A REPRESENTATION THEOREM FOR ORTHOGONALLY ADDITIVE POLYNOMIALS ON RIESZ SPACES

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Abstract. The aim of this article is to prove a representation theorem for orthogonally additive polynomials in the spirit of the recent theorem on representation of orthogonally additive polynomials on Banach lattices but for the setting of Riesz spaces. To this purpose the notion of p–orthosymmetric multilinear form is introduced and it is shown to be equivalent to the orthogonally additive property of the corresponding polynomial. Then the space of positive orthogonally additive polynomials on an Archimedean Riesz space taking values on an uniformly complete Archimedean Riesz space is shown to be isomorphic to the space of positive linear forms on the n-power in the sense of Boulabiar and Buskes of the original Riesz space.

1. Introduction

Given X, Y two Banach lattices, a n-homogeneous polynomial $P \in \mathcal{P}(^n X, Y)$ is said to be an orthogonally additive polynomial if $P(x+y) = P(x) + P(y)$ whenever x and y are disjoint elements of X. The first mathematician interested in that kind of polynomials was Sundaresan who, in 1991, obtained a representation theorem for polynomials on ℓ_p and on L^p. Pérez-García and Villanueva [12] proved the case $C(K)$ and the result was generalized to any Banach lattice by Benyamini, Lassalle and Llavona [2]. Carando, Lassalle and Zalduendo [7] found an independent proof for the case $C(K)$ and Ibort, Linares and Llavona [9] gave another one for the case ℓ_p.

In parallel to the study of the orthogonally additive polynomials in Banach lattices, there has been several authors interested in the study of an analogous concept for multilinear forms in Riesz spaces.

Buskes and van Rooij introduced in 2000 the concept of orthosymmetric bilinear mapping in a Riesz space (see [5]) They showed that the orthosymmetric positive mappings are symmetric which lead to a new proof of the commutativity of f-algebras.

In 2004 the same authors studied the representation of orthosymmetric mappings in Riesz spaces introducing the notion of the square of a Riesz space and giving several characterizations of it in their article [6]. This study was generalized to the n-linear case by Boulabiar and Buskes [3] in 2006.

The square works in a similar way as the concavification for Banach lattices, so it arises naturally raises the problem of the representation of orthogonally additive polynomials in Riesz spaces in analogy to the representation in Banach lattices.

The representation theorem [2] for orthogonally n-homogeneous additive polynomials on a Banach lattice E with values on a Banach space F, $\mathcal{P}_0(^n E, F)$, establishes an isometry with the space of F–valued continuous linear maps on

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the n-concavification of E, $\mathcal{L}(E_{(n)}, F)$, given by $P(f) = T(f^n)$ for all $f \in E$, $P \in \mathcal{P}_0(^nE, F)$ and $T \in \mathcal{L}(E_{(n)}, F)$. The aim of this article is to provide a representation theorem for orthogonally additive polynomials on Riesz spaces similar to this representation theorem.

Recall that a real vector space E together with an order relation $\leq$ compatible with the algebraic operations in E is called a Riesz space if, for every $x, y \in E$, there exists a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$. We define the absolute value of $x \in E$ as $|x| = x \vee (-x)$. An element $x \in E$ is said to be positive if $x \geq 0$. The positive cone is the space

$$E^+ = \{x \in E : x \geq 0\}.$$

Note that every $x \in E$ can be decomposed as the difference of two positive elements $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$. Furthermore, $|x| = x^+ + x^-$. A Riesz space will be called Archimedean if for every $x, y \in E^+$, such that $0 \leq nx \leq y$ for every $n \in \mathbb{N}$ we have that $x = 0$. For further information about Riesz spaces the reader is referred to the classical books [1] or [10].

A mapping between two Riesz spaces $P : E \longrightarrow F$ is said to be a n-homogeneous polynomial if there exists a n-linear form $A : E \times \ldots \times E \longrightarrow F$ such that $P(x) = A(x, \ldots, x)$. There is a natural one-one correspondence between polynomials and symmetric n-linear forms.

Given $E_1, \ldots, E_n, F$, Riesz spaces, a multilinear form $A : E_1 \times \ldots \times E_n \longrightarrow F$ is said to be positive if $A(x_1, \ldots, x_n) \geq 0$ for every $x_1, \ldots, x_n$ positive elements. A polynomial is positive if its associated symmetric multilinear mapping is positive.

2. Orthosymmetric Applications on Riesz spaces.

Our proof of the representation theorem relies in the connection between the orthogonally additive polynomials and the orthosymmetric multilinear mappings. Two ingredients will be essential: we will need that the equivalence of these concepts for a polynomial and its associated multilinear form and we will also need that the orthosymmetry and the positiveness guarantee the symmetry of a multilinear form.

Following Boulabiar and Buskes [3] a multilinear map $A : E \times \ldots \times E \longrightarrow F$ between two Riesz spaces E, F is said to be orthosymmetric if $A(x_1, \ldots, x_n) = 0$ whenever $x_1, \ldots, x_n \in E$ verifies $|x_i| \wedge |x_j| = 0$ for some $i, j \in \{1, \ldots, n\}$.

The next result shows the relation between the orthosymmetry and the additive orthogonality using the proof of the already quoted representation theorem [2].

Theorem 2.1. *Let E be a Riesz space, F a Banach space and $A : E \times \cdots \times E \longrightarrow F$ a symmetric positive multilinear form in E. A is orthosymmetric if and only if $P = \hat{A}$ is orthogonally additive.*

Proof. It is clear that if A is orthosymmetric, expanding

$$P(x + y) = A(x + y, \ldots, x + y)$$

for disjoint x and y, the unique nonzero terms are $A(x, \ldots, x)$ and $A(y, \ldots, y)$ so P is orthogonally additive.

Assume that P is orthogonally additive. Let $x_1, \ldots, x_n$ be elements of E with at least two disjoint. Consider $E_0 \subset E$ to be the ideal generated by $x_1, \ldots, x_n$ whose unit is $|x_1| + \cdots + |x_n|$. By Yosida's Representation Theorem (see for instance [10] Theorem 13.11), E_0 is Riesz isomorphic to a dense subspace $\hat{E_0}$ of $C(K)$ for certain compact Hausdorff K. Let $\tilde{P}$ be the polynomial on $\hat{E_0}$ defined by the composition of the previous isomorphism with P. As the isomorphism preserves the order, $\tilde{P}$ is positive and orthogonally additive. Extend $\tilde{P}$ by denseness to a positive orthogonally additive n-homogeneous polynomial on $C(K)$. For this purpose notice that if $|f| \wedge |g| = 0$ then there exist sequences $x_n, y_n \in \hat{E}_0$ converging to f and g respectively and such that for n large enough $|x_n| \wedge |y_n| = 0$. Then it is immediate to see that $\tilde{P}(f,g) = \lim_{n\to\infty} A(x_n + y_n, \dots, x_n + y_n) = \lim_{n\to\infty} A(x_n, \dots, x_n) + \lim_{n\to\infty} A(y_n, \dots, y_n) = \tilde{P}(f) + \tilde{P}(g)$ and the extension of the polynomial $\tilde{P}$ (denoted with the same symbol in what follows) is orthogonally additive. Note that this extension, being positive, will be also continuous (see [8] Proposition 4.1).

Hence, by the representation theorem of orthogonally additive polynomials [2] and taking into account that $C(K)_{(n)} = C(K)$ there exists $T \in \mathcal{L}(C(K)_{(n)}, F)$ such that

$$\tilde{P}(f) = T(f^n).$$

The symmetric n-linear form associated to $\tilde{P}$ is given by

$$(f_1, \dots, f_n) \longmapsto T(f_1 \cdots f_n),$$

then, this form as well as its restriction to $\hat{E}_0$ are orthosymmetric and by the Riesz isomorphism between E_0 and $\hat{E}_0$, A will be also orthosymmetric. □

In spite of this result we are still lacking of a natural and explicit representation of an orthogonally additive polynomial on a Riesz space as a linear form on a Riesz space. Such space would play the role of the n–concavification of Banach lattices used in the representation theorem for orthogonally additive polynomials on Banach lattices. The notion of n-power of Riesz spaces introduced by Boulabiar and Buskes [3] will be the appropriate tool for that.

That is the reason to introduce a new definition of orthosymmetry with the properties needed to actually prove our claims. The definition of orthosymmetry that we are going to use is more involved than the definition of orthosymmetry above but it will lead us to the proof the the main statement (see Theorem 2.5 below).

To begin with, we introduce the concept of p-disjoint elements:

Definition 2.2. *We will say that $x_1, \dots, x_n \in E$ are partitionally disjoint or p-disjoint it there exists a partition $I_1, \dots, I_m$, $2 \le m \le n$, of the index set $I = \{1, \dots, n\}$ such that the sets*

$$\{x_{i_k} : i_k \in I_k\}$$

are disjoint, that is $|x_{i_k}| \wedge |x_{i_l}| = 0$ whenever $k \neq l$.

Definition 2.3. *Given E, F Riesz spaces, a n-linear application $A : E \times \cdots \times E \longrightarrow F$ is said to be p-orthosymmetric if $A(x_1, \dots, x_n) = 0$ for $x_1, \dots, x_n$ p-disjoint.*

In order to prove that this new definition is coherent with the additive orthogonality of polynomials, we need a preliminary lemma:

Lemma 2.4. *Let E, F be Riesz spaces and $A : E \times \cdots \times E \longrightarrow F$ a symmetric n-linear form. The following assumptions are equivalent*

(1) *A is p-orthosymmetric.*
(2) *$A(x^i, y^{n-i}) = 0$ for x and y disjoint and $1 < i < n$.*

Proof. If A is p-orthosymmetric, it is obvious that $A(x^i, y^{n-i}) = 0$ for disjoint x and y with $1 < i < n$.

Conversely, let $\{x_1, \dots, x_n\}$ be p-disjoint. Assume that the partition in disjoint subsets is given by two elements, that is there exists some $1 < i < n$ such that the sets $\{x_1, \dots, x_i\}$ and $\{x_{i+1} \dots, x_n\}$ are disjoints. The general case is analogous.

The notation $B = A_{x_{i+1}, \dots, x_n}$ will represent the multilinear form B defined by $B(y_1, \dots, y_i) = A(y_1, \dots, y_i, x_{i+1}, \dots, x_n)$. By the Polarization Formula,

$$A(x_1,\dots,x_n)=B(x_1,\dots,x_i)=\frac{1}{i!2^i}\sum_{\varepsilon_j=\pm1}\varepsilon_1\cdots\varepsilon_i B(\varepsilon_1x_1+\cdots+\varepsilon_ix_i)^i$$

where $B(x)^i=B(x,\overbrace{\dots}^{i},x)$.

We conclude by showing that for each choice of signs $\varepsilon_j=\pm1$,

$$B(\varepsilon_1x_1+\cdots+\varepsilon_ix_i)^i=0.$$

In order to simplify, let $\varepsilon\mathbf{x}=\varepsilon_1x_1+\cdots+\varepsilon_ix_i$, note that

$$\begin{aligned}B(\varepsilon\mathbf{x})^i &= B(\varepsilon\mathbf{x},\dots,\varepsilon\mathbf{x})=A(\varepsilon\mathbf{x},\dots,\varepsilon\mathbf{x},x_{i+1},\dots,x_n)=\\ &\quad A_{\varepsilon\mathbf{x},\dots,\varepsilon\mathbf{x}}(x_{i+1},\dots,x_n)=C(x_{i+1},\dots,x_n)\end{aligned}$$

with the notations above.

Again by using the polarization formula, we get:

$$C(x_{i+1},\dots,x_n)=\frac{1}{(n-i)!2^{n-i}}\sum_{\delta_k=\pm1}\delta_1\cdots\delta_{n-i}C(\delta_1x_{i+1}+\cdots+\delta_{n-i}x_n)^{n-i}$$

as $\varepsilon_1x_1+\cdots+\varepsilon_ix_i$ and $\delta_1x_{i+1}+\cdots+\delta_{n-i}x_n$ are disjoint for every choice of signs $\varepsilon_j=\pm1$ and $\delta_k=\pm1$ our assumption allows us to conclude that the summands appearing in the previous expressions are zero and hence $A(x_1,\dots,x_n)=0$, as needed. □

The next result can be obtained as a straightforward generalization of Proposition 2.4 on [12].

Theorem 2.5. *Let E,F be Riesz spaces and $A:E\times\cdots\times E\longrightarrow F$ a symmetric multilinear form in E. A is p-orthosymmetric if and only if $P=\hat{A}$ is orthogonally additive.*

Finally, we will see that the p-orthosymmetry and the positiveness guarantee the symmetry as in the bilinear case. The following lemma generalizes Theorem 1 in [5]. To prove it we will follow a similar path from which it will be clear the reasons for our definition of orthosymmetry.

Lemma 2.6. *Let K be a compact Hausdorff space, E a uniformly dense Riesz subspace of $C(K)$, F an Archimedean Riesz space and $T:E\times\ldots\times E\longrightarrow F$ a positive p-orthosymmetric n-linear map. Let E^n the linear hull of the set*

$$\{f_1\cdots f_n : f_1,\dots,f_n\in E\}.$$

Then there exists a positive linear map $A:E^n\longrightarrow F$ such that

$$T(f_1,\dots,f_n)=A(f_1\cdots f_n)\ \textit{for every}\ f_1,\dots,f_n\in E.$$

Proof. As in [5] there is no loss of generality if we consider $E=C(K)$. Define $A(h)=T(\mathbf{1},\dots,\mathbf{1},h)$ where $\mathbf{1}$ stands for the function identically 1. It is clear that A is linear and positive. Note that as T is positive by [8] Proposition 4.1, it is continuous. We have to prove that if $f_1,\dots,f_n\in C(K)$ and $h=f_1\cdots f_n$, $T(f_1,\dots,f_n)=T(\mathbf{1},\dots,\mathbf{1},h)$.

Given $\varepsilon>0$, we will say, following [5], that $X\subset K$ is small if for every $x,y\in X$

$$|f_1(x)-f_1(y)|<\varepsilon,\dots,|f_n(x)-f_n(y)|<\varepsilon \text{ and } |h(x)-h(y)|<\varepsilon.$$

Take $u_1,\dots,u_N\in C(K)^+$ verifying $\sum u_n=\mathbf{1}$ and such that for every j, the set $S_j=\{x\in K : u_j(x)\neq 0\}$ is small and nonempty, and take $s_j\in S_j$, for $j=1,\dots,N$. Define for $i=1,\dots,n$,

$$f_i' = \sum_{j=1}^{N} f(s_j)u_j \text{ and } h' = \sum_{j=1}^{N} h(s_j)u_j.$$

Because

$$\begin{aligned}|T(f_1,\ldots,f_n) - T(\mathbf{1},\ldots,\mathbf{1},h))| &\leq |T(f_1,\ldots,f_n) - T(f_1',\ldots,f_n')| \\ &+ |T(f_1',\ldots,f_n') - T(\mathbf{1},\ldots,\mathbf{1},h'))| \\ &+ |T(\mathbf{1},\ldots,\mathbf{1},h')) - T(\mathbf{1},\ldots,\mathbf{1},h))|\end{aligned}$$

we have to bound these three terms. The second is immediate:

$$|T(f_1',\ldots,f_n') - T(\mathbf{1},\ldots,\mathbf{1},h'))| \leq \|h-h'\|_\infty |T(\mathbf{1},\ldots,\mathbf{1})| \leq \varepsilon T(\mathbf{1},\ldots,\mathbf{1}).$$

For the first

$$\begin{aligned}&|T(f_1,\ldots,f_n) - T(f_1',\ldots,f_n')| \leq |T(f_1,\ldots,f_n) - T(f_1',f_2\ldots,f_n)| + \\ &|T(f_1',f_2,\ldots,f_n) - T(f_1',f_2',f_3\ldots,f_n)| + \cdots + |T(f_1',\ldots,f_{n-1}',f_n) \\ &-T(f_1',\ldots,f_n')| = \sum_{i=1}^{n} |T(f_1',\ldots,f_{i-1}',f_i - f_i',\ldots,f_n)| \leq \\ &\sum_{i=1}^{n} \|f_1'\|_\infty \cdots \|f_i - f_i'\|_\infty \cdots \|f_n\|_\infty T(\mathbf{1},\ldots,\mathbf{1}) \leq \\ &\varepsilon T(\mathbf{1},\ldots,\mathbf{1}) \sum_{i=1}^{n} \|f_1\|_\infty \cdot \overbrace{\cdots}^{[i]} \cdot \|f_n\|_\infty\end{aligned}$$

With the notation $\overbrace{\cdots}^{[i]}$ we mean that in the product, the factor i does not appear. Finally

$$\begin{aligned}&|T(\mathbf{1},\ldots,\mathbf{1},h')) - T(\mathbf{1},\ldots,\mathbf{1},h))| \leq \\ &\sum_{j_1,\ldots,j_n} |f_1(s_{j_1})\cdots f_{n-1}(s_{j_{n-1}}) - f_1(s_{j_n})\cdots f_{n-1}(s_{j_n})||f_n(s_n)|T(u_{j_1},\ldots,u_{j_n}) \\ &\leq \|f_n\|_\infty \sum_{j_1,\ldots,j_n} |f_1(s_{j_1})\cdots f_{n-1}(s_{j_{n-1}}) - f_1(s_{j_1})\cdots f_{n-2}(s_{j_{n-2}})f_{n-1}(s_{j_n})|T_{j_1,\ldots,j_n} \\ &+\|f_n\|_\infty \sum_{j_1,\ldots,j_n} |f_1(s_{j_1})\cdots f_{n-2}(s_{j_{n-2}}) - f_1(s_{j_n})\cdots f_{n-2}(s_{j_n})||f_{n-1}(s_{j_n})|T_{j_1,\ldots,j_n} \\ &\leq \|f_1\|_\infty \overbrace{\cdots}^{[n-1]} \|f_n\|_\infty \sum_{j_1,\ldots,j_n} |f_{n-1}(s_{j_{n-1}}) - f_{n-1}(s_{j_n})|T_{j_1,\ldots,j_n} \\ &+\|f_{n-1}\|_\infty \|f_{n-1}\|_\infty \sum_{j_1,\ldots,j_n} |f_1(s_{j_1})\cdots f_{n-2}(s_{j_{n-2}}) - f_1(s_{j_n})\cdots f_{n-2}(s_{i_n})|T_{j_1,\ldots,j_n}\end{aligned}$$

Where the notation $T_{j_1,\ldots,j_n}$ stands for $T(u_{j_1},\ldots,u_{j_n})$. If we repeat the process, we get:

$$\begin{aligned}&|T(\mathbf{1},\ldots,\mathbf{1},h')) - T(\mathbf{1},\ldots,\mathbf{1},h))| \leq \\ &\sum_{i=1}^{n} \|f_1\|_\infty \overbrace{\cdots}^{[i]} \|f_n\|_\infty \sum_{j_1,\ldots,j_n} |f_i(s_{j_i}) - f_{n-1}(s_{j_n})|T(u_{j_1},\ldots,u_{j_n})\end{aligned}$$

Now, there are two options, if $u_{j_1}, \ldots, u_{j_n}$ are p-disjoint, then

$$T(u_{j_1}, \ldots, u_{j_n}) = 0,$$

otherwise there is a path connecting the set S_{j_i} and the set S_{j_n}, hence

$$|f_i(s_{j_i}) - f_{n-1}(s_{j_n})| \leq n\varepsilon$$

We conclude that

$$\begin{aligned}
&|T(\mathbf{1}, \ldots, \mathbf{1}, h')) - T(\mathbf{1}, \ldots, \mathbf{1}, h))| \leq \\
&\sum_{i=1}^{n} n\varepsilon \|f_1\|_\infty \overbrace{\cdots}^{[i]} \|f_n\|_\infty \sum_{j_1,\ldots,j_n} T(u_{j_1}, \ldots, u_{j_n}) \\
&\varepsilon T(\mathbf{1}, \ldots, \mathbf{1}) \sum_{i=1}^{n} n\|f_1\|_\infty \overbrace{\cdots}^{[i]} \|f_n\|_\infty
\end{aligned}$$

□

As a direct consequence of the previous lemma we obtain the following result:

Theorem 2.7. *Let E and F be Archimedean Riesz spaces. Let $T : E\times\ldots\times E \longrightarrow F$ be an p-orthosymmetric positive n-linear map. Then T is symmetric.*

The proof of this fact is an easy application of Yosida's Theorem which allows us to translate the problem to the $C(K)$ setting and using then Lemma 3.2.

3. Representation of orthogonally additive polynomials.

We will discuss now the notion of n–powers of Riesz spaces as presented by Boulabiar and Buskes that will be instrumental in what follows.

Definition 3.1 ([3]). *Let E be an Archimedean Riesz space and $n \geq 2$. The pair $(\bigodot_n E, \odot_n)$ it is said to be an n-power of E if*

1. *$\bigodot_n E$ is a Riesz space,*
2. *$\odot_n : E\times\ldots\times E \longrightarrow \bigodot_n E$ is a orthosymmetric n-morphism of Riesz spaces and,*
3. *for every F Archimedean Riesz space and every orthosymmetric n-morphism $T : E \times \ldots \times E \longrightarrow F$, there exists a unique Riesz homomorphism $T^{\odot_n} : \bigodot_n E \longrightarrow F$ such that $T = T^{\odot_n} \circ \odot_n$*

As those authors proved, the n-power is unique up to Riesz isomorphism. The results about the n-power can be translated in a straighforward way to the setting of p-orthosymmetry.

Lemma 3.2. *Let E be an Archimedean Riesz space and F a uniformly complete Archimedean Riesz space. The space $\mathcal{L}_o^+({}^nE, F)$ of positive p-orthosymmetric n-linear applications from E to F is isomorphic to the space $\mathcal{L}^+(\bigodot_n E, F)$ of positive linear forms from $\bigodot_n E$ to F.*

Proof. In this result, we generalize the ideas of [4] Theorem 3.1 to the n-linear case. If $T \in \mathcal{L}_o^+({}^nE, F)$ we will prove that there exists a unique $\Phi_T \in \mathcal{L}^+(\bigodot_n E, F)$ such that $\Phi_T \circ \bigodot_n = T$ and that the correspondence $T \longrightarrow \Phi_T$ is a Riesz isomorphism.

Given such T, let $\widetilde{T}$ be the unique positive operator $\widetilde{T} : \overline{\bigotimes}_n E \longrightarrow F$ verifying

$$T(x_1, \ldots, x_n) = \widetilde{T}(x_1 \otimes \ldots \otimes x_n).$$

If $\pi : \overline{\bigotimes}_n E \longrightarrow \bigodot_n E$ is the canonical projection, the operator $\Phi = \Phi_T$ that we need is the one satisfying $\widetilde{T} = \Phi \circ \pi$. The uniqueness of this operator is given by the uniqueness of $\widetilde{T}$. Moreover $\widetilde{T}$ it is positive since π is a Riesz homomorphism and $\widetilde{T}$ is positive. In particular, the correspondence $T \longmapsto \Phi$ respects the order.

To conclude the proof, we need to show that Φ is well defined. It is sufficient to show that the kernel of π is contained in the kernel of $\widetilde{T}$. We proceed as in [6] Theorem 4. Let $f_1 \otimes \ldots \otimes f_n \in \ker \pi$ then $f_1, \ldots, f_n$ are p-disjoint hence,

$$\widetilde{T}(f_1 \otimes \ldots \otimes f_n) = T(f_1, \ldots, f_n) = 0$$

since T is p-orthosymmetric.

Finally, observe that the correspondence $T \longmapsto \Phi_T$ is a Riesz isomorphism since T and Φ are positive.

□

Notice the relevance of the notion of p-orhosymmetry in the proof of the previous lemma. Then we obtain as a consequence of the previous results the representation theorem we were looking for:

Theorem 3.3 (Representation Theorem for positive orthogonally additive polynomials on Riesz spaces)**.** *Let E, F be Archimedean Riesz spaces with F uniformly complete and let $(\bigodot_n E, \odot_n)$ be the n-power of E. The space $\mathcal{P}_o^+({}^nE, F)$ of positive orthogonally additive n-homogeneous polynomials on E is isomorphic to the space $\mathcal{L}^+(\bigodot_n E, F)$ of positive linear applications from $\bigodot_n E$ to F.*

If we consider the case of f-algebras, the connection with the theorem of representation of orthogonally additive polynomials in Banach lattices becomes clearer:

Theorem 3.4. *Let E be a uniformly complete Archimedean Riesz subspace of a semiprime f-algebra A and F uniformly complete Archimedean. Then, for every positive orthogonally additive n-homogeneous polynomial $P \in \mathcal{P}_o^+({}^nE, F)$ there exists a unique positive linear application $L \in \mathcal{L}^+(E^n, F)$ such that $P(x) = L(x \cdots x)$ for every $x \in E$.*

Proof. By Theorem 3.3 [3], E^n with its product is the n-power of E and by the uniqueness given by Theorem 3.2 [3], there exists a Riesz isomporphism $i : E^n \longrightarrow \bigodot_n E$ such that $i(x_1 \cdots x_n) = x_1 \odot \ldots \odot x_n$. Define $L \in \mathcal{L}^+(E^n, F)$ as the linear application $L = \Phi_{\check{P}} \circ i$ (with the notation as in Lemma 3.2) then,

$$P(x) = \check{P}(x, \ldots, x) = \Phi_{\check{P}}(x \odot \ldots \odot x) = L(x \cdots x).$$

□

Recently Toumi [13] has obtained an alternative representation theorem for homogeneous orthogonally additive polynomials on Riesz spaces. Its relation with the results presented here will be discussed elsewhere.

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