Adjunctions

Ross Tate

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Definition (Adjunction for a 2-Category **C**)**.** A tuple $\langle \mathcal{C}, \mathcal{D}, f, g, \eta, \varepsilon, \mathfrak{f}, \mathfrak{g} \rangle$ whose components have the following types:

$\mathcal{C}$ is an object of **C**

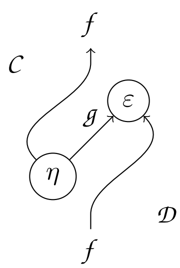
$\mathcal{D}$ is an object of **C**

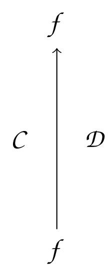
f is a morphism of **C** from $\mathcal{C}$ to $\mathcal{D}$

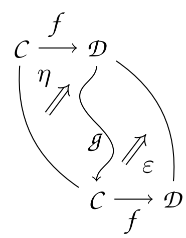
g is a morphism of **C** from $\mathcal{D}$ to $\mathcal{C}$

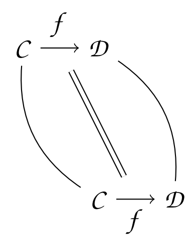
η is a 2-cell of **C** from $\mathcal{C}$ to $f\,;g$

ε is a 2-cell of **C** from $g\,;f$ to $\mathcal{D}$

$\mathfrak{f}$ is a proof that 

equals 

. In other words, 

equals 

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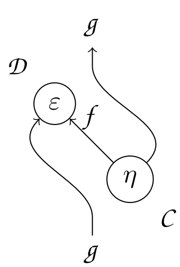
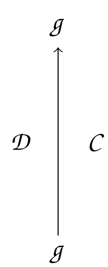
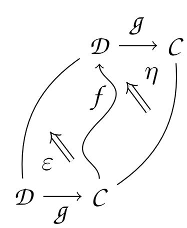
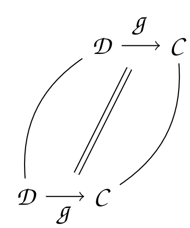
$\mathfrak{g}$ is a proof that 

equals 

. In other words, 

equals 

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Definition (Left/Right Adjoint)**.** f above is called the left adjoint, and g above is called the right adjoint. A morphism of a 2-category is a left/right adjoint if it is the left/right adjoint of some adjunction.

Exercise 1. Prove that there is a bijection between adjunctions in **Cat** and adjunctions via transpositions.

Example. Consider **Prost**. Suppose we had a pair of preordered sets $\langle C, \leq \rangle$ and $\langle D, \leq \rangle$, and we want to make an adjunction out of some relation-preserving functions $f : C \to D$ and $g : D \to C$. Then η exists if and only if $\forall c : C.\ c \leq g(f(c))$, and β exists if and only if $\forall d : D.\ f(g(d)) \leq g$. If η and β exist, then $\mathfrak{f}$ and $\mathfrak{g}$ are trivial since **Prost** is a *locally thin* 2-category. Such a situation is called a monotone Galois connection.