Chapter Nine: Music, Chords and Harmony

If the stars and planets are the gears of the universe, revolving in intricate ways in the skies, then music came to be seen from ancient times as a subtle reflection of this machinery, connecting it to the emotions and to the soul. The link was through the strange integral relationships, which they exhibit. In the case of the sky, we have wheels turning, the cycle of the day, of the month (from one full moon to the next), the year (the time from one vernal equinox to the next, i.e. from one season to the one next year). Integers appear when the cycles are compared, thus there seem to be 29 days in the lunar month (time from one full moon to the next), 365 days in a year (time from one vernal equinox to another, i.e. from one season to the next). But when more careful observations are made, the relations are more complex: there are really 29 ½ days in the lunar month or better 29 days 12 ¾ hours or better …. Likewise the year, not really 365 days but 365 ¼ days in a year, or better 365 days, 6 hours *less* about 11 minutes or ….Gears indeed.

What many early peoples noted was that when strings were plucked producing music, the sounds produced pleasing chords and tunes if the length of the strings had a proportion given by small integers, 2:1, 3:2, 4:3, 5:3, etc. Thus the quality of a tune made by plucking one or more strings was crucially affected by the ratio of the lengths of the string at the times plucked. The same went for blowing into or across holes in pipes and the pipe lengths. These relationships were apparently of great importance to Pythagoras (ca. 560-480 BCE), to the religious cult he started and to his later followers (though nothing really reliable is known about Pythagoras). The Pythagorean School divided up the areas of study into the *quadrivium*, the 4 subjects

- arithmetic
- geometry
- music
- astronomy

all of which contained number, the essence of the regularities of nature, all of which displayed the beauty of the universe. Put simply, even from our modern jaded perspective, is it not startling that strings with simple arithmetic ratios are exactly those which produce beautiful chords? Fortunately or unfortunately, there is a pretty simple explanation, which this Chapter will explain.

Moving ahead in history, this connection of integers with music was of great interest to Galileo also. He starts with

> <u>Salviati:</u> *Impelled by your queries I may give you some of my ideas concerning certain problems in music, a splendid subject, upon which so many eminent men have written: among these is Aristotle himself who has discussed numerous interesting acoustical questions. Accordingly, if on the basis of some easy and tangible experiments, I shall explain some striking phenomena in the domain of sound, I trust my explanations shall meet your approval.*
> <u>Sagredo:</u> *I shall receive them not only gratefully but eagerly. For, although I take pleasure in every kind of musical instrument and have paid considerable attention to harmony, I have never been able to fully understand why some combinations of*

musical tones are more pleasing than others, or why certain combinations not only fail to please but are even highly offensive.

Galileo knew, of course, that all music was produced by rapid vibrations of strings, or air in pipes and sought to make analogies with other oscillating systems, especially his favorite, the pendulum.

> If one bows the base string on a viola rather smartly and brings near it a goblet of fine, thin glass having the same tone [*tuono*] as that of the string, this goblet will vibrate and audibly resound. That the undulations of the medium are widely dispersed about the sounding body is evinced by the fact that a glass of water may be made to emit a tone merely by the friction of the finger-tip upon the rim of the glass; for in this water is produced a series of regular waves. The same phenomenon is observed to better advantage by fixing the base of the goblet upon the bottom of a rather large vessel of water filled nearly to the edge of the goblet; for if, as before, we sound the glass by friction of the finger, we shall see ripples spreading with the utmost regularity and with high speed to large distances about the glass. I have often remarked, in thus sounding a rather
>
> [143]
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> large glass nearly full of water, that at first the waves are spaced with great uniformity, and when, as sometimes happens, the tone of the glass jumps an octave higher I have noted that at this moment each of the aforesaid waves divides into two; a phenomenon which shows clearly that the ratio involved in the octave [*forma dell' ottava*] is two.
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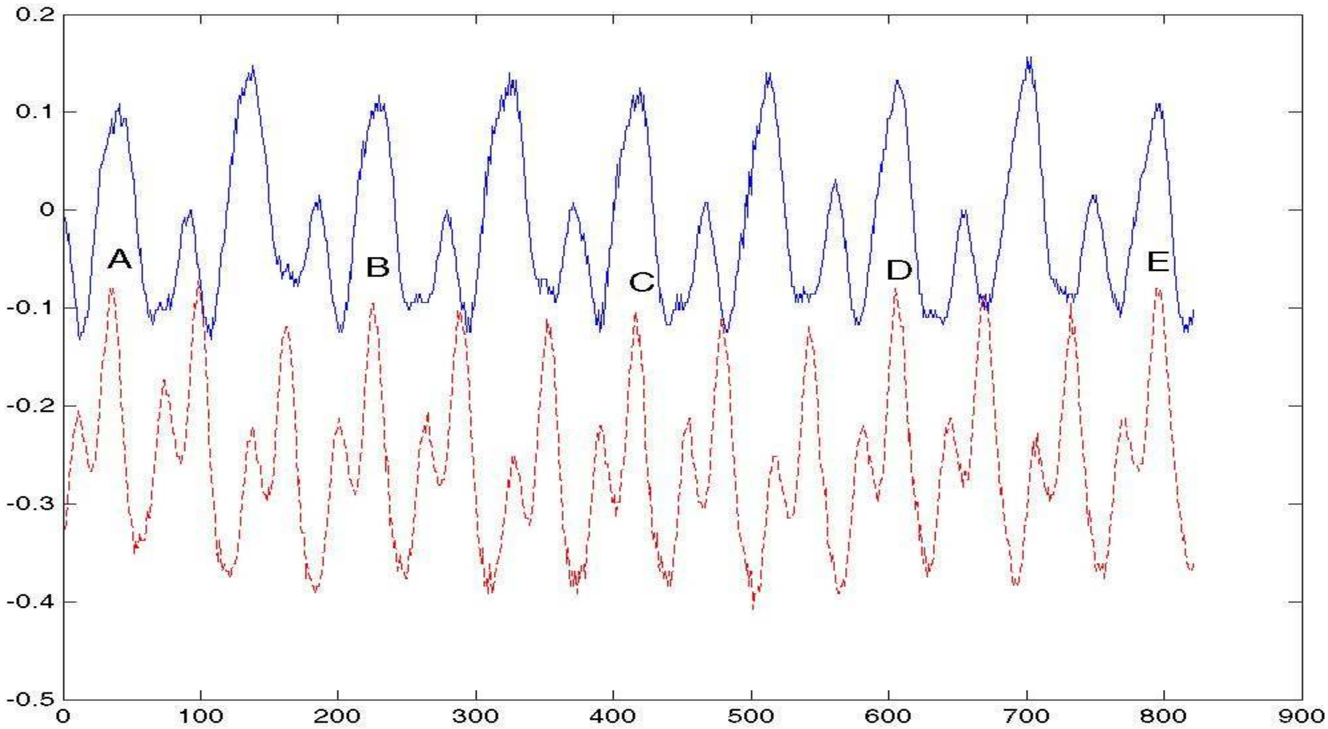
> SAGR. More than once have I observed this same thing, much to my delight and also to my profit. For a long time I have been perplexed about these different harmonies since the explanations hitherto given by those learned in music impress me as not sufficiently conclusive. They tell us that the diapason, i. e. the octave, involves the ratio of two, that the diapente which we call the fifth involves a ratio of 3:2, etc.; because if the open string of a monochord be sounded and afterwards a bridge be placed in the middle and the half length be sounded one hears the octave; and if the bridge be placed at 1/3 the length of the string, then on plucking first the open string and afterwards 2/3 of its length the fifth is given; for this reason they say that the octave depends upon the ratio of two to one [*contenuta tra'l due e l'uno*] and the fifth upon the ratio of three to two. This explanation does not impress me as sufficient to

On the right, Salviati is discussing his ideas about music and how, since it consists in vibrations, musical sounds from one object can excite another object into vibration. He discusses a specific set up in which a glass is placed in a large vessel, which is then filled nearly to the brim of the glass: the purpose is be able to see the vibration as waves in the water. Then he gets to the key point: if the tone changes from one note to another an octave higher, suddenly you see twice as many water waves, i.e. the frequency has doubled. Then he goes on to what musicians call the *fifth*, the note produced by a string $2/3^{rd}$'s the length of the original. But Sagredo is not convinced!

Well, why not jump ahead in time and look at what the air actually does when music is heard? Edison learned how to pick up the vibrations of air on a flexible membrane and, by fixing a small piece of iron to the membrane, transform the air pressure vibrations into vibrating electrical signals. Then, of course, we can put them in a computer and analyze them anyway we want. I recorded the voice of a female singer singing the major scale, *do, re, mi, fa, sol, la, ti, do* and here is a small part of this recording, showing *do* and *sol*:



I have drawn the vibration of *do* as a solid blue line oscillating around 0; and I moved *sol* down making it a dashed red line oscillating around -.25 simply in order to separate the two curves. Several things are immediately apparent: first of all, these waves are *not* sinusoidal! They are complex and fairly close to being periodic but not exactly periodic either. However, the blue curve for *do* shows 9 periods with major peaks interspersed with minor peaks, while the red curve shows 13 peaks. Look at the points marked A,B,C,D and E. At each letter both curves have peaks but between each pair, there is one extra peak for *do* and two for *sol*. In other words, two periods of *do* match three periods of *sol*. This is the 3:2 correspondence, which was discovered empirically by prehistoric musicians.

What we see is that the vibrations of the chord *do-sol* merge together into one shape that repeats itself every two periods of *do* and every three periods of *sol*. This is exactly what Galileo also claimed, as he describes on the next page, taken a few pages after the previous quote. Note that he guesses that the music consists in *pulses* of airwaves. I think he would have been thrilled to see the actual signals in the figure above.

Returning now to the original subject of discussion, I assert that the ratio of a musical interval is not immediately determined either by the length, size, or tension of the strings but rather by the ratio of their frequencies, that is, by the number of pulses of air waves which strike the tympanum of the ear, causing it also to vibrate with the same frequency. This fact established, we may possibly explain why certain pairs of notes, differing in pitch produce a pleasing sensation, others a less pleasant effect, and still others a disagreeable sensation. Such an explanation would be tantamount to an explanation of the more or less perfect consonances and of dissonances. The unpleasant sensation produced by the latter arises, I think, from the discordant vibrations of two different tones which strike the ear out of time [*sproporzionatamente*]. Especially harsh is the dissonance between notes whose frequencies are incommensurable; such a case occurs when one has two strings in unison and sounds one of them open, together with a part of the other

[147]

which bears the same ratio to its whole length as the side of a square bears to the diagonal; this yields a dissonance similar to the augmented fourth or diminished fifth [*tritono o semidiapente*].

Agreeable consonances are pairs of tones which strike the ear with a certain regularity; this regularity consists in the fact that the pulses delivered by the two tones, in the same interval of time, shall be commensurable in number, so as not to keep the ear drum in perpetual torment, bending in two different directions in order to yield to the ever-discordant impulses.

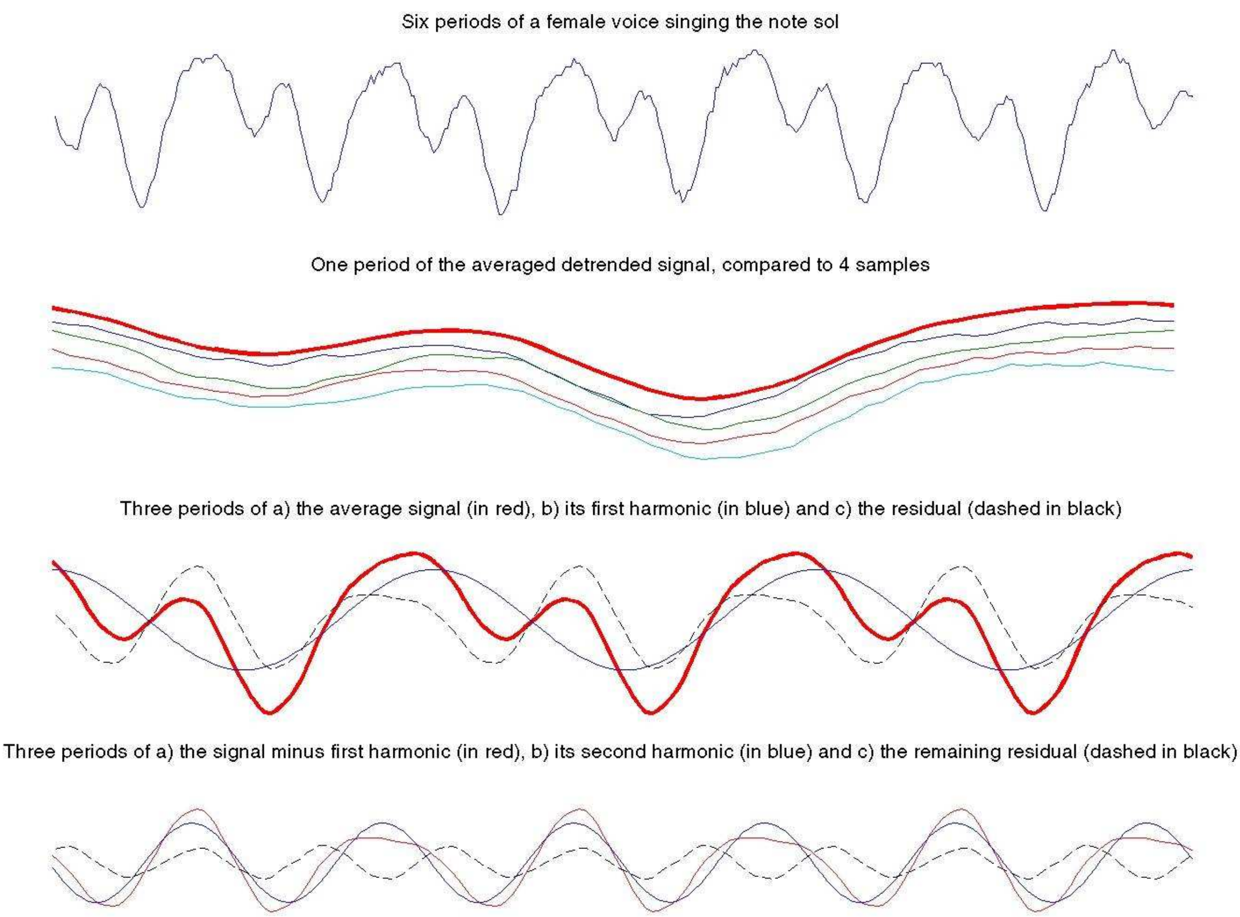
The first and most pleasing consonance is, therefore, the octave since, for every pulse given to the tympanum by the lower string, the sharp string delivers two; accordingly at every other vibration of the upper string both pulses are delivered simultaneously so that one-half the entire number of pulses are delivered in unison. But when two strings are in unison their vibrations always coincide and the effect is that of a single string; hence we do not refer to it as consonance. The fifth is also a pleasing interval since for every two vibrations of the lower string the upper one gives three, so that considering the entire number of pulses from the upper string one-third of them will strike in unison, i. e., between each pair of concordant vibrations there intervene two single vibrations; and when the interval is a fourth, three single vibrations intervene. In case the interval is a second where the ratio is 9/8 it is only every ninth vibration of the upper string which reaches the ear simultaneously with one of the lower; all the others are discordant and produce a harsh effect upon the recipient ear which interprets them as dissonances.

So far we have discussed three notes *do*, *sol* and the next *do*, one octave higher, whose three frequencies are in the ratio 2:3:4. Pursuing nice sounding chords leads to the whole major scale. Thus, we can add the note *mi* which has a frequency 5/4th's above the first *do* and this gives the 'major triad' *do-mi-sol* with frequency ratios 4:5:6. Then we can go backwards creating a triad just like this but starting at the high *do*. This gives two new notes called *fa* and *la*, so that the four notes *do, fa, la, do* have frequencies in the ratio 3:4:5:6. Lastly we add a higher frequency triad, which starts at *sol*: this is *sol, ti* and *re* one higher octave. Before you get totally confused, we make a chart:

NOTE	FREQ
do	1
re	9/8
mi	5/4
fa	4/3
sol	3/2
la	5/3
ti	15/8
high *do*	2
high *re*	9/4

Check that *do-mi-sol*, *fa-la*-high *do* and *sol-ti*-high *re* are all major triads. With numerology like this, no wonder Pythagoras thought numbers were magic.

In fact, Galileo only guessed half the story about why these chords sound nice. We mentioned above that the curves showing the air vibration were nowhere near sinusoidal curves. However, there is a very real sense in which they are made up of a combination of basic sinusoidal curves, added together. The components are (i) the sinusoid with the same period that approximates the curve best plus (ii) a sinusoid of *double* the frequency, i.e. half the period, that makes the best correction, then (iii) a sinusoid of triple the frequency or 1/3 the period which approximates what's left, etc. , continuing with higher and higher frequencies. These corrections are called the higher harmonics of the sound. Here's how this works:



On the top, you have the same voice as above singing *sol*, six periods being shown. Note that although the function has a basic period and looks like it repeats six times, there are small variations between periods. (This is less marked with a musical instrument.) On the second line, we show four examples of single periods of the voice and, in red, the average period. The average is much smoother because little tremolos in the voice have cancelled out. Then in the third line a single sinusoid has been matched to the average voice. The dashed line shows, however, the difference between the voice and its sinusoidal approximation. Remarkably, it seems to have twice the frequency. In the last graph, this residue has been approximated by a sinusoid of twice the frequency and the residue after subtracting that has been shown. The residue is very close to a sinusoid of triple the frequency. In this case, three harmonics suffice to reconstruct the voice almost exactly. Often still higher harmonics are needed.

Let's put this in formulas. Let $P(t)$ be the air pressure as a function of time. Then we model this by an exactly periodic function $Q(t)$, i.e. there is a period p such that $Q(t+p) \equiv Q(t)$, all t. P and Q will be very close to each other. We write Q as a sum of sinusoids like this:

$$\boxed{Q(t) = C_0 + C_1 \sin(2\pi ft + D_1) + C_2 \sin(4\pi ft + D_2) + C_3 \sin(6\pi ft + D_3) + \cdots}$$

This is a very important formula, so we have made it big and put it in a box. The C's and D's are constants. The frequency of the whole periodic signal Q is f and the sum is made up of terms $C\sin(2\pi nft + D)$ with frequencies nf, known as the n^{th} harmonic of Q. $P(t)$ will be given by such a formula too, but, because the human voice is complicated, you have to let the C's and D's vary a bit with time. For example, in the first figure above showing *do* and *sol*, you see a slow undulation superimposed on the periodic signal: this comes from C_0 changing slowly. And if you look over longer periods of time, you find that even the shape of the signal changes slowly: this is caused by the relative phases $D_1 - D_2$ and $D_1 - D_3$ changing slowly. Another effect is vibrato, where the frequency oscillates around a mean; this is modeled by having D_1 oscillate slowly. But for the female voice used in the last figure, the change isn't too great (see second plot in the figure) and we picked a musical note for which the above three terms are already a very good approximation of the full signal $P(t)$.

Another way to say it is that hidden in the sound of *sol* is already the note *sol* one octave higher (twice the frequency) and the note *re* two octaves higher. Why *re*? From the table above, its frequency is $9/8^{\text{th}}$'s the frequency of *do*, so two octaves higher, it is $9/2^{\text{th}}$'s the frequency and 9/2 = 3 x 3/2, triple the frequency of *sol*! So why do chords sound well together: their harmonics overlap and they are actually sharing these hidden parts of themselves.

Maybe you didn't want to take a course in music theory but it's hard to resist describing the next wrinkle, namely the black keys on the piano keyboard. The major scale is the white keys and they give *do* a special place, making it a kind of home base. But composers want to play with 'changing the key' in the middle of a piece, taking another note as home and making all the triads etc on top of this. The fractions now get to be quite messy and a remarkable discovery was made: if the frequencies of the major scale are fudged a bit and 5 new notes are added (the black keys), then you get a scale in which the frequency of each note has the same ratio to the frequency of the next note, namely $2^{1/12} \approx 1.06$. Why does this work? The key piece of number magic is that $2^{7/12} = 1.498\cdots$

so a note, which is indistinguishable from sol, occurs. In fact, here are all the notes in the so-called ‘tempered scale’ with their frequency ratios to compared to the ‘true’ scale:

Note	**tempered freq. ratio**	**true freq. ratio**	**error**
C (or *do*)	1.000	1	0%
C sharp (D flat)	1.059		
D (or *re*)	1.122	9/8 = 1.125	0.2%
D sharp (or E flat)	1.189		
E (or *mi*)	1.260	5/4 = 1.25	0.8%
F (or *fa*)	1.335	4/3 = 1.333	0.1%
F sharp (or G flat)	$1.414=\sqrt{2}$	(With C, Galileo’s Ex of a harsh dissonance)	
G (or *sol*)	1.498	3/2 = 1.5	0.1%
G sharp (or A flat)	1.587		
A (or *la*)	1.682	5/3 = 1.667	0.9%
A sharp (or B flat)	1.782		
B (or *ti*)	1.888	15/8 = 1.875	0.7%
C (one octave higher)	2.000	2	0%