**MTHSC 851 (Abstract Algebra)**
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**HW 8**
**Due Thursday April 2nd, 2009**

(1) Prove that there cannot be a nilpotent group $N$ generated by two elements with the property that every nilpotent group generated by two elements is a homomorphic image of $N$ (i.e., free objects do not always exist in the category $\mathfrak{C}$ of nilpotent groups).

(2) Let $G$ be a group with $X \subseteq G$, and let $A$ be the normal subgroup generated by $X$, i.e.,

$$A = \bigcap \{N \lhd G \colon X \subseteq N\} .$$

Let $Y = \{gxg^{-1} \mid x \in X,\ g \in G\}$. Show that $A = \langle Y \rangle$.

(3) Show that the Klein 4-group $V$ has presentation $\langle a, b \mid a^2 = b^2 = (ab)^2 = 1\rangle$.

(4) Show that the quaternion group $Q_2 = \{\pm 1, \pm i, \pm j, \pm k\}$ has presentation $\langle a, b \mid a^4 = 1,\ a^2 = b^2,\ ab = ba^3\rangle$.

(5) (a) Determine the group with presentation $\langle a, b \mid a^2 = 1,\ b^3 = 1,\ ab = ba\rangle$.
(b) If $G$ and $H$ each have more than one element, show that $G * H$ is an infinite group with center $\langle e \rangle$.
(c) Determine the group with presentation $\langle a, b \mid a^2 = 1,\ b^3 = 1\rangle$.

(6) Define the *generalized quaternion group* $Q_n$ by the presentation

$$Q_n = \langle a, b \mid a^{2n} = 1,\ b^2 = a^n,\ ab = ba^{-1}\rangle , \qquad \text{for } n \geq 1 .$$

(a) Show that $|Q_n| = 4n$.
(b) Show that $Q_1$ is cyclic and $Q_2$ is the quaternion group.
(c) Show that $Q_3$ is not isomorphic with either the dihedral group $D_6$ or the alternating group $A_4$.

(7) Let $G = \langle a, b \mid a^4 = b^3 = 1,\ ab = ba^3\rangle$. Show that $G$ is cyclic of order 6.

(8) Consider the Cartesian product $H = \mathbb{Z}_2 \times \mathbb{Z}_n$ (as a *set*). Define a binary operation on $H$ to be

$$(\bar{i}, \bar{j}) \cdot (\bar{k}, \bar{\ell}) = (\bar{i} + \bar{k}, (-1)^k \bar{j} + \bar{\ell}) .$$

(a) Show that $H$ is a group under this operation, and determine its order.
(b) Let $G = \langle a, b \mid a^n = 1,\ b^2 = 1,\ abab^{-1} = 1\rangle$. Show that $|G| \leq 2n$.
(c) Show that $H \cong G$.

(9) If $H$ and $K$ are subgroup of $G$, with $K \lhd G$, $K \cap H = 1$, and $KH = G$, then $G$ is called a semidirect product (or split extension) of $K$ by $H$.
(a) If $\sigma = (12) \in S_n$, $n \geq 2$, show that $S_n$ is a semidirect product of $A_n$ by $\langle \sigma \rangle$.
(b) Show that the dihedral group $D_n = \langle a, b \mid a^n = b^2 = 1,\ b^{-1}ab = a^{-1}\rangle$ is a semidirect product of $A = \langle a \rangle$ by $B = \langle b \rangle$.
(c) Show that the quaternion group $Q_2$ cannot be expressed as a semidirect product of two non-trivial subgroups.

(10) Suppose $K$ and $H$ are groups and $\phi \colon H \to \mathrm{Aut}(K)$ is a homomorphism. Let $G$ be the Cartesian product $K \times H$ as a set, but with binary operation $(x, y)(u, v) = (x \cdot \phi(y)u, yv)$. Show that $G$ is a group; denote it by $G = K \rtimes_\phi H$, and call it the *external semidirect product* of $K$ by $H$ relative to $\phi$. Show that $K_1 = \{(x, 1) \colon x \in K\} \lhd G$, $H_1 = \{(1, y) \colon y \in H\} \leq G$, and that $G$ is the semidirect product of $K_1$ by $H_1$ (as in the previous exercise).