Risk and Decision Analysis
5. Game Theory

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Post-graduation Course on
Complex Transport Infrastructure Systems

MIT|Portugal

Outline
Scope of game theory
Two-person zero sum games
Two-person general sum games
Noncooperative games
Two-person Cooperative Games
n-person Cooperative Games

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What is game theory?
What is the purpose?
Important elements
The strategic form
The extensive form
The coalitional form
What we will cover

What is game theory?

- Game theory is about

  mathematical modelling of strategic behavior.

- Strategic behavior arises whenever the outcome of an individual's actions depends on actions taken by other individuals.
- Examples include: store managers fixing prices, bidding in auctions, Voting at the United Nations, fair allocation of costs (or profits) when a group share a common facility.
- Game theory is an assemblage of ideas and theorems that attempts to provide a rational basis for resolving conflicts, with or without cooperation.

What is the purpose?

The content of Game Theory is a study of the following questions and related:

- What will each individual guess about the others' choices?
- What action will each person take? How should we react to them?
- What is the outcome of these actions? Is this outcome good for the group as a whole?
- Should the group act as a whole, how should the outcome be split among individuals?
- How do answers change, or may change, if each individual is unsure about the characteristics of others in the group?

Important elements

Game theory is a formal way to consider each of the following items:

- *group* In any game there is more than one decision maker; each decision-maker is referred to as a *player*.
- *interaction* What any individual player does directly affects at least one other player in the group.
- *strategic* An individual player accounts for this interdependence in deciding what action to take.
- *rational* While accounting for this interdependence each player will choose her best action.

The strategic form

Prisoner's dilemma

Two prisoners, Clavin and Klein, are hauled in for a suspected crime. The DA speaks to each prisoner separately. Each crook will be jailed for as many years as prescribed by the following table:

Calvin \ Klein	*Confess*	*Not Confess*
Confess	$5, 5$	$0, 15$
Not Confess	$15, 0$	$1, 1$

We say that this game is given in strategic (or normal) form.

The strategic form

Definition

The **strategic form** of a two-person game is defined by two sets X and Y and two real-valued functions $u_1(x, y)$ and $u_2(x, y)$ defined on $X \times Y$ such that:

- X is a nonempty set, the set of strategies of Player I.
- Y is a nonempty set, the set of strategies of Player II.
- $u_1(x, y)$ and $u_2(x, y)$ represent the payoffs to the players when x and y are the chosen strategies.

This definition generalizes in an obvious way to more than two players. In a game in strategic form, the players choose their strategies simultaneously, without knowing the choices of the other players.

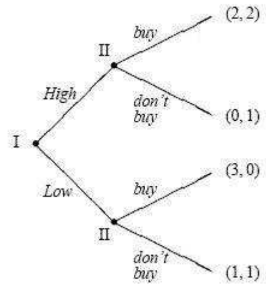
The extensive form

Quality with commitment

Consider a service provider (e.g. internet) who makes a first move, *High* or *Low* quality of service.

Then, the customer is informed about that choice and decide separately between *buy* and *don't buy* in each case.

The payoffs are given in the figure.



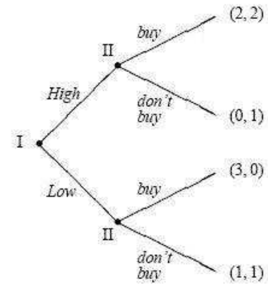
The extensive form

Quality with commitment (cont.)

We say that this game is given in extensive form.

Games in extensive form formalize interactions where the players can over time be informed about the actions of others.

In this case it is an extensive game of perfect information because every player is at any point aware of the previous choices of all other players.



The extensive form

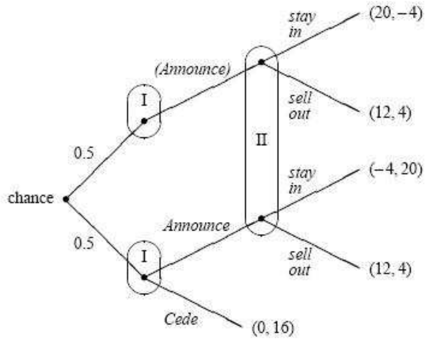
Microsoft *vs* Quitesmallersoft

A small startup has announced deployment of a key new technology.

With 50% chance, a large software company has the ability to produce a competing product.

The large company can either announce the release of the competing product (even if bluffing) or it can cede the market.

Then, the smaller company can either sell itself or it can remain independent and launch its product.



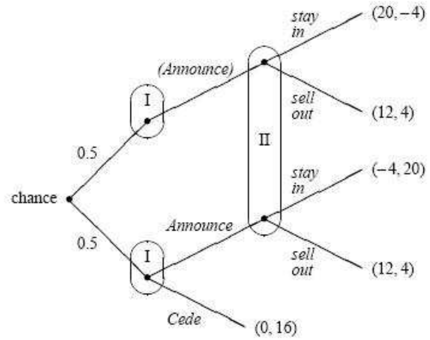
The extensive form

Microsoft *vs* Quitesmallersoft (cont.)

Extensive games of imperfect information model exactly which information is available to the players when they make a move.

The nodes enclosed by ovals are called information sets. A player cannot distinguish among the nodes in an information set.

Note that whether or not the large company is able to launch a competing product is *random*. This is modeled by chance moves.

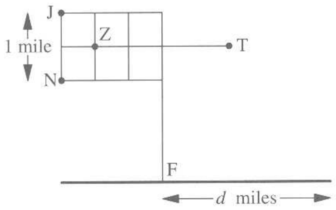


The coalitional form

Jed, Ned and Ted' car pool

Jed, Ned and Ted are neighbors. They work in the same office, at the same time, on the same days.

In order to save money they would like to form a car pool.



They must first agree on how to share the car pool's benefits.

The coalitional form

Jed, Ned and Ted' car pool (cont.)

Let Jed be Player 1, Ned Player 2 and Ted Player 3.

Let $c(S)$ denote the the cost of the coalition $S \subseteq \{1,2,3\}$ driving to work by sharing the same car.

Now, define $\bar{\nu}(S)$, the benefit of cooperation associated with the coalition $S \subseteq \{1,2,3\}$. Then,

$$\bar{\nu}(S) = \sum_{i \in S} c(\{i\}) - c(S)$$

so that by forming a car pool, J, N and T will save

$$\bar{\nu}(\{1,2,3\}) = \sum_{i=1}^{3} c(\{i\}) - c(\{1,2,3\}).$$

How should this money savings be split among them?

The coalitional form

Definition

- The coalitional form of an n-person game is given by the pair (N, ν), where
  - $N = \{1, 2, \ldots, n\}$ is the set of players;
  - ν is a real-valued function called the characteristic function of the game, defined on the set 2^N, of all coalitions (subsets of N).
- The real number $\nu(S)$ may be considered as the value, or worth, or power, of coalition S when its members act together as a unit.
- In general, but not always, the ν function will satisfy the superadditivity property

$$\nu(S) + \nu(T) \leq \nu(S \cup T), \qquad \text{for all disjoint } S, T.$$

so that all the players are better off forming coalitions. In particular, they will have the interest in forming the grand coalition, N.

The Program

1. Two-person zero sum games. These are the simplest games. We will analyse the so-called Matrix Games.
2. Two-Person general sum games. We will analyse the so-called Bimatrix Games.
   - Noncooperative theory. Nash equilibrium.
   - Cooperative theory: (1) side payments allowed; (2) Nash bargaining model.
3. Games in coalitional form. We will discuss ways to reach an agreement on a fair division.

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n-person Cooperative Games

The strategic form

The strategic form of a two-person zero sum game is given by a triplet (X, Y, A), where

- X is a nonempty set, the set of strategies of Player I.
- Y is a nonempty set, the set of strategies of Player II.
- A is a real-valued function defined on $X \times Y$.

Simultaneously, Player I chooses $x \in X$ and Player II chooses $y \in Y$.

Then, their choices are made known and I wins $A(x, y)$ from II.

Thus, $A(x, y)$ represents the winnings of I and the losses of II.

Example - Odd or Even

- Players I and II simultaneously call out numbers one or two. If the sum of the numbers is odd Player I wins the sum otherwise Player II wins the sum. Thus, $X = Y = \{1, 2\}$ and A is given (in euros) in the following table:

I \ II	1	2
1	-2	$+3$
2	$+3$	-4

- Suppose Player 1 uses the following mixed strategy: he calls 'one' with probability $3/5$ and 'two' with probability $2/5$. Then,
  - If II calls 'one', I wins $(-2)(3/5) + (+3)(2/5) = 0$, on average;
  - If II calls 'two', I wins $(+3)(3/5) + (-4)(2/5) = 1/5$, on average.

  Thus, through this mixed strategy, Player I is assured of at least breaking even on the average no matter what II does.

Pure Strategies and Mixed Strategies

- Elements of X and Y are called Pure Strategies.
- A probability distribution attached to the elements in X (or Y) is called a Mixed Strategy.
- What are the payoffs corresponding to a given probability distribution?
  In the same way you look at row x to identify the payoffs to Player I if he chooses the pure strategy x, you will look at the average payoffs to Player I if he chooses a mixed strategy.
- We will assume that payoffs reflect the **utility value** to the players.
- We will assume that players judge an outcome only on the basis of the **average utility** of the outcome.

The Minimax Theorem

- **The Minimax Theorem:** *For every two-person zero-sum game,*
  1. *there is a number V, called the value of the game,*
  2. *there is a mixed strategy for I that gives him at least an average gain of V no matter what II does, and*
  3. *there is a mixed strategy for II that gives him at most an average loss of V no matter what I does.*

  Such strategies are called minimax strategies.
- In the game of Odd-and-Even the minimax strategies are the **equalizing strategies**.

Saddle Points

- Consider the following game

$$A = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 1 \\ 3 & 1 & 2 & 2 \end{pmatrix} \qquad \begin{matrix} \text{row min} \\ 0 \\ 0 \\ 0 \\ 1 \end{matrix}$$

$$\text{col max} \quad 3 \quad 1 \quad 2 \quad 2$$

- For this game, $V = a_{42} = 1$ and the two minimax strategies are pure strategies.
- In general, we say that a_{ij} is a **saddle point** if
  - a_{ij} is the minimum of the ith row, and
  - a_{ij} is the maximum of the jth column.
- When a saddle point exist, it is the value of the game.

Iterated Removal of Dominated Strategies

- **Dominated Strategy:** We say the ith row of a matrix $A = (a_{ij})$ *dominates* the kth row if $a_{ij} \geq a_{kj}$ for all j. If the inequality is strict in all j then we say the the ith row *strictly dominates* the kth row.
- Similar definition for columns (but with inequality reversed).
- Dominated rows may be removed from the game. Everything that can be achieved with that row, can also be achieved without it.
- We may iterate this procedure as in the following example:

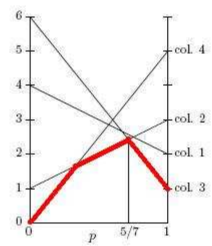
$$A = \begin{pmatrix} 2 & 0 & 4 \\ 1 & 2 & 3 \\ 4 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 0 \\ 1 & 2 \\ 4 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix}$$

From the last game we conclude that the optimal strategy for Player I must be $(0, 3/4, 1/4)$, the optimal strategy for Player II is $(1/4, 3/4, 0)$. The value is $7/4$.

Solving $2 \times n$ and $m \times 2$ games

Games with matrices of size $2 \times n$ or $m \times 2$ may be solved with the aid of a graphical interpretation.
Consider the following game

$$\begin{matrix} p \\ 1-p \end{matrix} \quad \begin{pmatrix} 2 & 3 & 1 & 5 \\ 4 & 1 & 6 & 0 \end{pmatrix}$$



The average payoffs for each of the pure strategies of Player II are drawn on the figure.

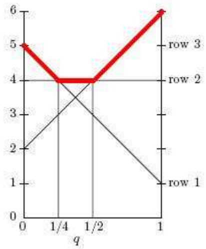
I's optimal strategy is $(5/7, 2/7)$ and the value is $17/7$.
Hence, the optimal strategy for Player I is the same as for the game

$$\begin{pmatrix} 3 & 1 \\ 1 & 6 \end{pmatrix}$$

Solving $2 \times n$ and $m \times 2$ games (cont.)

Similar reasoning applies to $m \times 2$ may be solved with the aid of a graphical interpretation. Consider the following game

$$\begin{matrix} q & 1-q \end{matrix}$$
$$\begin{pmatrix} 1 & 5 \\ 4 & 4 \\ 6 & 2 \end{pmatrix}$$



The average payoffs for each of the pure strategies of Player I are drawn on the figure.

From the graph we see that any value of q between $1/4$ and $1/2$ defines Player's 2 minimax strategy. The value of the game is 4.
The optimal strategy for Player I to play the pure strategy: row 2.

Best Responses

Consider an arbitrary two-person zero sum game (X, Y, A) where A is an $m \times n$ matrix. Let

$$X^* = \left\{ \boldsymbol{p} \equiv (p_1, \ldots, p_m) : p_i \geq 0, i = 1, \ldots, m \text{ and } \sum_{i=1}^{m} p_i = 1 \right\}$$
$$Y^* = \left\{ \boldsymbol{q} \equiv (q_1, \ldots, q_n) : q_j \geq 0, j = 1, \ldots, n \text{ and } \sum_{j=1}^{n} q_j = 1 \right\}.$$

Hence, X^* contains the mixed strategies for Player 1 and Y^* contains the mixed strategies for Player 2.

Best Responses (cont.)

- If Player II chooses a column using $\boldsymbol{q} \in Y^*$ and Player I chooses row i then, the average payoff to Player I is
$$\sum_{j=1}^{n} a_{ij} q_j = (A\boldsymbol{q})_i \, ,$$
- If Player I chooses a row using $\boldsymbol{p} \in X^*$ and Player II chooses column j then, the average payoff to Player I is
$$\sum_{i=1}^{m} a_{ij} p_i = \left(\boldsymbol{p}^T A\right)_j \, ,$$
- In general, if Player 1 uses $\boldsymbol{p} \in X^*$ and Player 2 uses $\boldsymbol{q} \in Y^*$ then, the average payoff to Player I is
$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j = \boldsymbol{p}^T A \boldsymbol{q}.$$

Best Responses (cont.)

- Suppose Player II announces that will use $\boldsymbol{q} \in Y^*$. Then, Player I's best response is to choose $i \in X$ (or, equivalently, $\boldsymbol{p} \in X^*$) that solves
$$\max_{1 \leq i \leq m} \sum_{j=1}^{n} a_{ij} q_j = \max_{\boldsymbol{p} \in X^*} \boldsymbol{p}^T A \boldsymbol{q}.$$
- Otherwise, Player I may plan for the worst through solving
$$\max_{\boldsymbol{p} \in X^*} \min_{\boldsymbol{q} \in Y^*} \boldsymbol{p}^T A \boldsymbol{q} = \max_{\boldsymbol{p} \in X^*} \min_{1 \leq j \leq n} \sum_{i=1}^{m} a_{ij} p_i = \underline{V}$$

which is called the lower value of the game. The vector $\boldsymbol{p} \in X^*$ where the maximum is achieved is called the minmax strategy for I.

Best Responses (cont.)

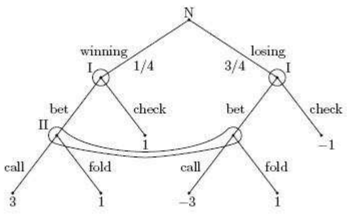
- Suppose Player I announces that will use $\boldsymbol{p} \in X^*$. Then, Player II's best response is to choose $j \in Y$ (or, equivalently, $\boldsymbol{q} \in Y^*$) that solves
$$\min_{1 \le j \le n} \sum_{i=1}^{m} a_{ij} p_i = \min_{\boldsymbol{q} \in Y^*} \boldsymbol{p}^T A \boldsymbol{q}.$$
- Otherwise, Player II may plan for the worst through solving
$$\min_{\boldsymbol{q} \in Y^*} \max_{\boldsymbol{p} \in X^*} \boldsymbol{p}^T A \boldsymbol{q} = \min_{\boldsymbol{q} \in Y^*} \max_{1 \le i \le m} \sum_{j=1}^{n} a_{ij} q_j = \overline{V}$$
which is called the upper value of the game. The vector $\boldsymbol{q} \in Y^*$ where the minimum is achieved is called the minmax strategy for II.

Reduction to Linear Programming

- Basically, it is a consequence of the *Equlibrium Theorem* stated before that for finite games
$$\overline{V} = \underline{V}$$
and that both players have minimax strategies.
- Both can be found through Linear Programming
- Let us solve an example using the `solver` of Excel ...

Basic Endgame in Poker

Both players put 1 euro.
Player I receives a winning card with prob. 1/4 or a losing card with prob. 3/4.
Then, knowing his card, Player 1 checks (his card is inspected) or bets (puts 2 more euros).
Then Player 2, not knowing the card, either fold (looses the pot) or call (puts 2 more euros and the card is inspected).



The extensive form of the game is drawn on the figure.

The Kuhn Tree

- A two-person game in extensive form is represented by a graph known as the Kuhn tree:
  - a finite tree with vertices T;
  - a payoff function that assigns a real number to each terminal vertex;
  - a partition of the rest of the vertices into two groups of information sets (one for each player);
  - from each information set, a set of edges corresponding to possible strategies.
- Knowing the Kuhn tree means knowing the rules of the game.
- Games in which both players know the Kuhn tree are called games of complete information.
- Recall that games of perfect information are games in which the information sets are single vertices.

Reduction to Strategic Form

- Any game in extensive form can be put in strategic form.
- If there are k information sets for Player 1 then a pure strategy for Player I is k-tuple $\boldsymbol{x} = (x_1, \ldots, x_k)$ where each component characterizes the choice of Player I facing a given information set.
- Proceed similarly with Player 2.
- Random payoffs are replaced by their average values.

Reduction to Strategic Form (cont.)

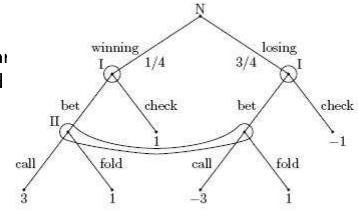
Player I has two information sets with two option in each. Hence, X contains

(b, b) : bet with winning card or losing card
(b, c) : bet with winning card, check with a losing car
(c, b) : check with winning card, bet with losing card
(c, c) : check with winning card or losing card

and Y contains

c : if I bets, call
f : if I bets, fold

Basic Endgame in Poker



The payoff matrix is
$$\begin{array}{c} \\ (b,b) \\ (b,c) \\ (c,b) \\ (c,c) \end{array} \begin{array}{c} \begin{array}{cc} c & f \end{array} \\ \begin{pmatrix} -3/2 & 1 \\ 0 & -1/2 \\ -2 & 1 \\ -1/2 & -1/2 \end{pmatrix} \end{array}$$
. Why?

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The strategic form

- A finite two-person general sum game in strategic form is defined similarly as in the zero-sum case, except that now payoffs are ordered pairs.
- Thus, a finite two-person game in strategic form can be represented by a so-called bimatrix. For example,

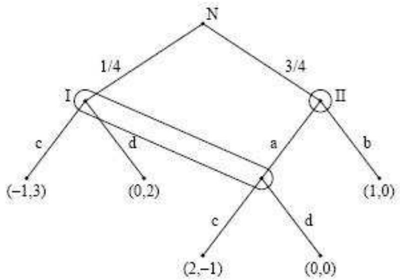
$$\begin{pmatrix} (1,4) & (2,0) & (-1,1) & (0,0) \\ (3,1) & (5,3) & (3,-2) & (4,4) \\ (0,5) & (-2,3) & (4,1) & (2,2) \end{pmatrix}$$

  where, the rows are the pure strategies of Player I and the columns are the pure strategies of Player II. If Player I chooses row 3 and Player II chooses column 2 then I receives -2 and II receives 3.
- The same game can be also represented by a pair of matrices (A, B) where

$$A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 3 & 5 & 3 & 4 \\ 0 & 2 & 4 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 0 & 1 & 0 \\ 1 & 3 & -2 & 4 \\ 5 & 3 & 1 & 2 \end{pmatrix}.$$

Reducing extensive to strategic form

- The extensive form of a general sum game may be defined as before, except that now payoffs are ordered pairs. See figure.
- The problem of reducing a general sum game in extensive form to one in strategic form is solved in similar manner as before.
- For the game in the figure, $X = \{c, d\}$ and $Y = \{a, b\}$. The corresponding bimatrix is



$$\begin{array}{c} \\ c \\ d \end{array} \begin{array}{c} \quad a \qquad\quad b \\ \begin{pmatrix} (5/4, 0) & (1/2, 3/4) \\ (0, 1/2) & (3/4, 1/2) \end{pmatrix}. \end{array}$$

Check!

Safety Levels

- Consider the following game

$$\begin{pmatrix} (2,0) & (1,3) \\ (0,1) & (3,2) \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}.$$

- Player I has a gurantee winning of 3/2, on the average, if he plays his *maxmin strategy* $(3/4, 1/4)$.
- Player II has a gurantee winning of 2, on the average, if he plays his *maxmin strategy* $(0, 1)$.
- We say that $3/2 = \text{Val}(A)$ is the safety level of Player I and $2 = \text{Val}(B^T)$ is the safety level of Player II.
- If both players use their maxmin strategies,

  Player I gets 3/2 while Player II gets $11/4 = (3/4)3 + (1/4)2$.

  Hence, Player II is happy. He gets more than his safety level.
- But Player I is unhappy. Can he do better?What if they could cooperate?

Cooperative vs Noncooperative Theory

- The analysis of general-sum games is more complex than zero-sum. In particular, the minimax theorem does not hold.
- The analysis divides into:
  - *Noncooperative Theory*: Either Players are unable to communicate bebore decisions are made or, if they do, they cannot make binding agreements.
  - *Cooperative Theory*: Players are allowed to make binding agreements. Theory further divides into
    - *Transferable Utility Games*: Payoffs have the same monetary units. Hence, it is a matter of fairly dividing an agreed outcome, *i.e.*, defining the *side payments*.
    - *NonTransferable Utility Games* Payoffs do have the same monetary units. Side payments are not allowed.

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Pure strategic equilibrium

A vector of pure strategies choices $(x_1, x_2, \ldots, x_n)$, with $x_i \in X_i$, is said be a Pure Strategic Equilibrium if, for all $i = 1, 2, \ldots, n$,

$$u_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \geq u_i(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n),$$

for all $x \in X$.

- Consider the following examples of bimatrix games

$$(a) \quad \begin{pmatrix} (3,3) & (0,0) \\ (0,0) & (5,5) \end{pmatrix} \qquad (b) \quad \begin{pmatrix} (3,3) & (4,3) \\ (3,4) & (5,5) \end{pmatrix}$$

- In both games, both $(3,3)$ and $(5,5)$ are PSE, there are no other.
- In (a), if they are allowed to communicate they will choose the second PSE because it will give them the maximum payoff.
- In (b), the first PSE is rather unstable because no player can be hurt by changing and, in fact, if they both change they will be better off.

(Mixed) strategic equilibrium

It is useful to extend this definition to allow for mixed strategies.

A vector of mixed strategies choices $(\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n)$, with

$$\boldsymbol{p}_i \in X_i^* \equiv \left\{ \boldsymbol{p} \equiv (p_1, \ldots, p_{m_i}) \colon p_i \geq 0, i = 1, \ldots, m_i \text{ and } \sum_{i=1}^{m_i} p_i = 1 \right\},$$

is said be a Strategic Equilibrium if, for all $i = 1, 2, \ldots, n$,

$$g_i(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{i-1}, \boldsymbol{p}_i, \boldsymbol{p}_{i+1}, \ldots, \boldsymbol{p}_n) \geq g_i(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{i-1}, \boldsymbol{p}, \boldsymbol{p}_{i+1}, \ldots, \boldsymbol{p}_n),$$

for all $\boldsymbol{p} \in X_i^*$.

Thus, any mixed strategy $\boldsymbol{p}_i$ in a strategic equilibrium $(\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n)$ is a Best Response to those of the other players.

Theorem. (Nash, 1950) *Every finite n-person game in strategic form has at least one strategic equilibrium.*

Example 1: A Coordination Game

- Consider the following game

$$\begin{pmatrix} (3,3) & (0,2) \\ (2,1) & (5,5) \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} 3 & 0 \\ 2 & 5 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 2 \\ 1 & 5 \end{pmatrix}.$$

- There are two pure PSE: $(3,3)$ and $(5,5)$.
- There is a third strategic equilibrium. Player I has the equalizing strategy $\boldsymbol{p} = (4/5, 1/5)$ for B and Player II has the equalizing strategy $\boldsymbol{q} = (5/6, 1/6)$ for A.
- If the players use these strategies, which define an SE, the average payoff is $(5/2, 13/5)$, the same as the safety levels.
- In general, any strategic equilibrium is at least as good as the safety levels.

Example 2: The Battle of the Sexes

Suppose the matrices are

$$\begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} (2,1) & (0,0) \\ (0,0) & (1,2) \end{pmatrix} \end{array} \text{ so, } A = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \end{array}, B = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \end{array}.$$

- The PSE are (a, a) and (b, b). But, Player I prefers the first and Player II prefers the second.
- Safety Levels. $v_I = v_{II} = 2/3$, the same for both players. Player's I minmax strategy is $(1/3, 2/3)$ while Player's II minmax strategy is $(2/3, 1/3)$.
- There is a third SE given by $\boldsymbol{p} = (2/3, 1/3)$ and $\boldsymbol{q} = (1/3, 2/3)$. The payoff is $(2/3, 2/3)$. Hence, it is worse than either of the PSE.

Example 3: The Prisoner's Dilemma

Consider the game with bimatrix

	cooperate	defect
cooperate	$(3,3)$	$(0,4)$
defect	$(4,0)$	$(1,1)$

- There is a unique PSE which is $(1,1)$.
- However, if both players use their dominated strategies, each player receives 3.

Finding all PSEs

For larger matrices it is not difficult to find all pure strategic equilibria. An example should make this clear:

$$\begin{pmatrix} (2,1) & (4,3) & (7^*,2) & (7^*,4) & (0,5^*) & (3,2) \\ (4^*,0) & (5^*,4) & (1,6^*) & (0,4) & (0,3) & (5^*,1) \\ (1,3^*) & (5^*,3^*) & (3,2) & (4,1) & (1^*,0) & (4,3^*) \\ (4^*,3) & (2,5^*) & (4,0) & (1,0) & (1^*,5^*) & (2,1) \end{pmatrix}$$

In this example there are two PSE with payoffs $(5,3)$ and $(1,5)$, respectively.

The Cournot Model of Duopoly

There are two competing firms producing a single homogeneous product. These firms must choose how much of the good to produce.
If Firm 1 produces q_1 and Firm 2 produces q_2 for a total of $Q = q_1 + q_2$, the price is

$$P(Q) = (a - Q)^+ \qquad (a > 0).$$

Hence, the payoffs for the two players are

$$u_1(q_1, q_2) = q_1(a - q_1 - q_2)^+ - cq_1$$

$$u_2(q_1, q_2) = q_2(a - q_1 - q_2)^+ - cq_2,$$

where c is the unit production cost. Assume $c < a$.

The Cournot Model of Duopoly (cont.)

To find a duopoly PSE, we look for a pure strategy for each player that is a best response to the other's strategy. Setting derivatives to zero,

$$\frac{\partial u_1}{\partial q_1}(q_1, q_2) = a - 2q_1 - q_2 - c = 0, \qquad (0 < q_1 + q_2 < a)$$

$$\frac{\partial u_2}{\partial q_2}(q_1, q_2) = a - q_1 - 2q_2 - c = 0, \qquad (0 < q_1 + q_2 < a).$$

Solving these equations simultaneously, we find

$$q_1^* = \frac{a-c}{3} \quad \text{and} \quad q_2^* = \frac{a-c}{3}$$

The payoff each player receives is $\dfrac{(a-c)^2}{9}$
Note: it may be shown that there are no more PSEs.

The Cournot Model of Duopoly (cont.)

It would be interesting to study the case in which the two firms cooperate, i.e., under a monopoly,

	Monopoly	Duopoly
Total production	$\frac{a-c}{2}$	$\frac{2(a-c)}{3}$
Total payoffs	$\frac{2(a-c)^2}{8}$	$\frac{2(a-c)^2}{9}$
Unit Cost	$\frac{a+2c}{3}$	$\frac{a+c}{2}$

Hence,

- if the firms were allowed to cooperate they could improve their profits.
- The consumer is better off under a duopoly than under a monopoly.

Outline

Scope of game theory

Two-person zero sum games

Two-person general sum games

Noncooperative games

Two-person Cooperative Games

n-person Cooperative Games

Cooperative Games

- In Noncooperative Theory, even if communication is allowed, players are forbidden to make binding agreements.
- In Cooperative Theory, communication is allowed and also allow binding agreements to be made.
- Hence, in Cooperative Theory, players can usually do much better. Recall the Prisoner's Dilemma game.
- We will analyse distinctly the Transferable Utility and NonTransferable Utility cases.

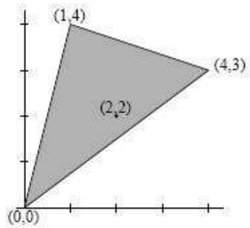
The NTU Feasible Set

- Consider a bimatrix game (A, B). Being a cooperative game the players may agree to achieve a payoff to be any of the points (a_{ij}, b_{ij}) or a probability mixture of all these points.
- **Def:** The NTU feasible set is the convex hull of the mn points (a_{ij}, b_{ij}) for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

As an example, consider the bimatrix game

$$\begin{pmatrix} (4,3) & (0,0) \\ (2,2) & (1,4) \end{pmatrix}$$

which has two PSE, upper left and lower right. The NTU feasible set is represented in the figure.



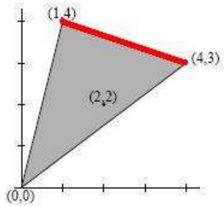
The NTU Feasible Set (cont.)

- If an agreement is to be reached, it must be such that no player can be made better off without making at least one other player worse off.
- **Def:** A feasible payoff vector, (v_1, v_2) is to be Pareto Optimal if the only feasible payoff vector (v_1', v_2') such that $v_1' \geq v_1$ and $v_2' \geq v_2$ is the vector $(v_1', v_2') = (v_1, v_2)$.

For the previous example,

$$\begin{pmatrix} (4,3) & (0,0) \\ (2,2) & (1,4) \end{pmatrix}$$

the Pareto Optimal payoffs are represented in the figure.



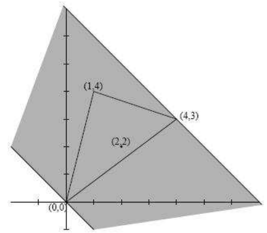
The TU Feasible Set

- Consider a bimatrix game (A, B). Now, each payoff vector (a_{ij}, b_{ij}) can be changed to $(a_{ij} + s, b_{ij} - s)$. The value s is a side payment to Player I.
- **Def:** The TU feasible set is the convex hull of points of the form $(a_{ij} + s, b_{ij} - s)$ for $i = 1, \dots, m$, for $j = 1, \dots, n$ and for any real number s.

As an example, consider the bimatrix game

$$\begin{pmatrix} (4,3) & (0,0) \\ (2,2) & (1,4) \end{pmatrix}.$$

The TU feasible set is represented in the figure.



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As an example, consider the bimatrix game

$$\begin{pmatrix} (4,3) & (0,0) \\ (2,2) & (1,4) \end{pmatrix}.$$

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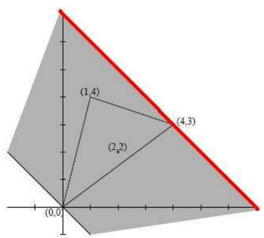


The TU problem

- Consider the following bimatrix game

$$\begin{pmatrix} (5,3) & (0,-4) \\ (0,0) & (3,6) \end{pmatrix}.$$

Try to reach an agreement!

- The TU problem is to choose the threats and proposed side payments judiciously.
- Next, we will analyse two-person Cooperative TU games. Later we will consider the n-person case.

Cooperative Strategy and Side Payments

- Consider a two-person Cooperative TU game with bimatrix (A, B).
- Rationality implies that players will agree to achieve the largest possible total payoff, *i.e.*,

$$\sigma = \max_i \max_j (a_{ij} + b_{ij}) .$$

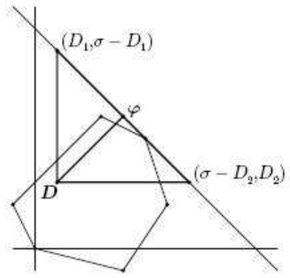
- Let $\boldsymbol{p}$ be a threat strategy for Player I and let $\boldsymbol{q}$ be a threat strategy for Player II. The resulting payoff vector is

$$D = D(p, q) = \left(p^T Aq, p^T Bq \right) = (D_1, D_2).$$

This is called the disagreemente (or threat) point. Hence, Player I will accept no less than D_1 and Player 2 will accept no less than D_2. Therefore, the mid point

$$\varphi = (\varphi_1, \varphi_2) = \left(\frac{\sigma + D_1 - D_2}{2}, \frac{\sigma + D_2 - D_1}{2} \right) \qquad (1)$$

Cooperative Strategy and Side Payments (cont.)



Cooperative Strategy and Side Payments (cont.)

- How to choose the threat strategies (*i.e.*, $\boldsymbol{p}$ and $\boldsymbol{q}$)?
- From (1), we see that Player I wants $D_1 - D_2$ to be maximum while Player II wants $D_1 - D_2$ to be minimum.
- This is in fact a zero-sum game with matrix $A - B$:

$$D_1 - D_2 = p^T Aq - p^T Bq = p^T (A - B) q.$$

- Let $\boldsymbol{p}^*$ and $\boldsymbol{q}^*$ denote optimal strategies of the game $A - B$. Then,

$$\varphi = (\varphi_1, \varphi_2) = \left(\frac{\sigma + \delta}{2}, \frac{\sigma - \delta}{2} \right) \qquad (2)$$

is the TU value (or TU solution), where $\delta = {p^*}^T (A - B) q^*$.
- The discrepancy between (2) and the chosen payoff $(a_{i_0 j_0}, b_{i_0 j_0})$ defines the side payment.

A TU game

- Consider the TU game with bimatrix

$$\begin{pmatrix} (0,0) & (6,2) & (-1,2) \\ (4,-1) & (3,6) & (5,5) \end{pmatrix}$$

- Cooperative strategy: I chooses row 2 and II chooses column 3, $\sigma = 10$.
- To determine the side payment, we must consider the zero-sum game with matrix

$$A - B = \begin{pmatrix} 0 & 4 & -3 \\ 5 & -3 & 0 \end{pmatrix}.$$

for which $p^* = (.3, .7)$ and $q^* = (0, .3, .7)$. Hence, $\delta = -.9$ so that

$$\varphi = \left(\frac{10 - .9}{2}, \frac{10 + .9}{2} \right) = (4.55, 5.45).$$

To arrive at this payoff from $(5, 5)$, requires a side payment of .45 from Player I to Player II.

The NTU problem

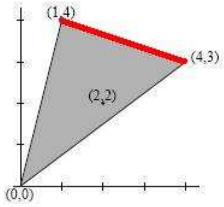
Now, we consider cooperative games in which side payments are forbidden.

It may be assumed that the utility scales are measured in noncomparable units.

Recall the bimatrix game shown before

$$\begin{pmatrix} (4,3) & (0,0) \\ (2,2) & (1,4) \end{pmatrix}.$$

The NTU feasible set and Pareto Optimal payoffs are represented in the figure.



Which of the Pareto points to agree upon?

The Nash Bargaining Model

We approach NTU games through the Nash Bargaining Model

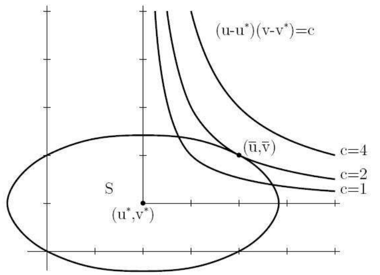
Let S be the NTU feasible set and let $(u^*, v^*) \in S$ be a threat point, *i.e.* the natural outcome if an agreement is not reached.
In the approach of Nash, a *fair* and *reasonable* outsome, or solution, of the game is a point $(\bar{u}, \bar{v}) = f(S, (u^*, v^*))$ to satisfy the following axioms:

1. **Feasibility**
2. **Pareto Optimality**
3. **Symmetry**
4. **Independence of irrelevant alternatives**
5. **Invariance under change of location and scale**

See explanation in class!

The Nash Bargaining Model (cont.)

Theorem (Nash, 1950) *There exists a unique function f satisfying the Nash axioms. Moreover, if there exists a point $(u, v) \in S$ such that $u > u^*$ and $v > v^*$, then $f(S, (u^*, v^*))$ is that point of S that maximizes $(u - u^*)(v - v^*)$ among points of S such that $u \geq u^*$ and $v \geq v^*$.*

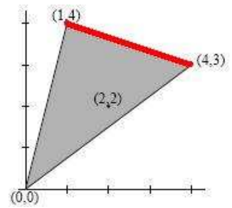


The Nash Bargaining Model (cont.)

Recall the bimatrix game shown before

$$\begin{pmatrix} (4,3) & (0,0) & (0,0) \\ (2,2) & (1,4) & (0,0) \\ (0,0) & (0,0) & (0,0) \end{pmatrix}.$$

The NTU feasible set and Pareto Optimal payoffs are represented in the figure.



- Let $(u^*, v^*) = (0, 0)$ and let S be the NTU feasible set.
- The NTU-solution is $(4, 3)$.

Outline

Scope of game theory

Two-person zero sum games

Two-person general sum games

Noncooperative games

Two-person Cooperative Games

n-person Cooperative Games

Characteristic Function Games

- We now consider *n*-person TU cooperative games, which allow for side payments to be made among the players.
- All players seek a fair distribution of a benefit and each player wants as much as possible of it.
- The fairness of a distribution is assumed to depend on the bargaining strengths of the various coalitions that could possibly form among some, but not all, of the players.
- But, a fundamental assumption is that a grand coalition of all players is formed - either voluntarily or enforced by an external agent or circumstance.
- The benefit is, usually, the savings or gain of the grand coalition.
- We will introduce a modeling framework, characteristic function games, and the solution concept the nucleolus.

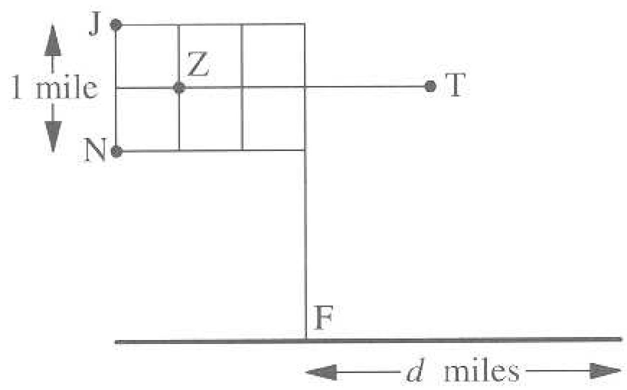
Characteristic functions

Jed, Ned and Ted's car pool problem

- Jed, Ned and Ted are neighbors. They work in the same office, at the same time, on the same days.
- In order to save money they would like to form a car pool.
- They must first agree on how to share the car pool's benefits.
- See map on the next slide.

Characteristic functions

Jed, Ned and Ted's map



Characteristic functions

The car pool cost functions

Let Jed be Player 1, Ned Player 2 and Ted Player 3.

Let $c(S)$ denote the the cost of the coalition $S \subseteq \{1,2,3\}$ driving to work by sharing the same car.

Assume that all players have identical cars and that the cost of driving to work, *including depreciation*, is k dollars per mile.

Then,

$$c(\{1\}) = (4+d)k, \quad c(\{1,2\}) = (4+d)k, \quad c(\{1,2,3\}) = (7+d)k,$$
$$c(\{2\}) = (3+d)k, \quad c(\{2,3\}) = (6+d)k,$$
$$c(\{3\}) = (3+d)k, \quad c(\{1,3\}) = (6+d)k.$$

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n-person Cooperative Games
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Characteristic functions
Imputations
Reasonable set
Rational core
Rational ϵ-core
Nucleolus

Characteristic functions

Jed, Ned and Ted's benefit of cooperation

Now, define $\bar{\nu}(S)$, the benefit of cooperation associated with the coalition $S \subseteq \{1,2,3\}$, through the following formula

$$\bar{\nu}(S) = \sum_{i \in S} c(\{i\}) - c(S).$$

Then,

$$\bar{\nu}(\{1\}) = 0, \quad \bar{\nu}(\{1,2\}) = (3+d)k, \quad \bar{\nu}(\{1,2,3\}) = (3+2d)k,$$
$$\bar{\nu}(\{2\}) = 0, \quad \bar{\nu}(\{2,3\}) = (d)k,$$
$$\bar{\nu}(\{3\}) = 0, \quad \bar{\nu}(\{1,3\}) = (1+d)k,$$

so that by forming a car pool, J, N and T will save $(3+2d)k$ dollars per trip.

How should this money savings be split among them?

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Characteristic functions

Jed, Ned and Ted's (normalized) benefit of cooperation

For completeness, we append $\bar{\nu}(\emptyset) = 0$. Hence, we have defined a characteristic function $\bar{\nu} \colon 2^N \to \mathbb{R}$.

The number $\bar{\nu}(S)$, the benefit that players in S can obtain if they cooperate with each other but not with the players outside S, is a measure of the bargaining strength of the coalition S.

It will be convenient to express this measure as a fraction of the strength of the grand coalition.

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Characteristic functions

Jed, Ned and Ted's (normalized) benefit of cooperation

Hence, define $\nu(S) = \bar{\nu}(S)/\bar{\nu}(\{1,2,3\})$, the normalized characteristic function.

Then, besides $\nu(\emptyset) = 0$, we have

$$\nu(\{1\}) = 0, \quad \nu(\{1,2\}) = \tfrac{3+d}{3+2d}, \quad \nu(\{1,2,3\}) = 1,$$
$$\nu(\{2\}) = 0, \quad \nu(\{2,3\}) = \tfrac{d}{3+2d},$$
$$\nu(\{3\}) = 0, \quad \nu(\{1,3\}) = \tfrac{1+d}{3+2d}.$$

Characteristic functions

Coalitional form

- The coalitional form of an n-person TU (cooperative) game is given by the pair (N, ν), where
  - $N = \{1, 2, \dots, n\}$ is the set of players;
  - ν is a real-valued function called the characteristic function of the game defined on the set 2^N, *i.e.*, the set of all possible coalitions with elements in N.
- The real number $\nu(S)$ may be considered as the value, or worth, or power, of coalition S when its members act together as a unit.
- In our development in class we will assume $\nu(N) = 1$.

Imputations

Jed, Ned and Ted's allocations

- Assume that the car pool will use Jed's car. So it is Jed who actually foots the bills.
- If x_1 is the fraction that Jed receives from the car pool's benefits $(3 + 2d)k$ then

$$0 \le x_1, x_2, x_3, \qquad x_1 + x_2 + x_3 = 1,$$

where, similarly, x_2 applies to Ned and x_3 applies to Ted.
- Then, Ned or Ted should pay Jed

$$c(\{i\}) - (3 + 2d)kx_i \qquad \text{(dollars per trip)}.$$

- Our task is to determine the fractions x_1, x_2 and x_3.
- We will refer to the vector $x \equiv (x_1, x_2, x_3)$ as an imputation.
- We will refer to the component x_i as Player i's allocation at x.

Imputations

Definition

- More generally, an imputation of an n-person cooperative game is an n-dimensional vector $x \equiv (x_1, x_2, \dots, x_n)$ such that

$$x_i \ge 0, \quad i = 1, 2, \dots, n, \tag{3a}$$

$$x_1 + x_2 + \dots + x_n = 1 \tag{3b}$$

- When x satisfies (3a) we say that x is individually rational.
- When x satisfies (3b) we say that x is group rational.

Reasonable set

So how much should Jed receive?

- So, what can be regarded as a fair distribution of the benefits of cooperation? For Jed, for example?
- For every subset T of $N \equiv \{1, 2, 3\}$ and containing 1, the value of

$$\nu(T) - \nu(T \setminus \{1\})$$

is like a marginal fraction value of Jed within coalition T.
- Let Π_1 is the family of whole subsets of N containing 1. Then,

$$\begin{aligned} x_1 &\le \max_{T \in \Pi_1} \{\nu(T) - \nu(T \setminus \{1\})\} \\ &= \max\{\nu(\{1,2,3\}) - \nu(\{2,3\}), \quad \nu(\{1,2\}) - \nu(\{2\}), \\ &\qquad \nu(\{1,3\}) - \nu(\{3\}), \quad \nu(\{1\}) - \nu(\emptyset)\}, \\ &= \max\left\{1 - \frac{3}{7}, \frac{4}{7} - 0, \frac{10}{21} - 0, 0 - 0\right\} = \frac{4}{7} \quad (d = 9) \end{aligned}$$

Reasonable set

Definition

The reasonable set is the set of imputations such that

$$x_i \leq \max_{T \in \Pi_i} \{\nu(T) - \nu(T \setminus \{i\})\},$$

for all $i = 1, 2, \ldots, n$, where $\Pi_i = \{S : i \in S \text{ and } S \subseteq N\}$.

In what follows we will represent the reasonable set by X.

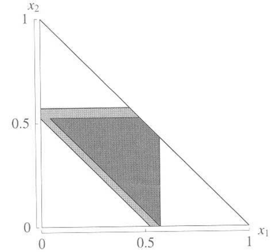
Reasonable set

Is the car pool problem solved?

In Jed, Ned and Ted's problem we have

$$X = \left\{ x \,\middle|\, \begin{array}{l} x_1 + x_2 + x_3 = 1, \\ 0 \leq x_1 \leq \frac{4}{7}, \quad 0 \leq x_2 \leq \frac{4}{7}, \quad 0 \leq x_3 \leq \frac{10}{21}. \end{array} \right\}$$

This set X is represented in light and dark shading in the figure.



So, the answer is No!
There are (still) too many points in the reasonable set.

Rational core

Definition

- The concept of reasonable set has enabled us to exclude the most unreasonable points from the set of imputations for the car pool.
- But infinitely many imputations may still remain.
- A concept that is useful in this regard is that of excess:

$$e(S, x) = \nu(S) - \sum_{i \in S} x_i \qquad (S \subseteq N, x \in X).$$

  Whenever $e(S, x) > 0$ the players in S regard the imputation x as unfair.
- The rational core of a game is the set

$$C^+(0) = \{x \in X : e(S, x) \leq 0, \text{ for } \textit{all} \text{ coalitions } S \neq \emptyset, N\}.$$

Rational core

Is the car pool problem solved?

For Jed, Ned and Ted's problem we have

$$e(\emptyset, x) = 0 = e(\{1,2,3\}, x) = 0, \tag{4a}$$

$$e(\{i\}, x) = -x_i, \quad i = 1, 2, 3, \tag{4b}$$

$$e(\{1,2\}, x) = \frac{4}{7} - x_1 - x_2, \tag{4c}$$

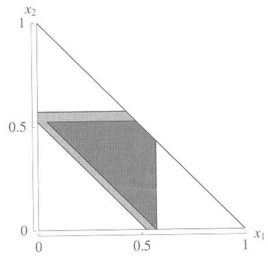
$$e(\{1,3\}, x) = \frac{10}{21} - x_1 - x_3, \tag{4d}$$

$$e(\{2,3\}, x) = \frac{3}{7} - x_2 - x_3, \tag{4e}$$

Rational core

Is the car pool problem solved? (cont.)

The rational core, *i.e.*, the points in X such that (4a)-(4e) holds, is represented in dark shading in the figure.



So, the answer is again No!
There are (still) too many points in the rational core.

Rational ϵ-core

Definition

- The rational ϵ-core of a game is the set of all imputations at which no coalition other than $\emptyset$ or N has a greater excess than ϵ. That is,

$$C^+(\epsilon) = \{x \in X : e(S,x) \le \epsilon, \text{ for } \textit{all} \text{ coalitions } S \neq \emptyset, N\}.$$

- Note that the rational core, if it exists, is a rational 0-core.
- The set $C^+(\epsilon)$ is characterized by a finite number of linear constraints on the variables $x_1, x_2, \ldots, x_n$ and ϵ.

Rational ϵ-core

Application to the car pool problem

For Jed, Ned and Ted's problem we have that $x \in C^+(\epsilon)$ if and only if $x \in X$ and

$$-x_i \le \epsilon, \quad i = 1,2,3, \tag{5a}$$

$$\frac{4}{7} - x_1 - x_2 \le \epsilon, \tag{5b}$$

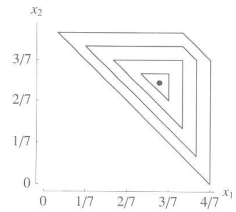
$$\frac{10}{21} - x_1 - x_3 \le \epsilon, \tag{5c}$$

$$\frac{3}{7} - x_2 - x_3 \le \epsilon, \tag{5d}$$

Rational ϵ-core

Application to car pool problem (cont.)

The rational ϵ-core, for $\epsilon \in \left\{0, -\frac{1}{21}, -\frac{2}{21}, -\frac{1}{7}, -\frac{11}{63}\right\}$, is represented in the figure.



- The smaller the ϵ, the smaller the set $C^+(\epsilon)$.
- The smaller ϵ for which $C^+(\epsilon)$ is nonempty is $-11/63$ and can be found by LP. (see class!)
- Since $C^+\left(-\frac{11}{63}\right) = \left\{\left(\frac{25}{63}, \frac{22}{63}, \frac{16}{63}\right)\right\}$, i.e., contains a single point, then

$\left(\frac{25}{63}, \frac{22}{63}, \frac{16}{63}\right)$ is the solution!

Rational ϵ-core

Least rational core

- Let ϵ_1 be the optimal value of the following LP,

$$\min_{\epsilon,x} \{\epsilon : x \in C^+(\epsilon)\} = \begin{cases} \min_x & \max \{e(S,x) : S \in N, S \neq \emptyset, N\} \\ \text{s.t.} & x \in X \end{cases}$$

- Hence, the value of ϵ_1 is simply the smallest maximum dissatisfaction among all coalitions that could possibly form.
- The set $X_1 = C^+(\epsilon_1)$ is called the least rational core.
- If $X_1 = \{x^*\}$, *i.e.*, contains a single point, then x^* is the solution of the game.
- However, X_1 may still contain infinitely many imputations. What to do then?

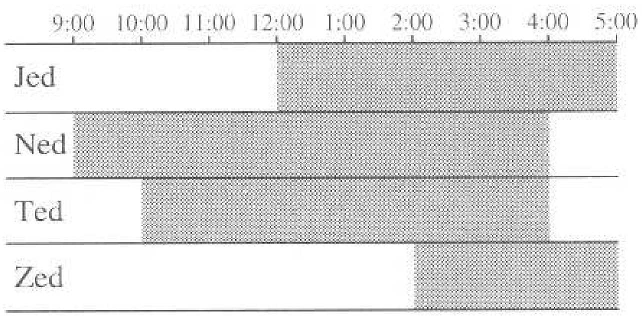
Nucleolus

The antique dealing problem

- Jed, Ned, Ted and Zed are antique dealers. They conduct their businesses in separate but adjoining rooms of a common premises.
- Their advertised office hours are shown in the figure, on the next slide.
- Because the dealers have other jobs, it is in the dealers' interest to pool their time in minding the store.
- The only constraint is that at least one of them should be in store during office hours.
- What are the fair allocations of store-minding duty?

Nucleolus

The antique dealing problem (cont.)



Nucleolus

Jed, Ned, Ted and Zed's benefit of cooperation

Let Jed be Player 1, Ned, Player 2, Ted, Player 3 and Zed, Player 4. Now, define $\bar{\nu}(S)$, the benefit of cooperation associated with the coalition $S \subseteq \{1,2,3,4\}$.Then,

$$\begin{array}{llll} \bar{\nu}(\{1\}) = 0, & \bar{\nu}(\{1,2\}) = 4, & \bar{\nu}(\{1,2,3\}) = 10, & \bar{\nu}(\{1,2,3,4\}) = 13, \\ \bar{\nu}(\{2\}) = 0, & \bar{\nu}(\{1,3\}) = 4, & \bar{\nu}(\{1,2,4\}) = 7, & \\ \bar{\nu}(\{3\}) = 0, & \bar{\nu}(\{1,4\}) = 3, & \bar{\nu}(\{1,3,4\}) = 7, & \\ \bar{\nu}(\{4\}) = 0, & \bar{\nu}(\{2,3\}) = 6, & \bar{\nu}(\{2,3,4\}) = 8, & \\ & \bar{\nu}(\{2,4\}) = 2, & & \\ & \bar{\nu}(\{3,4\}) = 2. & & \end{array}$$

so that, by cooperating, J, N, T and Z will save 13 office hours a day. How should this time savings be split among them?

Nucleolus

Jed, Ned, Ted and Zed's (normalized) benefit of cooperation

Now, define $\nu(S) = \bar{\nu}(S)/\bar{\nu}(\{1,2,3,4\})$, the normalized characteristic function. Then, besides $\nu(\emptyset) = 0$, we have

$$\nu(\{1\}) = 0, \quad \nu(\{1,2\}) = 4/13, \quad \nu(\{1,2,3\}) = 10/13, \quad \nu(\{1,2,3,4\}) = 1,$$

$$\nu(\{2\}) = 0, \quad \nu(\{1,3\}) = 4/13, \quad \nu(\{1,2,4\}) = 7/13,$$

$$\nu(\{3\}) = 0, \quad \nu(\{1,4\}) = 3/13, \quad \nu(\{1,3,4\}) = 7/13,$$

$$\nu(\{4\}) = 0, \quad \nu(\{2,3\}) = 6/13, \quad \nu(\{2,3,4\}) = 8/13,$$

$$\nu(\{2,4\}) = 2/13,$$

$$\nu(\{3,4\}) = 2/13.$$

and we have the coalitional game completely characterized.

Nucleolus

Solving the antique dealing problem

- First, we find the least rational core through solving the following linear program, *e.g.*, in Excel:

$$\min_{\epsilon,x} \{\epsilon \colon x \in X, e(S,x) \leq \epsilon \text{ for all } S \neq \emptyset, N\}. \tag{6}$$

- The optimal value is $\epsilon_1 = -3/26 = -0,11538\ldots$.
- Let $\bar{x}$ be an optimal x in (6). Is $\bar{x}$ unique? In other words, is $X_1 = C^+(\epsilon_1)$ singleton?
- How can we conclude this?

Nucleolus

Solving the antique dealing problem (cont.)

- Answer: Solve all the LPs,

$$\min_{x} \{e(S,x) \colon x \in X_1\}, \tag{7}$$

for every $S \neq \emptyset, N$.
- For some of these linear programs, the optimal x is not the $\bar{x}$ found before.

Thus, the set X_1 is not singleton.
- Note that whenever, for a given S, the optimal value of (7) is lower than ϵ_1 then it means that there is an imputation $x \in X_1$ for which the dissatisfaction for coalition S is lower than ϵ_1.
- As it may be checked, that is true for all coalitions except for coalitions $\{4\}$ and $\{1,2,3\}$.

Nucleolus

Solving the antique dealing problem (cont.)

- Now, let

$$\begin{aligned} \Sigma_1 &= \{S \neq \emptyset, N \colon e(S,x) < \epsilon_1 \text{ for some } x \in X_1\} \\ &= (2^N \setminus \{\emptyset, N\}) \setminus (\{4\} \cup \{1,2,3\}) \end{aligned}$$

- and solve the following linear program, *e.g.*, in Excel:

$$\min_{\epsilon,x} \{\epsilon \colon x \in X_1, e(S,x) \leq \epsilon \text{ for all } S \in \Sigma_1\}. \tag{8}$$

- The optimal value is $\epsilon_2 = -7/52 = -0,13462\ldots$.
- Let X_2 be the set of optimal x's in (8). Is X_2 singleton?

Nucleolus

Solving the antique dealing problem (cont.)

- Now, we solve the LPs,

$$\min_x \{e(S,x) \colon x \in X_2\}, \tag{9}$$

  for every $S \in \Sigma_1$.
  We observe that the optimal x is not the same in all of them.

  Thus, the set X_2 is not singleton.
- Moreover, we see that the optimal value of (9) is smaller than ϵ_2 for all coalitions in Σ_1 except for coalitions $\{1,4\}$ and $\{2,3,4\}$.

Nucleolus

Solving the antique dealing problem (cont.)

- Now, let

$$\Sigma_2 = \Sigma_1 \setminus (\{1,4\} \cup \{2,3,4\})$$

- and solve the following linear program, *e.g.*, in Excel:

$$\min_{\epsilon,x} \{\epsilon \colon x \in X_2, e(S,x) \leq \epsilon \text{ for all } S \in \Sigma_2\}. \tag{10}$$

- The optimal value is $\epsilon_3 = -15/104 = -0,14423\ldots$.
- Let X_3 be the set of optimal x's in (10). Is X_3 singleton?

Nucleolus

Solving the antique dealing problem (cont.)

- Now, from solving the LPs, $\min_x \{e(S,x) \colon x \in X_3\}$, for each $S \in \Sigma_2$,
- we observe that the optimal solution is the same in all.
  Hence, the optimal set X_3 is singleton!
- Since $X_3 = \left\{\left(\frac{1}{4}, \frac{33}{104}, \frac{33}{104}, \frac{3}{26}\right)\right\}$, then X_3 is the nucleolus, *i.e.*,

$$\left(\frac{1}{4}, \frac{33}{104}, \frac{33}{104}, \frac{3}{26}\right) \textit{ is the solution of the game.}$$

THE END