Ann. Math. Stat.
Dankook Univ.
Vol. 13. 1985

Some Open Problems in Hyperfunction Theory

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0. In this note I shall present some of the unsolved problems with which the theory of hyperfunctions is deeply concerned. Since these are gathered based on my personal knowledge, they are rather prejudiced, and even may contain trifling ones. I only expect that they may entertain those who are newly intending to study the theory of hyperfunctions. In the sequel we assume the knowledge on the standard terminology in the theory of hyperfunctions. See e.g. Komatsu [1], Morimoto [1] or Kaneko [1] as a reference book.

1. The theory of hyperfunctions of one independent variable is very easy to understand. We shall show that nevertheless we have some open problems concerning it, whereas there can be no more such possibility for distributions. This is, so to speak, because of the difficulty of the study of essential singularities of holomorphic functions compared to that of poles. Let f, g be two hyperfunctions with supports in $x \geq 0$. Then we can define their convolution by

$$(1) \qquad (f*g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy.$$

Most naturally, the right-hand side is interpreted as the integration along fibers of the hyperfunction $f(x-y)g(y)$ in two

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independent variables x, y whose support is proper with respect to y. If we wish to manage it within the framework of one variable, we can define $f*g$ locally as the difference of the boundary values of the holomorphic function

(2) $$\int_{\gamma} F(z-\zeta)G(\zeta)d\zeta,$$

where F, G are the defining functions of f, g respectively and γ is a path surrounding the positive real axis in the negative direction, with the endpoints lying sufficiently far from the origin compared to the region where z is now restricted (see fig. 1).

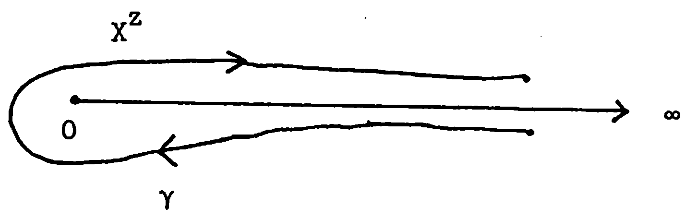

fig. 1

Problem A. Does $f*g=0$ imply either $f=0$ or $g=0$?

In the case of distributions this is known as Titchmarsh's theorem, and is used in the basis of Mikusiński's operational calculus (see e.g. Yousida [1]). Let C_+, $\mathcal{D}'_+$ and $\mathcal{B}_+$ denote the totality of continuous functions, distributions and hyperfunctions respectively, with supports in $x \geqq 0$. Then they constitute commutative algebras over $\mathbf{C}$ with respect to the convolution product.

We have

(3) $$C_+ \subset \mathcal{D}'_+ \subset \mathcal{B}_+.$$

By Titchmarsh's theorem we can construct the quotient field $\overline{C_+}$ of C_+. We see easily that $\mathcal{D}'_+ \subset \overline{C_+}$, where elements of $\mathcal{D}'_+$ correspond to those fractions whose denominators consist of polynomials of $x_+ = \max\{x,0\}$. We are therefore interested in the following.

Problem B. What is the relation between $\overline{C_+}$ and $\mathcal{B}_+$?

Of course $\mathcal{B}_+ \subset \overline{C_+}$ would imply the affirmative answer to Problem A. The significance of these problems is as follows: Mikusiński's interpretation of operational calculus is too abstract though it is completely powerful. Interpretation via $\mathcal{D}'_+$ is much more intuitive, but it can legitimize only finite order derivations. Therefore we expect that the use of $\mathcal{B}_+$ may give more concrete and yet sufficiently powerful interpretation. (Note that we can perform anyway a kind of hyperfunction theoretic operational calculus even if the answer to Problem A is negative. Note also that Yosida [1] contains no such viewpoints in spite of its subtitle.) These problems were proposed and tried by a graduate course student as his master's thesis more than ten years ago, but he himself failed to solve it.

Remark 1. For the distribution case the above mensioned Titchmarsh's theorem follows from the following more strong one of Titchmarsh:

$$\text{(4)} \qquad \mathrm{Ch\ supp}\ f*g = \mathrm{Ch\ supp}\ f + \mathrm{Ch\ supp}\ g$$

(see e.g. Hörmander [1], Theorem 4.3.3). This formula fails true for hyperfunctions. In fact, an example in Pólya [1] (p.597) shows that there exist hyperfunctions f, g such that the convex hull of their supports agrees with $[-1,0]$ and $[0,1]$ respectively, but supp $f*g$ reduces to the origin. (In Pólya's paper this assertion is expressed by means of the indicator diagram of the entire functions $\hat{f}(\zeta)$, $\hat{g}(\zeta)$ and $f(\zeta)g(\zeta)$, where $\hat{}$ denotes the Fourier transform:

$$\hat{f}(\zeta) = \int_{-\infty}^{\infty} e^{-ix\zeta} f(x)dx.$$

Needless to say, Pólya did not know the theory of hyperfunctions.)

Remark 2. Recall that we have the notion of Fourier hyperfunctions on the compactification $\mathbb{D} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ of the real line (Sato [1], Kawai [1], [2]; see also Chapter 8 of Kaneko [1] for more popularized exposition). Let $\mathcal{Q}_+$ denote the totality of Fourier hyperfunctions with supports in $0 \leqq x \leqq \infty$. Then there exists a natural surjection mapping $\mathcal{Q}_+ \to \mathcal{B}_+$ of restriction. IN other words, every element of $\mathcal{B}_+$ can be "extended" to an element of $\mathcal{Q}_+$ with the ambiguity of those elements of $\mathcal{Q}_+$ with supports in $\{-\infty\}$. The Fourier transformation maps $\mathcal{Q}_+$ injectively to a space of holomorphic functions on the lower half plane $\operatorname{Im} \zeta < 0$. Therefore the convolution algebra $\mathcal{Q}_+$ is isomorphic to a usual algebra of holomorphic functions and hence contains no zero divisor. In terms of $\mathcal{Q}_+$ Problem A may be rewritten as follows:

Problem A'. Does $\mathcal{Q}_+$ contain elements f, g such that $\operatorname{supp} f \neq \{+\infty\}$, $\operatorname{supp} g \neq \{+\infty\}$ but that $\operatorname{supp} f*g = \{+\infty\}$?

2. In Kaneko [2] (corollary 1.8) we proved that a hyperfunction of one variable $f(x)$ with support in $[-1,1]$ can always be represented in the form

$$f(x) = J_1(D)\mu_1 + J_2(D)\mu_2 + J_3(D)\mu_3,$$

where μ_j are measures with supports in $[-1,1]$ and $J_j(D)$ are local operators, i.e. infinite order differential operators corresponding to the convolution operators by hyperfunctions with supports concentrated at the origin (and $J_j(\zeta)$ are the Fourier image of them). For the necessity to further discussion let us recall the outline of the proof: First we regularize f to a continuous function g as $f(x) = J_1(D)g(x)$ regardless to the support. Here $J_1(D)$ is in general an elliptic local operator of the type

(5) $$J_1(\zeta) = \prod_{n=1}^{\infty}\left(1 + \frac{\zeta^2}{(n\varphi(n))^2}\right),$$

with a function $\varphi(t)$ of $t \geqq 1$ monotone increasing to $+\infty$. Then we cut the support of $g(x)$ to $[-1,1]$. The ambiguity at the end $x = \pm 1$ gives the other two terms $J_2(D)\delta(x+1)$, $J_3(D)\delta(x-1)$. Examining this proof M. Sato conjectured the following.

Conjecture C. Every hyperfunction $f(x)$ of one variable with support in $[-1,1]$ can be represented by two terms as

$$f(x) = J_1(D)\mu_1 + J_2(D)\mu_2.$$

The background of this conjecture is as follows: We could prove this conjecture if in the above we could choose $J_1(D)$ among the hyperbolic local operators, because then supp g will spread out only on one side, say $x \geqq -1$, and the ambiguity would appear only at one endpoint $x = 1$. This is not true in general because the general growth order

$$O(e^{\varepsilon|\zeta| + |\mathrm{Im}\zeta|}) \qquad \text{for any} \quad \varepsilon > 0$$

of the Fourier image of hyperfunctions cannot be dominated by a hyperbolic local operator of the type

(6) $$J_1(\zeta) = \prod_{n=1}^{\infty}\left(1 + \frac{i\zeta}{n\varphi(n)}\right).$$

But Sato conjectures that gathering the factors of $\hat{f}(\zeta)$ corresponding to the zeros apart from the real axis, we can first execute the factorization $\hat{f}(\zeta) = J_0(\zeta)\hat{g}(\zeta)$, where $\hat{g}(\zeta)$ is now so moderate as to be dominated by (6). Then the above argument will apply to $g(x)$ to obtain the expression by two terms. Thus the following is much more fundamental:

Conjecture D. Let $f(x)$ be a hyperfunction with support in $[-1,1]$. Then there exists a local operator $J_0(D)$ and a hyperfunction $g(x)$ with support in $[-1,1]$ such that $\hat{f}(\zeta)=J_0(\zeta)\hat{g}(\zeta)$ and that

$$(7)\qquad |\hat{g}(\xi)| \leqq \prod_{n=1}^{\infty} \left|1+\frac{i\xi}{n\rho(n)}\right| \quad \text{on} \quad \xi\in R.$$

This is rather a problem in the theory of Functions. It often happens that a problem in the new theory evokes a new problem in the classical theory. We must not neglect the education of classical mathematics!

Remark 3. The growth condition (7) is equivalent to the following

$$\int_{-\infty}^{\infty} \frac{\log|\hat{g}(\xi)|}{1+|\xi|^2}\, d\xi < +\infty.$$

That is to say, $g(x)$ reduces to a non-quasi-analytic functional.

Remark 4. As a for a distribution $f(x)$ it is known even in the case of $n\geqq 2$ variables that if its support is contained in a regular compact set K then $f(x)$ can be represented as a finite sum of derivatives of measures with supports contained in K (see Schwartz[1], Théorème XXXIV). For a hyperfunction $f(x)$ I once tried to prove a similar representation on interpreting the derivatives in the sense of local operators, but succeeded in it only when $f(x)$ is in some subclass of non-quasi-analytic functionals. This is because such $f(x)$ can be regularized by a local operator of hyperbolic type (see Kaneko [2]). Now I rather conjecture the contrary:

Problem E. Give an example of a hyperfunction with compact convex support K such that it cannot be represented as a finite sum of the form $\Sigma J_j(D)\mu_j$ by measures with supports in K.

3. Let S be a hyperfunction with compact support. The solvability of the convolution equation $S*u=f$ in the space of hyperfunctions, or equivalently the surjectivity of

$S^*: \mathcal{B}(R^n) \to \mathcal{B}(R^n)$ was studied in the basic paper of Kawai [1], [2]. Following Ehrenpreis' theory he presented the condition (S) (S implying the initial of "slowly decreasing") as a sufficient condition for the surjectivity. It therefore remains to show that

Problem F. Is the condition (S) also necessary for the surjectivity of the convolution operator S^*?

In the case of distributions the necessity of some condition is clear from the fact that the convolution operator by a function S of class C_0^∞ can never act surjectively on $\mathcal{D}'(R^n)$. In the case of hyperfunctions, however, we do not have a regularizer with compact support. Hence we do not yet know the solution to the following.

Problem G. Give an example of S^* which is not surjective on $\mathcal{B}(R^n)$.

Kawai (and aoso Schapira [1]) showed that every distribution S with compact support satisfies condition (S), hence acts surjectively on $\mathcal{B}(\mathbb{R}^n)$. This can be generalized to any S of non-quasi-analytic functional (c.f. Ehrenpreis [1], Proposition 4.5). Kawai also showed that local operators act surjectively on $\mathcal{B}(\mathbb{R}^n)$. Therefore in the case of one variable Sato's conjecture D will imply the following.

Conjecture H. In the case of one variable the convolution operator S^* is always surjective on $\mathcal{B}(R)$ for any hyperfunction S with compact support.

This means that the condition (S) is automatic and of no use in the case of one variable. One may therefore expect the following instead of Problem G:

Conjecture G'. The convolution operator S^* is always surjective on $\mathcal{B}(\mathbb{R}^n)$ also in the case of $n \geqq 2$ variables.

The convolution operator S^* acts also on the space of Fourier hyperfunctions $\mathcal{Q}(\mathbf{D}^n)$. Hence the above problems have their conterparts on this space. According to the study of Kawai [1] the condition (S') for the solvability in this case seems a little different from the former one (S). But actually we do not know if it is really different or even substantial for the surjectivity (though for $n=1$ Kawai [1] proves the necessity of (S')). Further, for this case in order to clarify the matter M. Sato proposed to study convolution operators with non-compact supports. Then we have a regularizer, i.e. rapidly decreasing real analytic functions such as e^{-x^2}, hence the surjectivity surely requires some non-empty condition to S. It will be a good problem for those newly intending the study of the theory of Fourier hyperfunctions:

Problem I. Let S be a rapidly decreasing Fourier hyperfunction. Find a necessary and (or) sufficient condition for S so that $S^*: \mathcal{Q}(\mathbf{D}^n) \to \mathcal{Q}(\mathbf{D}^n)$ is surjective.

At the first try we may put the restriction that S is a rapidly decreasing real analytic function of modified type outside a compact set. This assumption will make us possible to imitate the case of S with compact support.

4. Concerning convolution operators we leave still other problems. For example, Kawai([2], §4) determined elliptic (i.e. analytic hypoelliptic) convolution operators and declared that he found much more abundant class of elliptic convolution operators than Ehrenpreis did, because Ehrenpreis ([1], Theorem 5.16) found after all that an elliptic convolution operator S* in the distribution category is nothing but the composition of an elliptic differential operator and a translation. However, as a matter of fact Kawai only showed that there exists an elliptic local operator J(D) of really infinite order. Therefore we do not know any example of convolution operator which is essentially

different from a differential operator or a translation.

Conjecture J. An elliptic hyperfunction convolution operator is always the composition of an elliptic local operator and a translation.

5. The general theory of linear partial differential equation $P(D)$ with constant coefficients in the framework of hyperfunctions was mostly done up to the time of Kawai [1], [2]. It remains, however, some problems still open. Such is the characterization of evolution operators:

Problem K. Find the necessary and sufficient condition for $P(D)$ so that it has a fundamental solution E with support contained in a closed half space $\langle x,\vartheta\rangle \geqq 0$.

This problem was proposed and solved by Hörmander for the case of distributions (see [1], Theorem 12.8.1). We may follow his argument based on the a priori estimate (though I did not dare to do it). A new, hyperfunction theoretic approach would be more interesting. Recall here the general principle that to solve a same problem the case of hyperfunctions is much easier than the case of distributions and the solution may be written in terms of the principal part of $P(D)$ only. Note also that the operator $\partial_1 - \partial_2^2$ on R^2 is of evolution type only to the direction $\vartheta = (1,0)$ in the distribution sense, but to every direction in the hyperfunction sense because it has a hyperfunction fundamental solution with support contained in the half line $\{x_1 = 0\ ,\ \pm x_2 \geqq 0\}$.

6. In the hyperfunction category, the problem of hyperbolicity may seem to be already well clarified even in the micro-local level. When looking at the details, however, there still remain very fundamental problems. For example, we have the following attractive.

Problem L. Find the necessary and sufficient condition for the local initial data $u_0(x')$, $u_1(x')$ so that the Cauchy problem

$$(8)\qquad \begin{cases} (D_1^2+\cdots+D_{n-1}^2-D_n^2)u=0 \\ u(0,x')=u_0(x'),\ \dfrac{\partial u}{\partial x_1}(0,x')=u_1(x') \end{cases}$$

has a local hyperfunction solution.

We have the well known sufficient condition

$$(9)\qquad \text{S.S.}u_j(x')\subset \mathbb{R}^{n-1}\times\{\xi_2^2+\cdots+\xi_{n-1}^2<\xi_n^2\},\quad j=0,1$$

(I-hyperbolicity of Kawai [3]). Also we know that if $\text{S.S.}u_j(x')$ contain a direction in the elliptic region $\xi_2^2+\cdots+\xi_{n-1}^2>\xi_n^2$, then $u_j(x')$ must satisfy a relation written by a pseudo-differential equation there, and micro-locally on a neighborhood of this direction this relation is necessary and sufficient for the solvability of the Cauchy problem (Kataoka [1], Proposition 1.3). What remains to be clarified is therefore the condition which $u_j(x')$ should satisfy at the "glancing region" $\xi_2^2+\cdots+\xi_{n-1}^2=\xi_n^2$. We have yet no idea on what kind of language this condition will be expressed.

Remark 5. The condition for the micro-local solvability of the non-characteristic Cauchy problem for an operator $P(D)$ with constant coefficients

$$(19)\qquad \begin{cases} P(D)u=0 \\ \dfrac{\partial^j u}{\partial x_1^j}(0,x')=u_j(x'),\quad 0\leqq j\leqq m-1, \end{cases}$$

for the hyperfunction data satisfying $\text{S.S.}u_j(x')\subset\mathbb{R}^{n-1}\times I$, ($I$ being an open subset of S^{n-2}) is easily seen to be that of I-hyperbolicity, i.e. that the equation $P_m(\zeta_1,\xi')=0$ for ζ_1 has only real roots when $\xi'\in I$. For the distribution case the answer to the corresponding problem seems to be unknown. There are some algebraic difficulty in micro-localizing Gårding's argument for the necessity part.

7. We can consider the solvability of the same problems in only one side $x_1 > 0$ or $x_1 < 0$. In this case the initial value problem reduces to the unilateral non-characteristic boundary value problem, and this is more basic because the solvability of the Cauchy problem is equivalent to the solvability of the unilateral boundary value problems to both sides. As for the formulation of boundary value problem see e.g. Komatsu-Kawai [1] for the hyperfunction category and Kaneko [3], [4] for the distribution category. The compatibility of both theory is remarked in Kaneko [4]. That is, if a local distribution solution $u(x)$ of a linear partial differential equation $P(x, D)u = 0$ with C^∞ coefficients which is defined on $x_1 > 0$ is prolongeable as a distribution to $x_1 \leqq 0$, then we can consider the boundary values

$$u_j(x') = \lim_{\varepsilon \downarrow 0} \frac{\partial^j u}{\partial x_1^j}(\varepsilon, x'), \qquad 0 \leqq j \leqq m-1,$$

in the sense of distributions, but if $P(x,D)$ has real analytic coefficients, these values agree with the boundary values $\partial^j u/\partial x_1^j|_{x_1 \to +0}$ in the sense of hyperfunction boundary value theory. The converse is not known even for an operator with constant coefficients:

Problem M. Let u be a local distribution solution of $P(x,D)u = 0$ on $x_1 > 0$. Assume that its boundary values $\partial^j u/\partial x_1^j|_{x_1 \to +0}$, $0 \leqq j \leqq m-1$, in the sense of hyperfunction boundary value theory, are all distributions by chance. Then can the solution u be prolonged as a distribution to $x_1 \leqq 0$?

8. Likewise we leave the problem of characterization of partially (semi-)hyperbolic operators. Leray in [1] introduced the notion of partial hyperbolicity to characterize the operators for which the non-characteristic Cauchy problem

$$(11)\qquad \begin{cases} P(x,D)u=0 \\ \dfrac{\partial^j u}{\partial x_1^j}(0,x')=u_j(x'), \quad 0\leqq j\leqq m-1, \end{cases}$$

is solvable (in some Gevrey class) for any data $u_j(x')$ which are holomorphic in a part of variables $x''=(x_2,\cdots,x_\ell)$ in the initial plane. According to him P is called partially hyperbolic modulo the linear subvarieties $x''=\text{const.}$ if the equation $P_m(x,\zeta_1,\xi')=0$ for ζ_1 has only real roots when ξ' is real and $\xi''=(\xi_2,\cdots,\xi_\ell)=0$. It seems to me, however, that this condition of Leray rather concerns solvability for those data which are entire holomorphic in x''. See Kaneko [6], pp.428-429. Leray himself requires some additional condition on the characteristic roots to establish the solvability of this Cauchy problem. We therefore introduce the following:

Conjecture N. The necessary and sufficient condition on an operator P with constant coefficients for the local solvability of the Cauchy problem (10) for any hyperfunction initial data $u_j(x')$, $0\leqq j\leqq m-1$, containing $x''=(x_2,\cdots x_\ell)$ as holomorphic parameters, is that the roots of $P_m(\zeta_1,\zeta')=0$ for ζ_1 satisfy

$$(12)\qquad |\mathrm{Im}\zeta_1|\leqq b|\mathrm{Im}\zeta'|+c|\zeta''|, \quad \text{when} \quad \zeta'\in \mathbf{C}^{n-1},$$

with some constants b, $c>0$.

The notion of hyperfunctions with holomorphic parameters is just the hyperfunction variant of partially holomorphic property fot the usual functions or distributions. Note that condition (12) is in general stronger than the condition

$$(13)\qquad \mathrm{Im}\zeta_1=0 \quad \text{for} \quad \zeta' \ \text{real and} \ \zeta''=0$$

of Leray, though they agree when the roots are simple. In Kaneko [6], Corollary 2.8 the sufficiency of the above condition is proved for the case $\ell=n-1$, and it may be easily generalized to an arbitrary ℓ. So we are mainly interested in the necessity part. We may likely cons der the solvability of the unilateral boundary value problem on $\pm x_1 > 0$. (Then the corresponding condition must be (12) with $\pm \mathrm{Im}\, \zeta_1$ in place of $|\mathrm{Im}\, \zeta_1|$ in the right-hand side). These problems may of course be considered for operators with variable coefficients. In that case we would have to be content with partial results. C.f. Hamada-Leray-Wagschal[1].

The distribution variant of these problems require further conditions on the lower order terms and seems not yet studied at all.

9. We shall finish this note by showing one of the main research subjects of the author. Let $P(x,D)$ be a linear partial differential operator with real analytic coefficients whose principal part may completely vanish at the origin. Find the necessary and sufficient condition for each of the following:

Problem O. When $P(x,D)u = 0$ has a hyperfunction solution with support concentrated at the origin?

Problem P. When $P(x,D)u = 0$ has a hyperfunction solution u such that sing supp $u = \{0\}$?

Problem Q. When $P(x,D)u = 0$ has a hyperfunction solution u with the irremovable isolated singularity 0 (i.e. $P(x,D)u = 0$ on $\Omega \setminus \{0\}$ for some neighborhood Ω of 0 but never $P(x,D)\tilde{u}=0$ for any extension $\tilde{u}$ of u to Ω)?

Problem R. When $P(x,D)u = 0$ has a real analytic solution u with the irremovable isolated singularity 0?

In Problem R we can understand the irremovability both in the sense of hyperfunction solutions or of real analytic solutions. This ambiguity is taken over by virtue of the answer to Problem P. Problem O solves the uniqueness of the extension as a hyperfunction solution in the removable case. Concerning these problems an interesting partial answer will also be welcomed.

Remark. Via the duality argument Problem O is equivalent to the following.

Problem 0'. When $P(x,D)$ acts surjectively on the germs of holomorphic functions at the origin?

Hence in view of the Cauchy-Kowalewski theorem we see from this the well known partial answer to Problem 0: $P(x,D)u = 0$ has no solution with support concentrated at the origin if the principal part $P_m(x,D)$ of P has a term different from zero at the origin (or equivalently if P has at least one non-characteristic direction at the origin). Thus we are essentially concerned with the case $P_m(0,D) = 0$. This problem is not at all easy even for operators of first order with two independent variables. See Oshima [1] for a partial answer. It is related with the famous classification problem of the vector fields in the analytic category. Hence the problem of small divisor equally interferes with us, and a complete answer to such problems may require mathematics of the 21-th century! To be collected by a similar line is the problem of classification of the diffractive points for the wave equation in the analytic category (Oshima [2]).

Very close to this is the following.

Problem S. Find the necessary and sufficient condition for a linear partial differential operator in two independent variables

$$a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y} + c(x,y)$$

to act surjectively on the germ of hyperfunctions at 0.

Miwa [1] studied the case when a and b are real valued homogeneous linear functions of x, y and $c = 0$. (He also considered some cases of three independent variables.) Of course the problem has a sense for a general operator $P(x,D)$, but it should be noted that the problem is sufficiently difficult already in the above form.

As for further partial results concerning these problems see Kawai [4], Kaneko [5], [7], [8], Ôaku [1], [2]. These papers will also serve to find more fresh research problems.

10. In concluding we again remark that these problems are rather particular. We expect anyway that the reader will utilize them to enrich the theory of hyperfunctions by bringing in what are more important than the problems themselves. For the specialists in the theory of linear partial differential equations we introduce the following opinion of Prof. S. Matsuura of about ten years ago: "Hyperfunctions will be the most natural tools to treat analytic solutions." Perhaps even now this may be a common understanding for most of the non-specialists of hyperfunction theory. As such the above mentioned researches on continuation of real analytic solutions may have a little confirmative value. Another such example is the researches on the existence of global real analytic solutions whose hyperfunction theoretic treatment was originated by T. Kawai. (See Kaneko [9] and articles cited there. The reader may also find several fresh research problems in this field.) However we rather expcet that the reader can now well share with us the recognition that hyperfunctions have their own "raison d'être" apart from their utility to other fields. Finally we wish that the reader will confirm this and seek more advanced research problems for himself in the active reports and papers such as are given in the proceedings of the RIMS symposiums (Sûrikaiseki-kenkyûsho Kôkyûroku) on hyperfunctions (No's 108,114,145,162,168,192,201,209,225,226,227,238,248,266,281, 287,295,324,341,355,361,410,416,431,459,468,497,508,533,558) or in the following ones:

"Hyperfunctions and Pseudo-differential Equations (Proceedings of Katata Symposium 1971)", Lecture Notes in Math. No. 287, Springer, 1973.

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