A Quick Glimpse of Symplectic Topology

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Definition of SYMPLECTIC

Popularity: Bottom 20% of words

1 : relating to or being an intergrowth of two different minerals (as in ophicalcite, myrmekite, or micropegmatite)

2 : relating to or being a bone between the hyomandibular and the quadrate in the mandibular suspensorium of many fishes that unites the other bones of the suspensorium

Why *Symplectic*?

On page 165 of his book "The Classical Groups" Hermann Weyl starts a chapter on the Symplectic Group. In a footnote he writes:

* The name "complex group" formerly advocated by me in allusion to line complexes, as these are defined by the vanishing of antisymmetric bilinear forms, has become more and more embarrassing through collision with the word "complex" in the connotation of complex number. I therefore propose to replace it by the corresponding Greek adjective "symplectic." Dickson calls the group the "Abelian linear group" in homage to Abel who first studied it.

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Some Definitions

Definition 1:

A symplectic form on a real vector space V is a non-degenerate antisymmetric bilinear form

$$\omega \colon V \times V \longrightarrow \mathbb{R}$$

Exercise: Such a thing exists if and only if the dimension of V is even! In fact, there is always a basis for which the matrix of the symplectic form becomes:

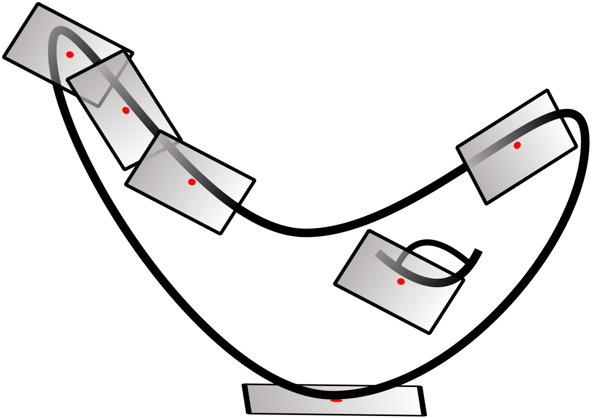
$$\begin{pmatrix} \mathbf{0} & \mathbf{I_n} \\ -\mathbf{I_n} & \mathbf{0} \end{pmatrix}$$

Here is (the) one on $\mathbb{R}^4$:

$$\omega((v_1, v_2, v_3, v_4), (w_1, w_2, w_3, w_4)) = v_1 w_3 + v_2 w_4 - v_3 w_1 - v_4 w_2$$

Definition 2:

A symplectic structure ω on a manifold M is a smoothly-varying choice of symplectic form on each tangent space. We also require ω to be "closed".



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- Complex projective varieties
- many more...

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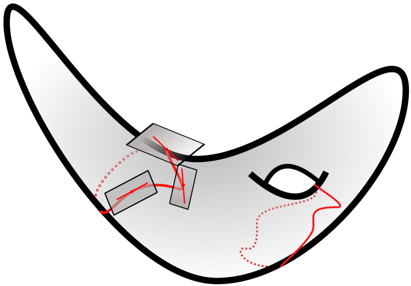
Why I like it...

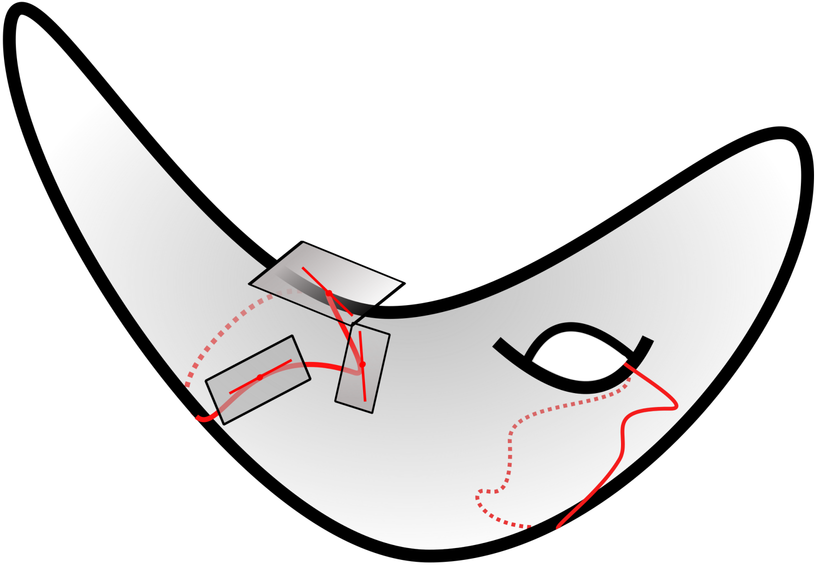
- *Flexibility* vs. *rigidity*
- Employs differential geometry, functional analysis, homological algebra, sheaf theory...

Lagrangian submanifolds

Definition 3

• A Lagrangian subspace of (V^{2n}, ω) is an $n-$dimensional subspace $L \leq V$ such that for every $v, w \in L$ one has $\omega(v, w) = 0$.
• A Lagrangian submanifold of (M^{2n}, ω) is an $n-$dimensional submanifold $L \subseteq M$ such that for every $x \in L$ the tangent space $T_x L$ is a Lagrangian subspace of $T_x M$.





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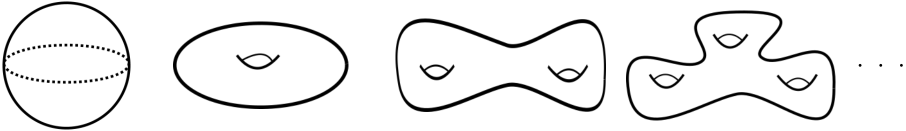

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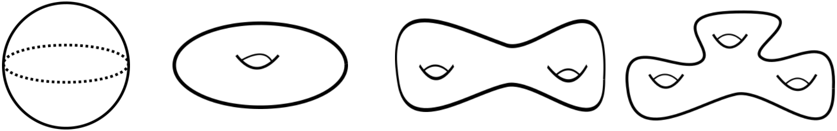
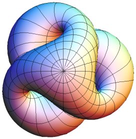


. . .

   b) Non-orientable

$\mathbb{R}P^2$



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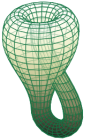

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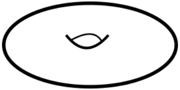

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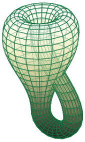

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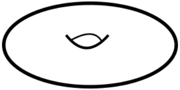

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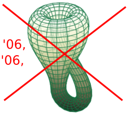
Shevchishin, '06,
Nemirovsky, '06,
Mohnke



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