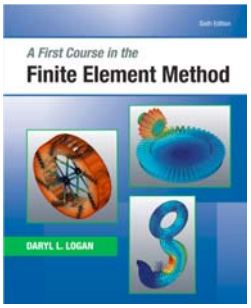
Chapter 9 – Axisymmetric Elements



Learning Objectives

- To review the basic concepts and theory of elasticity equations for axisymmetric behavior.
- To derive the axisymmetric element stiffness matrix, body force, and surface traction equations.
- To demonstrate the solution of an axisymmetric pressure vessel using the stiffness method.
- To compare the finite element solution to an exact solution for a cylindrical pressure vessel.
- To illustrate some practical applications of axisymmetric elements.

Axisymmetric Elements

Introduction

In previous chapters, we have been concerned with line or one-dimensional elements (Chapters 2 through 5) and two-dimensional elements (Chapters 6 through 8).

In this chapter, we consider a special two-dimensional element called the ***axisymmetric element.***

This element is quite useful when symmetry with respect to geometry and loading exists about an axis of the body being analyzed.

Problems that involve soil masses subjected to circular footing loads or thick-walled pressure vessels can often be analyzed using the element developed in this chapter.

Axisymmetric Elements

Introduction

We begin with the development of the stiffness matrix for the simplest axisymmetric element, the triangular torus, whose vertical cross section is a plane triangle.

We then present the longhand solution of a thick-walled pressure vessel to illustrate the use of the axisymmetric element equations.

This is followed by a description of some typical large-scale problems that have been modeled using the axisymmetric element.

Axisymmetric Elements

Derivation of the Stiffness Matrix

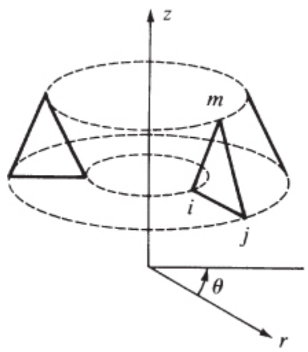
In this section, we will derive the stiffness matrix and the body and surface force matrices for the axisymmetric element.

However, before the development, we will first present some fundamental concepts prerequisite to the understanding of the derivation.

Axisymmetric Elements

Derivation of the Stiffness Matrix

Axisymmetric elements are triangular tori such that each element is symmetric with respect to geometry and loading about an axis such as the *z* axis.



Hence, the *z* axis is called the *axis of symmetry* or the *axis of revolution.*

Each vertical cross section of the element is a plane triangle.

The nodal points of an axisymmetric triangular element describe circumferential lines.

Axisymmetric Elements

Derivation of the Stiffness Matrix

In plane stress problems, stresses exist only in the *x-y* plane.

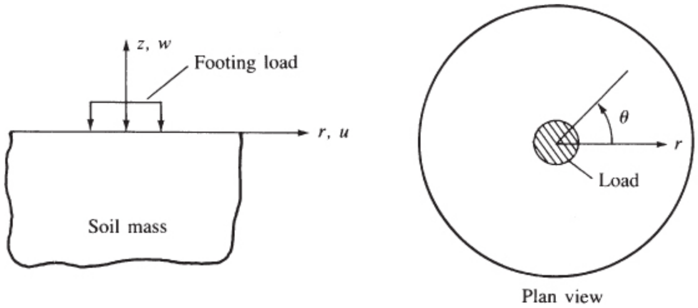
In axisymmetric problems, the radial displacements develop circumferential strains that induce stresses σ_r, σ_θ, σ_z and τ_{rz} where *r*, θ, and *z* indicate the radial, circumferential, and longitudinal directions, respectively.

Triangular torus elements are often used to idealize the axisymmetric system because they can be used to simulate complex surfaces and are simple to work with.

Axisymmetric Elements

Derivation of the Stiffness Matrix

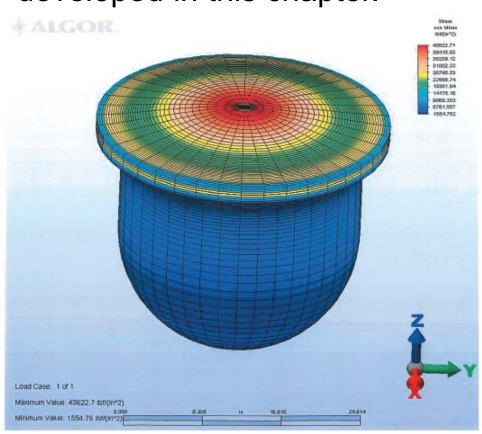
For instance, the axisymmetric problem of a semi-infinite half-space loaded by a circular area (circular footing) can be solved using the axisymmetric element developed in this chapter.

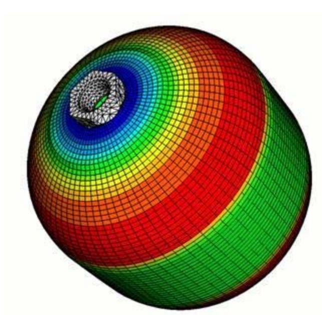


Axisymmetric Elements

Derivation of the Stiffness Matrix

For instance, the axisymmetric problem of a domed pressure vessel can be solved using the axisymmetric element developed in this chapter.

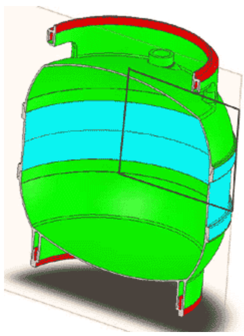


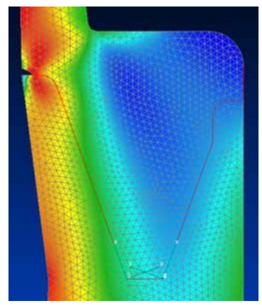


Axisymmetric Elements

Derivation of the Stiffness Matrix

For instance, the axisymmetric problem of stresses acting on the barrel under an internal pressure loading.

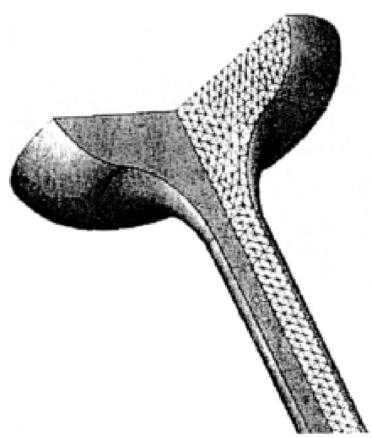




Axisymmetric Elements

Derivation of the Stiffness Matrix

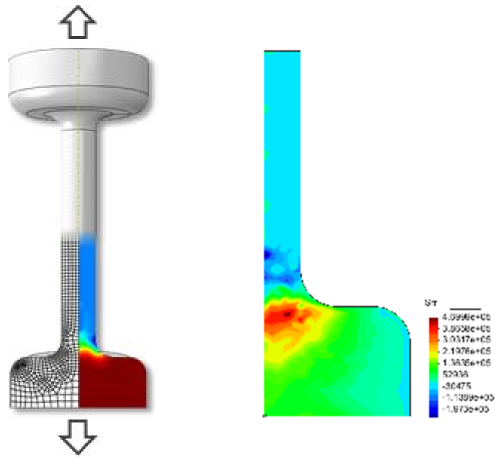
For instance, the axisymmetric problem of an engine valve stem can be solved using the axisymmetric element developed in this chapter.



Axisymmetric Elements

Derivation of the Stiffness Matrix

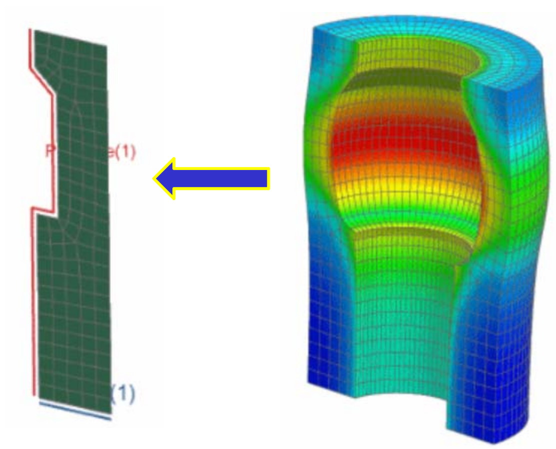
For instance, an axisymmetric specimen loaded under tension-compression.



Axisymmetric Elements

Derivation of the Stiffness Matrix

An axisymmetric domain.



Axisymmetric Elements

Derivation of the Stiffness Matrix

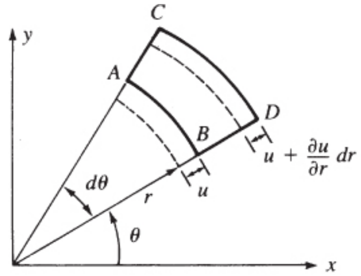
Because of symmetry about the *z* axis, the stresses are independent of the θ coordinate.

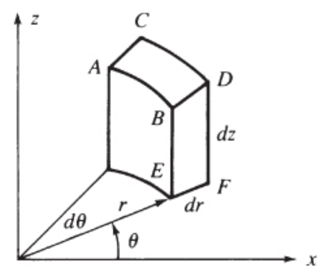
Therefore, all derivatives with respect to θ vanish, and the displacement component *v* (tangent to the θ direction), the shear strains $\gamma_{r\theta}$ and $\gamma_{\theta z}$ and the shear stresses $\tau_{r\theta}$ and $\tau_{\theta z}$ are all zero.

Axisymmetric Elements

Derivation of the Stiffness Matrix

Consider an axisymmetric ring element and its cross section to represent the general state of strain for an axisymmetric problem.



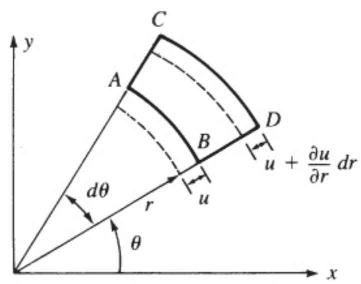


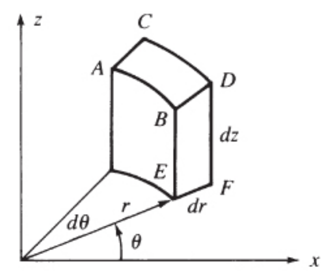
Axisymmetric Elements

Derivation of the Stiffness Matrix

The displacements can be expressed for element *ABCD* in the plane of a cross-section in cylindrical coordinates.

We then let *u* and *w* denote the displacements in the radial and longitudinal directions, respectively.



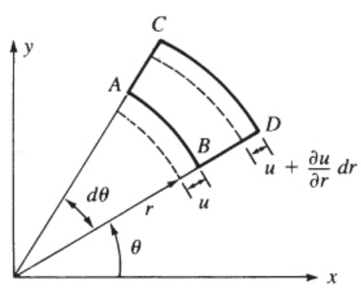


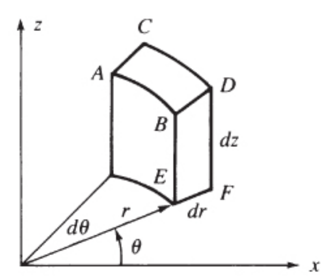
Axisymmetric Elements

Derivation of the Stiffness Matrix

The side *AB* of the element is displaced an amount *u,* and side *CD* is then displaced an amount *u* + (∂*u*/ ∂*r*) in the radial direction.

The normal strain in the radial direction is then given by: $\varepsilon_r = \dfrac{\partial u}{\partial r}$



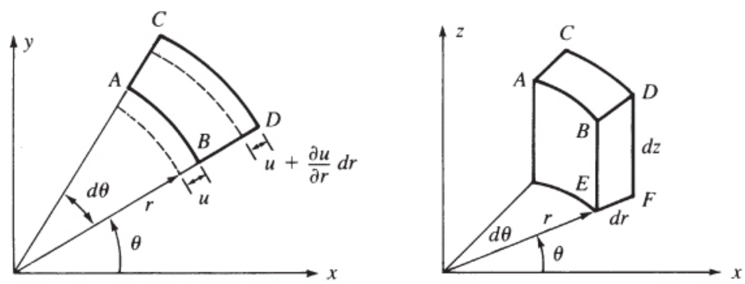


Axisymmetric Elements

Derivation of the Stiffness Matrix

The strain in the tangential direction depends on the tangential displacement *v* and on the radial displacement *u.*

However, for axisymmetric deformation behavior, recall that the tangential displacement *v* is equal to zero.



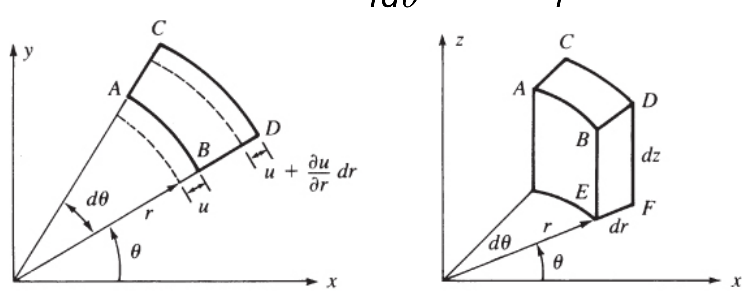
Axisymmetric Elements

Derivation of the Stiffness Matrix

The tangential strain is due only to the radial displacement.

Having only radial displacement *u,* the new length of the arc *AB* is $(r + u)d\theta$, and the tangential strain is then given by:

$$\varepsilon_\theta = \frac{(r+u)d\theta - rd\theta}{rd\theta} = \frac{u}{r}$$



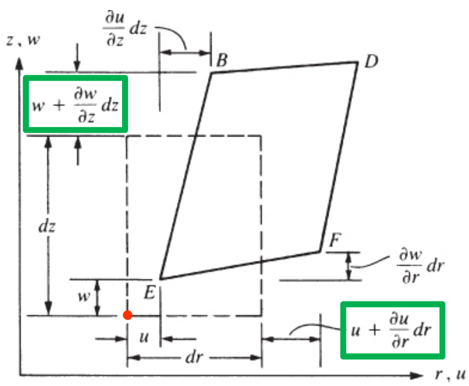
Axisymmetric Elements

Derivation of the Stiffness Matrix

Consider the longitudinal element *BDEF to* obtain the longitudinal strain and the shear strain.

The element displaces by amounts *u* and *w* in the radial and longitudinal directions at point *E.*

The element displaces additional amounts: $(\partial w/\partial z)dz$ along line *BE* and $(\partial u/\partial r)dr$ along line *EF.*



Axisymmetric Elements

Derivation of the Stiffness Matrix

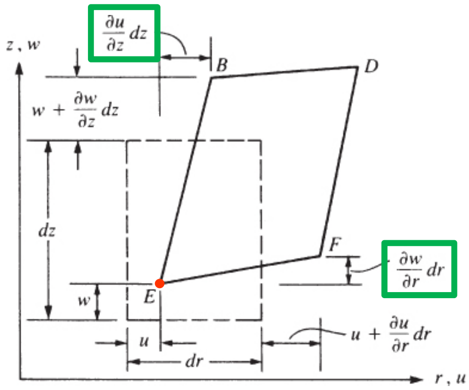
Furthermore, observing lines *EF* and *BE,* we see that point F moves upward an amount $(\partial w/\partial r)dr$ with respect to point *E* and point *B* moves to the right an amount $(\partial u/\partial z)dz$ with respect to point *E.*

The longitudinal normal strain is given by:

$$\varepsilon_z = \frac{\partial w}{\partial z}$$

The shear strain in the *r-z* plane is:

$$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$



Axisymmetric Elements

Derivation of the Stiffness Matrix

Summarizing the strain-displacement relationships gives:

$$\varepsilon_r = \frac{\partial u}{\partial r} \qquad \varepsilon_\theta = \frac{u}{r} \qquad \varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

The isotropic stress-strain relationship, obtained by simplifying the general stress-strain relationships, is:

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 & 0 \\ \nu & 1-\nu & 0 & 0 \\ 0 & 0 & 1-\nu & 0 \\ 0 & 0 & 0 & 0.5-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix}$$

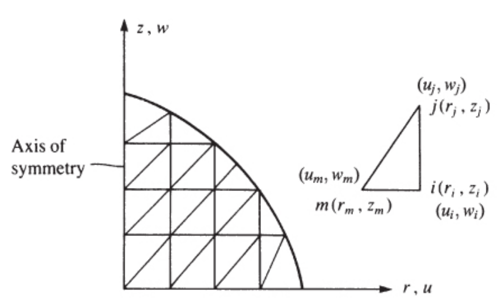
Axisymmetric Elements

Derivation of the Stiffness Matrix

The procedure to derive the element stiffness matrix and element equations is identical to that used for the plane-stress in Chapter 6.

Step 1 - Discretize and Select Element Types

An axisymmetric solid is shown discretized below, along with a typical triangular element.



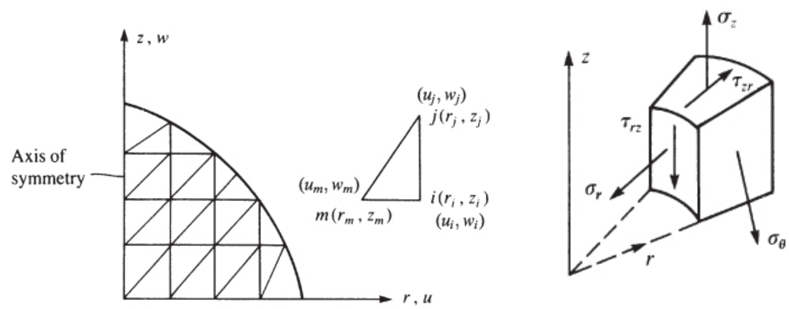
Axisymmetric Elements

Derivation of the Stiffness Matrix

The procedure to derive the element stiffness matrix and element equations is identical to that used for the plane-stress in Chapter 6.

Step 1 - Discretize and Select Element Types

The stresses in the axisymmetric problem are:



Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 2 - Select Displacement Functions

The element displacement functions are taken to be:

$$u(r,z) = a_1 + a_2 r + a_3 z$$

$$w(r,z) = a_4 + a_5 r + a_6 z$$

The nodal displacements are:

$$\{d\} = \begin{Bmatrix} d_i \\ d_j \\ d_m \end{Bmatrix} = \begin{Bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{Bmatrix}$$

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 2 - Select Displacement Functions

The function *u* evaluated at node *i* is: $u(r_i, z_i) = a_1 + a_2 r_i + a_3 z_i$

The general displacement function is then expressed in matrix form as:

$$\{\Psi_i\} = \begin{Bmatrix} a_1 + a_2 r + a_3 z \\ a_4 + a_5 r + a_6 z \end{Bmatrix} = \begin{bmatrix} 1 & r & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & r & z \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 2 - Select Displacement Functions

By substituting the coordinates of the nodal points into the equation we can solve for the ***a***'s*:*

$$\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix} = \begin{bmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_m & z_m \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \qquad \Rightarrow \quad \{a\} = [x]^{-1}\{u\}$$

$$\begin{Bmatrix} w_i \\ w_j \\ w_m \end{Bmatrix} = \begin{bmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_m & z_m \end{bmatrix} \begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix} \qquad \Rightarrow \quad \{a\} = [x]^{-1}\{w\}$$

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 2 - Select Displacement Functions

Performing the inversion operations we have:

$$[x]^{-1} = \frac{1}{2A}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \qquad 2A = \begin{vmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_m & z_m \end{vmatrix}$$

$$2A = r_i\left(z_j - z_m\right) + r_j\left(z_m - z_i\right) + r_m\left(z_i - z_j\right)$$

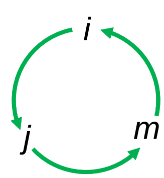
where A is the area of the triangle

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 2 - Select Displacement Functions

$$[x]^{-1} = \frac{1}{2A}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}$$



$$\alpha_i = r_j z_m - z_j r_m \qquad \beta_i = z_j - z_m \qquad \gamma_i = r_m - r_j$$

$$\alpha_j = r_m z_i - z_m r_i \qquad \beta_j = z_m - z_i \qquad \gamma_j = r_i - r_m$$

$$\alpha_m = r_i z_j - z_i r_j \qquad \beta_m = z_i - z_j \qquad \gamma_m = r_j - r_i$$

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 2 - Select Displacement Functions

The values of ***a*** may be written matrix form as:

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}$$

$$\begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \begin{Bmatrix} w_i \\ w_j \\ w_m \end{Bmatrix}$$

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 2 - Select Displacement Functions

Expanding the above equations:

$$\{u\} = \{1 \quad r \quad z\} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

Substituting the values for ***a*** into the above equation gives:

$$\{u\} = \frac{1}{2A} [1 \quad r \quad z] \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}$$

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 2 - Select Displacement Functions

We will now derive the *u* displacement function in terms of the coordinates *r* and *z*.

$$\{u\} = \frac{1}{2A}\begin{bmatrix} 1 & r & z \end{bmatrix}\begin{bmatrix} \alpha_i u_i + \alpha_j u_j + \alpha_m u_m \\ \beta_i u_i + \beta_j u_j + \beta_m u_m \\ \gamma_i u_i + \gamma_j u_j + \gamma_m u_m \end{bmatrix}$$

Multiplying the matrices in the above equations gives:

$$u(r,z) = \frac{1}{2A}\Big\{\left(\alpha_i + \beta_i r + \gamma_i z\right)u_i + \left(\alpha_j + \beta_j r + \gamma_j z\right)u_j + \left(\alpha_m + \beta_m r + \gamma_m z\right)u_m\Big\}$$

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 2 - Select Displacement Functions

We will now derive the *w* displacement function in terms of the coordinates *r* and *z*.

$$\{w\} = \frac{1}{2A}\begin{bmatrix} 1 & r & z \end{bmatrix}\begin{bmatrix} \alpha_i w_i + \alpha_j w_j + \alpha_m w_m \\ \beta_i w_i + \beta_j w_j + \beta_m w_m \\ \gamma_i w_i + \gamma_j w_j + \gamma_m w_m \end{bmatrix}$$

Multiplying the matrices in the above equations gives:

$$w(r,z) = \frac{1}{2A}\Big\{\left(\alpha_i + \beta_i r + \gamma_i z\right)w_i + \left(\alpha_j + \beta_j r + \gamma_j z\right)w_j + \left(\alpha_m + \beta_m r + \gamma_m z\right)w_m\Big\}$$

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 2 - Select Displacement Functions

The displacements can be written in a more convenience form as:

$$u(r,z) = N_i u_i + N_j u_j + N_m u_m$$

$$w(r,z) = N_i w_i + N_j w_j + N_m w_m$$

where:

$$N_i = \frac{1}{2A}\left(\alpha_i + \beta_i r + \gamma_i z\right)$$

$$N_j = \frac{1}{2A}\left(\alpha_j + \beta_j r + \gamma_j z\right)$$

$$N_m = \frac{1}{2A}\left(\alpha_m + \beta_m r + \gamma_m z\right)$$

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 2 - Select Displacement Functions

The elemental displacements can be summarized as:

$$\left\{\Psi_i\right\} = \begin{Bmatrix} u(r,z) \\ w(r,z) \end{Bmatrix} = \begin{Bmatrix} N_i u_i + N_j u_j + N_m u_m \\ N_i w_i + N_j w_j + N_m w_m \end{Bmatrix}$$

$$\{\Psi\} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{Bmatrix}$$

$$\{\Psi\} = [N]\{d\}$$

Axisymmetric Elements

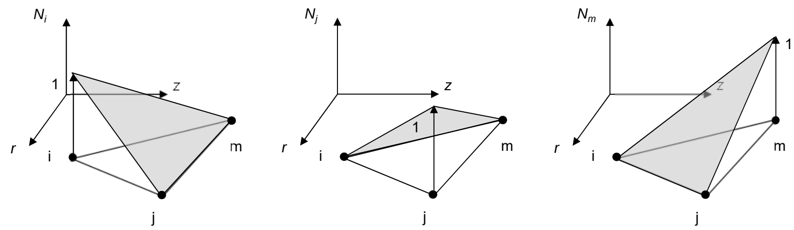
Derivation of the Stiffness Matrix

Step 2 - Select Displacement Functions

In another form the equations are:

$$[N] = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix}$$

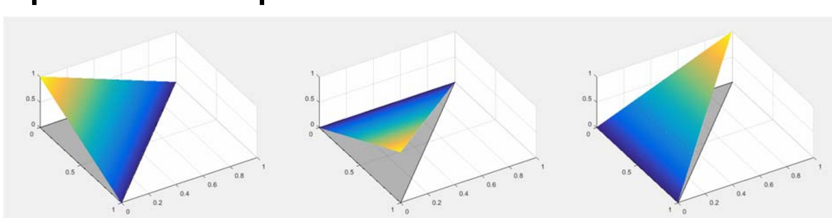
The linear triangular shape functions are illustrated below:



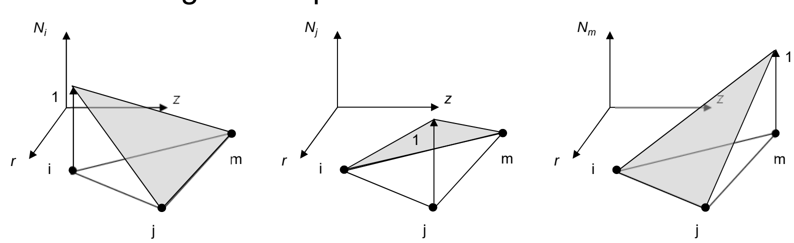
Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 2 - Select Displacement Functions



The linear triangular shape functions are illustrated below:



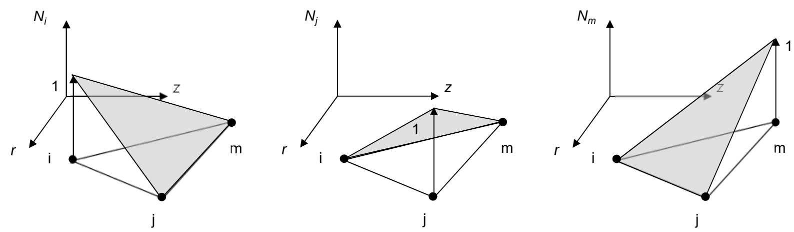
Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 2 - Select Displacement Functions

So that *u* and *w* will yield a constant value for rigid-body displacement, $N_i + N_j + N_m = 1$ for all *r* and *z* locations on the element.

The linear triangular shape functions are illustrated below:



Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 2 - Select Displacement Functions

So that *u* and *w* will yield a constant value for rigid-body displacement, $N_i + N_j + N_m = 1$ for all *r* and *z* locations on the element.

For example, assume all the triangle displaces as a rigid body in the *x* direction: $u = u_0$

$$\{\Psi\} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} u_0 \\ 0 \\ u_0 \\ 0 \\ u_0 \\ 0 \end{Bmatrix}$$

$$u_0 = u_0\left(N_i + N_j + N_m\right)$$

$$\Rightarrow N_i + N_j + N_m = 1$$

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 2 - Select Displacement Functions

So that *u* and *w* will yield a constant value for rigid-body displacement, $N_i + N_j + N_m = 1$ for all *r* and *z* locations on the element.

For example, assume all the triangle displaces as a rigid body in the *z* direction: $w = w_0$

$$\{\Psi\} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} 0 \\ w_0 \\ 0 \\ w_0 \\ 0 \\ w_0 \end{Bmatrix}$$

$$w_0 = w_0\left(N_i + N_j + N_m\right)$$

$$\Rightarrow N_i + N_j + N_m = 1$$

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

Elemental Strains*:* The strains over a two-dimensional element are:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{u}{r} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial r} \end{Bmatrix} = \begin{Bmatrix} a_2 \\ a_6 \\ \dfrac{a_1}{r} + a_2 + \dfrac{a_3 z}{r} \\ a_3 + a_5 \end{Bmatrix}$$

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

***Elemental Strains**:* The strains over a two-dimensional element are:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{u}{r} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial r} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \dfrac{1}{r} & 1 & \dfrac{z}{r} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

Substituting our approximation for the displacement gives:

$$\frac{\partial u}{\partial r} = u_{,r} = \frac{\partial}{\partial r}\left(N_i u_i + N_j u_j + N_m u_m\right)$$

$$u_{,r} = N_{i,r} u_i + N_{j,r} u_j + N_{m,r} u_m$$

where the comma indicates differentiation with respect to that variable.

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

The derivatives of the interpolation functions are:

$$N_{i,r} = \frac{1}{2A}\frac{\partial}{\partial r}\left(\alpha_i + \beta_i r + \gamma_i z\right) = \frac{\beta_i}{2A}$$

$$N_{j,r} = \frac{\beta_j}{2A} \qquad N_{m,r} = \frac{\beta_m}{2A}$$

Therefore:

$$\frac{\partial u}{\partial r} = \frac{1}{2A}\left(\beta_i u_i + \beta_j u_j + \beta_m u_m\right)$$

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

In a similar manner, the remaining strain terms are approximated as:

$$\frac{\partial w}{\partial z} = \frac{1}{2A}\left(\gamma_i w_i + \gamma_j w_j + \gamma_m w_m\right)$$

$$\frac{u}{r} = \frac{1}{2A}\left[\left(\frac{\alpha_i}{r} + \beta_i + \frac{\gamma_i z}{r}\right)u_i + \left(\frac{\alpha_j}{r} + \beta_j + \frac{\gamma_j z}{r}\right)u_j + \left(\frac{\alpha_m}{r} + \beta_m + \frac{\gamma_m z}{r}\right)u_m\right]$$

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = \frac{1}{2A}\left(\beta_i u_i + \gamma_i w_i + \beta_j u_j + \gamma_j w_j + \beta_m u_m + \gamma_m w_m\right)$$

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

We can write the strains in matrix form as:

$$\begin{Bmatrix}\varepsilon_r\\ \varepsilon_z\\ \varepsilon_\theta\\ \gamma_{rz}\end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \frac{\alpha_i}{r}+\beta_i+\frac{\gamma_i z}{r} & 0 & \frac{\alpha_j}{r}+\beta_j+\frac{\gamma_j z}{r} & 0 & \frac{\alpha_m}{r}+\beta_m+\frac{\gamma_m z}{r} & 0 \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}\begin{Bmatrix}u_i\\ v_i\\ u_j\\ v_j\\ u_m\\ v_m\end{Bmatrix}$$

$$\{\varepsilon\} = \begin{bmatrix} B_i & B_j & B_m \end{bmatrix}\begin{Bmatrix}d_i\\ d_j\\ d_m\end{Bmatrix}$$

$$\{\varepsilon\} = [B]\{d\}$$

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

We can write the strains in matrix form as:

$$\{\varepsilon\} = \begin{Bmatrix}\varepsilon_r\\ \varepsilon_z\\ \varepsilon_\theta\\ \gamma_{rz}\end{Bmatrix} = \begin{bmatrix} [B_i] & [B_j] & [B_m] \end{bmatrix}\begin{Bmatrix}u_i\\ w_i\\ u_j\\ w_j\\ u_m\\ w_m\end{Bmatrix}$$

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

Stress-Strain Relationship: The in-plane stress-strain relationship is:

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} \qquad \{\sigma\} = [D][B]\{d\}$$

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 & 0 \\ \nu & 1-\nu & 0 & 0 \\ 0 & 0 & 1-\nu & 0 \\ 0 & 0 & 0 & 0.5-\nu \end{bmatrix}$$

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 4 - Derive the Element Stiffness Matrix and Equations

The stiffness matrix can be defined as:

$$[k] = \int_V [B]^T [D][B] dV$$

For a circumferential differential element the integral becomes:

$$[k] = 2\pi \int_A [B]^T [D][B]\, r\, dr\, dz$$

After integrating along the circumferential boundary, the [B] matrix is a function of *r* and *z.*

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 4 - Derive the Element Stiffness Matrix and Equations

Therefore, [*k*] is a function of *r* and *z* and is of order 6 x 6.

We can evaluate [*k*] by one of three methods:

1. Numerical integration (Gaussian quadrature) as discussed in Chapter 10.

2. Explicit multiplication and term-by-term integration.

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 4 - Derive the Element Stiffness Matrix and Equations

Therefore, [*k*] is a function of *r* and *z* and is of order 6 x 6.

We can evaluate [*k*] by one of three methods:

3. Evaluate [*B*] for a centroidal point $(\bar{r}, \bar{z})$ of the element

$$\left[B(\bar{r}, \bar{z})\right] = \left[\bar{B}\right]$$

$$r = \bar{r} = \frac{r_i + r_j + r_m}{3} \qquad z = \bar{z} = \frac{z_i + z_j + z_m}{3}$$

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 4 - Derive the Element Stiffness Matrix and Equations

Therefore, [*k*] is a function of *r* and *z* and is of order 6 x 6.

We can evaluate [*k*] by one of three methods:

3. Evaluate [*B*] for a centroidal point $(\bar{r}, \bar{z})$ of the element

   As a first approximation: $[k] = 2\pi \bar{r} A \left[\bar{B}\right]^T [D] \left[\bar{B}\right]$

   If the triangular subdivisions are consistent with the final stress distribution (that is, small elements in regions of high stress gradients), then acceptable results can be obtained by Method 3.

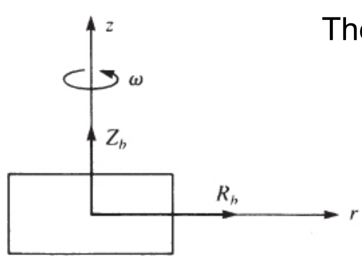
Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 4 - Derive the Element Stiffness Matrix and Equations

Distributed Body Forces

Loads such as gravity (in the direction of the *z* axis) or centrifugal forces in rotating machine parts (in the direction of the *r* axis) are considered to be body forces.



The body forces can be found by:

$$\{f_b\} = 2\pi \int_A [N]^T \begin{Bmatrix} R_b \\ Z_b \end{Bmatrix} r\, dr\, dz$$

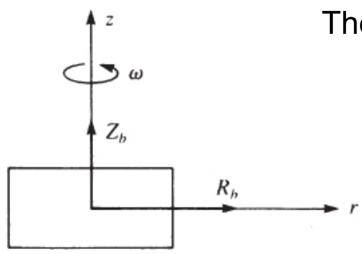
Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 4 - Derive the Element Stiffness Matrix and Equations

Distributed Body Forces

Where $R_b = \omega^2 \rho r$ for a machine part moving with a constant angular velocity ω about the z axis, with material mass density ρ and radial coordinate r, and Z_b is the body force per unit volume due to the force of gravity.



The body forces can be found by:

$$\{f_b\} = 2\pi \int_A [N]^T \begin{Bmatrix} R_b \\ Z_b \end{Bmatrix} r\, dr\, dz$$

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 4 - Derive the Element Stiffness Matrix and Equations

Distributed Body Forces

Considering the body force at node i, we have

$$\{f_{bi}\} = 2\pi \int_A [N_i]^T \begin{Bmatrix} R_b \\ Z_b \end{Bmatrix} r\, dr\, dz \qquad [N_i]^T = \begin{bmatrix} N_i & 0 \\ 0 & N_i \end{bmatrix}$$

Multiplying and integrating yields

$$\{f_{bi}\} = \frac{2\pi}{3} \begin{Bmatrix} \bar{R}_b \\ Z_b \end{Bmatrix} A\bar{r}$$

The origin of the coordinates is the centroid of the element, and R_b is the radially directed body force per unit volume evaluated at the centroid of the element.

Axisymmetric Elements

Derivation of the Stiffness Matrix

Step 4 - Derive the Element Stiffness Matrix and Equations

Distributed Body Forces

The body forces at nodes *j* and *m* are identical to those given for node *i*. Hence, for an element, we have

$$\{f_b\} = \frac{2\pi A\bar{r}}{3}\begin{Bmatrix} \bar{R}_b \\ Z_b \\ \bar{R}_b \\ Z_b \\ \bar{R}_b \\ Z_b \end{Bmatrix} \qquad \bar{R}_b = \omega^2\rho\bar{r}$$

Axisymmetric Elements

Derivation of the Stiffness Matrix

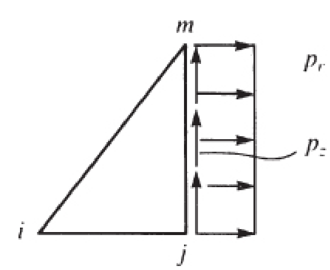
Step 4 - Derive the Element Stiffness Matrix and Equations

Surface Forces

Surface forces can be found by $\{f_s\} = \int_S [N_s]^T \{T\} dS$

Where again $[N_s]$ denotes the shape function matrix evaluated along the surface where the surface traction acts.

For example, along the vertical face *jm* of an element, let uniform loads p_r, and p_z, be applied along surface $r = r_j$.



Axisymmetric Elements

Derivation of the Stiffness Matrix

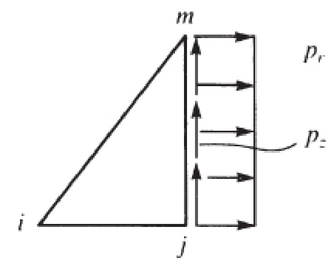
Step 4 - Derive the Element Stiffness Matrix and Equations

Surface Forces

For instance, for node j, substituting N_j gives

$$\{f_{sj}\} = \int_{z_j}^{z_m} \frac{1}{2A}\begin{bmatrix} \alpha_j + \beta_j r + \gamma_j z & 0 \\ 0 & \alpha_j + \beta_j r + \gamma_j z \end{bmatrix}\begin{Bmatrix} p_r \\ p_z \end{Bmatrix} 2\pi r_j \, dz$$

Evaluated at $r = r_j$ and z



Axisymmetric Elements

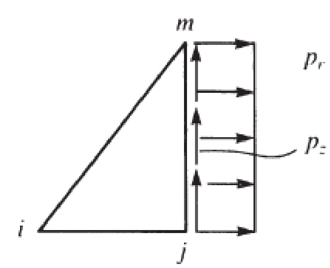
Derivation of the Stiffness Matrix

Step 4 - Derive the Element Stiffness Matrix and Equations

Surface Forces

Integrating the equations explicitly along with similar evaluations for f_{si} and f_{sm} the total distribution of surface force to nodes i, j, and m is

$$\{f_s\} = \frac{2\pi r_j \left(z_m - z_j\right)}{2}\begin{Bmatrix} 0 \\ 0 \\ p_r \\ p_z \\ p_r \\ p_z \end{Bmatrix}$$



Axisymmetric Elements

Derivation of the Stiffness Matrix

Steps 5 - 7

Steps 5 through 7, which involve assembling the total stiffness matrix, total force matrix, and total set of equations; solving for the nodal degrees of freedom; and calculating the element stresses, are analogous to those of Chapter 6 for the CST element.

Axisymmetric Elements

Derivation of the Stiffness Matrix

Steps 5 - 7

The stresses are not constant in each element.

They are usually determined by one of two methods that we use to determine the LST element stresses.

1. Either we determine the centroidal element stresses, or
2. We determine the nodal stresses for the element and then average them.

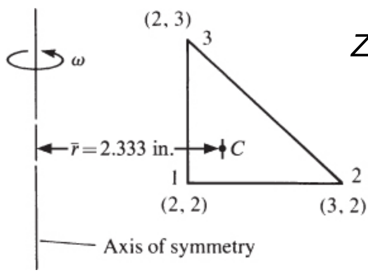
The latter method has been shown to be more accurate in some cases.

Axisymmetric Elements

Example 1

For the element of an axisymmetric body rotating with a constant angular velocity ω = 100 rev/min, evaluate the approximate body force matrix.

Include the weight of the material, where the weight density ρ_w = 0.283 lb./in.^{3}. Dimensions are inches.



$$Z_b = 0.283 \text{ lb./in.}^3$$

Axisymmetric Elements

Example 1

Let evaluate the approximate body force matrix.

The body forces per unit volume evaluated at the centroid of the element are:

$$\bar{R}_b = \omega^2 \rho \bar{r}$$

$$= \left[(100\,\text{rpm})\left(2\pi\,\tfrac{\text{rad}}{\text{rev}}\right)\left(\frac{1\,\text{min}}{60\,\text{sec}}\right)\right]^2 \frac{0.283 \text{ lb./in.}^3}{\left(32.2\,\frac{\text{ft.}}{\text{s}^2} \times \frac{12\,\text{in.}}{\text{ft.}}\right)}(2.333\,\text{in.})$$

$$= 0.187\,\text{lb./in.}^3$$

$$\frac{2\pi A\bar{r}}{3} = \frac{2\pi\left(0.5\,\text{in.}^2\right)(2.333\,\text{in.})}{3} = 2.44\,\text{in.}^2$$

Axisymmetric Elements

Example 1

Let evaluate the approximate body force matrix.

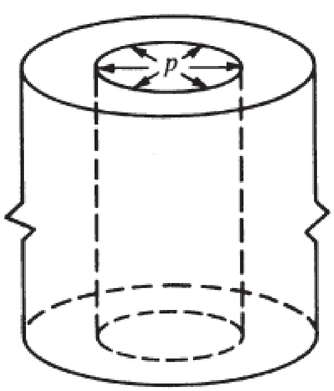
The body forces per unit volume evaluated at the centroid of the element are:

$$\{f_b\} = \frac{2\pi A\bar{r}}{3}\begin{Bmatrix} \bar{R}_b \\ Z_b \\ \bar{R}_b \\ Z_b \\ \bar{R}_b \\ Z_b \end{Bmatrix} = 2.44\,\text{in.}^3\begin{Bmatrix} 0.187 \\ -0.283 \\ 0.187 \\ -0.283 \\ 0.187 \\ -0.283 \end{Bmatrix}\frac{\text{lb.}}{\text{in.}^3} = \begin{Bmatrix} 0.457 \\ -0.691 \\ 0.457 \\ -0.691 \\ 0.457 \\ -0.691 \end{Bmatrix}\text{lb.}$$

Axisymmetric Elements

Example 2

For the long, thick-walled cylinder under internal pressure *p* equal to 1 psi, determine the displacements and stresses.

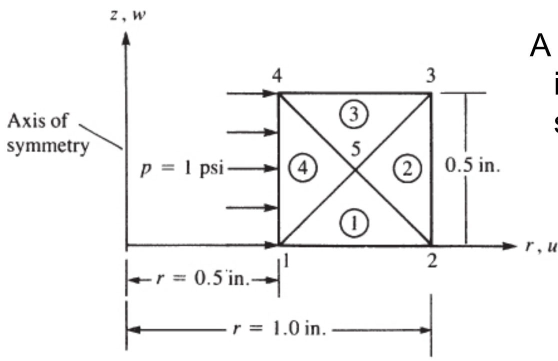


Axisymmetric Elements

Example 2

First discretize the cylinder into four triangular elements.

A horizontal slice of the cylinder represents the total cylinder behavior.



A coarse mesh of elements is used for simplicity's sake

Axisymmetric Elements

Example 2

The governing global matrix equation is:

$$\begin{Bmatrix} F_{1r} \\ F_{1z} \\ F_{2r} \\ F_{2z} \\ F_{3r} \\ F_{3z} \\ F_{4r} \\ F_{4z} \\ F_{5r} \\ F_{5z} \end{Bmatrix} = [K] \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \\ u_4 \\ w_4 \\ u_5 \\ w_5 \end{Bmatrix}$$

[K] is a matrix of order 10 x 10

Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix

The [K] matrix is assembled in the usual manner by superposition of the individual element stiffness matrices.

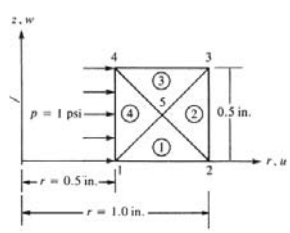
For simplicity's sake, we will evaluate [B] for a centroidal point $(\bar{r}, \bar{z})$ of the element.

$$[k] = 2\pi\bar{r}A\left[\bar{B}\right]^T [D]\left[\bar{B}\right]$$

Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix: Element 1



$$r = \bar{r} \qquad z = \bar{z}$$

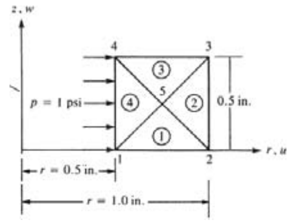
$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.00 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$\left[\bar{B}\right] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \frac{\alpha_i}{\bar{r}} + \beta_i + \boxed{\frac{\gamma_i \bar{z}}{\bar{r}}} & 0 & \frac{\alpha_j}{\bar{r}} + \beta_j + \boxed{\frac{\gamma_j \bar{z}}{\bar{r}}} & 0 & \frac{\alpha_m}{\bar{r}} + \beta_m + \boxed{\frac{\gamma_m \bar{z}}{\bar{r}}} & 0 \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}$$

Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix: Element 1



$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.00 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$\alpha_i = r_j z_m - z_j r_m = (1.0)(0.25) - (0.0)(0.75) = 0.25 \text{in.}^2$$

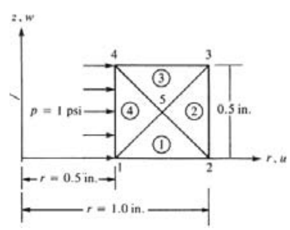
$$\alpha_j = r_m z_i - z_m r_i = (0.75)(0.0) - (0.25)(0.5) = -0.125 \text{in.}^2$$

$$\alpha_m = r_i z_j - z_i r_j = (0.5)(0.0) - (0.0)(1.0) = 0$$

Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix: Element 1



$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.00 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$\beta_i = z_j - z_m = -0.25 \text{in}^2 \qquad \gamma_i = r_m - r_j = -0.25 \text{in.}^2$$

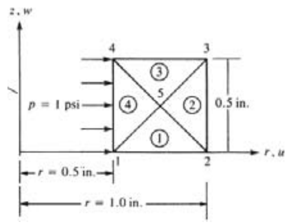
$$\beta_j = z_m - z_i = 0.25 \text{in}^2 \qquad \gamma_j = r_i - r_m = -0.25 \text{in.}^2$$

$$\beta_m = z_i - z_j = 0 \qquad \gamma_m = r_j - r_i = 0.5 \text{in.}^2$$

Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix: Element 1



$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.00 \\ 0.25 \end{Bmatrix} \text{in.}$$

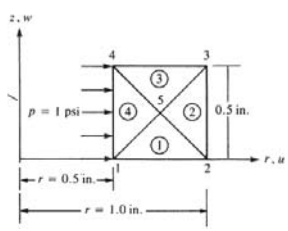
$$\bar{r} = \sum_{i=1}^{3} \frac{r_i}{3} = 0.75 \text{in.} \qquad \bar{z} = \sum_{i=1}^{3} \frac{z_i}{3} = 0.0833 \text{in.}$$

$$A = \frac{1}{2}(0.5)(0.25) = 0.0625 \text{in.}^2$$

Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix: Element 1



$$r = \bar{r} \qquad z = \bar{z}$$

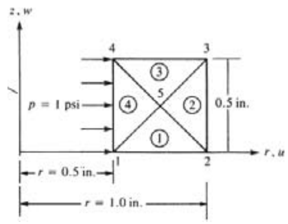
$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.00 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$\left[\bar{B}\right] = \frac{1}{0.125} \begin{bmatrix} -0.25 & 0 & 0.25 & 0 & 0 & 0 \\ 0 & -0.25 & 0 & -0.25 & 0 & 0.5 \\ 0.0556 & 0 & 0.0556 & 0 & 0.0556 & 0 \\ -.025 & -0.25 & -0.25 & 0.25 & 0.5 & 0 \end{bmatrix} \frac{1}{\text{in.}}$$

Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix: Element 1



$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.00 \\ 0.25 \end{Bmatrix} \text{in.}$$

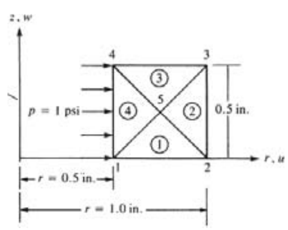
$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 & 0 \\ \nu & 1-\nu & 0 & 0 \\ 0 & 0 & 1-\nu & 0 \\ 0 & 0 & 0 & 0.5-\nu \end{bmatrix}$$

Assume that $E = 30 \times 10^6$ psi and $\nu = 0.3$

Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix: Element 1



$$r = \bar{r} \qquad z = \bar{z}$$

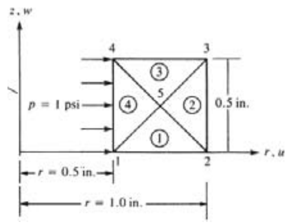
$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.00 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$[D] = 57.7\left(10^6\right) \begin{bmatrix} 0.7 & 0.3 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix} psi$$

Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix: Element 1



$$r = \bar{r} \qquad z = \bar{z}$$

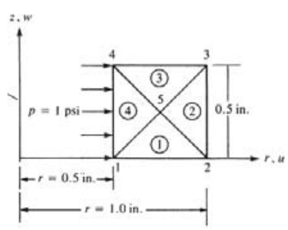
$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.00 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$\left[\bar{B}\right]^T_{6x4}\left[D\right]_{4x4} = \frac{57.7\left(10^6\right)}{0.125}\begin{bmatrix} -0.158 & -0.0583 & -0.0361 & -0.05 \\ -0.075 & -0.175 & -0.075 & -0.05 \\ 0.192 & 0.0917 & 0.114 & -0.05 \\ -0.075 & -0.175 & -0.075 & 0.05 \\ 0.0167 & 0.0166 & 0.0388 & 0.1 \\ 0.15 & 0.35 & 0.15 & 0 \end{bmatrix}$$

Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix: Element 1



$$r = \bar{r} \qquad z = \bar{z}$$

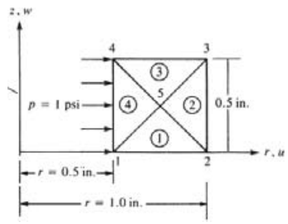
$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.00 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$\begin{array}{c} \qquad\qquad i = 1 \qquad\qquad\qquad j = 2 \qquad\qquad\qquad m = 5 \\ \left[k^{(1)}\right] = \left(10^6\right)\left[\begin{array}{cc|cc|cc} 54.46 & 29.45 & -31.63 & 2.26 & -29.37 & -31.71 \\ 29.45 & 61.17 & -11.33 & 33.98 & -31.72 & -95.15 \\ \hline -31.63 & -11.33 & 72.59 & -38.52 & -20.31 & 49.84 \\ 2.26 & 33.98 & -38.52 & 61.17 & 22.66 & -95.15 \\ \hline -29.37 & -31.72 & -20.31 & 22.66 & 56.72 & 9.06 \\ -31.71 & -95.15 & 49.84 & -95.15 & 9.06 & 190.31 \end{array}\right] \dfrac{\text{lb.}}{\text{in.}} \end{array}$$

Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix: Element 2



$$r = \bar{r} \qquad z = \bar{z}$$

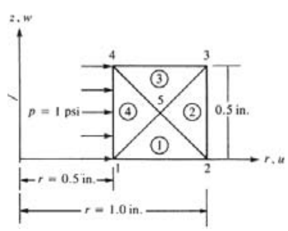
$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.50 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$[\bar{B}] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \frac{\alpha_i}{\bar{r}} + \beta_i + \boxed{\frac{\gamma_i \bar{z}}{\bar{r}}} & 0 & \frac{\alpha_j}{\bar{r}} + \beta_j + \boxed{\frac{\gamma_j \bar{z}}{\bar{r}}} & 0 & \frac{\alpha_m}{\bar{r}} + \beta_m + \boxed{\frac{\gamma_m \bar{z}}{\bar{r}}} & 0 \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}$$

Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix: Element 2



$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.50 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$\alpha_i = r_j z_m - z_j r_m = (1.0)(0.25) - (0.5)(0.75) = -0.125 \text{in.}^2$$

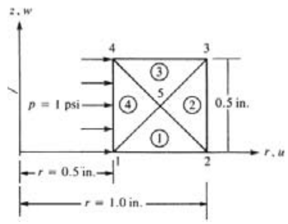
$$\alpha_j = r_m z_i - z_m r_i = (0.75)(0.0) - (0.25)(1.0) = -0.25 \text{in.}^2$$

$$\alpha_m = r_i z_j - z_i r_j = (1.0)(0.5) - (0.0)(1.0) = 0.5 \text{in.}^2$$

Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix: Element 2



$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.50 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$\beta_i = z_j - z_m = 0.25 \text{in}^2 \qquad \gamma_i = r_m - r_j = -0.25 \text{in.}^2$$

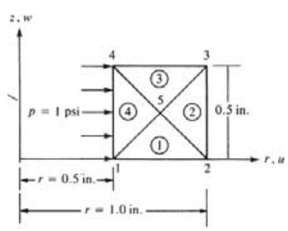
$$\beta_j = z_m - z_i = 0.25 \text{in}^2 \qquad \gamma_j = r_i - r_m = 0.25 \text{in.}^2$$

$$\beta_m = z_i - z_j = -0.5 \text{in}^2 \qquad \gamma_m = r_j - r_i = 0$$

Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix: Element 2



$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.50 \\ 0.25 \end{Bmatrix} \text{in.}$$

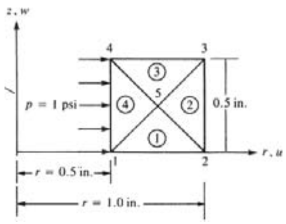
$$\bar{r} = \sum_{i=1}^{3} \frac{r_i}{3} = 0.9167 \text{in.} \qquad \bar{z} = \sum_{i=1}^{3} \frac{z_i}{3} = 0.25 \text{in.}$$

$$A = \frac{1}{2}(0.5)(0.25) = 0.0625 \text{in.}^2$$

Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix: Element 2



$$r = \bar{r} \qquad z = \bar{z}$$

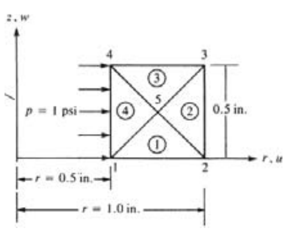
$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.50 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$\begin{array}{c} \quad i=2 \qquad\qquad\qquad j=3 \qquad\qquad\qquad m=5 \\ \left[k^{(2)}\right] = \left(10^6\right)\left[\begin{array}{rr|rr|rr} 85.75 & -46.07 & 52.52 & 12.84 & -118.92 & 33.23 \\ -46.07 & 74.77 & -12.84 & -41.54 & 45.32 & -33.23 \\ \hline 52.52 & -12.84 & 85.74 & 46.07 & -118.92 & -33.23 \\ 12.84 & -41.54 & 46.07 & 74.77 & -45.21 & -33.23 \\ \hline -118.92 & 45.32 & -118.92 & -45.21 & 216.41 & 0 \\ 33.23 & -33.23 & -33.23 & -33.23 & 0 & 66.46 \end{array}\right] \frac{\text{lb.}}{\text{in.}} \end{array}$$

Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix: Element 3



$$r = \bar{r} \qquad z = \bar{z}$$

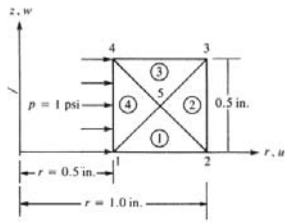
$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ 0.50 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 0.50 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$\left[\bar{B}\right] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \frac{\alpha_i}{\bar{r}} + \beta_i + \boxed{\frac{\gamma_i \bar{z}}{\bar{r}}} & 0 & \frac{\alpha_j}{\bar{r}} + \beta_j + \boxed{\frac{\gamma_j \bar{z}}{\bar{r}}} & 0 & \frac{\alpha_m}{\bar{r}} + \beta_m + \boxed{\frac{\gamma_m \bar{z}}{\bar{r}}} & 0 \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}$$

Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix: Element 3



$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ 0.50 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 0.50 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$\alpha_i = r_j z_m - z_j r_m = (0.5)(0.25) - (0.5)(0.75) = -0.25 \text{in.}^2$$

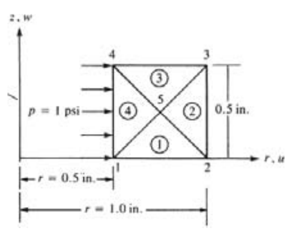
$$\alpha_j = r_m z_i - z_m r_i = (0.75)(0.5) - (0.25)(1.0) = 0.125 \text{in.}^2$$

$$\alpha_m = r_i z_j - z_i r_j = (1.0)(0.5) - (0.5)(0.5) = 0.25 \text{in.}^2$$

Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix: Element 3



$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ 0.50 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 0.50 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$\beta_i = z_j - z_m = 0.25 \text{in}^2 \qquad \gamma_i = r_m - r_j = 0.25 \text{in.}^2$$

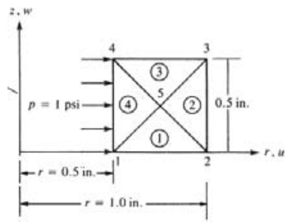
$$\beta_j = z_m - z_i = -0.25 \text{in}^2 \qquad \gamma_j = r_i - r_m = 0.25 \text{in.}^2$$

$$\beta_m = z_i - z_j = 0 \qquad \gamma_m = r_j - r_i = -0.5 \text{in.}^2$$

Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix: Element 3



$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ 0.50 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 0.50 \\ 0.25 \end{Bmatrix} \text{in.}$$

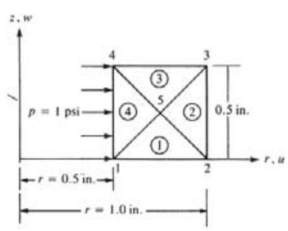
$$\bar{r} = \sum_{i=1}^{3} \frac{r_i}{3} = 0.75 \text{in.} \qquad \bar{z} = \sum_{i=1}^{3} \frac{z_i}{3} = 0.417 \text{in.}$$

$$A = \frac{1}{2}(0.5)(0.25) = 0.0625 \text{in.}^2$$

Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix: Element 3



$$r = \bar{r} \qquad z = \bar{z}$$

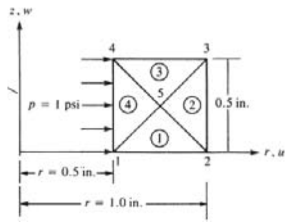
$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ 0.50 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 0.50 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$\left[k^{(3)}\right] = \left(10^6\right) \begin{array}{c} \begin{array}{cccccc} \quad i = 3 & & \quad j = 4 & & \quad m = 5 & \end{array} \\ \left[\begin{array}{cc|cc|cc} 72.58 & 38.52 & -31.63 & 11.33 & -20.31 & -49.84 \\ 38.52 & 61.17 & -2.26 & 33.98 & -22.66 & -95.15 \\ \hline -31.63 & -2.26 & 54.46 & -29.45 & -29.37 & 31.72 \\ 11.33 & 33.98 & -29.45 & 61.17 & 31.72 & -95.15 \\ \hline -20.31 & -22.66 & -29.37 & 31.72 & 56.72 & -9.06 \\ -49.84 & -95.15 & 31.72 & -95.15 & -9.06 & 190.31 \end{array}\right] \end{array} \frac{\text{lb.}}{\text{in.}}$$

Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix: Element 4



$$r = \bar{r} \qquad z = \bar{z}$$

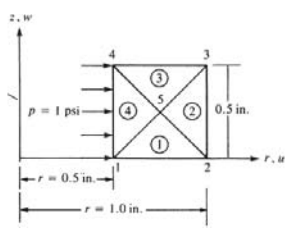
$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 0.75 \\ 0.50 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.25 \\ 0.50 \end{Bmatrix} \text{in.}$$

$$[\bar{B}] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \frac{\alpha_i}{\bar{r}} + \beta_i + \boxed{\frac{\gamma_i \bar{z}}{\bar{r}}} & 0 & \frac{\alpha_j}{\bar{r}} + \beta_j + \boxed{\frac{\gamma_j \bar{z}}{\bar{r}}} & 0 & \frac{\alpha_m}{\bar{r}} + \beta_m + \boxed{\frac{\gamma_m \bar{z}}{\bar{r}}} & 0 \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}$$

Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix: Element 4



$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 0.75 \\ 0.50 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.25 \\ 0.50 \end{Bmatrix} \text{in.}$$

$$\alpha_i = r_j z_m - z_j r_m \quad = (0.75)(0.5) - (0.25)(0.5) = 0.25 \text{in.}^2$$

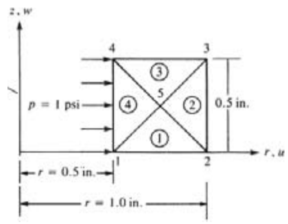
$$\alpha_j = r_m z_i - z_m r_i \quad = (0.5)(0.0) - (0.5)(0.5) = -0.25 \text{in.}^2$$

$$\alpha_m = r_i z_j - z_i r_j \quad = (0.5)(0.25) - (0.0)(0.75) = 0.125 \text{in.}^2$$

Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix: Element 4



$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 0.75 \\ 0.50 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.25 \\ 0.50 \end{Bmatrix} \text{in.}$$

$$\beta_i = z_j - z_m = -0.25\,\text{in}^2 \qquad \gamma_i = r_m - r_j = -0.25\,\text{in.}^2$$

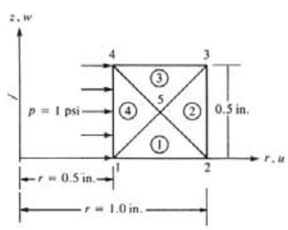
$$\beta_j = z_m - z_i = 0.5\,\text{in}^2 \qquad \gamma_j = r_i - r_m = 0$$

$$\beta_m = z_i - z_j = -0.25\,\text{in.}^2 \qquad \gamma_m = r_j - r_i = 0.25\,\text{in.}^2$$

Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix: Element 4



$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 0.75 \\ 0.50 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.25 \\ 0.50 \end{Bmatrix} \text{in.}$$

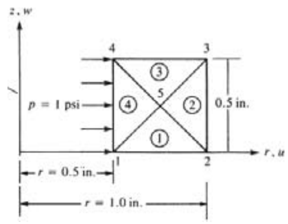
$$\bar{r} = \sum_{i=1}^{3} \frac{r_i}{3} = 0.5833\,\text{in.} \qquad \bar{z} = \sum_{i=1}^{3} \frac{z_i}{3} = 0.25\,\text{in.}$$

$$A = \frac{1}{2}(0.5)(0.25) = 0.0625\,\text{in.}^2$$

Axisymmetric Elements

Example 2

Assemblage of the Stiffness Matrix: Element 4



$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 0.75 \\ 0.50 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.25 \\ 0.50 \end{Bmatrix} \text{in.}$$

$$\begin{array}{c} \\ \left[k^{(4)}\right] = \left(10^6\right) \end{array} \begin{array}{c} \begin{array}{ccc} \quad i = 1 \quad & \quad j = 5 \quad & \quad m = 4 \quad \end{array} \\ \left[\begin{array}{cc|cc|cc} 41.53 & 21.90 & -66.45 & -21.14 & 20.39 & -0.75 \\ 21.92 & 47.57 & -36.24 & -21.14 & 0.75 & -26.43 \\ \hline -66.45 & -36.24 & 169.14 & 0 & -66.45 & 36.24 \\ -21.14 & -21.14 & 0 & 42.28 & 21.14 & -21.14 \\ \hline 20.39 & 0.75 & -66.45 & 21.14 & 41.53 & -21.90 \\ -0.75 & -26.43 & 36.24 & -21.14 & -21.90 & 47.57 \end{array}\right] \end{array} \frac{\text{lb.}}{\text{in.}}$$

Axisymmetric Elements

Example 2

Using superposition of the element stiffness matrices, where we rearrange the elements of each stiffness matrix in order of increasing nodal degrees of freedom, we can obtain the global stiffness matrix.

Axisymmetric Elements

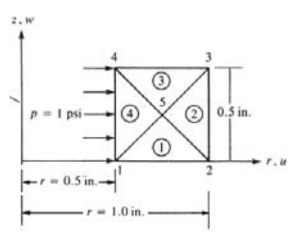
Example 2

$$
[K] = (10^6)\begin{array}{c}
\begin{array}{cccccccccc} \quad & 1 & \quad & \quad & 2 & \quad & \quad & 3 & \quad & \quad\ 4 \quad\quad\quad 5 \end{array} \\
\left[\begin{array}{cc|cc|cc|cc|cc}
95.99 & 51.35 & -36.63 & 2.26 & 0 & 0 & 20.39 & -0.75 & -95.82 & -52.86 \\
51.35 & 108.74 & -11.33 & 33.98 & 0 & 0 & 33.98 & -26.43 & -67.96 & -116.3 \\
\hline
-36.63 & -11.33 & 158.34 & -84.59 & 52.52 & 12.84 & 0 & 0 & -139.2 & 83.07 \\
2.26 & 33.98 & -84.59 & 135.94 & -12.84 & -41.54 & 0 & 0 & 67.98 & -128.4 \\
\hline
0 & 0 & 52.52 & -12.84 & 158.33 & 84.59 & -31.63 & 11.33 & -139.2 & -83.07 \\
0 & 0 & 12.84 & -41.54 & 84.59 & 135.94 & -2.26 & 33.98 & -67.98 & -128.4 \\
\hline
20.39 & 33.98 & 0 & 0 & -31.63 & -2.26 & 95.99 & -51.35 & -95.82 & 52.86 \\
-0.75 & -26.43 & 0 & 0 & 11.33 & 33.98 & -51.35 & 108.74 & 67.96 & -116.3 \\
\hline
-95.82 & -67.96 & -139.2 & 67.98 & -139.2 & -67.98 & -95.82 & 67.96 & 498.99 & 0 \\
-52.86 & -116.3 & 83.07 & -128.4 & -83.07 & -128.4 & 52.86 & -116.3 & 0 & 489.36
\end{array}\right]
\end{array}
\frac{lb.}{in.}
$$

Axisymmetric Elements

Example 2

The applied nodal forces are given as:



$$
\{f_s\} = \frac{2\pi r_i (z_j - z_m)}{2}\begin{Bmatrix} p_r \\ p_z \\ 0 \\ 0 \\ p_r \\ p_z \end{Bmatrix} = \begin{Bmatrix} 0.785 \\ 0 \\ 0 \\ 0 \\ 0.785 \\ 0 \end{Bmatrix} \text{lb.}
$$

$$
F_{1r} = F_{4r} = \frac{2\pi(0.5\text{in.})(0.5\text{in.})}{2}(1psi) = 0.785\,lb.
$$

Axisymmetric Elements

Example 2

The resulting equations are:

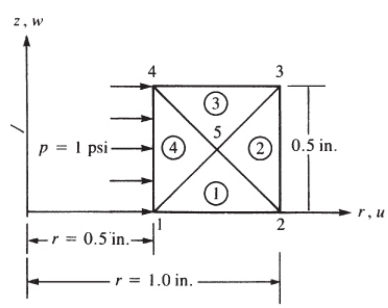
$$\left(10^6\right)\frac{lb.}{in.}\begin{bmatrix} 95.99 & 51.35 & -36.63 & 2.26 & 0 & 0 & 20.39 & -0.75 & -95.82 & -52.86 \\ 51.35 & 108.74 & -11.33 & 33.98 & 0 & 0 & 33.98 & -26.43 & -67.96 & -116.3 \\ -36.63 & -11.33 & 158.34 & -84.59 & 52.52 & 12.84 & 0 & 0 & -139.2 & 83.07 \\ 2.26 & 33.98 & -84.59 & 135.94 & -12.84 & -41.54 & 0 & 0 & 67.98 & -128.4 \\ 0 & 0 & 52.52 & -12.84 & 158.33 & 84.59 & -31.63 & 11.33 & -139.2 & -83.07 \\ 0 & 0 & 12.84 & -41.54 & 84.59 & 135.94 & -2.26 & 33.98 & -67.98 & -128.4 \\ 20.39 & 33.98 & 0 & 0 & -31.63 & -2.26 & 95.99 & -51.35 & -95.82 & 52.86 \\ -0.75 & -26.43 & 0 & 0 & 11.33 & 33.98 & -51.35 & 108.74 & 67.96 & -116.3 \\ -95.82 & -67.96 & -139.2 & 67.98 & -139.2 & -67.98 & -95.82 & 67.96 & 498.99 & 0 \\ -52.86 & -116.3 & 83.07 & -128.4 & -83.07 & -128.4 & 52.86 & -116.3 & 0 & 489.36 \end{bmatrix}\begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \\ u_4 \\ w_4 \\ u_5 \\ w_5 \end{Bmatrix} = \begin{Bmatrix} 0.785 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.785 \\ 0 \\ 0 \\ 0 \end{Bmatrix} lb.$$

Axisymmetric Elements

Example 2

The nodal displacements are:

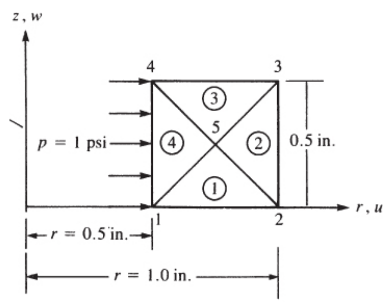
$$\begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \\ u_4 \\ w_4 \\ u_5 \\ w_5 \end{Bmatrix} = \begin{Bmatrix} 0.0322 \\ 0.00115 \\ 0.0219 \\ 0.00206 \\ 0.0219 \\ -0.00206 \\ 0.0322 \\ -0.00115 \\ 0.0244 \\ 0.0 \end{Bmatrix} \times \left(10^{-6}\right) \text{in.}$$



Axisymmetric Elements

Example 2

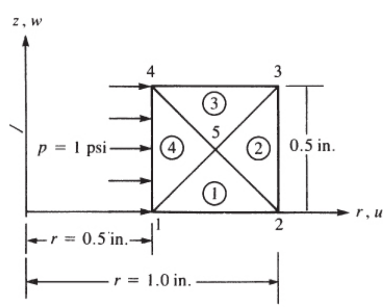
The results for nodal displacements are as expected because radial displacements at the inner edge are equal ($u_1 = u_4$) and those at the outer edge are equal ($u_2 = u_3$).



Axisymmetric Elements

Example 2

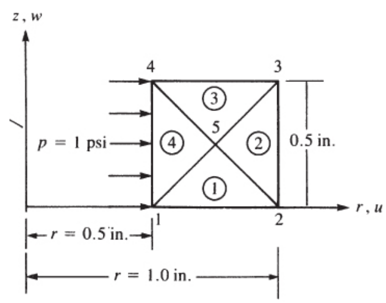
In addition, the axial displacements at the outer nodes and inner nodes are equal but opposite in sign ($w_1 = -w_4$ and $w_2 = -w_3$) as a result of the Poisson effect and symmetry.



Axisymmetric Elements

Example 2

Finally, the axial displacement at the center node is zero (w_5 = 0), as it should be because of symmetry.



Axisymmetric Elements

Example 2

Determine the stresses in each element as: $\{\sigma\} = [D][\bar{B}]\{d\}$

For Element 1:

$$[D] = 57.7(10^6)\begin{bmatrix} 0.7 & 0.3 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix} psi$$

$$[\bar{B}] = \frac{1}{0.125}\begin{bmatrix} -0.25 & 0 & 0.25 & 0 & 0 & 0 \\ 0 & -0.25 & 0 & -0.25 & 0 & 0.5 \\ 0.0556 & 0 & 0.0556 & 0 & 0.0556 & 0 \\ -.025 & -0.25 & -0.25 & 0.25 & 0.5 & 0 \end{bmatrix} \tfrac{1}{\text{in.}}$$

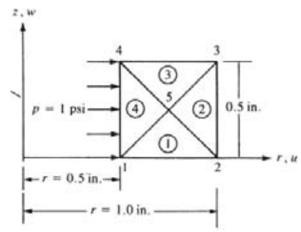
Axisymmetric Elements

Example 2

Determine the stresses in each element as: $\{\sigma\} = [D][\bar{B}]\{d\}$

For Element 1:

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{xy} \end{Bmatrix} = [D][\bar{B}] \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_5 \\ w_5 \end{Bmatrix} = \begin{Bmatrix} -0.338 \\ -0.0126 \\ 0.942 \\ -0.1037 \end{Bmatrix} psi$$



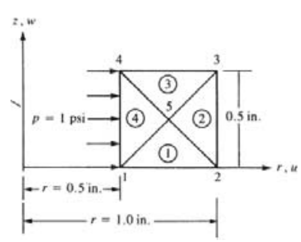
Axisymmetric Elements

Example 2

Determine the stresses in each element as: $\{\sigma\} = [D][\bar{B}]\{d\}$

For Element 2:

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{xy} \end{Bmatrix} = [D][\bar{B}] \begin{Bmatrix} u_2 \\ w_2 \\ u_3 \\ w_3 \\ u_5 \\ w_5 \end{Bmatrix} = \begin{Bmatrix} -0.105 \\ -0.0747 \\ 0.690 \\ 0.0 \end{Bmatrix} psi$$



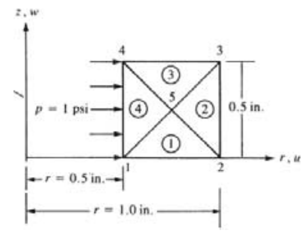
Axisymmetric Elements

Example 2

Determine the stresses in each element as: $\{\sigma\} = [D]\left[\bar{B}\right]\{d\}$

For Element 3:

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{xy} \end{Bmatrix} = [D]\left[\bar{B}\right]\begin{Bmatrix} u_3 \\ w_3 \\ u_4 \\ w_4 \\ u_5 \\ w_5 \end{Bmatrix} = \begin{Bmatrix} -0.337 \\ -0.0125 \\ 0.942 \\ 0.1037 \end{Bmatrix} psi$$



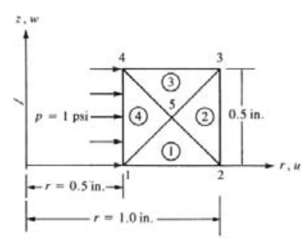
Axisymmetric Elements

Example 2

Determine the stresses in each element as: $\{\sigma\} = [D]\left[\bar{B}\right]\{d\}$

For Element 4:

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{xy} \end{Bmatrix} = [D]\left[\bar{B}\right]\begin{Bmatrix} u_1 \\ w_1 \\ u_5 \\ w_5 \\ u_4 \\ w_4 \end{Bmatrix} = \begin{Bmatrix} -0.470 \\ 0.1493 \\ 1.426 \\ 0.0 \end{Bmatrix} psi$$



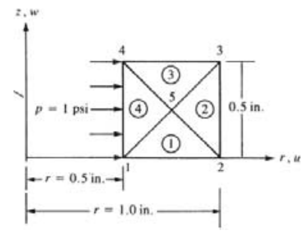
Axisymmetric Elements

Example 2

Determine the stresses in each element as: $\{\sigma\} = [D][\bar{B}]\{d\}$

For Element 1:

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} -0.338 \\ -0.0126 \\ 0.942 \\ -0.1037 \end{Bmatrix} psi$$



For Element 3:

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} -0.337 \\ -0.0125 \\ 0.942 \\ 0.1037 \end{Bmatrix} psi$$

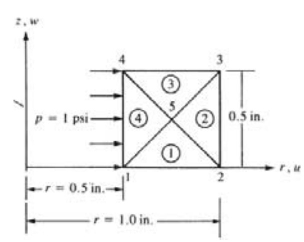
Axisymmetric Elements

Example 2

Determine the stresses in each element as: $\{\sigma\} = [D][\bar{B}]\{d\}$

For Element 4:

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} -0.470 \\ 0.1493 \\ 1.426 \\ 0.0 \end{Bmatrix} psi$$

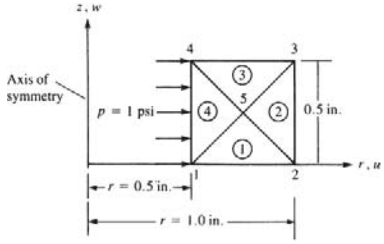


For Element 2:

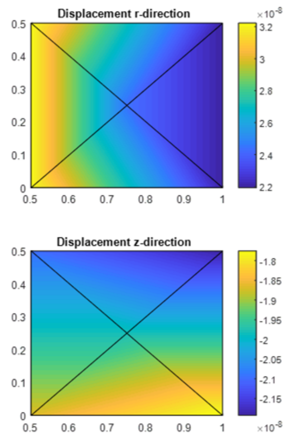
$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} -0.105 \\ -0.0747 \\ 0.690 \\ 0.0 \end{Bmatrix} psi$$

Axisymmetric Elements

Example 2 – Using Matlab code

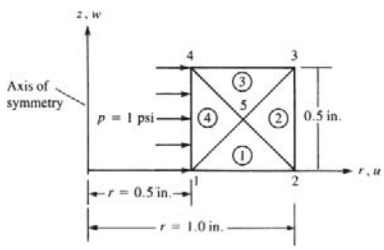


A coarse mesh of 4 CST elements

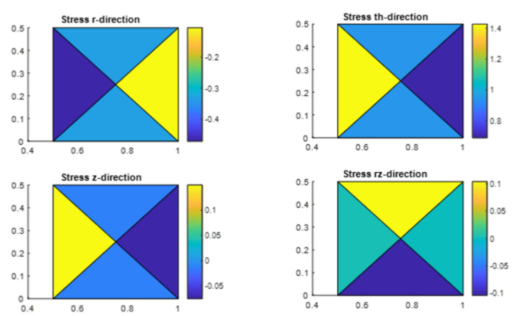


Axisymmetric Elements

Example 2 – Using Matlab code

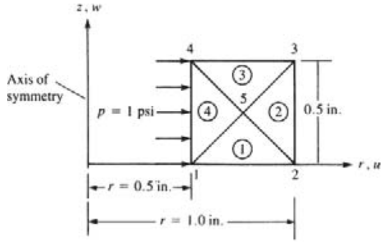


A coarse mesh of 4 CST elements

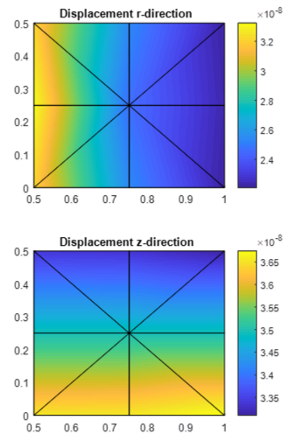


Axisymmetric Elements

Example 2 – Using Matlab code

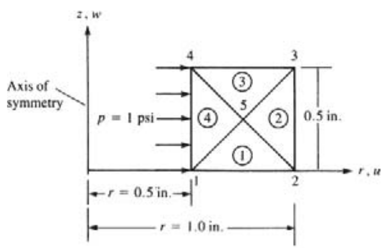


A coarse mesh of 8 CST elements

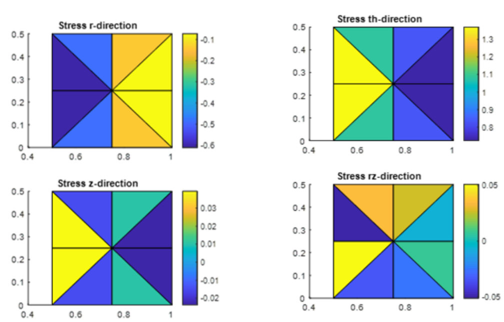


Axisymmetric Elements

Example 2 – Using Matlab code

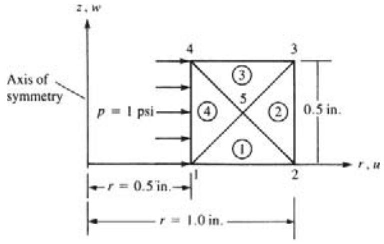


A coarse mesh of 8 CST elements

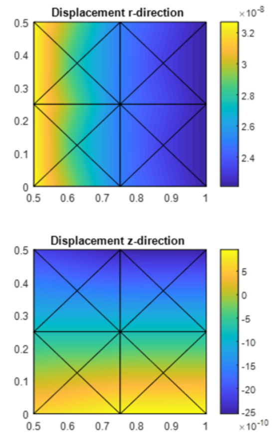


Axisymmetric Elements

Example 2 – Using Matlab code

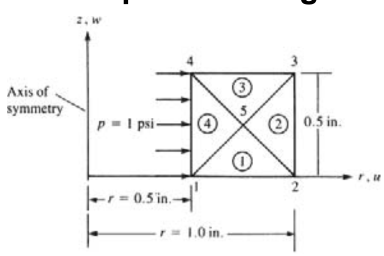


A coarse mesh of 16 CST elements

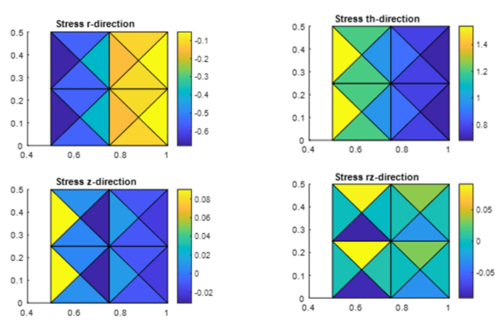


Axisymmetric Elements

Example 2 – Using Matlab code



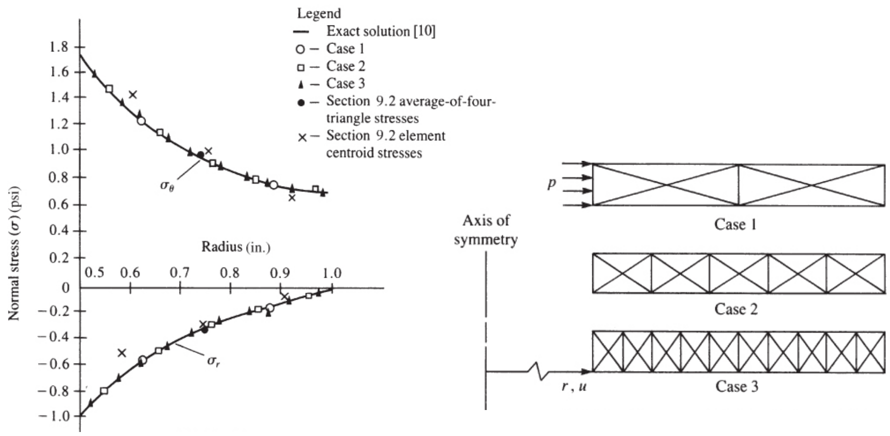
A coarse mesh of 16 CST elements



Axisymmetric Elements

Example 2

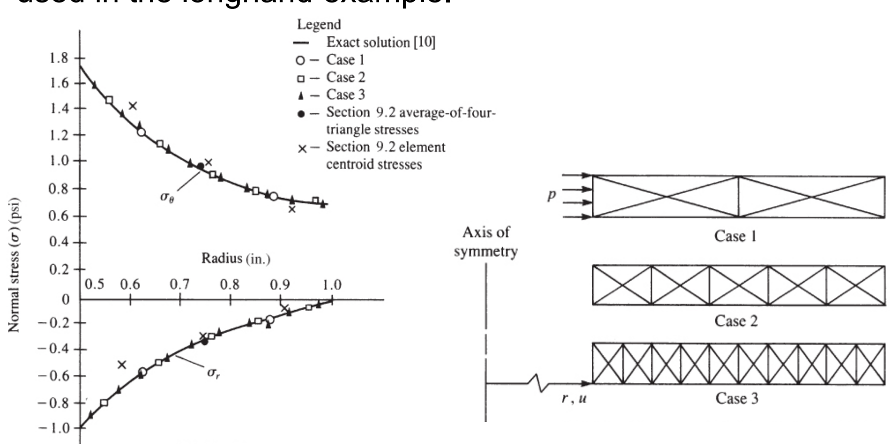
The figure below shows the exact solution along with the results determined here and the other results.



Axisymmetric Elements

Example 2

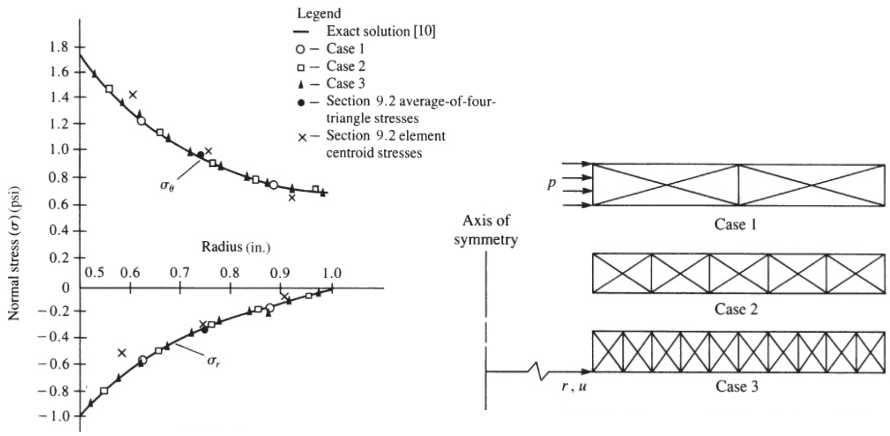
Observe that agreement with the exact solution is quite good except for the limited results due to the very coarse mesh used in the longhand example.



Axisymmetric Elements

Example 2

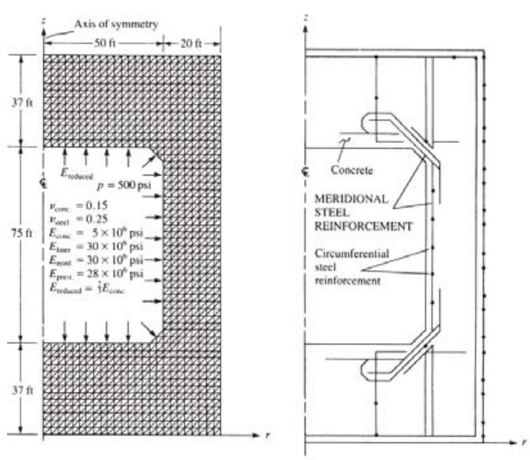
Stresses have been plotted at the center of the quadrilaterals and were obtained by averaging the stresses in the four connecting triangles.



Axisymmetric Elements

Applications

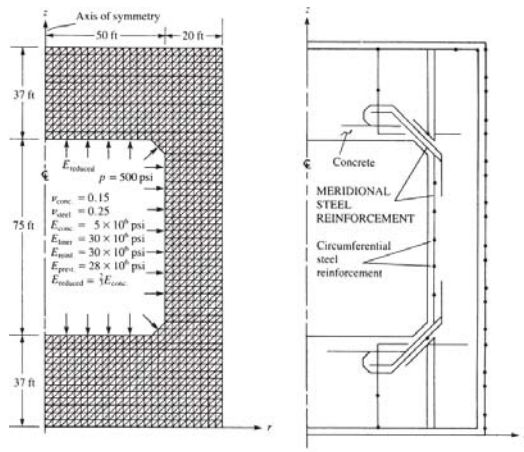
Consider the finite element model of a steel-reinforced concrete pressure vessel.



Axisymmetric Elements

Applications

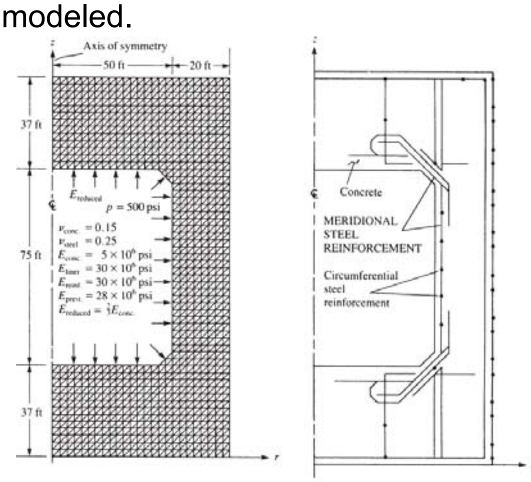
The vessel is a thick-walled cylinder with flat heads.



Axisymmetric Elements

Applications

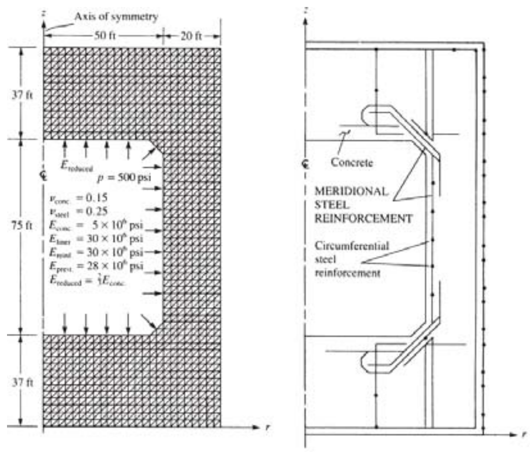
An axis of symmetry (the *z* axis) exists such that only one-half of the *r*-*z* plane passing through the middle of the structure need be modeled.



Axisymmetric Elements

Applications

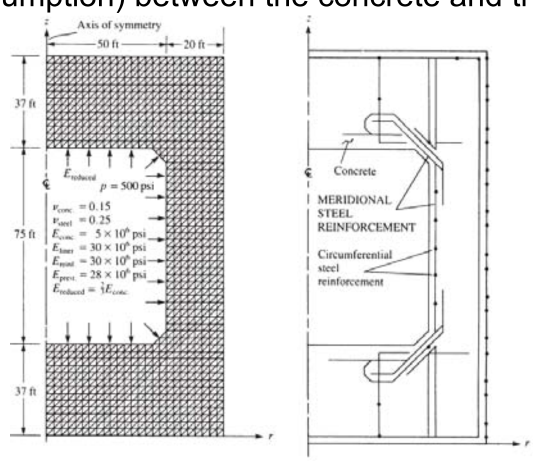
The concrete was modeled by using the axisymmetric triangular element developed in this chapter.



Axisymmetric Elements

Applications

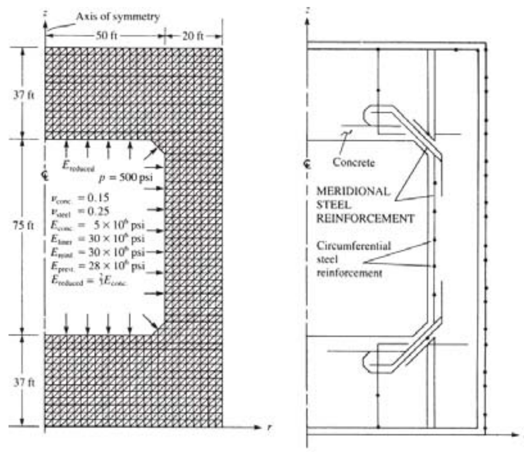
The steel elements were laid out along the boundaries of the concrete elements so as to maintain continuity (or perfect bond assumption) between the concrete and the steel.



Axisymmetric Elements

Applications

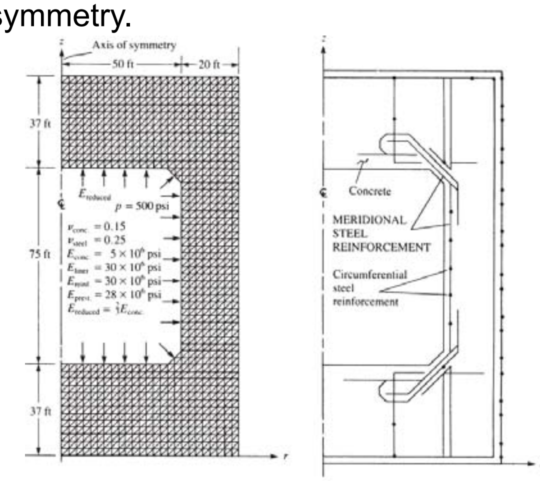
The vessel was then subjected to an internal pressure as shown in the figure.



Axisymmetric Elements

Applications

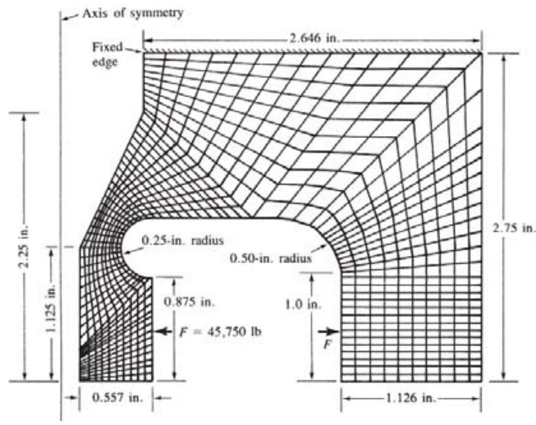
Note that the nodes along the axis of symmetry should be supported by rollers preventing motion perpendicular to the axis of symmetry.



Axisymmetric Elements

Applications

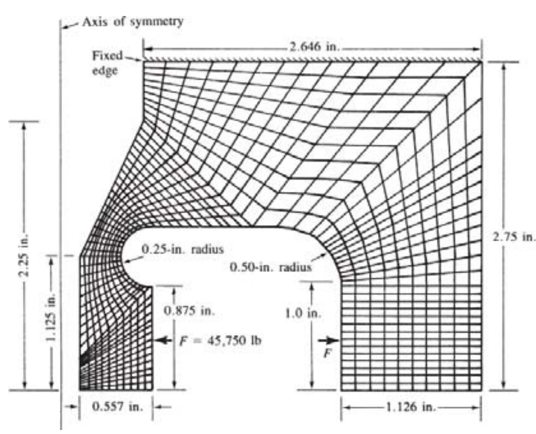
The figure below shows a finite element model of a high-strength steel die used in a thin-plastic-film-making process



Axisymmetric Elements

Applications

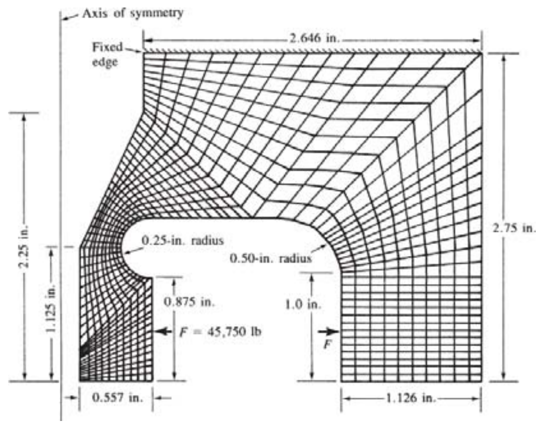
The die is an irregularly shaped disk. An axis of symmetry with respect to geometry and loading exists as shown.



Axisymmetric Elements

Applications

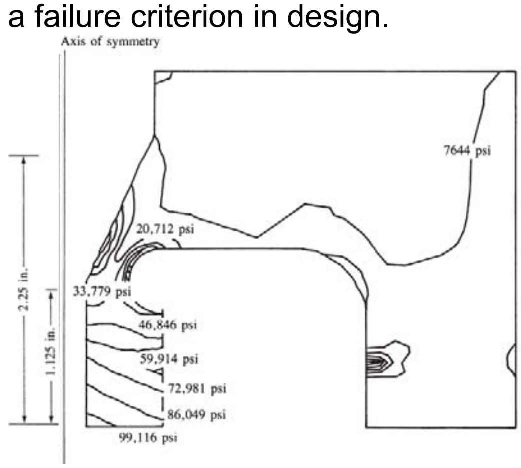
The die was modeled by using simple quadrilateral axisymmetric elements. The locations of high stress were of primary concern.



Axisymmetric Elements

Applications

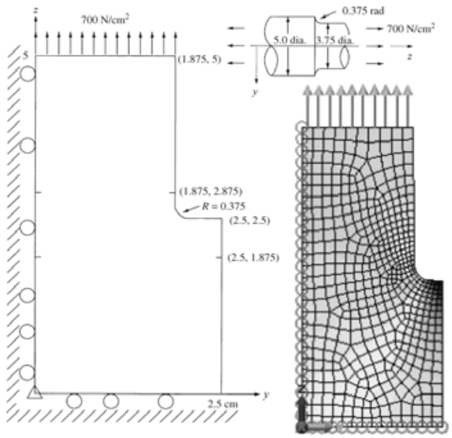
The figure shows a plot of the von Mises stress contours for the die. The von Mises (or equivalent, or effective) stress is often used as a failure criterion in design.



Axisymmetric Elements

Applications

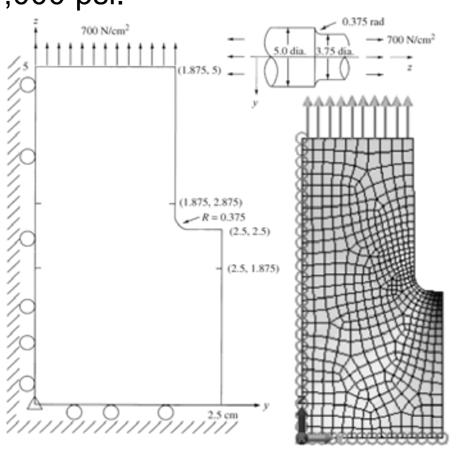
The figure shows a stepped 4130 steel shaft with a fillet radius subjected to an axial pressure of 1,000 psi in tension.



Axisymmetric Elements

Applications

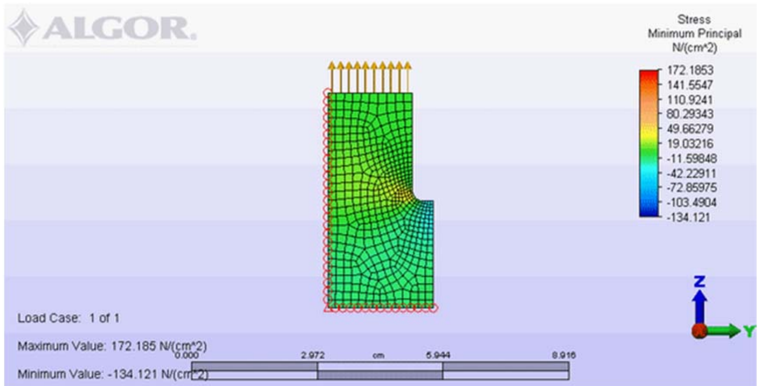
Fatigue analysis for reversed axial loading required an accurate stress concentration factor to be applied to the average axial stress of 1,000 psi.



Axisymmetric Elements

Applications

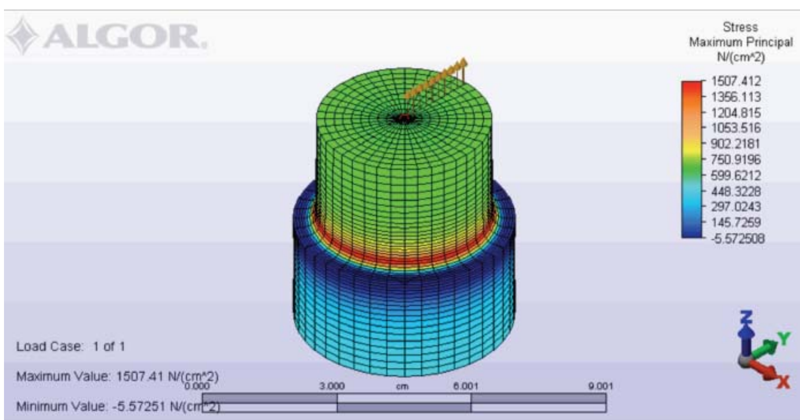
Fatigue analysis for reversed axial loading required an accurate stress concentration factor to be applied to the average axial stress of 1,000 psi.



Axisymmetric Elements

Applications

The figure below shows the resulting maximum principal stress plot using a computer program.



Axisymmetric Elements

Problems

18. Work problems ***9.2***, ***9.3***, and ***9.5*** in your textbook.

19. Work problem ***9.16*** in your textbook using Camp's axisymmetric CST Matlab code.

End of Chapter 9