# Investigating some self-similar groups via nilpotent quotients

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## Abstract

We survey the successful computational investigation of certain finitely generated, infinitely presented groups. We show how their nilpotent quotients can be used to analyze their Schur multipliers and their outer automorphism groups. As application, we investigate the generalized Fabrykowski-Gupta groups $\Gamma_p$ for some small primes $p$ and propose conjectures about their lower central series, their Schur multipliers and their automorphism groups.

## 1 Introduction

The Burnside problem, posed by William Burnside in 1902, is one of the oldest and most influential problems in group theory. It asks whether a finitely generated group in which every element has finite order is finite. This general question was answered negatively by Golod and Šafarevič in 1964. Among the first explicit counter-examples to this problem are the Grigorchuk group [11] and the Gupta-Sidki group [15].

Both groups can be realized as groups of automorphisms of certain infinite regular trees. Subgroups of the automorphism group of infinite regular trees have subsequently proved to be a rich source for groups with interesting properties such as infinite torsion groups and groups with an intermediate word growth. Most of the currently known examples of such groups act on the infinite tree in a special form: they are so-called *self-similar groups*. We refer to the book by Nekrashevych [20] for further details.

For many self-similar groups there are presentations with finitely many generators and infinitely many relators known. In particular, Bartholdi [1] developed a mechanism to determine such presentations for certain self-similar groups. The infinitely many relators of these presentations can be described in a finite form using an action of a finitely generated monoid of endomorphisms on a finite set of relations. Group presentations with such relations are nowadays known as *finite $L$-presentations* in honor of Lyseniok's construction of such a presentation for the Grigorchuk group [19].

Many computational problems are known to be undecidable in general for finitely presented groups. Hence computations with finitely presented groups are rather limited in general. Computations with infinitely presented groups are certainly not easier. Nonetheless the finite $L$-presentations proved to be a computationally useful tool in the investigation of various interesting examples of self-similar groups; see for example [4] and [16].

Our first aim here is to give a brief overview on the recently developed computational methods for investigating groups given by a finite $L$-presentation. This includes the algorithm in [4] to determine the quotients of the lower central series of such a group and the method of [16] to approximate its Schur multiplier if the $L$-presentation is invariant; see Section 2 for a definition. Further, we discuss an approach to approximate the automorphism group of a finitely $L$-presented group. All our algorithms are implemented in the computer algebra system Gap [10] and are or will be made available as part of the NQL-package [17].

As application, we show how to investigate an interesting family of self-similar groups: the *generalized Fabrykowski-Gupta groups* $\Gamma_p$, for $p$ an odd prime. These have been defined in [13] and finite $L$-presentations for them are determined in [4]. The group $\Gamma_3$ is the Fabrykowski-Gupta group: a group with an intermediate word growth [8, 9]. For all other primes, the groups $\Gamma_p$ are not very well investigated so far, but they seem promising examples for groups with interesting properties. Here, we exhibit our computational results on the shape of their lower central series quotients extending the results of [4], their Schur multipliers as obtained in [16], and, additionally, on their automorphism groups.

## 2 Finite $L$-presentations

Let $F$ be a finitely generated free group over the alphabet $\mathcal{X}$ and suppose that $\mathcal{Q}, \mathcal{R} \subset F$ are finite subsets of the free group $F$ and $\Phi \subset \mathrm{End}(F)$ is a finite set of endomorphisms of $F$. Then the quadruple $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ is a *finite $L$-presentation*. It defines the *finitely $L$-presented group*

$$G = \left\langle \mathcal{X} \;\middle|\; \mathcal{Q} \cup \bigcup_{\varphi \in \Phi^*} \mathcal{R}^{\varphi} \right\rangle,$$

where $\Phi^*$ denotes the free monoid generated by $\Phi$; that is, the closure of $\Phi \cup \{\mathrm{id}\}$ under composition. A finite $L$-presentation $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ is *invariant* if every endomorphism $\varphi \in \Phi$ induces an endomorphism of $G$; that is, if the normal closure of $\mathcal{Q} \cup \bigcup_{\varphi \in \Phi^*} \mathcal{R}^{\varphi}$ in $F$ is $\varphi$-invariant. For example, every finite $L$-presentation of the form $\langle \mathcal{X} \mid \emptyset \mid \Phi \mid \mathcal{R} \rangle$ is invariant.

Invariant finite $L$-presentations generalize finite presentations as every finite presentation $\langle \mathcal{X} \mid R \rangle$ translates to the invariant finite $L$-presentation $\langle \mathcal{X} \mid \emptyset \mid \emptyset \mid R \rangle$. Further, many self-similar groups can be defined by finite invariant $L$-presentations. A famous example is the Grigorchuk group; see [19]. The Gupta-Sidki group has a finite $L$-presentation, but no invariant $L$-presentation is known at current.

### 2.1 Example: The generalized Fabrykowski-Gupta groups

The generalized Fabrykowski-Gupta groups $\Gamma_p$ are defined in [13] as groups of automorphisms of the infinite $p$-ary tree. In [4] a finite invariant $L$-presentation for $\Gamma_p$ on two generators is determined. We denote the generators by $a$ and $r$. Let

$s_i = r^{a^i}$ for $0 \le i < p$ and $s = s_1$. Reading indices modulo $p$, we define

$$\mathcal{R} = \left\{ a^p, \left[ s^{r^\ell}, s_j^{s_{j-1}^m} \right], s^{-r^{\ell+1}} s^{r^\ell r^{s_{p-1}^m}} \;\middle|\; 2 < j < p;\ 0 \le \ell, m < p \right\}.$$

Further, let $\varphi$ be the endomorphism of the free group on $\{a, r\}$ defined by $\varphi(a) = r^{a^{-1}}$ and $\varphi(r) = r$. Then we define the generalized Fabrykowski-Gupta group by the finite $L$-presentation

$$\Gamma_p = \langle \{a, r\} \mid \emptyset \mid \{\varphi\} \mid \mathcal{R} \rangle.$$

A slightly shorter presentation for $\Gamma_p$ can be obtained from the FR-package [3] of the computer algebra system Gap [10].

## 3 Computing nilpotent quotients

Let $G$ be a group given by a finite $L$-presentation and $c \in \mathbb{N}$. Then the algorithm in [4] computes a consistent polycyclic presentation for the quotient $G/\gamma_{c+1}G$, where $\gamma_{c+1}G$ denotes the $(c+1)$-st subgroup of the lower central series of $G$; that is, $\gamma_1 G = G$ and $\gamma_{i+1}G = [\gamma_i G, G]$ for $i \in \mathbb{N}$. The algorithm generalizes the nilpotent quotient algorithm for finitely presented groups as described in [21]. It uses induction on $c$. To describe the induction step, we assume that we have already determined a consistent polycyclic presentation for $G/\gamma_c G$ and we wish to extend this to $G/\gamma_{c+1}G$.

As $G$ is given by a finite $L$-presentation, we can consider $G$ as $G \cong F/K$ for a finitely generated free group $F$ and a normal subgroup $K$ which is generated as normal subgroup by the (possibly infinitely many) relations of $G$. Then

$$G/\gamma_c G \cong F/K\gamma_c F.$$

We define the *covering group* of $G/\gamma_c G$ by

$$(G/\gamma_c G)^* = F/[K\gamma_c F, F].$$

This is a central extension of $G/\gamma_c G$ and hence is a nilpotent group. It is also finitely generated as quotient of the finitely generated group $F$. Hence $(G/\gamma_c G)^*$ is polycyclic. The first step in the nilpotent quotient algorithm is to determine a consistent polycyclic presentation for this covering group from a consistent polycyclic presentation of $G/\gamma_c G$.

If the given $L$-presentation is invariant, then the finitely many relators in $\mathcal{Q}$ and $\mathcal{R}$ translate to elements of $(G/\gamma_c G)^*$ and, similarly, the endomorphisms in $\Phi$ translate. This yields an (infinite) generating set for the image $U$ of $K$ in the covering group. As the covering group is polycyclic, it follows that ascending chains of subgroups terminate. Hence we can use a spinning algorithm to determine a finite generating set for $U$ from the given infinite one. Now we apply general algorithmic methods for polycyclic groups [18] to determine a consistent polycyclic presentation for $(G/\gamma_c G)^*/U \cong F/K\gamma_{c+1}F \cong G/\gamma_{c+1}G$.

If the considered finite $L$-presentation $F/K$ for $G$ is not invariant, then we determine $S \subseteq F$ and $L \leq F$ such that $K = \langle S \cup L \rangle^F$, $S$ is a finite set and $F/L$ is given by an invariant $L$-presentation; For example, we can choose $S = \mathcal{Q}$ and $L$ the normal closure of $\cup_{\varphi \in \Phi^*} \mathcal{R}^\varphi$. Then we apply the algorithm for invariant $L$-presentations to $H \cong F/L$ and obtain $G/\gamma_{c+1}G$ as quotient of $H/\gamma_{c+1}H$.

### 3.1 Example: The generalized Fabrykowski-Gupta groups

Many self-similar groups exhibit periodic patterns in their lower central series sections. A prominent example is the Grigorchuk group whose lower central series sections have been determined in [22] using theoretical means. As a result it is known that this group has finite width.

Here we extend our computations from [4] of the lower central series quotients of the groups $\Gamma_p$. Note that the lower central series quotients of $\Gamma_3$ have been determined in [2] and our computations support that result. We summarize our experimental evidence in the following table. All determined sections $\gamma_c\Gamma_p/\gamma_{c+1}\Gamma_p$ are elementary abelian $p$-groups. Hence their isomorphism type is determined by their rank. We list the sequences of their ranks in the following table writing $n^{[m]}$ if the rank $n$ appears in $m$ consecutive places in the sequence. The last column of the table exhibits the maximal class $c$ for which we determined $\Gamma_p/\gamma_{c+1}\Gamma_p$.

| $p$ | $\mathrm{rk}_p(\gamma_c\Gamma_p/\gamma_{c+1}\Gamma_p)$ | class |
|---|---|---|
| 3 | $2, 1^{[1]}, 2^{[1]}, 1^{[1]},\ 2^{[3]}, 1^{[3]},\ 2^{[9]},\ 1^{[9]}, 2^{[27]}, 1^{[27]}, 2^{[65]}$ | 147 |
| 5 | $2, 1^{[3]}, 2^{[1]}, 1^{[13]}, 2^{[5]}, 1^{[65]}, 2^{[25]}, 1^{[26]}$ | 139 |
| 7 | $2, 1^{[5]}, 2^{[1]}, 1^{[33]}, 2^{[7]}, 1^{[68]}$ | 115 |
| 11 | $2, 1^{[9]}, 2^{[1]}, 1^{[89]}$ | 100 |

Let $f_p(\ell) = p + (p^2 - 2p - 1)(p^{\ell+1} - 1)/(p - 1)$ and $g_p(\ell) = f_p(\ell) + p^{\ell+1}$. Then the above table suggests that

$$\mathrm{rk}_p(\gamma_c\Gamma_p/\gamma_{c+1}\Gamma_p) = \begin{cases} 2, & \text{if } c \in \{1, p\} \text{ or } f_p(\ell) \leq c < g_p(\ell) \text{ for some } \ell \in \mathbb{N}_0, \\ 1, & \text{otherwise.} \end{cases}$$

Our computational evidence also strongly supports the conjecture from [4] that the Fabrykowski-Gupta groups $\Gamma_p$ have width 2 for all odd primes $p$.

## 4 Investigating Schur multipliers

The Schur multiplier $M(G)$ of a group $G$ is an invariant of the group which can be defined as the second homology group $H_2(G, \mathbb{Z})$. It is of central interest in the area of self-similar groups. One reason is that finitely presented groups have a finitely generated Schur multiplier. Hence proving that a Schur multiplier is not finitely generated also proves that the considered group is not finitely presented. This strategy has been applied successfully in [12] to show that the Grigorchuk group is not finitely presented.

In [16] an approach for investigating the Schur multiplier of a group $G$ given by an invariant $L$-presentation is described. The main aim of this approach is to collect computational evidence towards proving that the Schur multiplier of the considered group is not finitely generated. We recall the main ideas of this algorithm here briefly.

Let $F$ be a finitely generated free group and $K$ be a normal subgroup of $F$ so that $G \cong F/K$. Then the Hopf formula yields that $M(G)$ is isomorphic to $(K \cap F')/[K, F]$. Additionally, as $G/\gamma_c G \cong F/K\gamma_c F$, we obtain that $M(G/\gamma_c G)$ is isomorphic to $(K\gamma_c F \cap F')/[K\gamma_c F, F]$. These isomorphisms induce

$$\varphi_c \colon M(G) \to M(G/\gamma_c G),\ g[K, F] \mapsto g[K\gamma_c F, F].$$

The map $\varphi_c$ is a homomorphism of abelian groups. It yields a sequence of subgroups

$$M(G) \geq \ker \varphi_1 \geq \ker \varphi_2 \geq \dots .$$

The homomorphism $\varphi_c$ and the quotient $M_c(G) = M(G)/\ker \varphi_c$ can be computed with the method in [16] for fixed $c$. We consider the isomorphism types of these quotients as a sequence in $c$. As shown in [16], this approach often yields periodic structures in the resulting sequence of quotients of $M(G)$.

### 4.1 Example: The generalized Fabrykowski-Gupta groups

All computed quotients $M_c(\Gamma_p)$ are elementary abelian $p$-groups. We exhibit their ranks in the following table using the notation of Section 3.1.

| $p$ | $\mathrm{rk}_p(M_c(\Gamma_p))$ |
|---|---|
| 3 | $0^{[2]}, 1^{[3]}, 2^{[0]}, 3^{[9]}, 4^{[1]}, 5^{[26]}, 6^{[4]}, 7^{[77]}, 8^{[13]}, 9^{[12]}$ |
| 5 | $0^{[1]}, 1^{[4]}, 2^{[2]}, 3^{[20]}, 4^{[10]}, 5^{[100]}, 6^{[1]}$ |
| 7 | $0^{[1]}, 1^{[2]}, 2^{[6]}, 3^{[2]}, 4^{[14]}, 5^{[42]}, 6^{[14]}, 7^{[34]}$ |
| 11 | $0^{[1]}, 1^{[2]}, 2^{[2]}, 3^{[2]}, 4^{[10]}, 5^{[2]}, 6^{[22]}, 7^{[22]}, 8^{[22]}, 9^{[13]}$ |

These computational results suggest that the groups $\Gamma_p$ have an infinitely generated Schur multiplier and thus are not finitely presentable.

## 5 Investigating automorphism groups

The full automorphism group of the underlying infinite regular tree induces automorphisms of a self-similar group. The resulting group is often the full automorphism group [5]. Explicit descriptions of automorphism groups of various prominent self-similar groups have also been obtained: in [14] it is proved that the outer automorphism group of the Grigorchuk group is an elementary abelian 2-group of infinite rank and the outer automorphism group of the Gupta-Sidki group is a split extension of an elementary abelian 3-group of infinite rank with the Klein four group.

Our aim in this section is to approximate the outer automorphism group of a finitely $L$-presented group using nilpotent quotients. We show that the application of the Fabrykowski-Gupta groups of our approach allows to guess the broad structure of their outer automorphism groups.

For every group $G$, the lower central series term $\gamma_c G$ is a fully invariant subgroup of $G$. Thus every $\alpha \in \mathrm{Aut}(G)$ induces an automorphism $\alpha_c$ of $G/\gamma_c G$ and we obtain a homomorphism $\varphi_c\colon \mathrm{Aut}(G) \to \mathrm{Aut}(G/\gamma_c G)$, $\alpha \mapsto \alpha_c$. The homomorphism $\varphi_c$ maps the inner automorphisms $\mathrm{Inn}(G)$ onto $\mathrm{Inn}(G/\gamma_c G)$. Hence $\varphi_c$ induces a homomorphism

$$\nu_c\colon \mathrm{Out}(G) \to \mathrm{Out}(G/\gamma_c G)$$

for every positive integer $c$. We define $\mathcal{I}_c := \mathrm{im}(\nu_c)$ and obtain

$$\mathrm{Out}(G)/\ker(\nu_c) \cong \mathcal{I}_c \leq \mathrm{Out}(G/\gamma_c G).$$

The same argument as above shows that for every $d \leq c$ there is a natural homomorphism

$$\mu_{c,d}\colon \mathrm{Out}(G/\gamma_c G) \to \mathrm{Out}(G/\gamma_d G).$$

We define $\mathcal{J}_{c,d} := \mathrm{im}(\mu_{c,d})$. As $\nu_d = \nu_c \circ \mu_{c,d}$, it follows that $\mathcal{I}_d \leq \mathcal{J}_{c,d}$ for every $c \geq d$. Hence we obtain a sequence of subgroups

$$\mathcal{I}_d \leq \ldots \leq \mathcal{J}_{c,d} \leq \mathcal{J}_{c-1,d} \leq \ldots \leq \mathcal{J}_{d,d} = \mathrm{Out}(G/\gamma_d G).$$

In our applications we fix a possibly large $c$ and compute $\mathcal{J}_{c,d}$ for $1 \leq d \leq c$. If $c$ is large with respect to $d$, then $\mathcal{J}_{c,d}$ yields a useful approximation of the image $\mathcal{I}_d$.

### 5.1 Example: The generalized Fabrykowski-Gupta groups

The lower central series quotients of $\Gamma_p$ for small $p$ are all finite $p$-groups and thus we can use the algorithm of [6] to determine $\mathrm{Aut}(G/\gamma_c G)$. This automorphism group is soluble for almost all $c$; in this case we determine a polycyclic presentation for it using a method implemented in [7]. This allows to compute a polycyclic presentation for its quotient modulo the inner automorphism group $\mathrm{Out}(G/\gamma_c G)$. We can now determine $\mathcal{J}_{c,d}$ for various values on $d$ as subgroup of $\mathrm{Out}(G/\gamma_c G)$.

#### 5.1.1 The group $\Gamma_3$

We consider $c = 51$ and determine $\mathcal{J}_{c,d}$ for $1 \leq d \leq 41$. We obtain that $\mathcal{J}_{c,d}$ is an elementary abelian 3-group of rank

$$\mathrm{rk}_3(\mathcal{J}_{c,d}) = \begin{cases} 0, & \text{if } d \in \{1,2\} \\ 1, & \text{if } d \in \{3,4\} \\ 2, & \text{if } d \in \{5,\ldots,10\} \\ 3, & \text{if } d \in \{11,\ldots,28\} \\ 4, & \text{if } d \in \{29,\ldots,32\} \\ 5, & \text{if } d \in \{33,\ldots,35\} \\ 6, & \text{if } d \in \{36,\ldots,41\}. \end{cases}$$

This induces the conjecture that the outer automorphism group of $\Gamma_3$ is an elementary abelian 3-group of infinite rank.

### 5.1.2 The group $\Gamma_5$

We consider $c = 79$ and determine $\mathcal{J}_{c,d}$ for $1 \leq d \leq 70$. We obtain that $\mathcal{J}_{c,d}$ is an elementary abelian 5-group of rank

$$\mathrm{rk}_5(\mathcal{J}_{c,d}) = \begin{cases} 0, & \text{if } d \in \{1, \ldots, 4\} \\ 1, & \text{if } d \in \{5, \ldots, 18\} \\ 2, & \text{if } d \in \{19, \ldots, 50\} \\ 3, & \text{if } d \in \{51, \ldots, 55\} \\ 4, & \text{if } d \in \{56, \ldots, 70\}. \end{cases}$$

This induces the conjecture that the outer automorphism group of $\Gamma_5$ is an elementary abelian 5-group of infinite rank.

### 5.1.3 The group $\Gamma_7$

We consider $c = 77$ and determine $\mathcal{J}_{c,d}$ for $1 \leq d \leq 63$. Let $C_n^m$ denote the direct product of $m$ copies of the cyclic group of order $n$ and by $D_{2n}$ the dihedral group with $2n$ elements. Then

$$\mathcal{J}_{c,d} \cong \begin{cases} C_2, & \text{if } d \in \{1, \ldots, 6\} \\ D_{14} & \text{if } d \in \{7, \ldots, 40\} \\ C_7 \times D_{14} & \text{if } d \in \{41\} \\ C_7^2 \times D_{14} & \text{if } d \in \{42, \ldots, 56\} \\ C_7^3 \times D_{14} & \text{if } d \in \{57, \ldots, 63\}. \end{cases}$$

This induces the conjecture that $\mathrm{Out}(\Gamma_7)$ is a direct product of an elementary abelian 7-group of infinite rank by the dihedral group $D_{14}$.

### 5.1.4 The group $\Gamma_{11}$

We consider $c = 85$ and determine $\mathcal{J}_{c,d}$ for $1 \leq d \leq 66$. Then

$$\mathcal{J}_{c,d} \cong \begin{cases} C_{10} & \text{if } d \in \{1, \ldots, 10\} \\ C_{11} \rtimes C_{10} & \text{if } d \in \{11, \ldots, 22\} \\ C_{11}^2 \rtimes C_{10} & \text{if } d \in \{23, \ldots, 33\} \\ C_{11}^3 \rtimes C_{10} & \text{if } d \in \{34, \ldots, 44\} \\ C_{11}^4 \rtimes C_{10} & \text{if } d \in \{45, \ldots, 55\} \\ C_{11}^5 \rtimes C_{10} & \text{if } d \in \{56, \ldots, 66\}. \end{cases}$$

This induces the conjecture that $\mathrm{Out}(\Gamma_{11})$ is a split extension of an elementary abelian 11-group of infinite rank by the cyclic group $C_{10}$. Further, the cyclic group $C_{10}$ acts diagonally on $C_{11}^m$.

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