_{eng}}{dt} = \frac{d\left(\frac{l_f - l_0}{l_0}\right)}{dt} = \frac{1}{l_0}\frac{dl}{dt} = \frac{v}{l_0}$$

True strain rate

$$\dot{\varepsilon} == \frac{d\left[ln\left(\frac{l}{l_0}\right)\right]}{dt} = \frac{1}{l}\frac{dl}{dt} = \frac{v}{l}$$

Suggested readings

- http://www.mech.utah.edu/~brannon/public/Mohrs_Circle.pdf
- Schaum's Outline of Strength of Materials, Fifth Edition (Schaum's Outline Series) Fifth (5th) Edition Paperback – September 12, 2010
- Strength of Materials (Dover Books on Physics) Reprinted Edition by J. P. Den Hartog, ISBN-10: 0486607550