Chapter 8
Is there an Axiom for everything ?

Jean-Yves Béziau

We first start by clarifying what axiomatizing everything can mean. We then study a famous case of axiomatization, the axiomatization of natural numbers, where two different aspects of axiomatization show up, the model-theoretical one and the proof-theoretical one. After that we discuss a case of axiomatization in a sense opposed to the one of arithmetic, the axiomatization of the notion of order, where the idea is not to catch a specific structure, but a notion. A third mathematical case is then examined, the one of identity, a simple and obvious notion, but that cannot be axiomatized in first-order logic. We then move on to more general notions: the axiomatization of causality and the universe. To end with we deal with an even more tricky question: the axiomatization of reasoning itself. In conclusion we discuss in the light of our investigations the relation between axiomatization and understanding.

8.1 Axiomatizing Everything

One may want to axiomatize everything. Is it possible? To answer this question we need to understand what it means. This can be understood in two different ways:

1. Given *anything*, to axiomatize it.
2. A single axiomatization for the *whole thing*.

(1) is weaker than (2) and can be seen as a particular case. "Single axiomatization" is ambiguous; the extreme case is *one* axiom describing everything, the world, the whole reality, like a fundamental equation explaining the universe.

In this paper we will examine if (1) and (2) are possible or not. To do that we need to understand what we have on both sides: *Axiom* and *Everything*. But the clue is the relation between the two, which can be qualified as *axiomatizing*. Although this

Jean-Yves Béziau
University of Brazil, Rio de Janeiro, Brazilian Academy of Philosophy, Brazilian Research Center, e-mail: jyb@ufrj.br

notion is well-known, especially through the promotion of the so-called "Axiomatic Method" (see e.g. Gonseth (1939) and Hintikka (2011)), there are many confusions surrounding it. As often, the confusion is due to the mixture of different meanings attached to one word. Alfred Tarski (1944) analyzes the situation in the following way:

> We should reconcile ourselves with the fact that we are confronted, not with one concept, but with several different concepts which are denoted by one word; we should try to make these concepts as clear as possible (by means of definition, or of an axiomatic procedure, or in some other way); to avoid further confusions, we should agree to use different terms for different concepts; and then we may proceed to a quiet and systematic study of all concepts involved, which will exhibit their main properties and mutual relations.

This is here in relation with the concept of truth, and there are indeed some axiomatic theories of truth helping to clarify this concept (see Halbach (2011)). Regarding the concept of axiomatization, it is not clear if we can axiomatize it, but at least we can give definitions of it.

It is good to start with a very general view, encompassing the different possible understandings, before making specific distinctions. We can say that *axiomatizing is finding some simple and obvious truths from which it is possible to master a certain field*. It is a way to concentrate the understanding of a field in a few statements, using some basic notions and properties that can grasp the rest. It can be seen as a kind of reduction, but not necessarily a negative reduction, a reductionism. We can metaphorically compare that to condensation or to be at the top of a mountain from where it is possible to see a whole region.

The axiomatic method started in Ancient Greece but it was developed in a much more sophisticated way in modern logic. To properly understand how it works, one needs to have a basic knowledge of the four theories forming the basis of modern logic: set theory, proof theory, recursion theory and model theory. We will here explain the axiomatic method from this perspective. But since our paper is for a wide audience and is rather philosophical we will not give too many technical details. Nevertheless what we are saying can be precisely developed at a technical level and we are giving precise references supporting what we are saying.

8.2 Axiomatizing Natural Numbers

Let us start with a basic, central and critical example: the axiomatization of the natural numbers. By contrast to geometry, arithmetic was axiomatized only at the end of the 19th century. This was done independently and in different ways by Charles Sanders Peirce (1881), Richard Dedekind (1888) and Giuseppe Peano (1889).

We have an intuitive idea of the natural numbers since childhood through their names, enumerating them: 0, 1, 2, 3 . . . , making operations with them and using them to order things. In modern mathematics natural numbers are considered as forming a structure,[1] which can be presented in different ways. The following one is quite close to our intuition about them:

$$\mathcal{N} = \langle \mathbb{N}; <, +, \times \rangle.$$

It is a set with a binary relation and two binary functions. The structure of natural numbers can be considered in other ways, for example adding a unary function of succession and a constant for the number zero:

$$\mathcal{N} = \langle \mathbb{N}; 0, s, <, +, \times \rangle.$$

From a *model-theoretical* point of view, axiomatizing the natural numbers means finding a set of axioms that characterizes the structure $\mathcal{N}$ in the sense that this structure is the only structure which verifies, obeys, is a *model* of these axioms. Here "only" means up to isomorphism. And since isomorphism depends on one-to-one correspondence, this makes sense only for a given cardinality. A set of axioms is called a *theory* and a theory is said to be *categorical* for a given cardinality if all models of this theory of this cardinality are isomorphic. Taking in account these details and using the related terminology we can say that axiomatizing the natural numbers means, from a model-theoretical point of view, finding a categorical theory for $\mathcal{N}$ (Categorical relatively to denumerability, since the natural numbers are typically denumerable). We will talk of *MTC-axiomatization* (MTC being an abbreviation for "model-theoretical categorical").

Although model-theoretical axiomatization was already quite clear with the work of David Hilbert in 1899 on axiomatization of geometry (Hilbert, 1899) and the concept of categoricity was introduced in 1904 by Oswald Veblen (Veblen, 1904), MTC-axiomatization was made perfectly precise only with the work of Tarski in model theory in the 1950s, developing in a systematic way the relation between a theory and its models (Tarski, 1954a,b, 1955).

If we have a theory such that $\mathcal{N}$ is a model of this theory, but there is also a quite different structure model of this theory, then we will not say that this theory is a MTC-axiomatization of $\mathcal{N}$. In 1934 Thoralf Skolem showed that the basic axiomatization for natural numbers, *PA* (*Peano Arithmetic*), has some non-standard models, in which there are non-standard numbers coming after all the usual natural numbers (Skolem, 1934). According to that, *PA* does not properly MTC-axiomatized the natural numbers. This result is an application of the compactness theorem, a central theorem of first-order logic.

Axiomatization in modern logic goes hand to hand with *formalization* in the sense that the axioms are formulated in a precise language having some precise properties. Axioms and other statements are expressed through *formulas*. The language of first-

[1] The notion of structure was promoted as the central notion of mathematics by Bourbaki, cf. the Chapter 4 of the book *Théorie des ensembles* 1970, entitled *Structures*. See also (Bourbaki, 1948) and (Corry, 1996). And Bourbaki stressed that the structure of natural numbers is not at all the simplest structure, it is a mix /combination of different structures (*carrefour de structures* in French). See our recent paper (Beziau, 2017b), pointing the many aspects of the number 1 according to different structures it is merged in.

order logic is the main language of modern logic. There are quantifiers, connectives, relations and functions between/over objects. But in first-order logic there are no relations or functions between/over sets of objects. And in first-order logic it is not allowed to quantify over relations or functions. Basic topological concepts are typically not first-order.

In second-order logic it is possible to do so and by doing that to have a MTC-axiomatization of natural numbers. The reason why it is often not considered as satisfactory has to do with mechanization of reasoning, a concept that has been studied systematically in recursion theory. Recursion theory gives a precise definition of computability, corresponding to the informal notion of "algorithm". Second-order logic is strongly not mechanizable in particular due to the fact that the compactness theorem, according to which if a formula is a consequence of a theory it is a consequence of a finite subtheory of this theory, is not valid.

The notion of consequence can be understood in two different ways: proof-theoretically (symbolized by "$\vdash$") or model-theoretically (symbolized by "$\vDash$"). "$T \vDash F$" is understood as "All models of T are models of F" and $T \nvDash F$ as "There is a model of T which is not a model of F".[2] When we have a structure which is a model of F, we say that F is true in this structure. If we have a structure which is not a model of F, we say that F is false in this structure. By definition of classical negation (symbolized by "$\neg$"), it is equivalent as saying that $\neg F$ is true in this structure.

A theory is said to be *incomplete* when we have a formula F such that neither this formula nor its negation are a consequence of T. This formula is said to be *independent*. These notions make sense both from a proof-theoretical and model-theoretical perspectives. From the model-theoretical point of view, this means that there is a model of T in which F is true and a model of T in which F is false. So if we have a theory which is incomplete, this theory cannot be considered as a proper MTC-axiomatization of a given structure, because it has two models which essentially differ as shown by the independent formula, true in one model and false in another model.

On the other hand we may have a theory in first-order logic having different models, but complete, because the difference between models cannot necessarily be expressed in first-order logic. The incompleteness of *first-order arithmetic* (the theory formulated in first-order logic to axiomatize the structure of natural numbers $\mathcal{N}$) cannot be deduced from Skolem's theorem about non-standard models. But the fact that $\mathcal{N}$ is not MTC-axiomatizable in first-order logic can be deduced from Gödel's proof-theoretical incompleteness theorem of arithmetic (1931), via the completeness theorem establishing a correspondence between proof-theory and model-theory, a theorem also proved by Gödel (in 1930).

A structure which is MTC-axiomatizable is complete. And in first-order logic a complete theory is *decidable*, in the sense that we have an algorithm to know if a formula is a consequence or not of this theory. But a theory can also be incomplete and decidable; this is for example the case of the theory of dense order as proved by Robert Vaught in 1954. In classical propositional logic, the empty theory is incomplete and

[2] This definition was given by Tarski in 1936, although at this time the notion of model was not yet completely clear. Tarski was also not yet using the symbol "$\vDash$".

decidable: we can use truth-tables to check if a formula is a tautology or not and there are formulas, such that atomic propositions, which are independent.

Axiomatization was traditionally conceived from a *proof-theoretical* point of view, in the sense that we can *prove* all truths about a given field from these axioms. Proving meaning here a step-by-step deduction where every step is clearly explained and justified. This is how axiomatization appears in the 1657 book by Blaise Pascal, who was the first to clearly describe and analyze the procedure.[3] Incompleteness can be seen as a serious drawback for proof-theoretical axiomatization. If we have an incomplete theory T for the natural numbers, there is an independent formula F. This formula expressing a statement about natural numbers is true or false. If it is true, we would like it to be a proof-theoretical consequence of T, and if it is false we would like its negation $\neg F$ to be a proof-theoretical consequence of T. But it does not work. Incompleteness can be seen here as a discrepancy between truth and proof as argued by Tarski in his famous paper "Truth and Proof" (1969).

Summarizing: from a model-theoretical point of view, the fact that it is not possible to find a first-order complete theory for the structure of the natural numbers $\mathcal{N}$ means that we cannot find some axioms formulated in first-order language that precisely catch $\mathcal{N}$; from a proof-theoretical point of view it means that we cannot prove all the truths about natural numbers from a first-order set of axioms.[4]

On the other hand the structure of natural numbers $\mathcal{N}$ is MTC-axiomatizable in second-order logic. This means that we have a clear understanding of $\mathcal{N}$, caught by a few propositions expressed in a precise language, the language of second-order logic, despite the fact that our reasoning about $\mathcal{N}$ is not mechanizable, in the sense that it cannot be fully described by a recursively enumerable system like first-order logic.

Gödel himself made the following comments: "In 1678 Leibniz made a claim of the universal characteristic. In essence it does not exist: any systematic procedure for solving problems of all kinds must be nonmechanical." (Wang, 1997, p. 6.3.16); "My incompleteness theorem makes it likely that mind is not mechanical" (Wang, 1997, p. 6.1.9).

The fact that our mind is not mechanical is rather good news, we are not reducible to computers. Axiomatizability in higher logic shows that we can understand things which are beyond computability. But can we understand everything, can we axiomatize everything?

---

[3] Tarski was much influenced by Blaise Pascal, in particular when writing "Sur la méthode déductive" (1937).

[4] More exactly: from a recursive set of axioms. This means we should be able to identify these axioms, we have to exclude the case where we have any infinite set of axioms, like the extreme case of all formulas true in $\mathcal{N}$, which trivially is a complete theory for $\mathcal{N}$.

8.3 Axiomatizing the Notion of Order

The relation of order between natural numbers is a *discrete* total order with first element and without last element. Discrete means that between a natural number and its immediate successor, say between 7 and 8, there is no other natural numbers. Another example of order is the strict order among rational numbers. It is radically different in the sense that between two rational numbers there is always another one, it is called a linear *dense* order. This order obeys the following axioms:

$\forall x \quad \neg(x < x)$	irreflexivity
$\forall x \forall y \forall z \quad (x < y \wedge y < z) \rightarrow x < z$	transitivity
$\forall x \forall y \quad (x \neq y \wedge x < y) \rightarrow \neg(y < x)$	antisymmetry
$\neg \exists x \forall y \quad (x < y)$	no first element
$\neg \exists x \forall y \quad (y < x)$	no last element
$\forall x \forall y \quad (x < y \rightarrow \exists z (x < z \wedge z < y))$	density
$\forall x \forall y \quad x \neq y \rightarrow (x < y \vee y < x)$	totality or linearity

It has been shown (result originally due to Cantor) that these axioms have only one denumerable model (up to isomorphism).[5] For this reason we can say that the notion of dense order without end points is MTC-axiomatizable (in first-order logic).

From these axioms we can extract two axioms, for which we will use the sign "*R*" rather than "<":

$\forall x \forall y \forall z \quad (xRy \wedge yRz) \rightarrow xRz$	transitivity
$\forall x \forall y \quad (x \neq y \wedge xRy) \rightarrow \neg yRx$	antisymmetry

The symbol "<" leads us to think of irreflexive (or strict) order by contrast to the symbol "≤". If we don't have the axiom $\forall x \neg(x < x)$ of irreflexivity we need a notation leaving space for our imagination, leaving the door open to various *interpretations* (a canonical concept of model theory). A relation obeying the two above axioms is called a *relation of order*.[6] Can we say that this theory, the conjunction of these two axioms, axiomatizes the notion of order?

This theory has many different models. The order can be dense or not, can have a first element or not, can be partial or not. Is the fact that we have many different models of this theory a problem for talking about axiomatization? Not necessarily. We have caught something common to all these relations of order, avoiding things opposite to this notion such as cycles. We can talk about *MT-axiomatization*, removing the question of categoricity.

---

[5] The axiomatization presented here is not independent in the sense that for example the axiom of antisymmetry is a consequence of the axioms of irreflexivity and transitivity.

[6] Sometimes a relation of order is defined as a relation also being reflexive. This is not a very good choice, because then the notion of strict order is contradictory.

MT-axiomatization is important if we want to generalize axiomatization to non-mathematical notions, for example the notion of animal. There are many "non-isomorphic" animals, different in shapes, internal features, behaviors, nevertheless one may look for axioms characterizing the very nature of the notion of animal, if any. For the relation of order we have two axioms expressed by two first-order formulas. If we put a conjunction between the two, this can be reduced to only one formula:

$$\forall x \forall y \forall z (xRy \wedge yRz) \rightarrow xRz \wedge \forall x \forall y (x \neq y \wedge xRy) \rightarrow \neg yRx$$

This is rather artificial. These two axioms correspond to two different ideas that are put together with a conjunction. This is not the same as one single axiom corresponding to a single idea from which two or more axioms can be deduced, a synthetic axiom.

In set theory there is the *axiom of abstraction* saying that any property/formula determines a set:

$$\exists x \forall y \quad (y \in x \leftrightarrow Fy)$$

The problem is that not only other basic intuitive axioms about sets are consequence of this axiom, but also all formulas, in another words: this axiom is trivial, is inconsistent. This is the famous Russell's paradox.

Reduction to a single axiom, even if it is not trivial, may appear as rather meaningless, a formal artificial game, at best showing capacity of high intellectual gymnastic. For example, Tarski (1938) provided the following single axiom for abelian group, using division as the unique primitive relation:

$$x/(y/(z/(x/y))) = z$$

This reduction is a kind of reductionism by opposition to the usual set of axioms where each axiom has a clear and definite meaning.

A more consistent and meaningful example is from the field of logic: all the properties of classical negation can be put in only one intuitive axiom, the strong reduction to the absurd. This axiom can then be decomposed in various axioms, each having a distinct meaning like elimination of double negation, *ex-falso sequitur quod libet*, etc. (for details about that, see Beziau (1994)).

It is also important to stress that one axiom may have many different equivalent formulations, the most famous example being the axiom of choice (cf. Rubin and Rubin (1963; 1985)). Although all these formulations are equivalent, they correspond to different ideas. What all these ideas have in common is not clear, we cannot really say that there is one unique idea beyond / behind all these formulations. At best we can say that one formulation is more typical, more representative. This variation of formulations, of meanings, can also manifest not for only one axiom, but for a set of axioms. A notable example is the case of Boolean algebra, which can be seen either as a distributive complemented lattice or as an idempotent ring. This was discovered by Marshall Stone (1935) who was amazed by the coincidence of these two perspectives, these two different axiomatizations of the same structure.

These considerations about MT-axiomatization are important for answering the question about a single axiom for everything. This singleness can be seen as a weird singularity! If we axiomatize time using the notion of order, it is not possible to characterize it with only one axiom, considering that antisymmetry and transitivity are two different ideas that can hardly be subsumed by a unique third idea expressed in one axiom. And even if we can find one single axiom corresponding to one single idea, this axiom can be considered as equivalent to another axiom corresponding to another idea, as shown by the case of the axiom of choice. We may have different equivalent perspectives on the same reality which itself is beyond a unique particular characterization.

8.4 Axiomatizing Identity

The notion of identity seems obvious but it can be understood in different ways. Let us here consider identity as a relation between things (objectual identity). Two things can be more or less identical depending for example of how many properties they share. A definition of objectual identity, attributed to Leibniz, is that if two things have the same properties they are identical. There is a more radical notion of identity, that we will call *trivial identity*, represented by the diagram in Fig. 8.1. The relation

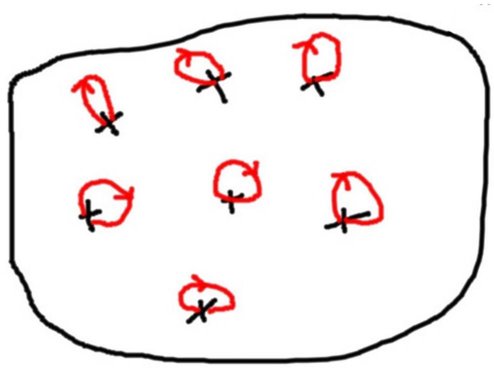

Fig. 8.1 Illustration of the identity relation.

of identity is here relating each object to itself and that's it! According to the above diagram the relation of identity does not hold between different objects.

The problem is that this relation of trivial identity cannot be model-theoretically axiomatized in first-order logic (see e.g. Hodges (1983)).

But there is another problem, despite the fact that we can visually represent this notion in a diagram, it is not clear that we can directly phrase it. We can say:

Every object is identical to itself and different from the others.

But this means:

Every object is identical to itself and not identical to non-identical objects.

The second part of the proposition is a tautology and since the conjunction of a proposition p with a tautology is equivalent to the proposition p, this formulation of the axiom of identity is equivalent to:

Every object is identical to itself.

And such an axiom is nothing else than the axiom of reflexivity and does not exclude reflexive relations which are not trivial identity, where two different elements can be in relation. From this perspective we can say that the relation of trivial identity corresponds to a certain situation, which we can understand through a picture but that we cannot directly phrase (for more details see Beziau (2015b)).

However it is possible to formulate it in an indirect way, which can be expressed in second-order logic, MTC-axiomatizing it, saying that it is the least reflexive relation (see Manzano (1996)).

This example shows different levels of understanding: at one level we pictorially understand something but cannot phrase it. At this level we cannot therefore properly talk about axiomatization. Understanding does not necessarily reduce to axiomatization. In the case of identity we are lucky that at a second level we can axiomatize it, but there can be phenomena not axiomatizable at any level, and which cannot be phrased.

8.5 Axiomatizing Causality

Leibniz has promoted the famous dictum *Nihil est sine ratione*, called the "principle of sufficient reason", which often is interpreted as *Nothing is without a cause* and that we can positively state as *Everything has a cause*.

Is it true that everything has a cause? To answer this question we must investigate what causality is. There are different ways to do that. One of them is to try to axiomatize causality. Is it possible to axiomatize causality? To do so, like for any axiomatization, we have to choose a conceptual framework.[7] One pretty natural option is to consider that causality is a binary relation between events, we can write "$a \hookrightarrow b$", read as "a causes b" or "b is caused by a".

[7] David Hilbert in his famous paper on axiomatic thought (1918) is using the expression "conceptual framework" (in German : *Fachwerk von Begriffen*), but he identifies it to the notion of theory. We prefer to use the word "theory" as referring to a set of statements that in particular can be considered as axioms.

Axiomatizing causality then means describing some basic properties of this binary relation. There are different options that we will not discuss here, just presenting one option among others (for more details see Beziau (2015a)). Our objective is not to defend one particular vision of causality but to show that there are intrinsic problems which are quite independent of a particular choice.

We consider the properties described by the following formulas:

$$\forall x \quad \neg(x \hookrightarrow x) \qquad \text{irreflexivity} \qquad (8.1)$$

$$\forall x \forall y \quad (x \neq y \wedge x \hookrightarrow y) \rightarrow \neg(y \hookrightarrow x) \qquad \text{antisymmetry} \qquad (8.2)$$

$$\exists x \exists y \exists z \quad (x \hookrightarrow y) \wedge (y \hookrightarrow z) \wedge \neg(x \hookrightarrow z) \qquad \text{non-transitivity} \qquad (8.3)$$

$$\forall x \exists y \quad (y \hookrightarrow x) \qquad \text{everything has a cause} \qquad (8.4)$$

The main problem is that we have some models obeying these axioms where the relation can be seen as something radically different from causality, like the binary relation of immediate succession among the integers, i.e. aRb *iff* $b = a + 1$.

That is another central point of the question of axiomatization: even if we do not limit axiomatization to MTC-axiomatization, extending it to MT-axiomatization, we don't want to catch things that are of a nature very different from what we have in mind. It is not clear in this case how to avoid that, if we can do that by adding further axioms.

There is another problem in this axiomatization of causality. It is with axiom 8.3, the axiom of non-transitivity, which is existential/negative. Generally when axiomatizing something we are looking for universal features.[8]

As we were saying, the above theory is just a possible axiomatization of causality. One may want for example to put an axiom saying that there is a first cause, but, in any case, whatever axioms we choose, it seems the same problem will repeat: we will have some models very different in nature of what we want to axiomatize.

Axiomatizing causality can be seen as a way to axiomatize everything. *Nihil est sine ratione* is indeed a very general principle that can be viewed as a key for the understanding of reality. But if we formalize it as axiom 8.4 above, it does not really make sense by itself, we need to add other axioms. And even adding further axioms, it is not clear at all that we are succeeding to axiomatize causality.

Rougier in his 1920 book *Les paralogismes du rationalism* (*Paralogisms of rationalism*) strongly criticizes such kind of general principles considering them as meaningless and Leibniz is one of his favorite scapegoats when criticizing rationalism.[9] One of his criticisms is about the principle "The whole is bigger than the part", explaining how modern set theory resolves Galileo's paradox by making the distinc-

[8] Rolando Chuaqui and Patrick Suppes (1995) have shown that classical mechanics can be axiomatized by formulas with only universal quantifiers.

[9] Rougier (1889-1992) was a good friend of Moritz Schlick and one of the main promoters of the Vienna Circle. He organized in 1935 at the Sorbonne in Paris a big congress on scientific philosophy with the participation of Schlick, Carnap, Neurath, Russell, Tarski, Lindenbaum, etc. The result was a 8-volume book: *Actes du Congrès International de Philosophie Scientifique – Sorbonne, Paris, 1935*.

tion between inclusion and one-to-one correspondence. However we can say that in this case axiomatization (axiomatic set theory) helps to clarify our conceptualization.

8.6 Axiomatizing Reality

One of the difficulties for axiomatizing reality is that is has many aspects. This is what we can see looking around us here on earth, not even travelling through the universe. We may want to reduce reality to few objects, few phenomena, but that's not so easy without a heavy reductionism, like physicalism.

We can axiomatize physics, Hilbert himself did a lot of work in that direction (see Corry (2004)). But to axiomatize the reducibility, let's say of biological phenomena, not to say psychological phenomena, to physical phenomena is not that easy. We can argue that physicalism will be seriously supported only when such kind of axiomatization is provided.

Einstein is reported to have claimed: "The grand aim of all science is to cover the greatest number of empirical facts by logical deduction from the smallest possible number of hypotheses or axioms." (Barnett, 1948) The axiomatization mentioned here is not what we have called model-theoretical axiomatization, it is rather proof-theoretical axiomatization.

However the theory of general relativity can be model-theoretically axiomatized in various ways in first-order logic (see Andréka, Madarsz, and Németi (2007)). As it is known, Gödel (1949) has shown that Einstein general theory of relativity has some non-standard models, so it is not a MTC-axiomatization. It does not fully grasp reality because it admits different incompatible models. To solve the problem it is necessary to reduce the number of models. If it possible to do that in a natural way, by finding a categorical axiomatization of the universe based on some intuitive axioms, not by artificially adding an *ad hoc* axiom eliminating rotating universe, is an open question. And someone who believes that our universe is really rotating should find an axiomatization which does not admit a standard model according to which we cannot come back in the past.

Axiomatizing the (physical) universe is something, axiomatizing (biological) life is another thing. Physics can in some sense be seen as something merely geometrical, part of mathematics. It is easier to fix a conceptual framework as the basis for an axiomatization of physics, than of biology. Axiomatization of biology is still something very much experimental, although it was initiated at the beginning of the 20th century. Tarski first love was neither mathematics, nor logic, but biology and since he was found of the axiomatic method he encouraged people to axiomatize biology in particular Joseph Henri Woodger (see Woodger (1937)).

Someone may want to develop a general theory/axiomatization of everything using a very abstract theory not directly ontologically committed with the nature of its objects, for example set theory. But even in the case of someone convincingly succeeding to argue that everything in reality is (or can be interpreted as) a set, does this mean that set theory can axiomatize reality? There is an ambiguity here. If we just stay at the mathematical level, we can say that the notion of a group can be conceived and defined using set theory, but it does not make properly sense to say that an axiomatic theory of set, let's say *ZF*, axiomatizes group theory, in particular the axioms of group theory are not a consequence of the axioms of *ZF* (for more details see Beziau (2002)).

8.7 Axiomatizing Reasoning

Human beings have been characterized as rational animals, logical animals, animals able to reason. Reasoning is a basic feature of human beings and reasoning can be considered as the backbone of thought. "Logic is the anatomy of thought" is a dictum attributed to Locke, logic has been named the *Art of Thinking* (cf. Arnauld and Nicole (1662)) and Boole, considered as one of the main originators of modern logic, wrote a famous book called *The Laws of Thought* (1854) (see Beziau (2010, 2017a)).

From this perspective axiomatizing reasoning may be interpreted as axiomatizing human beings and also as indirectly axiomatizing reality, considering that reasoning is the basic tool to capture, describe, understand reality.

But can we axiomatize reasoning? Can we find some axioms describing the reality of reasoning, the way we are reasoning? To answer this question we will here again consider axiomatization from a model-theoretical point of view. Can we find some axioms, whether in first-order logic or in second-order logic, which are a MTC-axiomatization of reasoning?

As for other fields, we have to set up a conceptual framework. We consider the one directly inspired by Tarski's theory of consequence operator initiated in (Tarski, 1928), where we have an abstract consequence relation, that we will express using the notation "$T \Vdash F$". Proof-theoretical consequence (symbolized by "$\vdash$") and model-theoretical consequence (symbolized by "$\vDash$") can both be seen as particular models of this abstract consequence relation. By modeling model-theoretical consequence we are at a meta-level. We can also say that we are axiomatizing axiomatization (see Beziau (2006)).

We may fix some general axioms for this abstract consequence relation, such as the three following ones (due to Tarski):

$T \Vdash F$ when F is an element of T	reflexivity
If $T \Vdash F$ and T is included in U, then $U \Vdash F$	monotonicity
If $T \Vdash F$ for every F in U and $U \Vdash G$, then $T \Vdash G$	transitivity

These axioms make sense if we have in view a proof-theoretical notion of consequence defined using what is called an "Hilbert system". They also hold for a model-theoretical notion of consequence defined using the basic idea of model inclusion (cf. Tarski (1936)). So these axioms are very general. They hold not only independently of a specific logical language corresponding to a given conceptual framework, but also independently of the way the consequence relation is originally conceived.

These axioms admit many different models, so they are not going on the direction of a MTC-axiomatization of reasoning. This can be done by adding further axioms.

For example with the two following axioms

$$T \Vdash p \rightarrow q \text{ iff } T, p \Vdash q$$
$$\text{If } T, \neg p \Vdash q \text{ and } T, \neg p \Vdash \neg q \text{ then } T \Vdash p$$

we succeed to fully axiomatize classical propositional logic. This does not mean that we have axiomatized reasoning, unless we believe that reasoning reduces to classical propositional logic. Hard to believe! At best one may believe that reasoning reduces to the one described by first-order logic and we can add further axioms about quantifiers to axiomatize this system.

But many other different logical systems have been developed in the last 100 years and it is not clear what the correct one is, if any. Anyway, a general theory of abstract consequence relation allows us to master all these systems, by MT-axiomatizing them.

However there may be doubts about the validity of the three Tarski's axioms. In particular the second one does not hold for logical systems which have been qualified as "non-monotonic" for this very reason. Shall we delete this axiom? And then do not we have an axiomatization which is too general, not characterizing reasoning, admitting models having nothing to do with reasoning?

It is very difficult to sustain that we can axiomatize reasoning unless we believe that there is a very specific right kind of reasoning that can be described within a precise conceptual framework.

8.8 Axiomatizing and Understanding

To conclude we can say that the axiomatic method, that we can summarize in one word, *axiomatizing*, is a key to understanding, not only by catching a structure up to isomorphism, but also by catching a notion having many different aspects, like a relation of order.

And it is important to make a clear distinction between axiomatization and mechanization of thought. Understanding certainly does not reduce to computability. Our analysis of axiomatization also clearly shows that understanding does not reduce to the singularity of the singleness of an axiom. Although a key feature of axiomatizing is to make a reduction to some few statements about few concepts, the axiomatic method is not monotheist. Moreover, there are different ways of understanding which, although equivalent, have each their own value. The axiom method is not one-sided.

Finally, there are "things", like reasoning itself, that we cannot properly axiomatize and that maybe are beyond our understanding, although understanding does not necessarily reduce to axiomatization.

Acknowledgements

The origin of this paper is a talk presented at the meeting *Kurt Gödel's Legacy: Does Future lie in the Past?* which took place July 25-27, 2019 in Vienna, Austria. Thanks to the organizers of the event and also thanks to Jan Zygmunt, Robert Purdy and Jorge Petrucio Viana for useful comments and information.

References

Actes du Congrès International de Philosophie Scientifique – Sorbonne, Paris, 1935 (1936).

Andréka, H., J. Madarsz, and I. Németi (2007). „Logic of space-time and Relativity theory". In: *Handbook of Spatial Logics*. Ed. by M. Aiello, I. Pratt-Hartmann, and J. van Benthem. Springer, Heidelberg, pp. 607–711.

Arnauld, A. and P. Nicole (1662). *La logique ou l'art de penser*. Savreux, Paris.

Barnett, L. (1948). *The Universe and Dr Einstein*. William Morrow and Company, New York.

Beziau, J.-Y. (1994). „Théorie législative de la négation pure". In: *Logique et Analyse* 147-148, pp. 209–225.

– (2002). „La théorie des ensembles et la théorie des catégories: présentation de deux soeurs ennemies du point de vue de leurs relations avec les fondements des mathématiques". In: *Boletín de la Asociación Matemática Venezolana* 9, pp. 45–53.

– (2006). „Les axiomes de Tarski". In: *La philosophie en Pologne 1918-1939*. Ed. by R. Pouivet and M. Rebuschi. Vrin, Paris, pp. 135–149.

– (2010). „Logic is not logic". In: *Abstracta* 6, pp. 73–102.

– (2015a). „Modelling causality". In: *Conceptual Clarifications Tributes to Patrick Suppes (1922-2014)*. College Publications, London, pp. 187–205.

– (2015b). „Panorama de l'identité". In: *Al Mukhatabat* 14, pp. 205–219.

– (2017a). „Being aware of rational animals". In: *Representation and Reality: Humans, Animals and Machines*. Ed. by G. Dodig-Crnkovic and R. Giovagnoli. Springer International Publishing, Cham, pp. 319–331.

– (2017b). „MANY 1 - A Transversal Imaginative Journey across the Realm of Mathematics". In: *Special Issue on Mathematical Pluralism of the Journal of Indian Council of Philosophical Research* 34.2. M.Chakraborty and M.Friend (eds), pp. 259–287.

Boole, G. (1854). *An Investigation of the laws of thought, on which are founded the mathematical theories of logic and probabilities*. London and Cambridge: Macmillan & Co.

Bourbaki, N. (1948). „L'architecture des mathématiques —La mathématique ou les mathématiques". In: *Les grands courants de la pensée mathématique*. Ed. by F. le Lionnais. Cahier du Sud, pp. 35–47.

– (1970). *Théorie des ensembles*. Hermann, Paris.

Chuaqui, R. and P. Suppes (1995). „Free-variable axiomatic foundations of infinitesimal analysis: A fragment with finitary consistency proof“. In: *The Journal of Symbolic Logic* 60, pp. 122–159.

Corry, L. (1996). *Modern algebra and the rise of mathematical structures*. (2nd edition 2004). Birkhäuser, Basel.

– (2004). *David Hilbert and the axiomatization of physics (1898-1918)*. Springer, Dordrecht.

Dedekind, R. (1888). *Was sind und was sollen die Zahlen?* Braunschweig: Friedrich Vieweg & Sohn.

Gödel, K. (1930). „Die Vollständigkeit der Axiome des logischen Funktionenkalküls“. In: *Monatshefte für Mathematik und Physik* 37, pp. 349–60.

– (1931). „Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I“. In: *Monatshefte für Mathematik und Physik* 38, pp. 173–98.

– (1949). „An example of a new type of cosmological solutions of Einstein’s field equations of gravitation“. In: *Reviews of Modern Physics* 21, pp. 447–450.

Gonseth, F. (1939). „La méthode axiomatique“. In: *Bulletin de la Société Mathématique de France* 67, pp. 43–63.

Halbach, V. (2011). *Axiomatic theories of truth*. Cambridge: Cambridge University Press.

Hilbert, D. (1899). *Grundlagen der Geometrie*. Leipzig: Teubner.

– (1918). „Axiomatisches Denken“. In: *Mathematische Annalen* 78, pp. 405–415.

Hintikka, J. (2011). „What is the axiomatic method?“ In: *Synthese* 183, pp. 69–85.

Hodges, W. (1983). „Elementary predicate logic“. In: *Handbook of Philosophical Logic I*. Ed. by D. Gabbay and F. Guenthner. Dordrecht, Reidel.

Manzano, M. (1996). *Extensions of first order logic*. Cambridge: Cambridge University Press.

Pascal, B. (1657). *De l’esprit géométrique et de l’art de persuader*.

Peano, G. (1889). *Arithmetices Principia, Novo Methodo Exposita, Fratres Bocca*. Rome.

Peirce, C. (1881). „On the logic of number“. In: *American Journal of Mathematics* 4.1, pp. 85–95.

Rougier, L. (1920). *Les paralogismes du rationalisme*. Paris: Félix Alcan.

Rubin, H. and J. Rubin (1963). *Equivalents of the axiom of choice I*. Amsterdam: North Holland.

– (1985). *Equivalents of the axiom of choice II*. Amsterdam: North Holland.

Skolem, T. (1934). „Über die Nicht-charakterisierbarkeit der Zahlenreihe mittels endlich oder abzählbar unendlich vieler Aussagen mit ausschliesslich Zahlenvariablen“. In: *Fundamenta Mathematicae* 23, pp. 150–161.

Stone, M. (1935). „Subsumption of the theory of Boolean algebras under the theory of rings“. In: *Proceedings of National Academy of Sciences* 21, pp. 103–105.

Tarski, A. (1928). „Remarques sur les notions fondamentales de la méthodologie des mathématiques“. In: *Annales de la Société Polonaise de Mathématique* 7, pp. 270–272.

Tarski, A. (1936). „O pojęciu wynikania logicznego“. In: *Przegad Filozoficzny* 39, pp. 58–68.

– (1937). „Sur la méthode déductive“. In: *Travaux du IXe Congrès International de Philosophie, VI*. Hermann, Paris, pp. 95–103.

– (1938). „Ein Beitrag zur Axiomatik der Abelschen Gruppen“. In: *Fundamenta Mathematicae* 30, pp. 253–256.

– (1944). „The Semantic conception of truth and the foundations of semantics“. In: *Philosophy and Phenomenological Research* 4, pp. 341–376.

– (1954a). „Contributions to the theory of models. I“. In: *Indagationes Mathematicae* 16, pp. 572–581.

– (1954b). „Contributions to the theory of models. II“. In: *Indagationes Mathematicae* 16, pp. 582–588.

– (1955). „Contributions to the theory of models. III“. In: *Indagationes Mathematicae* 17, pp. 56–64.

– (1969). „Truth and proof“. In: *Scientific American* 220.6, pp. 63–77.

Vaugth, R. (1954). „Applications of the Löwenheim-Skolem-Tarski theorem to problems of completeness and decidability“. In: *Indagationes Mathematicae* 16, pp. 467–472.

Veblen, O. (1904). „A system of axioms for geometry“. In: *Transactions of the American Mathematical Society* 5, pp. 343–384.

Wang, H. (1997). *A logical journey - From Gödel to philosophy*. Cambridge, Mass: MIT Press.

Woodger, J. (1937). *The Axiomatic Method in Biology, with appendices by Alfred Tarski and W.J. Floyd*. Cambridge University Press, Cambridge.