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On the Existence of Nash Equilibria in Generalized Noncooperative Games and Applications

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Abstract. In this paper we introduce the concepts of generalized noncooperative games and Nash equilibria in topological vector spaces. Some sufficient conditions for the existence of Nash equilibria in such games are established. By applying these results to quasiconvex and convex vector optimization problems, we have obtained some new results on the existence of optimal solutions which extend some previously corresponding results.

Keywords: Noncooperative games, Nash equilibria, vector optimization, Pareto minimal point, upper and lower semicontinuous vector functions, upper and lower semicontinuous set-valued maps.

1. Introduction

In the classical sense, a noncooperative game $\Omega = (X_i, f_i)_{i\in I}$ consists of a finite set I of players and for each $i \in I$, a set X_i of strategies and a loss function $f_i : X \to \mathbb{R}$ of i^{th} player, where $X := \prod_{i\in I} X_i$. For $x = (x_j)_{j\in I} \in X, i \in I, y_i \in X_i$, we define $x^i := (x_j)_{j\in I\setminus\{i\}}$ and denote by (x^i, y_i) the vector x with i^{th} component replaced by y_i. A point $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ is called a Nash equilibrium of the game Ω if for each $i \in I$,

$$f_i(\bar{x}) \leqslant f_i(\bar{x}^i, x_i),\ \forall x_i \in X_i.$$

It is well known (see [7]) that if for each $i \in I$,

i) X_i is a nonempty convex and compact subset of some finite dimensional Euclide space,

ii) f_i is continuous on X and convex in the i^{th} argument,
then Ω has a Nash equilibrium.

In this paper, we shall generalize the concepts of noncooperative game and Nash equilibrium in the following cases:

a) The index set I is infinite.
b) For each $i \in I$, X_i is a subset of some real topological vector space.
c) At each time, the set of strategies of each player is changed by the strategies of other players.
d) The loss function f_i has value in a real topological vector space which is ordered by a convex cone.

These new concepts are suggested by concepts of equilibria in abstract economies [5]. Besides, in practice, there are several models in which at each time the set of strategies of each player is specified by strategies of other players, for instance, in chess play. Hence the strategy sets of players are not fixed but often changed.

The purpose of this paper is to establish some sufficient conditions for the existence of Nash equilibria in such games and to show some applications in vector optimization problems. Our paper is organized as follows. In the next section, we present some preliminaries concerning cone orders in topological vector spaces, continuity and convexity of vector functions with respect to a cone, continuity and generalized fixed point theorems for set-valued maps. In Sec. 3, we establish some sufficient conditions for the existence of Nash equilibria in generalized games in finite and infinite dimensions cases. Applying these results to vector optimization, we obtain some new results on the existence of optimal solutions of quasiconvex and convex vector optimization problems.

2. Preliminaries

Let Y be a real topological vector space, $C \subsetneq Y$ a convex cone. We define an order '$\preceq_C$' in Y as follow

$$x, y \in Y, x \preceq_C y \Leftrightarrow y - x \in C.$$

When no confusion occurs, we write '$\preceq$' instead of '$\preceq_C$'. We also write $x \prec y$ if $x \preceq y$ and $x \neq y$. If $\text{int}C \neq \emptyset$ then $x \ll y$ means $y - x \in \text{int}C$. The cone C is called pointed if $C \cap (-C) = \{0\}$.

Lemma 2.1. *Assume that the cone $C \subsetneq Y$ is convex with nonempty interior. Then*

i) $0 \notin \text{int}C$.
ii) $\text{int}C = \lambda\, \text{int}C,\ \forall \lambda > 0$.
iii) $C + \text{int}C = \text{int}C$.
iv) $\{0\} \cup \text{int}C$ *is pointed.*

Proof. i) If $0 \in \text{int}C$ then $\text{int}C$ is an absorbent neighborhood of 0. Since C is a cone then $C = Y$, which contradicts assumptions.

ii) Let $\lambda > 0$. Then $\lambda \ \text{int}C \subset C$ and $\lambda \ \text{int}C$ is open. Hence $\lambda\text{int}C \subset \text{int}C$. Conversely, as above we have $\text{int}C = \lambda \ (\frac{1}{\lambda}\text{int}C) \subset \lambda \ \text{int}C$.

iii) Let $x \in C, y \in \text{int}C$. Since C is convex then $\frac{1}{2}(x+y) \in \text{int}\ C$. Apply *ii*), we get $x + y \in \text{int}\ C$. The converse inclusion is obvious.

iv) Suppose the contrary, that $\{0\} \cup$ int C is not pointed. Then there exists $x \in \text{int}\ C \cap (-\text{int}\ C)$. By *iii*), $0 = x + (-x) \in \text{int}\ C$ which contradicts *i*). The proof is complete. ∎

Definition 2.2. ([9, Definition 2.1]) *We say C satisfies condition* $(*)$ *if there exists a convex cone $K \subsetneq Y$ with nonempty interior such that*

$$C \setminus \{0\} \subset \text{int}K.$$

We recall that a set $B \subset C$ is called a base of C if $0 \notin B$ and for every $x \in C \setminus \{0\}$, there are unique $y \in B, t > 0$ such that $x = ty$.

Proposition 2.3. ([9, Proposition 2.3]) *Let Y be a real locally convex Hausdorff space. If C has a convex compact base then it satisfies condition* $(*)$.

By Lemma 2.1 *i*), C satisfies condition (*) only if C is pointed. The converse is not true in general. However, when Y is finite dimensional then every closed, convex and pointed cone has a convex and compact base (see [5, Remark 1.6]), hence it satisfies condition (*).

Definition 2.4. ([6, Definition 2.1]) *Let $A \subset Y$ and $x \in A$. We say that*

i) *x is an efficient (or Pareto-minimal) point of A if $y \preceq x$ for some $y \in A$ implies $x \preceq y$. The set of efficient points of A is denoted by* $\text{Min}(A|C)$.

ii) *Suppose that* $\text{int}C \neq \emptyset$, *$x$ is a weakly efficient point of A if* $x \in \text{Min}(A|\{0\} \cup \text{int}C)$. *The set of weakly efficient points of A is denoted by* $\text{WMin}(A|C)$.

If no confusion occurs, we write $\text{Min}(A), \text{WMin}(A)$ *instead of* $\text{Min}(A|C)$, $\text{WMin}(A|C)$.

Lemma 2.5. *Let $A \subset Y$ and $x \in A$. Then*

i) *If C is pointed then*

$$x \in \text{Min}A \Leftrightarrow y \not\prec x, \forall y \in A.$$

ii) *If* int $C \neq \emptyset$ *then*

$$x \in \text{WMin}A \Leftrightarrow y \not\ll x, \forall y \in A.$$

iii) *If $K \subsetneq Y$ is a convex cone with nonempty interior such that $C \setminus \{0\} \subset intK$, then*

$$x \in \text{WMin}(A|K) \Rightarrow x \in \text{Min}(A|C).$$

Proof. The proof is straightforward. ∎

Now, let E be another real topological vector space, D be a nonempty subset of E, $f : D \to Y$ be a vector function and $x \in D$. We say f attains Pareto-minimum (respectively, weakly minimum) at x if $f(x) \in \text{Min}f(D)$ (respectively, $f(x) \in \text{WMin}f(D)$).

Definition 2.6. ([9, Definition 2.6]) *We say that*

i) *f is lower semicontinuous, in brief l.s.c., (resp., upper semicontinuous, in brief u.s.c.) at x with respect to C if for any neighborhood V of $f(x)$, there exists a neighborhood W of x such that $y \in W \cap D$ implies $f(y) \in V + C$ (resp., $f(y) \in V - C$).*

ii) *f is continuous with respect to C at x if it is l.s.c. and u.s.c. with respect to C at this point.*

iii) *f is continuous (resp., l.s.c., u.s.c.) with respect to C if it is continuous (resp., l.s.c., u.s.c.) with respect to C at every $x \in D$.*

From the definition we see that if f is continuous in the usual sense then it is continuous with respect to any cone. The converse is true if Y is Hausdorff and C has a closed convex bounded base (see [6, Proposition 1.8]).

Assume that $D \subset E$ is convex. Then the vector function $f : D \to Y$ is called convex (with respect to C) if for every $x, y \in D, \lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y \preceq \lambda f(x) + (1 - \lambda)f(y).$$

Let $A \subset Y$ and $a \in Y$, we say that a is an upper bound of A if

$$x \preceq a, \forall x \in A.$$

The set of upper bounds of A is denoted by UbA.

Definition 2.7. *A vector function $f : D \to Y$ is called quasiconvex (with respect to C) if for every $x, y \in D, \lambda \in [0, 1]$,*

$$f(\lambda x + (1 - \lambda)y \preceq a, \forall a \in \mathrm{Ub}\{f(x), f(y)\}.$$

Let $a \in Y$. We denote by $\mathrm{lev}_a f$ the level set

$$\mathrm{lev}_a f := \{x \in D|\ f(x) \preceq a\}.$$

Lemma 2.8. *Let $D \subset E$ be a nonempty convex set and let $f : D \to Y$ be a vector function. We have*

i) *If f is convex then it is also quasiconvex.*

ii) *If f is convex (resp., continuous) with respect to C then it is also convex (resp., continuous) with respect to any cone $K \supset C$.*

iii) *f is quasiconvex if and only if $\mathrm{lev}_a f$ is convex for all $a \in Y$.*

Proof. This is immediate from definition. ■

Now, we assume that int$C \neq \emptyset$. Let $x \in D$. We define $\bar{P}(x) := \{y \in D|f(y) \preceq f(x)\}, P(x) := \{y \in D|f(y) \ll f(x)\}$.

Lemma 2.9. *If $f :\ D \to Y$ is quasiconvex, then for every $x \in D$, we have*

i) *$\bar{P}(x)$ and $P(x)$ are convex,*

ii) *$x \in \bar{P}(x)$ and $x \notin P(x)$.*

Proof. i) Let $y, z \in P(x), \lambda \in (0,1)$ be arbitrary. Then $f(y)-f(x), f(z)-f(x) \in -\text{int}C$, hence, $0 \in (f(y)-f(x)+\text{int}C) \cap (f(z)-f(x)+\text{int}C)$. By the absorption of neighborhoods of the origin and by Lemma 2.1, there exists $a \in -\text{int}C \cap [(f(y)-f(x)+\text{int}C) \cap (f(z)-f(x)+\text{int}C)]$. Then $f(y)-f(x), f(z)-f(x) \preceq a$, or, $f(y), f(z) \preceq f(x)+a$. By the quasiconvexity of f, we have

$$f(\lambda y + (1-\lambda)z) \preceq f(x) + a.$$

Since $a \in -\text{int}C$ then $f(\lambda y + (1-\lambda)z) \ll f(x)$. Hence, $P(x)$ is convex. The convexity of $\bar{P}(x)$ is immediate from Lemma 2.8 (*iii*).

ii) This is immediate from definition and from the fact that $0 \notin \text{int}C$. ■

Let F be another topological vector space, $\alpha : E \to F$ be a set-valued map and $x \in \text{dom}\alpha$, where $\text{dom}\alpha := \{x \in E | \alpha(x) \neq \emptyset\}$. Let $D \subset F$, we denote $\alpha^{-1}(D) := \{x \in E | \alpha(x) \cap D \neq \emptyset\}$.

Definition 2.10. ([1]). *We say that*

i) *α is lower semicontinuous, in brief l.s.c., at x if for any open set $V \subset F$ such that $V \cap \alpha(x) \neq \emptyset$, $\alpha^{-1}(V)$ is a neighborhood of x.*

ii) *α is upper semicontinuous, in brief u.s.c., at x if for any neighborhood V of $\alpha(x)$, there exists a neighborhood W of x such that $\alpha(y) \subset V, \ \forall y \in W$.*

iii) *α is continuous at x if it is l.s.c. and u.s.c. at x.*

iv) *α is continuous (resp., l.s.c., u.s.c.) if it is continuous (resp., l.s.c., u.s.c.) at every $x \in \text{dom}\alpha$.*

v) *α has nonempty (resp., closed, convex) value if $\alpha(x)$ is nonempty (resp., closed, convex) for all $x \in E$.*

Proposition 2.11. [5] *Let I be a finite or countably infinite index set, $X = \prod_{i \in I} X_i$, where X_i is a nonempty convex compact subset of some finite dimensional Euclide space; for each $i \in I$, let $\varphi_i : X \to X_i$ be a set-valued map with convex values and l.s.c. Then there exists $\bar{x} \in X$ such that, for each i, either $\bar{x}_i \in \varphi_i(\bar{x})$ or $\varphi_i(\bar{x}) = \emptyset$.*

Let $\{E_i\}_{i \in I}$ be an arbitrary family of real topological vector spaces, $X_i \subset E_i$, be nonempty sets, $\alpha_i : X \to X_i$ be set-valued maps, for every $i \in I$, where $X := \prod_{i \in I} X_i$. Let $j \in I$.

Definition 2.12. [3] *We say that*

i) *α_j is KF on X if $x_j \notin \text{co}\alpha_j(x)$, for all $x = (x_i)_{i \in I} \in X$ and $\alpha_j^{-1}(y_j) \cap K$ is open in K for any nonempty compact subset $K \subset X$, and for any $y_j \in X_j$, where $\text{co}\alpha_j(x)$ denotes the convex hull of $\alpha_j(x)$.*

ii) *α_j is KF-majorized at $x \in X$ if there exist a KF map $\beta_j : X \to X_j$ and a neighborhood V of x such that $\alpha_j(y) \subset \beta_j(y)$, for all $y \in V$.*

iii) *α_j is KF-majorized if it is KF-majorized at every $x \in \text{dom}\alpha_j$.*

If α_j is KF on X then obviously it is KF-majorized.

Lemma 2.13. *Assume that* $\alpha_j : X \to X_j$ *is KF on* X. *Let* $D_i \subset X_i$ *be nonempty sets, for every* $i \in I$. *If* $\alpha_j(D) \subset D_j$, *then* $\beta_j : D \to D_j$ *is KF on* D, *where,* $D := \prod_{i\in I} D_i$ *and* $\beta_j := \alpha_j|_D$.

Proof. Let $x = (x_i)_{i\in I} \in D$. Since α_j is KF then $x_j \notin \mathrm{co}\alpha_j(x) = \mathrm{co}\beta_j(x)$. Moreover, for any $y_j \in D_j$ and for any nonempty compact subset $K \subset D$, we have

$$\beta_j^{-1}(y_j) \cap K = \alpha_j^{-1}(y_j) \cap D \cap K = \alpha_j^{-1}(y_j) \cap K.$$

Since α_j is KF and K is compact in X then from equality above, $\beta_j^{-1}(y_j) \cap K$ is open in K. The proof is complete. ∎

Proposition 2.14. *Assume that* $\{E_i\}_{i\in I}$ *is a family of real locally convex Hausdorff spaces. If for each* $i \in I$, *we have*
i) X_i *is nonempty, convex and compact,*
ii) α_i *is KF-majorized,*
iii) $\mathrm{dom}\alpha_i$ *is open in* X,
then there exists $\bar{x} \in X$ *such that* $\alpha_i(\bar{x}) = \emptyset$, *for all* $i \in I$.

Proof. The proof is immediate by Theorem 3 of [4]. ∎

Remark 2.15. We note that Proposition 2.14 is still valid if ii) and iii) are replaced by
ii') α_i is KF, for every $i \in I$.

Indeed, from the remark after Definition 2.12, we have ii')⇒ ii). Moreover, from the equality

$$\mathrm{dom}\alpha_i = \{x \in X | \alpha_i(x) \neq \emptyset\} = \bigcup_{y_i \in \alpha_i(X)} (\alpha_i^{-1}(y_i) \cap X),$$

and from the fact that $(\alpha_i^{-1}(y_i) \cap X)$ is open in X, since X is compact and α_i is KF, we have iii).

3. Nash Equilibria of Generalized Noncooperative Games and Applications

A noncooperative game $\Omega = (X_i, f_i)_{i\in I}$ consists of a set I of agents (or players) and for each $i \in I$, a set X_i of strategies and a loss function $f_i : X \to Y$ of i^{th} player, where $X := \prod_{i\in I} X_i$ and Y is a real topological vector space which is ordered by a convex cone $C \subsetneq Y$ with nonempty interior.

For $x = (x_j)_{j\in I} \in X, i \in I, y_i \in X_i$, we define $x^i := (x_j)_{j\in I\setminus\{i\}}$ and denote by (x^i, y_i) the vector x with i^{th} component is replaced by y_i.

Definition 3.1. [9] *We say that*

i) *a point* $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ *is a Nash equilibrium of the game* Ω (*with respect to* C) *if*

$$\forall i \in I, f_i(\bar{x}) \in \mathrm{Min}(\{f_i(\bar{x}^i, x_i)|x_i \in X_i\}|C).$$

ii) *a point* $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ *is a weakly Nash equilibrium of the game* Ω (*with respect to* C) *if*

$$\forall i \in I, f_i(\bar{x}) \in \mathrm{WMin}(\{f_i(\bar{x}^i, x_i)|x_i \in X_i\}|C).$$

For each $i \in I$, giving a constraint correspondence (set-valued map) $\alpha_i : X \to X_i$, we have a generalized noncooperative game $\Omega' = (X_i, \alpha_i, f_i)_{i\in I}$. We could see that, in practice, there are several games in which the strategy set of each player at each time is changed by the strategies of other players, for instant, chess play. This fact is described by constraint correspondences $\alpha_i : X \to X_i,\ i \in I$, where $\alpha_i(x)$ symbolizes the set of strategies that i^{th} player can choose if other players play strategies x^i. In chess play, for $i = 1, 2$, we denote by X_i the set of all the moves that i^{th} player can choose at any time, by x_i the move that i^{th} player chooses at a given time and by $\alpha_1(x_1, x_2)$ the set of all the moves that the 1^{st} player can choose if the 2^{nd} player chooses x_2. Then $|X_1| = 64^{16}$ while $|\alpha_1(x_1, x_2)|$ is always less than $|X_1|$, where $|A|$ denotes the number of elements of A.

Definition 3.2. *We say that*
1) *a point* $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ *is a Nash equilibrium of the generalized game* Ω' (*with respect to* C) *if for every* $i \in I$,
i) $\bar{x}_i \in \alpha_i(\bar{x})$,
ii) $f_i(\bar{x}) \in \mathrm{Min}(\{f_i(\bar{x}^i, x_i)|x_i \in \alpha_i(\bar{x})\}|C)$.
2) *a point* $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ *is a weakly Nash equilibrium of the generalized game* Ω' (*with respect to* C) *if for every* $i \in I$,
i) $\bar{x}_i \in \alpha_i(\bar{x})$,
ii) $f_i(\bar{x}) \in \mathrm{WMin}(\{f_i(\bar{x}^i, x_i)|x_i \in \alpha_i(\bar{x})\}|C)$.

Since the cone $\{0\} \cup \mathrm{int}C$ is pointed then by Lemma 2.5, 2. ii) is equivalent to
2. ii') $f_i(\bar{x}^i, x_i) \not\ll f_i(\bar{x}),\ \forall x_i \in \alpha_i(\bar{x})$.

Now, we consider the generalized game $\Omega' = (X_i, \alpha_i, f_i)_{i\in I}$. Assume that I is finite or countably infinite and for each $i \in I$, X_i is a subset of some finite dimensional Euclidean space.

Theorem 3.3. *If for each* $i \in I$,
i) X_i *is nonempty, convex and compact,*
ii) f_i *is continuous with respect to* C *and quasiconvex in the* i^{th} *argument,*
iii) α_i *is continuous and has nonempty, convex, closed values,*
then Ω' *has a weakly Nash equilibrium. In addition, if* C *satisfies condition* $(*)$ *and* f_i *is convex in the* i^{th} *argument, for every* $i \in I$, *then* Ω' *has a Nash equilibrium.*

To prove the theorem, we need the following lemmas.

Lemma 3.4. *Let $f, g : X \to Y$ be u.s.c. vector functions (with respect to C). For each $i \in I$, define $h_i : X \times X_i \to Y$ as follows*

$$h_i(x, y_i) := f(x^i, y_i) + g(x), \forall x \in X, y_i \in X_i.$$

Then h is also u.s.c..

Proof. Let $(x, y_i) \in X \times X_i$ and let $V \subset Y$ be an arbitrary neighborhood of the origin. Then there exists another neighborhood W of the origin such that

$$W + W \subset V.$$

Since f, g are u.s.c. then there exist neighborhoods U_1 of (x^i, y_i) and U_2 of x such that

$$z \in U_1 \Rightarrow f(z) \in f(x^i, y_i) + W - C$$

$$z \in U_2 \Rightarrow g(z) \in g(x) + W - C$$

By definition of product topology, there exist finite sets $J \subset I \setminus \{i\}, L \subset I$ such that

$$U_1 \supset \prod_{j \in I \setminus J \cup \{i\}} X_j \times \prod_{j \in J} U_{1j} \times U_{1i},$$

$$U_2 \supset \prod_{j \in I \setminus L} X_j \times \prod_{j \in L} U_{2j},$$

where, U_{1j} is a neighborhood of x_j, for all $j \in J$, U_{1i} is a neighborhood of y_i, U_{2j} is a neighborhood of x_j, for all $j \in L$. Put

$$U := \Big[(\prod_{j \in I \setminus J} X_j \times \prod_{j \in J} U_{1j}) \cap (\prod_{j \in I \setminus L} X_j \times \prod_{j \in L} U_{2j}) \Big] \times U_{1i}.$$

Then U is a neighborhood of (x, y_i) and

$$(x', y'_i) \in U \Rightarrow ({x'}^i, y'_i) \in U_1, x' \in U_2.$$

Hence, $h_i(x', y'_i) = f({x'}^i, y'_i) + g(x') \in (f(x^i, y_i) + W - C) + (g(x) + W - C) \subset (f(x^i, y_i) + g(x) + V) - C = (h_i(x, y_i) + V) - C$. The proof is complete. ∎

For each $i \in I, x \in X$, put $P_i(x) := \{y_i \in X_i | f_i(x^i, y_i) \ll f_i(x)\}$. Since f_i is quasiconvex in the i^{th} argument, then by Lemma 2.9, $P_i(x)$ is convex and $x_i \notin P_i(x)$. Denote by grafP_i the graph of P_i.

Lemma 3.5. grafP_i *is open in $X \times X_i$.*

Proof. Since f_i is continuous with respect to C, then by Lemma 3.4 we obtain the upper semicontinuity of the following function

$$h(x, y_i) := f_i(x^i, y_i) - f_i(x).$$

Let $(x, y_i) \in \text{graf}P_i$ be arbitrary. Then $f_i(x^i, y_i) - f_i(x) \in -\text{int}C$. By the upper semicontinuity of h, there exist neighborhoods U of x and V of y_i such that

$$x' \in U, y_i' \in V \Rightarrow f_i(x'^i, y_i') - f_i(x') \in -\text{int}C - C = -\text{int}C,$$

hence, $y_i' \in P_i(x')$, or, $(x', y_i') \in \text{graf}P_i$. Thus, (x, y_i) is an interior point of $\text{graf}P_i$. The proof is complete. ∎

Remark 3.6. From the proofs above, we see that Lemma 3.4 and Lemma 3.5 are still valid if X_i is an arbitrary topological space, for every $i \in I$.

Lemma 3.7. *For each $i \in I$, the set-valued map $x \in X \to \alpha_i(x) \cap P_i(x) \subset X_i$ is l.s.c..*

Proof. Let $x \in \text{dom}(\alpha_i \cap P_i)$ be arbitrary. Let U be an open subset of X_i such that $U \cap (\alpha_i \cap P_i)(x) \neq \emptyset$. Let $y_i \in U \cap (\alpha_i \cap P_i)(x)$. Since $\text{graf}P_i$ is open (Lemma 3.5) then there exist open neighborhoods V of x and W of y_i such that $V \times W \subset \text{graf}P_i$. Put $U' = U \cap W$. Since $V \times U' \subset V \times W \subset \text{graf}P_i$ then

$$V \subset P_i^{-1}(z_i), \forall z_i \in U'. \qquad (a)$$

On the other hand, $U' \cap \alpha_i(x) \neq \emptyset$ then by the continuity assumption of α_i, $\alpha_i^{-1}(U')$ is a neighborhood of x. Hence, $\alpha_i^{-1}(U') \cap V$ is a neighborhood of x. By (a), we have

$$\alpha_i^{-1}(U') \cap V \subset (\alpha_i \cap P_i)^{-1}(U').$$

Hence, $(\alpha_i \cap P_i)^{-1}(U)$ is a neighborhood of x. Thus, $(\alpha_i \cap P_i)$ is l.s.c. at x. The proof is complete. ∎

Lemma 3.8. *Let $G \subset X$ be open in X and let $A : G \to X_i, B : X \to X_i$ be set-valued maps such that A, B are l.s.c. and $B(x) \subset A(x)$, for all $x \in G$. Then the set-valued map $C : X \to X_i$ defined by*

$$C(x) := \begin{cases} A(x), & x \in G, \\ B(x), & x \notin G, \end{cases}$$

is l.s.c..

Proof. Since A is l.s.c. on G and G is open in X, then C is l.s.c. at every $x \in G \cap \text{dom}C$. Now, let $x \in \text{dom}C \setminus G$ and let U be an open subset in X_i such that $U \cap C(x) \neq \emptyset$. Since B is l.s.c. at x then there exists a neighborhood V of x such that

$$x' \in V \Rightarrow B(x') \cap U \neq \emptyset.$$

Since $B(x) \subset C(x)$, for all $x \in X$, then $C(x') \cap U \neq \emptyset$, for all $x' \in V$. Hence, C is l.s.c. at x. The proof is complete. ∎

Proof of Theorem 3.3. For each $i \in I$, put $G_i := \{x \in X |\ x_i \notin \alpha_i(x)\}$. Let $x \in G_i$ be arbitrary. Since α_i is u.s.c. with nonempty closed convex values, then there exist an open neighborhood U_x of x in X and an open convex V_x in X_i such that

$$x' \in U_x \Rightarrow x'_i \notin V_x,\ \alpha_i(x') \subset V_x. \tag{b}$$

Hence, G_i is open in X. Since G_i is metrizable then it is paracompact. Hence, the open covering $(U_x)_{x\in G_i}$ of G_i has a locally finite closed refinement $(W_j)_{j\in J}$, i.e.,

• $(W_j)_{j\in J}$ is a covering of G_i and W_j is closed in G_i for every $j \in J$,

• for each $x \in G_i$, there exists a neighborhood intersecting only finitely many W_j,

• there exists a mapping $\pi : J \to G_i$ such that $W_j \subset U_{\pi(j)}$, for every $j \in J$.

Define the set-valued map $\delta_i : G_i \to X_i$ by

$$\delta_i(x) := \bigcap_{x\in W_j} V_{\pi(j)}.$$

Then δ_i has the following properties:

• *δ_i is l.s.c.* Let $x \in G_i$ be arbitrary and let $V \subset X_i$ be open such that $\delta_i(x)\cap V \neq \emptyset$. Then there exists a neighborhood W of x intersecting only finitely many $W_j, j \in J'$, where J' is some finite index set. Since W_j is closed in G_i then without loss of generality, we may assume $x \in W_j$, for all $j \in J'$. Then for each $x' \in W$ one has $\{j|x' \in W_j\} \subset J'$. Then by the definition of δ_i, we get $\delta_i(x) \subset \delta_i(x')$. Hence, $\delta_i(x') \cap V \neq \emptyset$, for all $x' \in W$, i.e., ${\delta_i}^{-1}(V)$ is a neighborhood of x. Thus, δ_i is l.s.c. at x.

• *For every $x \in G_i$, $\delta_i(x)$ is convex, $x_i \notin \delta_i(x)$ and $\alpha_i(x) \subset \delta_i(x)$.* Let $x \in G_i$. As above, one has $\delta_i(x) = \bigcap_{j\in J'} V_{\pi(j)}$. For each $j \in J'$, $V_{\pi(j)}$ is convex then $\delta_i(x)$ is convex. On the other hand, $x \in W_j \subset U_{\pi(j)}$ then from (b), one has

$$x_i \notin V_{\pi(j)} \supset \alpha_i(x).$$

Now, consider the set-valued map $\varphi_i : X \to X_i$ defined by

$$\varphi_i(x) = \begin{cases} \delta_i(x), & x \in G_i,\\ \alpha_i(x) \cap P_i(x), & x \notin G_i. \end{cases}$$

From properties of δ_i, α_i, P_i, by applying Lemmas 3.7, 3.8, then φ_i is l.s.c. with convex values. Hence, by Proposition 2.11, there exists $\bar{x} \in X$ such that

$$\varphi_i(\bar{x}) = \emptyset \text{ or } \bar{x}_i \in \varphi_i(\bar{x}),\ \forall i \in I.$$

Since $x_i \notin \varphi_i(x)$, for all $x \in X, i \in I$, then by the definition of φ_i and the relationship above, one has for every $i \in I$, $\bar{x}_i \in \alpha_i(\bar{x})$,

$$f_i(\bar{x}^i, x_i) \not\ll f_i(\bar{x}),\ \forall x_i \in \alpha_i(\bar{x}).$$

Thus, $\bar{x}$ is a weakly Nash equilibrium of the generalized game Ω'.

In addition, assume that C satisfies condition (*) and f_i is convex in the i^{th} argument. Then there exists a convex cone $K \subsetneq Y$ such that $C \setminus \{0\} \subset$

int$K$. By Lemma 2.8, f_i is continuous with respect to K and quasiconvex in the i^{th} argument with respect to K. Applying the results above, Ω' has a weakly Nash equilibrium $\bar{x}$ with respect to K, hence by Lemma 2.5. *iii*), $\bar{x}$ is a Nash equilibrium of Ω' with respect to C. The proof is complete. ■

By taking $\alpha_i(x) = X_i$, for all $x \in X, i \in I$, we have immediately

Corollary 3.9. *If for each $i \in I$,*
i) *X_i is nonempty, convex and compact,*
ii) *f_i is continuous with respect to C and quasiconvex in the i^{th} argument,*
then the game $\Omega = (X_i, f_i)_{i\in I}$ has a weakly Nash equilibrium. In addition, if C satisfies condition $(*)$ and f_i is convex in the i^{th} argument, for every $i \in I$, then Ω has a Nash equilibrium .

When I has only one element, we obtain

Corollary 3.10. *Assume E is a finite dimensional Euclide space, $X \subseteq E$ is a nonempty and convex subset and $f : X \to Y$ is a vector function. If*
i) *X is compact,*
ii) *f is continuous with respect to C and quasiconvex,*
then f attains a weak minimum on X. In addition, if C satisfies condition $(*)$ and f is convex, then f attains a Pareto-minimum on X.

Now we consider the generalized game $\Omega' = (X_i, \alpha_i, f_i)_{i\in I}$, where I is an arbitrary index set, for each $i \in I$, X_i is a subset of some real locally convex Hausdorff space E_i. Let $x \in X, i \in I$. Denote $\bar{P}_i(x) := \{y_i \in X_i | f_i(x^i, y_i) \preceq f_i(x)\}$, $P_i(x) := \{y_i \in X_i | f_i(x^i, y_i) \ll f_i(x)\}$.

Theorem 3.11. *If for each $i \in I$,*
i) *X_i is nonempty and convex,*
ii) *f_i is continuous with respect to C and quasiconvex in the i^{th} argument,*
iii) *$(\alpha_i)^{-1}(y_i) \cap K$ is open in K, for any $y_i \in X_i$ and for any nonempty, convex, compact subset $K \subseteq X$,*
furthermore,
iv) *there exists $\bar{x} \in X$ such that for each $i \in I$,*
- *$\bar{P}_i(\bar{x})$ is compact,*
- *$P_i(x) \subset \bar{P}_i(\bar{x}), \forall x \in \prod_{j\in I} \bar{P}_j(\bar{x})$,*
- *$x_i \in \alpha_i(x), \forall x \in \prod_{j\in I} \bar{P}_j(\bar{x})$,*

then Ω' has a weakly Nash equilibrium.

Proof. For each $i \in I$, we define a set-valued map $Q_i : X \to X_i$ by

$$Q_i(x) := \alpha_i(x) \cap P_i(x), \forall x \in X.$$

By Lemma 2.9, $P_i(x)$ is convex and $x_i \notin P_i(x)$, then from the definition of Q_i, one has $x_i \notin \text{co}Q_i(x)$, for all $x \in X, i \in I$. Now, let $K \subset X$ be an arbitrary nonempty compact subset and let $y_i \in X_i$. We have

$$Q_i^{-1}(y_i) = \alpha_i^{-1}(y_i) \cap P_i^{-1}(y_i).$$

Moreover, $P_i^{-1}(y_i)$ is open in X since grafP_i is open in $X \times X_i$ (Remark 3.6). This and *iii*) imply that $Q_i^{-1}(y_i) \cap K$ is open in K. Hence, Q_i is KF on X, for all $i \in I$.

Next, we shall prove for each $i \in I$,

$$Q_i(\prod_{j\in I} \bar{P}_j(\bar{x})) \subset \bar{P}_i(\bar{x}). \tag{a}$$

Indeed, from the definition of Q_i and from the second condition of iv), for all $x \in \prod_{j\in I} \bar{P}_j(\bar{x})$, one has

$$Q_i(x) \subset P_i(x) \subset \bar{P}_i(\bar{x}),$$

this implies (a). Then by Lemma 2.13, Q_i is KF on $\prod_{j\in I} \bar{P}_j(\bar{x})$. The sets $\bar{P}_j(\bar{x})$ are compact and by Lemma 2.9, are convex and nonempty. Hence by Proposition 2.14 and Remark 2.15, there exists $x^* \in \prod_{j\in I} \bar{P}_j(\bar{x})$ such that

$$Q_i(x^*) = \emptyset, \forall i \in I.$$

Then for every $i \in I$, one has

$$x^*{}_i \in \alpha_i(x^*),$$

$$f_i(x^{*i}, x_i) \not\ll f_i(x^*), \forall x_i \in \alpha_i(x^*).$$

Hence x^* is a weakly Nash equilibrium of Ω'. The proof is complete. ■

By taking $\alpha_i(x) := X_i$, for all $x \in X, i \in I$, one has

Corollary 3.12. *Assume that the game $\Omega = (X_i, f_i)_{i\in I}$ satisfies, for each $i \in I$,*
i) *X_i is nonempty and convex,*
ii) *f_i is continuous with respect to C; quasiconvex in the i^{th} variable, furthermore,*
iii) *there exists $\bar{x} \in X$ such that for each $i \in I$,*
- *$\bar{P}_i(\bar{x})$ is compact,*
- *$P_i(x) \subset \bar{P}_i(\bar{x}), \forall x \in \prod_{j\in I} \bar{P}_j(\bar{x})$.*

Then Ω has a weakly Nash equilibrium.

When I has only one element, we get immediately the following corollary which extends the corresponding result of Tan and Tinh [9].

Corollary 3.13. *Assume that E is a locally convex Hausdorff space, $X \subseteq E$ is a nonempty convex subset and $f : X \to Y$ is a vector function. If*

i) *f is continuous with respect to C and quasiconvex,*
ii) *there exists $\bar{x} \in X$ such that the set $\{x \in X | f(x) \preceq f(\bar{x})\}$ is compact,*
then f attains a weak minimum on X.

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