Alethic-Deontic Logic: Some Theorems

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Abstract
The purpose of this paper is to prove some theorems in alethic-deontic logic. Alethic-deontic logic is a kind of bimodal logic that combines ordinary alethic (modal) logic and deontic logic. Ordinary alethic logic is a branch of logic that deals with modal concepts, such as necessity and possibility, modal sentences, arguments and systems. Deontic logic is the logic of norms. It is about normative words, such as "ought", "right" and "wrong", normative sentences, arguments and systems. Alethic-deontic logic contains both modal and normative concepts and can be used to study how these interact. This paper contains some interesting theorems that can be proved in alethic-deontic logic. I will show that all primitive deontic operators are redundant when prefixed to the alethic operators in some systems. I will prove that necessarily equivalent sentences have the same deontic status in many systems. I will establish that the set of sentences in some alethic-deontic systems can be partitioned into five, mutually exclusive, exhaustive subsets. Finally, I will show that there are exactly ten distinct modalities in some alethic-deontic systems.

1. Introduction

The purpose of this paper is to prove some theorems in alethic-deontic logic. Alethic-deontic logic is a kind of bimodal logic that combines ordinary alethic (modal) logic and deontic logic. Ordinary alethic logic is a branch of logic that deals with modal concepts, such as necessity and possibility, modal sentences, arguments and systems. For some introductions, see e.g. Chellas (1980), Blackburn, de Rijke, & Venema (2001), Blackburn, van Benthem, Wolter (eds.) (2007), Fitting & Mendelsohn (1998), Gabbay (1976), Gabbay & Guenthner (2001), Kracht (1999), Garson (2006), Girle (2000), Lewis & Langford (1932), Popkorn (1994), Segerberg (1971), and Zeman (1973). Deontic logic is the logic of norms. It deals with normative words, such as "ought", "right" and "wrong", normative sentences, arguments and systems. Introductions to this branch can be found in e.g. Gabbay, Horty, Parent, van der Meyden & van der Torre (eds.) (2013), Hilpinen (1971), (1981), Rönnedal (2010), and Åqvist (1987), (2002). Alethic-deontic logic contains both modal and normative concepts and can be used to study how these interact. In the paper Rönnedal (2012) I say more about various bimodal systems. Alan R. Anderson was perhaps the first philosopher to combine alethic and deontic logic (see Anderson (1956)). Fine & Schurz (1996), Gabbay & Guenthner (2001), Gabbay, Kurucz, Wolter, Zakharyaschev (2003), Kracht (1999), and Kracht & Wolter (1991) include more information about how to combine various logical systems.

This paper contains some interesting theorems that can be proved in alethic-deontic logic. It is divided into 6 sections. Section 2 contains an introduction to the systems we study in this paper. In section 3 I will prove that all primitive deontic operators are redundant when prefixed to the alethic operators in some systems. Section 4 includes a proof of the fact that necessarily equivalent sentences have the same deontic status in many systems. In section 5 I will establish that the set of sentences in some alethic-deontic systems can be partitioned into five, mutually exclusive, exhaustive subsets. Finally, in section 6, I will prove that there are exactly ten distinct modalities in some alethic-deontic systems.

2. Alethic-deontic logic

In this section I will briefly describe the alethic-deontic logics that we will study in this essay. In the paper Rönnedal (2012) I say more about them and about bimodal systems in general. For more background information, see Rönnedal (2012b).

Syntax

Alphabet. (i) A denumerably infinite set Prop of proposition letters p, q, r, s, t, p_1, q_1, r_1, s_1, t_1, p_2, q_2, r_2, s_2, t_2…, (ii) the usual primitive truth-functional connectives, (iii) the modal operators $\Box$ and $\Diamond$, (iv) the deontic operators O and P, and (v) the brackets (,).

Language. The language L is the set of well-formed formulas (wffs) generated by the usual clauses for proposition letters and propositionally compound sentences, and the following clauses: (i) if A is a wff, then $\Box$A, $\Diamond$A, OA and PA are wffs, and (iii) nothing else is a wff.

Capital letters A, B, C … are used to represent arbitrary (not necessarily atomic) formulas of the object language. $\vdash$ A, says that A is a theorem (in some system determined by the context). Outer brackets around sentences are usually dropped if the result is not ambiguous.

Definitions. $\ominus A = \Box\neg A$, $FA = O\neg A$, $KA = PA \wedge P\neg A$, and $NA = (OA \vee O\neg A)$. $\bot$ (falsum) and $\top$ (verum) are defined as usual.

The translationfunction t. To understand the intended interpretation of the formal language in this essay we can use the following translation function. t($\neg$A) = It is not the case that t(A). t(A $\rightarrow$ B) = If t(A), then t(B). And similarly for all other propositional connectives. t($\Box$A) = It is necessary that t(A). t($\Diamond$A) = It is possible that t(A). t($\ominus$A) = It is impossible that t(A).

t(OA) = It ought to be the case that t(A). t(PA) = It is permitted that t(A). t(KA) = It is optional (deontically contingent) that t(A). t(NA) = It is non-optional (deontically non-contingent) that t(A). If t(p) is a sentence in English, we can use t to translate a formal sentence whose only atomic proposition letter is p into English. For instance, let t(p) be "You give money to every poor person in the whole world". Then the t-translation of "$\neg\Diamond p \rightarrow \neg Op$" is "If it is not the case that it is possible that you give money to every poor person in the whole world, then it is not the case that it ought to be the case that you give money to every poor person in the whole world". So, "$\neg\Diamond p \rightarrow \neg Op$" is the "contraposition" of one version of the so-called "ought implies can principle".

There seem to be several different kinds of necessity and possibility: logical, metaphysical, natural, historical etc. It might be plausible to use different logical systems to symbolise these different kinds. However, we can use the same symbols in each case.

Semantics

We use the same kind of semantics as in Rönnedal (2012). The only difference is that we treat $\diamondsuit$, F, K and N as defined operators in this essay. The fundamental concepts are the same, the truth-conditions for various sentences are the same, the classifications of different systems are the same.

Proof theory

We will use two kinds of proof theories in this essay: one axiomatic and one that is based on semantic tableaux. Both are described in Rönnedal (2012). All fundamental axioms and tableau rules that are used in our proofs in the present paper are also described in that essay. All other rules are easily derived from the axioms and the primitive rules together with the definitions. In Rönnedal (2012) we called some axioms a-axioms (a as in "alethic), some b-axioms, and some ab-axioms. We will call the b-axioms "d-axioms" in this essay (d as in "deontic"), and the ab-axioms "ad-axioms". And similarly for the tableau rules.

3. Redundant operators

Theorem 1. The deontic operators O and P are redundant when prefixed to the alethic operators $\Box$ and $\Diamond$ in **(i)** every tableau system that contains T-a4, T-a5, T-dD and T-MO (as primitive or derived rules), and in **(ii)** every axiomatic system that contains a4, a5, dD and MO (as axioms or theorems). I.e. $\vdash \otimes A \leftrightarrow *\otimes A$, where $\otimes = \Box$ or $\Diamond$ and $* =$ O or P, holds in the indicated systems.

Proof. **(i)** To prove this we must show that all of the following sentences are theorems in every tableau system that includes the tableau rules T-a4, T-a5, T-dD and T-MO: □A ↔ O□A (O□R), □A ↔ P□A (P□R), ◇A ↔ O◇A (O◇R), ◇A ↔ P◇A (P◇R). We begin by proving O□R. O□R states that it is necessary that A if and only if it is obligatory that it is necessary that A.

O□R   □A ↔ O□A

(1) ¬(□A ↔ O□A), 0

↙ ↘

(2) □A, 0 [1, ¬↔]	(14) ¬□A, 0 [1, ¬↔]
(3) ¬O□A, 0 [1, ¬↔]	(15) O□A, 0 [1, ¬↔]
(4) P¬□A, 0 [3, ¬O]	(16) ◇¬A, 0 [14, ¬□]
(5) 0s1 [4, P]	(17) 0r1 [16, ◇]
(6) ¬□A, 1 [4, P]	(18) ¬A, 1 [16, ◇]
(7) ◇¬A, 1 [6, ¬□]	(19) 0s2 [T-dD]
(8) 1r2 [7, ◇]	(20) □A, 2 [15, 19, O]
(9) ¬A, 2 [7, ◇]	(21) 0r2 [19, T-MO]
(10) 0r1 [5, T-MO]	(22) 2r1 [17, 21, T-a5]
(11) 0r2 [8, 10, T-a4]	(23) A, 1 [20, 22, □]
(12) A, 2 [2, 11, □]	(24) * [18, 23]
(13) * [9, 12]	

Hence, we see that O may be deleted when prefixed to a sentence of the form □A. Next we turn to P□R.

P□R   □A ↔ P□A

(1) ¬(□A ↔ P□A), 0

↙ ↘

(2) □A, 0 [1, ¬↔]	(14) ¬□A, 0 [1, ¬↔]
(3) ¬P□A, 0 [1, ¬↔]	(15) P□A, 0 [1, ¬↔]
(4) O¬□A, 0 [3, ¬P]	(16) ◇¬A, 0 [14, ¬□]
(5) 0s1 [T-dD]	(17) 0s1 [15, P]
(6) ¬□A, 1 [4, 5, O]	(18) □A, 1 [15, P]
(7) ◇¬A, 1 [6, ¬□]	(19) 0r2 [16, ◇]
(8) 1r2 [7, ◇]	(20) ¬A, 2 [16, ◇]
(9) ¬A, 2 [7, ◇]	(21) 0r1 [17, T-MO]
(10) 0r1 [5, T-MO]	(22) 1r2 [19, 21, T-a5]
(11) 0r2 [8, 10, T-a4]	(23) A, 2 [18, 22, □]
(12) A, 2 [2, 11, □]	(24) * [20, 23]
(13) * [9, 12]	

According to P□R, it is necessary that A if and only if it is permitted that it is necessary that A. Consequently, P may be deleted when prefixed to a sentence of the form □A.

The following tableau establishes that $O\Diamond R$ is a theorem in the indicated systems.

$O\Diamond R$ $\quad \Diamond A \leftrightarrow O\Diamond A$

$$(1)\ \neg(\Diamond A \leftrightarrow O\Diamond A),\ 0$$

$$\swarrow \qquad \searrow$$

(2) $\Diamond A$, 0 [1, $\neg\leftrightarrow$]	(14) $\neg\Diamond A$, 0 [1, $\neg\leftrightarrow$]
(3) $\neg O\Diamond A$, 0 [1, $\neg\leftrightarrow$]	(15) $O\Diamond A$, 0 [1, $\neg\leftrightarrow$]
(4) $P\neg\Diamond A$, 0 [3, $\neg O$]	(16) $\Box\neg A$, 0 [14, $\neg\Diamond$]
(5) 0s1 [4, P]	(17) 0s1 [T-dD]
(6) $\neg\Diamond A$, 1 [4, P]	(18) $\Diamond A$, 1 [15, 17, O]
(7) $\Box\neg A$, 1 [6, $\neg\Diamond$]	(19) 1r2 [18, $\Diamond$]
(8) 0r2 [2, $\Diamond$]	(20) A, 2 [18, $\Diamond$]
(9) A, 2 [2, $\Diamond$]	(21) 0r1 [17, T-MO]
(10) 0r1 [5, T-MO]	(22) 0r2 [19, 21, T-a4]
(11) 1r2 [8, 10, T-a5]	(23) $\neg A$, 2 [16, 22, $\Box$]
(12) $\neg A$, 2 [7, 11, $\Box$]	(24) * [20, 23]
(13) * [9, 12]	

According to $O\Diamond R$, it is possible that A if and only if it is obligatory that it is possible that A. O may thus be deleted when prefixed to a sentence of the form $\Diamond A$. Finally, we prove $P\Diamond R$.

$P\Diamond R$ $\quad \Diamond A \leftrightarrow P\Diamond A$

$$(1)\ \neg(\Diamond A \leftrightarrow P\Diamond A),\ 0$$

$$\swarrow \qquad \searrow$$

(2) $\Diamond A$, 0 [1, $\neg\leftrightarrow$]	(14) $\neg\Diamond A$, 0 [1, $\neg\leftrightarrow$]
(3) $\neg P\Diamond A$, 0 1, $\neg\leftrightarrow$]	(15) $P\Diamond A$, 0 [1, $\neg\leftrightarrow$]
(4) $O\neg\Diamond A$, 0 [3, $\neg P$]	(16) $\Box\neg A$, 0 [14, $\neg\Diamond$]
(5) 0r1 [2, $\Diamond$]	(17) 0s1 [15, P]
(6) A, 1 [2, $\Diamond$]	(18) $\Diamond A$, 1 [15, P]
(7) 0s2 [T-dD]	(19) 1r2 [18, $\Diamond$]
(8) $\neg\Diamond A$, 2 [4, 7, O]	(20) A, 2 [18, $\Diamond$]
(9) $\Box\neg A$, 2 [8, $\neg\Diamond$]	(21) 0r1 [17, T-MO]
(10) 0r2 [7, T-MO]	(22) 0r2 [19, 21, T-a4]
(11) 2r1 [5, 10, T-a5]	(23) $\neg A$, 2 [16, 22, $\Box$]
(12) $\neg A$, 1 [9, 11, $\Box$]	(24) * [20, 23]
(13) * [6, 12]	

$P\Diamond R$ says that it is possible that A if and only if it is permitted that it is possible that A. We conclude that P may be deleted when prefixed to a sentence of the form $\Diamond A$ in the systems we have mentioned.

(ii) follows immediately from (i) and the soundness and completeness theorems found in Rönnedal (2012) and (2012b). However, in section 6 we will also consider some explicit axiomatic arguments. ■

The proof of theorem 1 is now finished. Due to this theorem, we may always delete the deontic operators O and P when they are prefixed to $\Box$ or $\Diamond$ in any part of any formula in the indicated systems. O$\Box$R, P$\Box$R etc. can be viewed as a kind of reduction principles similar to other well known principles of this sort.

What happens if we add F, K and N to our language? Does any of the following “reduction laws” hold: $\Box A \leftrightarrow F\Box A$, $\Diamond A \leftrightarrow F\Diamond A$, $\Box A \leftrightarrow K\Box A$, $\Diamond A \leftrightarrow K\Diamond A$, $\Box A \leftrightarrow N\Box A$, $\Diamond A \leftrightarrow N\Diamond A$? The answer to this question is no. However we have: $F\Box A \leftrightarrow \neg\Box A$, $F\Diamond A \leftrightarrow \neg\Diamond A$, $K\Box A \leftrightarrow \bot$, $K\Diamond A \leftrightarrow \bot$, $N\Box A \leftrightarrow \top$, and $N\Diamond A \leftrightarrow \top$. So, in one sense F, K and N are redundant when prefixed to $\Box$ or $\Diamond$. However, if we delete them from a formula this might affect the formula’s truth-value.

4. Necessarily equivalent sentences and deontic status

Theorem 2. Necessarily equivalent sentences have the same deontic status with respect to O, P, F, K and N in **(i)** every tableau system that contains T-MO, and in **(ii)** every axiomatic system that contains MO. More precisely: $\vdash \Box(A \leftrightarrow B) \to (*A \leftrightarrow *B)$, where * = O, P, F, K and N, hold in the indicated systems. If two sentences do not have the same deontic status with respect to O, P, F, K and N in a tableau system that contains T-MO or in an axiomatic system that contains MO, then they are not necessarily equivalent. More precisely: $\vdash \neg(*A \leftrightarrow *B) \to \neg\Box(A \leftrightarrow B)$, where * = O, P, F, K and N, holds in **(iii)** every tableau system that contains T-MO, and in **(iv)** every axiomatic system that contains MO.[1]

Proof. **(i)** To prove this theorem we have to show that all of the following sentences are theorems in every tableau system that contains T-MO: $\Box(A \leftrightarrow B) \to (PA \leftrightarrow PB)$ (PE), $\Box(A \leftrightarrow B) \to (OA \leftrightarrow OB)$ (OE), $\Box(A \leftrightarrow B) \to (FA \leftrightarrow FB)$ (FE), $\Box(A \leftrightarrow B) \to (KA \leftrightarrow KB)$ (KE), and $\Box(A \leftrightarrow B) \to (NA \leftrightarrow NB)$ (NE). The tableau proofs of (OE) and (NE) are left to the reader. However, we will see how it is easy (by an axiomatic argument) to establish (NE) once we have proven (KE). All we have to do then is to prove (PE), (FE), and (KE). Let us begin with (PE).

(PE) says that if it is necessary that t(A) if and only if t(B), then it is permitted that t(A) if and only if it is permitted that t(B), according to the t-translation of this sentence.

[1] Of course, $\Box(A \leftrightarrow B) \to (\Box A \leftrightarrow \Box B)$ and $\Box(A \leftrightarrow B) \to (\Diamond A \leftrightarrow \Diamond B)$ are also theorems in every alethic-deontic system described in Rönnedal (2012).

$\Box(A \leftrightarrow B) \rightarrow (PA \leftrightarrow PB)$

(1) $\Box(A \leftrightarrow B) \rightarrow (PA \leftrightarrow PB)$, 0
(2) $\Box(A \leftrightarrow B)$, 0
(3) $\neg(PA \leftrightarrow PB)$, 0
↙ ↘

(4) PA, 0	(18) ¬PA, 0
(5) ¬PB, 0	(19) PB, 0
(6) O¬B, 0	(20) O¬A, 0
(7) 0s1	(21) 0s1
(8) A, 1	(22) B, 1
(9) ¬B, 1	(23) ¬A, 1
(10) 0r1	(24) 0r1
(11) $A \leftrightarrow B$, 1	(25) $A \leftrightarrow B$, 1
↙ ↘	↙ ↘

(12) A, 1	(15) ¬A, 1	(26) A, 1	(29) ¬A, 1
(13) B, 1	(16) ¬B, 1	(27) B, 1	(30) ¬B, 1
(14) *	(17) *	(28) *	(31) *

Next we continue with (FE). The t-translation of (FE) looks like this: If it is necessary that t(A) if and only if t(B), then it is forbidden that t(A) if and only if it is forbidden that t(B).

$\Box(A \leftrightarrow B) \rightarrow (FA \leftrightarrow FB)$

(1) $\neg(\Box(A \leftrightarrow B) \rightarrow (FA \leftrightarrow FB))$, 0
(2) $\Box(A \leftrightarrow B)$, 0
(3) $\neg(FA \leftrightarrow FB)$, 0
↙ ↘

(4) FA, 0	(19) ¬FA, 0
(5) ¬FB, 0	(20) FB, 0
(6) O¬A, 0	(21) O¬B, 0
(7) PB, 0	(22) PA, 0
(8) 0s1	(23) 0s1
(9) B, 1	(24) A, 1
(10) ¬A, 1	(25) ¬B, 1
(11) 0r1	(26) 0r1
(12) $A \leftrightarrow B$, 1	(27) $A \leftrightarrow B$, 1
↙ ↘	↙ ↘

(13) A, 1	(16) ¬A, 1	(28) A, 1	(31) ¬A, 1
(14) B, 1	(17) ¬B, 1	(29) B, 1	(32) ¬B, 1
(15) *	(18) *	(30) *	(33) *

Since we have not introduced any special rules for deontic contingency and non-contingency, or optionality and non-optionality, we must first translate the sentence (KE) to be able to decide whether it is valid or not by use of the tableau method. By definition, $\Box(A \leftrightarrow B) \rightarrow (KA \leftrightarrow KB)$ is equivalent to $\Box(A \leftrightarrow B) \rightarrow ((PA \wedge P\neg A) \leftrightarrow (PB \wedge P\neg B))$. To prove that the former is a theorem in our indicated systems, it thus suffices to prove the latter.

If we apply the t-translation function to (KE), it says that if it is necessary that t(A) if and only if t(B), then it is optional that t(A) if and only if it is optional that t(B). Here is the tableau proof.

$\Box(A \leftrightarrow B) \rightarrow (KA \leftrightarrow KB)$

(1) $\neg(\Box(A \leftrightarrow B) \rightarrow ((PA \wedge P\neg A) \leftrightarrow (PB \wedge P\neg B)))$, 0
(2) $\Box(A \leftrightarrow B)$, 0
(3) $\neg((PA \wedge P\neg A) \leftrightarrow (PB \wedge P\neg B))$, 0
$\swarrow$ $\searrow$
(4) $PA \wedge P\neg A$, 0 | (34) $\neg(PA \wedge P\neg A)$, 0
(5) $\neg(PB \wedge P\neg B)$, 0 | (35) $PB \wedge P\neg B$, 0
(6) PA, 0 | (36) PB, 0
(7) $P\neg A$, 0 | (37) $P\neg B$, 0
(8) 0s1 | (38) 0s1
(9) A, 1 | (39) B, 1
(10) 0s2 | (40) 0s2
(11) $\neg A$, 2 | (41) $\neg B$, 2
(12) 0r1 | (42) 0r1
(13) 0r2 | (43) 0r2
(14) $A \leftrightarrow B$, 1 | (44) $A \leftrightarrow B$, 1
(15) $A \leftrightarrow B$, 2 | (45) $A \leftrightarrow B$, 2
$\swarrow$ $\searrow$ | $\swarrow$ $\searrow$
(16) A, 1 | (31) $\neg A$, 1 | (46) A, 1 | (61) $\neg A$, 1
(17) B, 1 | (32) $\neg B$, 1 | (47) B, 1 | (62) $\neg B$, 1
$\swarrow$ $\searrow$ | (33) * | $\swarrow$ $\searrow$ | (63) *
(18) A, 2 | (21) $\neg A$, 2 | (48) A, 2 | (51) $\neg A$, 2
(19) B, 2 | (22) $\neg B$, 2 | (49) B, 2 | (52) $\neg B$, 2
(20) * | $\swarrow$ $\searrow$ | (50) * | $\swarrow$ $\searrow$
(23) $\neg PB$, 0 | (27) $\neg P\neg B$, 0 | (53) $\neg PA$, 0 | (57) $\neg P\neg A$, 0
(24) $O\neg B$, 0 | (28) $O\neg\neg B$, 0 | (54) $O\neg A$, 0 | (58) $O\neg\neg A$, 0
(25) $\neg B$, 1 | (29) $\neg\neg B$, 2 | (55) $\neg A$, 1 | (59) $\neg\neg A$, 2
(26) * | (30) * | (56) * | (60) *

I will omit the tableau proof of (NE). It is similar to the one just given. Instead we establish the result by the following reasoning. 1. $\Box(A \leftrightarrow B) \rightarrow$

$(KA \leftrightarrow KB)$ [From the tableau proof above]. 2. $\Box(A \leftrightarrow B) \rightarrow (\neg KA \leftrightarrow \neg KB)$ [1, propositional logic]. 3. $\Box(A \leftrightarrow B) \rightarrow (NA \leftrightarrow NB)$ [by 2 and the definitions of K and N]. We know that this kind of reasoning (and the kind exhibited in the proof of KE) is correct, since we have proved that all our axiomatic systems as well as our tableau systems are sound and complete with respect to the same semantics (Rönnedal (2012), (2012b)).

(ii) follows immediately from (i) and the soundness and completeness theorems found in Rönnedal (2012) and (2012b).

(iii) and **(iv).** This is essentially the contrapositive of part (i) and part (ii), respectively. (Details are easy and are left to the reader.) ■

5. A partition of the sentences in T-dD and dD systems

Theorem 3. The set of all sentences (in our formal language) can be partitioned into the following, mutually exclusive, exhaustive subsets in **(i)** every tableau system that includes T-dD (as a primitive or derived rule), and in **(ii)** every axiomatic system that includes dD (as an axiom or theorem).

$\Box A \wedge OA$	$OA \wedge \neg\Box A$	$PA \wedge P\neg A$	$FA \wedge \neg\diamond A$	$FA \wedge \diamond A$

Proof. **(i)** To prove this theorem we must show that every sentence is contained in at least one of these categories and that it is contained in at most one of them. This amounts to proving the following theorems.

$P = (\Box A \wedge OA) \vee (OA \wedge \neg\Box A)$
$\quad \vee\ (PA \wedge P\neg A) \vee (FA \wedge \neg\diamond A) \vee (FA \wedge \diamond A).$
$A = (\Box A \wedge OA) \rightarrow$
$\quad (\neg(OA \wedge \neg\Box A) \wedge \neg(PA \wedge P\neg A) \wedge \neg(FA \wedge \neg\diamond A) \wedge \neg(FA \wedge \diamond A)).$
$B = (OA \wedge \neg\Box A) \rightarrow$
$\quad (\neg(\Box A \wedge OA) \wedge \neg(PA \wedge P\neg A) \wedge \neg(FA \wedge \neg\diamond A) \wedge \neg(FA \wedge \diamond A)).$
$C = (PA \wedge P\neg A) \rightarrow$
$\quad (\neg(OA \wedge \neg\Box A) \wedge \neg(\Box A \wedge OA) \wedge \neg(FA \wedge \neg\diamond A) \wedge \neg(FA \wedge \diamond A)).$
$D = (FA \wedge \neg\diamond A) \rightarrow$
$\quad (\neg(OA \wedge \neg\Box A) \wedge \neg(PA \wedge P\neg A) \wedge \neg(\Box A \wedge OA) \wedge \neg(FA \wedge \diamond A)).$
$E = (FA \wedge \diamond A) \rightarrow$
$\quad (\neg(OA \wedge \neg\Box A) \wedge \neg(PA \wedge P\neg A) \wedge \neg(FA \wedge \neg\diamond A) \wedge \neg(\Box A \wedge OA)).$

A is equivalent to the conjunction of the following sentences:

A1 $(\Box A \wedge OA) \rightarrow \neg(OA \wedge \neg\Box A)$, A2 $(\Box A \wedge OA) \rightarrow \neg(PA \wedge P\neg A)$, A3 $(\Box A \wedge OA) \rightarrow \neg(FA \wedge \neg\diamond A)$, A4 $(\Box A \wedge OA) \rightarrow \neg(FA \wedge \diamond A)$.

B is equivalent to the conjunction of the following sentences:

B1 $(OA \wedge \neg\Box A) \rightarrow \neg(\Box A \wedge OA)$, B2 $(OA \wedge \neg\Box A) \rightarrow \neg(PA \wedge P\neg A)$, B3 $(OA \wedge \neg\Box A) \rightarrow \neg(FA \wedge \neg\text{⟠}A)$, B4 $(OA \wedge \neg\Box A) \rightarrow \neg(FA \wedge \text{⟠}A)$.

C is equivalent to the conjunction of the following sentences: C1 $(PA \wedge P\neg A) \rightarrow \neg(OA \wedge \neg\Box A)$, C2 $(PA \wedge P\neg A) \rightarrow \neg(\Box A \wedge OA)$, C3 $(PA \wedge P\neg A) \rightarrow \neg(FA \wedge \neg\text{⟠}A)$, C4 $(PA \wedge P\neg A) \rightarrow \neg(FA \wedge \text{⟠}A)$.

D is equivalent to the conjunction of the following sentences: D1 $(FA \wedge \neg\text{⟠}A) \rightarrow \neg(OA \wedge \neg\Box A)$, D2 $(FA \wedge \neg\text{⟠}A) \rightarrow \neg(PA \wedge P\neg A)$, D3 $(FA \wedge \neg\text{⟠}A) \rightarrow \neg(\Box A \wedge OA)$, D4 $(FA \wedge \neg\text{⟠}A) \rightarrow \neg(FA \wedge \text{⟠}A)$.

E is equivalent to the conjunction of the following sentences: E1 $(FA \wedge \text{⟠}A) \rightarrow \neg(OA \wedge \neg\Box A)$, E2 $(FA \wedge \text{⟠}A) \rightarrow \neg(PA \wedge P\neg A)$, E3 $(FA \wedge \text{⟠}A) \rightarrow \neg(FA \wedge \neg\text{⟠}A)$, E4 $(FA \wedge \text{⟠}A) \rightarrow \neg(\Box A \wedge OA)$.

So, it is enough that we show that P, A1-A4, B1-B4, C1-C4, D1-D4 and E1-E4 are theorems in our systems. A1, B1, D4 and E3 are true by propositional logic alone. We continue to prove A3 and A4.

A3 $(\Box A \wedge OA) \rightarrow \neg(FA \wedge \neg\text{⟠}A)$

(1) $\neg((\Box A \wedge OA) \rightarrow \neg(FA \wedge \neg\text{⟠}A)), 0$
(2) $\Box A \wedge OA, 0$ [1, $\neg\rightarrow$]
(3) $\neg\neg(FA \wedge \neg\text{⟠}A), 0$ [1, $\neg\rightarrow$]
(4) $FA \wedge \neg\text{⟠}A, 0$ [3, $\neg\neg$]
(5) $\Box A, 0$ [2, $\wedge$]
(6) $OA, 0$ [2, $\wedge$]
(7) $FA, 0$ [4, $\wedge$]
(8) $\neg\text{⟠}A, 0$ [4, $\wedge$]
(9) 0s1 [T-dD]
(10) A, 1 [6, 9, O]
(11) $\neg A$, 1 [7, 9, F']
(12) * [10, 11]

A4 $(\Box A \wedge OA) \rightarrow \neg(FA \wedge \text{⟠}A)$

(1) $\neg((\Box A \wedge OA) \rightarrow \neg(FA \wedge \text{⟠}A)), 0$
(2) $\Box A \wedge OA, 0$ [1, $\neg\rightarrow$]
(3) $\neg\neg(FA \wedge \text{⟠}A), 0$ [1, $\neg\rightarrow$]
(4) $FA \wedge \text{⟠}A, 0$ [3, $\neg\neg$]
(5) $\Box A, 0$ [2, $\wedge$]
(6) $OA, 0$ [2, $\wedge$]
(7) $FA, 0$ [4, $\wedge$]
(8) $\text{⟠}A, 0$ [4, $\wedge$]
(9) 0s1 [T-dD]
(10) A, 1 [6, 9, O]
(11) $\neg A$, 1 [7, 9, F']
(12) * [10, 11]

B3 $(OA \wedge \neg\Box A) \rightarrow \neg(FA \wedge \neg\text{⟠}A)$

(1) $\neg((OA \wedge \neg\Box A) \rightarrow \neg(FA \wedge \neg\text{⟠}A)), 0$
(2) $OA \wedge \neg\Box A, 0$ [1, $\neg\rightarrow$]
(3) $\neg\neg(FA \wedge \neg\text{⟠}A), 0$ [1, $\neg\rightarrow$]
(4) $FA \wedge \neg\text{⟠}A, 0$ [3, $\neg\neg$]
(5) $OA, 0$ [2, $\wedge$]
(6) $\neg\Box A, 0$ [2, $\wedge$]
(7) $FA, 0$ [4, $\wedge$]
(8) $\neg\text{⟠}A, 0$ [4, $\wedge$]

B4 $(OA \wedge \neg\Box A) \rightarrow \neg(FA \wedge \text{⟠}A)$

(1) $\neg((OA \wedge \neg\Box A) \rightarrow \neg(FA \wedge \text{⟠}A)), 0$
(2) $OA \wedge \neg\Box A, 0$ [1, $\neg\rightarrow$]
(3) $\neg\neg(FA \wedge \text{⟠}A), 0$ [1, $\neg\rightarrow$]
(4) $FA \wedge \text{⟠}A, 0$ [3, $\neg\neg$]
(5) $OA, 0$ [2, $\wedge$]
(6) $\neg\Box A, 0$ [2, $\wedge$]
(7) $FA, 0$ [4, $\wedge$]
(8) $\text{⟠}A, 0$ [4, $\wedge$]

(9) 0s1 [T-dD]
(10) A, 1 [5, 9, O]
(11) ¬A, 1 [7, 9, F’]
(12) * [10, 11]

(9) 0s1 [T-dD]
(10) A, 1 [5, 9, O]
(11) ¬A, 1 [7, 9, F’]
(12) * [10, 11]

D1 is equivalent to B3 and E1 is equivalent to B4. Consequently, D1 and E1 are theorems in all T-dD systems. For the tableaux above prove that both B3 and B4 are theorems in systems of this kind.

A2 (□A ∧ OA) → ¬(PA ∧ P¬A)

(1) ¬((□A ∧ OA) → ¬(PA ∧ P¬A)), 0
(2) □A ∧ OA, 0 [1, ¬→]
(3) ¬¬(PA ∧ P¬A), 0 [1, ¬→]
(4) PA ∧ P¬A, 0 [3, ¬¬]
(5) □A, 0 [2, ∧]
(6) OA, 0 [2, ∧]
(7) PA, 0 [4, ∧]
(8) P¬A, 0 [4, ∧]
(9) 0s1 [8, P]
(10) ¬A, 1 [8, P]
(11) A, 1 [6, 9, O]
(12) * [10, 11]

B2 (OA ∧ ¬□A) → ¬(PA ∧ P¬A)

(1) ¬((OA ∧ ¬□A) → ¬(PA ∧ P¬A)), 0
(2) OA ∧ ¬□A, 0 [1, ¬→]
(3) ¬¬(PA ∧ P¬A), 0 [1, ¬→]
(4) PA ∧ P¬A, 0 [3, ¬¬]
(5) OA, 0 [2, ∧]
(6) ¬□A, 0 [2, ∧]
(7) PA, 0 [4, ∧]
(8) P¬A, 0 [4, ∧]
(9) 0s1 [8, P]
(10) ¬A, 1 [8, P]
(11) A, 1 [5, 9, O]
(12) * [10, 11]

We now know that A2 and B2 are theorems in all T-dD systems. C2 is the contrapositive of A2 and C1 is the contrapositive of B2. It follows that also C2 and C1 are theorems in all systems of this kind.

D2 (FA ∧ ¬⟠A) → ¬(PA ∧ P¬A)

(1) ¬((FA ∧ ¬⟠A) → ¬(PA ∧ P¬A)), 0
(2) FA ∧ ¬⟠A, 0 [1, ¬→]
(3) ¬¬(PA ∧ P¬A), 0 [1, ¬→]
(4) PA ∧ P¬A, 0 [3, ¬¬]
(5) FA, 0 [2, ∧]
(6) ¬⟠A, 0 [2, ∧]
(7) PA, 0 [4, ∧]
(8) P¬A, 0 [4, ∧]
(9) 0s1 [7, P]
(10) A, 1 [7, P]
(11) ¬A, 1 [5, 9, F’]
(12) * [10, 11]

E2 (FA ∧ ⟠A) → ¬(PA ∧ P¬A)

(1) ¬((FA ∧ ⟠A) → ¬(PA ∧ P¬A)), 0
(2) FA ∧ ⟠A, 0 [1, ¬→]
(3) ¬¬(PA ∧ P¬A), 0 [1, ¬→]
(4) PA ∧ P¬A, 0 [3, ¬¬]
(5) FA, 0 [2, ∧]
(6) ⟠A, 0 [2, ∧]
(7) PA, 0 [4, ∧]
(8) P¬A, 0 [4, ∧]
(9) 0s1 [7, P]
(10) A, 1 [7, P]
(11) ¬A, 1 [5, 9, F’]
(12) * [10, 11]

These tableaux prove that D2 and E2 are theorems in all T-dD systems. D2 is the contrapositive of C3 and E2 is the contrapositive of C4. Hence, C3 and C4 are also theorems in every T-dD system. D3 is the contrapositive of A3 and E4 is the contrapositive of A4. It follows that D3 and E3 are theorems in all T-dD systems.

We have now established A, B, C, D and E. All that is left to prove is that (P) is a theorem in all tableau systems that include T-dD.

Consider (P) $(\Box A \wedge OA) \vee (OA \wedge \neg\Box A) \vee (PA \wedge P\neg A) \vee (FA \wedge \neg\Diamond A) \vee (FA \wedge \Diamond A)$. We can prove this sentence directly by constructing a semantic tableau that starts with the negation of (P). But the proof is quite long. So, I will combine axiomatic and tableau techniques. First we show that $OA \vee (PA \wedge P\neg A) \vee FA$ is a theorem in every system that includes T-dD. Indeed, this sentence is a theorem even in the weakest so-called normal deontic logic, sometimes called OK. The theorem states that everything is obligatory, optional or forbidden.

P $\quad OA \vee ((PA \wedge P\neg A) \vee FA)$

(1) $\neg(OA \vee ((PA \wedge P\neg A) \vee FA))$, 0
(2) $\neg OA$, 0 [1, $\neg\vee$]
(3) $\neg((PA \wedge P\neg A) \vee FA)$, 0 [1, $\neg\vee$]
(4) $\neg(PA \wedge P\neg A)$, 0 [3, $\neg\vee$]
(5) $\neg FA$, 0 [3, $\neg\vee$]
(6) $P\neg A$, 0 [2, $\neg O$]
(7) PA, 0 [5, $\neg F$]
(8) 0s1 [7, P]
(9) A, 1 [7, P]
(10) 0s2 [6, P]
(11) $\neg A$, 2 [6, P]
↙ ↘

(12) $\neg PA$, 0 [4, $\neg\wedge$]	(16) $\neg P\neg A$, 0 [4, $\neg\wedge$]
(13) $O\neg A$, 0 [12, $\neg P$]	(17) $O\neg\neg A$, 0 [16, $\neg P$]
(14) $\neg A$, 1 [8, 13, O]	(18) $\neg\neg A$, 2 [10, 17, O]
(15) * [9, 14]	(19) * [11, 18]

Now we can reason as follows. We know that A is equivalent to $(A \wedge B) \vee (A \wedge \neg B)$. Hence, OA is equivalent to $(OA \wedge \Box A) \vee (OA \wedge \neg\Box A)$ and FA is equivalent to $(FA \wedge \neg\Diamond A) \vee (FA \wedge \Diamond A)$. So, we substitute OA by $(OA \wedge \Box A) \vee (OA \wedge \neg\Box A)$ and FA by $(FA \wedge \neg\Diamond A) \vee (FA \wedge \Diamond A)$ in $OA \vee (PA \wedge P\neg A) \vee FA$. It follows that $(\Box A \wedge OA) \vee (OA \wedge \neg\Box A) \vee (PA \wedge P\neg A) \vee (FA \wedge \neg\Diamond A) \vee (FA \wedge \Diamond A)$ is a theorem. Our proof of theorem 3 is now finished. ■

6. Modalities in some systems

Theorem 4. In theorem 4 we use a language without the defined concepts in section 2. **(i)** In every axiomatic (normal) alethic-deontic logic that contains the deontic system OS5+, the alethic system S5[2], MO, OC, ad4 $OA \rightarrow \Box OA$, and ad5 $PA \rightarrow \Box PA$ (as axioms and/or theorems) there are at most ten distinct modalities: A, $\neg A$, $\Diamond A$, $\Box A$, PA, OA, $\neg\Diamond A$, $\neg\Box A$, $\neg PA$ and $\neg OA$. **(ii)** In the axiomatic alethic-deontic system aS5dOS5+adMOOC45[3] there are exactly ten distinct modalities: A, $\neg A$, $\Diamond A$, $\Box A$, PA, OA, $\neg\Diamond A$, $\neg\Box A$, $\neg PA$ and $\neg OA$. **(iii)** In every normal alethic-deontic tableau system that includes T-aT, T-aB, T-a4, T-dD, T-d4, T-d5, T-MO, T-OC, T-ad4, and T-ad5 (as primitive and/or derived rules) there are at most ten distinct modalities: A, $\neg A$, $\Diamond A$, $\Box A$, PA, OA, $\neg\Diamond A$, $\neg\Box A$, $\neg PA$ and $\neg OA$. **(iv)** In the alethic-deontic tableau system T-aTB4dD45adMOOC45[4] there are exactly ten distinct modalities: A, $\neg A$, $\Diamond A$, $\Box A$, PA, OA, $\neg\Diamond A$, $\neg\Box A$, $\neg PA$ and $\neg OA$. **(v)** In the axiomatic alethic-deontic system aS5dOS5+adMOOC45 and in the alethic-deontic tableau system T-aTB4dD45adMOOC45 (and in all systems that are deductively equivalent) a string of modal operators reduces to its innermost modality. E.g. all of the following equivalences hold in these systems: $OO\Diamond A \leftrightarrow \Diamond A$, $\neg\Box\Box OPA \leftrightarrow \neg PA$, $\Box\Box\neg\Diamond POOA \leftrightarrow \neg OA$, $P\Box\Diamond\Diamond\neg P\neg PA \leftrightarrow PA$, $\Diamond\Box OP\Box A \leftrightarrow \Box A$, $\neg\Diamond\Box\Box\neg OA \leftrightarrow OA$, $\neg P\neg\Box\Box\Box\neg\Diamond A \leftrightarrow \neg\Diamond A$, $\neg O\neg\Diamond\Diamond\Diamond\Box O\neg\Box A \leftrightarrow \neg\Box A$.

Proof. **(i)** The proof of this theorem consists of two parts. First we show that the systems we are interested in contain the following reduction laws:

> (i) $A \leftrightarrow \neg\neg A$, (ii) $\Diamond A \leftrightarrow \Diamond\Diamond A$, (iii) $\Box A \leftrightarrow \Box\Box A$, (iv) $\Diamond A \leftrightarrow \Box\Diamond A$, (v) $\Box A \leftrightarrow \Diamond\Box A$, (vi) $PA \leftrightarrow PPA$, (vii) $OA \leftrightarrow OOA$, (viii) $PA \leftrightarrow OPA$, (ix) $OA \leftrightarrow POA$, (x) $\Diamond A \leftrightarrow P\Diamond A$, (xi) $\Diamond A \leftrightarrow O\Diamond A$, (xii) $\Box A \leftrightarrow P\Box A$, (xiii) $\Box A \leftrightarrow O\Box A$, (xiv) $PA \leftrightarrow \Diamond PA$, (xv) $PA \leftrightarrow \Box PA$, (xvi) $OA \leftrightarrow \Diamond OA$, (xvii) $OA \leftrightarrow \Box OA$.

Then we prove that given these reduction laws any modality is equivalent to one of the ten modalities mentioned above. To prove the first step, we use axiomatic techniques. It is a well known fact that (ii)-(v) hold in S5 and it has been established that (vi)-(ix) are theorems in OS5+.[5] All that remains to show then is (i) and (x)-(xvii). (i) is simply the law of double negation and (xii)-(xvii) can be obtained in the following manner.

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[2] See e.g. Rönnedal (2010), chapter 7, for more information about OS5+ (also called KD45). S5 is described in almost every introduction to (alethic) modal logic.

[3] This system contains some redundancy and there are a number a different deductively equivalent systems. Therefore, the conclusion also holds in many "other" systems.

[4] See the comment in footnote 3.

[5] See e.g. Rönnedal (2010), pp. 260-265.

(x) $\Diamond A \leftrightarrow P\Diamond A$

1. $\Box\Diamond A \rightarrow P\Diamond A$	[OC’ $\Diamond$A/A]
2. $\Diamond A \rightarrow \Box\Diamond A$	[S5]
3. $\Diamond A \rightarrow P\Diamond A$	[1, 2, PL]
4. $P\Diamond A \rightarrow \Diamond\Diamond A$	[MO’ $\Diamond$A/A]
5. $\Diamond\Diamond A \rightarrow \Diamond A$	[S5]
6. $P\Diamond A \rightarrow \Diamond A$	[4, 5, PL]
7. $\Diamond A \leftrightarrow P\Diamond A$	[3, 6, PL]

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1. $P\Box A \rightarrow \Diamond\Box A$	[MO’ $\Box$A/A]

(xi) $\Diamond A \leftrightarrow O\Diamond A$

2. $\Diamond\Box A \rightarrow \Box A$	[S5]
3. $P\Box A \rightarrow \Box A$	[1, 2, PL]
4. $\neg\Box A \rightarrow \neg P\Box A$	[3, PL]
5. $\neg\Box\neg A \rightarrow \neg P\Box\neg A$	[4, ¬A/A]
6. $\Diamond A \rightarrow \neg P\neg\Diamond A$	[5, $\Box\Diamond$I]
7. $\Diamond A \rightarrow O\Diamond A$	[6, OPI]
8. $O\Diamond A \rightarrow \Diamond\Diamond A$	[OC $\Diamond$A/A]
9. $\Diamond\Diamond A \rightarrow \Diamond A$	[S5]
10. $O\Diamond A \rightarrow \Diamond A$	[8, 9, PL]
11. $\Diamond A \leftrightarrow O\Diamond A$	[7, 10, PL]

Theorem 4**(i)**, part (x)-(xi)

(xii) $\Box A \leftrightarrow P\Box A$

1. $\Box\Box A \rightarrow P\Box A$	[OC’ $\Box$A/A]
2. $\Box A \rightarrow \Box\Box A$	[S5]
3. $\Box A \rightarrow P\Box A$	[1, 2, PL]
4. $P\Box A \rightarrow \Diamond\Box A$	[MO’ $\Box$A/A]
5. $\Diamond\Box A \rightarrow \Box A$	[S5]
6. $P\Box A \rightarrow \Box A$	[4, 5, PL]
7. $\Box A \leftrightarrow P\Box A$	[3, 6, PL]

(xiii) $\Box A \leftrightarrow O\Box A$

1. $\Box\Box A \rightarrow O\Box A$	[MO $\Box$A/A]
2. $\Box A \rightarrow \Box\Box A$	[S5]
3. $\Box A \rightarrow O\Box A$	[1, 2, PL]
4. $O\Box A \rightarrow \Diamond\Box A$	[OC $\Box$A/A]
5. $\Diamond\Box A \rightarrow \Box A$	[S5]
6. $O\Box A \rightarrow \Box A$	[4, 5, PL]
7. $\Box A \leftrightarrow O\Box A$	[3, 6, PL]

Theorem 4**(i)**, part (xii)-(xiii)

(xiv) $PA \leftrightarrow \Diamond PA$

1. $PA \rightarrow \Diamond PA$	[aT’ PA/A]
2. $OA \rightarrow \Box OA$	[ad4]
3. $\neg\Box OA \rightarrow \neg OA$	[2, PL]
4. $\neg\Box O\neg A \rightarrow \neg O\neg A$	[3, ¬A/A]
5. $\neg\Box\neg PA \rightarrow PA$	[4, OPI]
6. $\Diamond PA \rightarrow PA$	[5, $\Box\Diamond$I]
7. $PA \leftrightarrow \Diamond PA$	[1, 6, PL]

(xvi) $OA \leftrightarrow \Diamond OA$

1. $OA \rightarrow \Diamond OA$	[aT’ OA/A]
2. $PA \rightarrow \Box PA$	[ad5]
3. $\neg\Box PA \rightarrow \neg PA$	[2, PL]
4. $\neg\Box P\neg A \rightarrow \neg P\neg A$	[3, ¬A/A]
5. $\neg\Box\neg OA \rightarrow OA$	[4, OPI]
6. $\Diamond OA \rightarrow OA$	[5, $\Box\Diamond$I]
7. $OA \leftrightarrow \Diamond OA$	[1, 6, PL]

Theorem 4**(i)**, part (xiv), (xvi)

Note that $\Diamond PA \rightarrow PA$ is the dual of $OA \rightarrow \Box OA$ and that $\Diamond OA \rightarrow OA$ is the dual of $PA \rightarrow \Box PA$.

(xv) $PA \leftrightarrow \Box PA$		(xvii) $OA \leftrightarrow \Box OA$	
1. $PA \rightarrow \Box PA$	[ad5]	1. $OA \rightarrow \Box OA$	[ad4]
2. $\Box PA \rightarrow PA$	[aT PA/A]	2. $\Box OA \rightarrow OA$	[aT OA/A]
3. $PA \leftrightarrow \Box PA$	[1, 2, PL]	3. $OA \leftrightarrow \Box OA$	[1, 2, PL]

Theorem 4**(i)**, part (xv), (xvii)

We have now proved that all of the reduction laws mentioned above hold in every system that contains S5, OS5+, MO, OC, $OA \rightarrow \Box OA$, and $PA \rightarrow \Box PA$.[6] It remains to show that any system that contains these laws have at most the ten distinct modalities listed in the theorem. We do this by systematically adding single modalities to the empty modality and show how the result ultimately reduces to one of the ten modalities.

So, start with the empty modality (0) *A. Add one modality. Then we get (1) $\neg A$, (2) $\Diamond A$, (3) $\Box A$, (4) PA, or (5) OA.

Add one modality to (1). Then we get (1.1) $\neg\neg A$, which is equivalent to (1), (1.2) $\Diamond\neg A$, which is equivalent to $\neg\Box A$, (1.3) $\Box\neg A$, which is equivalent to $\neg\Diamond A$, (1.4) $P\neg A$, which is equivalent to $\neg OA$, or (1.5) $O\neg A$, which is equivalent to $\neg PA$. Add one modality to (1.2). Then we get (1.2.1) $\neg\Diamond\neg A$, which is equivalent to $\Box A$ (see (3)), or we get (1.2.2) $\Diamond\Diamond\neg A$, (1.2.3) $\Box\Diamond\neg A$, (1.2.4) $P\Diamond\neg A$, or (1.2.5) $O\Diamond\neg A$, all of which are equivalent to (1.2). Add one modality to 1.3. Then we get (1.3.1) $\neg\Box\neg A$, which is equivalent to $\Diamond A$ (see (2)), or we get (1.3.2) $\Diamond\Box\neg A$, (1.3.3) $\Box\Box\neg A$, (1.3.4) $P\Box\neg A$, or (1.3.5) $O\Box\neg A$, all of which are equivalent to (1.3). Add one modality to (1.4). Then we get (1.4.1) $\neg P\neg A$, which is equivalent to OA (see (5)), or we get (1.4.2) $\Diamond P\neg A$, (1.4.3) $\Box P\neg A$, (1.4.4) $PP\neg A$, or (1.4.5) $OP\neg A$, all of which are equivalent to (1.4). Add one modality to (1.5). Then we get (1.5.1) $\neg O\neg A$, which is equivalent to PA (see (2)), or we get (1.5.2) $\Diamond O\neg A$, (1.5.3) $\Box O\neg A$, (1.5.4) $PO\neg A$, or (1.5.5) $OO\neg A$, all of which are equivalent to (1.5).

Add one modality to (2). Then we get (2.1) $\neg\Diamond A$ (see (1.3)), (2.2) $\Diamond\Diamond A$, (2.3) $\Box\Diamond A$, (2.4) $P\Diamond A$, or (2.5), all of which are equivalent to (2).

Add one modality to (3). Then we get (3.1) $\neg\Box A$ (see (1.2)), (3.2) $\Diamond\Box A$, (3.3) $\Box\Box A$, (3.4) $P\Box A$, or (3.5) $O\Box A$, all of which are equivalent to (3).

Add one modality to (4). Then we get (4.1) $\neg PA$ (see (1.5)), (4.2) $\Diamond PA$, (4.3) $\Box PA$, (4.4) PPA, or (4.5) OPA, all of which are equivalent to (4).

[6] In the proofs above, PL means that the step follows by propositional logic. S5 means that the sentence is a theorem in the alethic system S5. aT' is the dual of aT, MO' is the dual of MO, and OC' is the dual of OC. $\Box\Diamond$I includes the usual relationships between $\Box$ and $\Diamond$ and OPI the usual relationships between O and P.

Add one modality to (5). Then we get (5.1) ¬OA (see (1.4)), (5.2) ◇OA, (5.3) □OA, (5.4) POA, or (5.5) OOA, all of which are equivalent to (5). This takes care of all the possibilities.

(ii) To prove (ii) we must show that the system aS5dOS5+adMOOC45 doesn't contain any more reduction laws, i.e. we must show that it is not the case that □A ↔ ◇A, OA ↔ PA, □A ↔ OA, ◇A ↔ PA etc. We can do this by describing a countermodel for each equivalence of this kind. Details are left to the reader.

(iii) Follows directly from (i) by the soundness and completeness theorems in Rönnedal (2012). See also Rönnedal (2012b).

(iv) To prove part (iv) we can use the same countermodels as in (ii).

(v) Follows from the previous parts of this theorem.

The proof of theorem 4 is now finished. ■

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