Serre Dimension of Monoid Algebras

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Abstract

Let R be a commutative Noetherian ring of dimension d, M a commutative cancellative torsion-free monoid of rank r and P a finitely generated projective $R[M]$-module of rank t.

(1) Assume M is Φ-simplicial seminormal. (i) If $M \in \mathcal{C}(\Phi)$, then *Serre dim* $R[M] \leq d$. (ii) If $r \leq 3$, then *Serre dim* $R[int(M)] \leq d$.

(2) If $M \subset \mathbb{Z}_+^2$ is a normal monoid of rank 2, then *Serre dim* $R[M] \leq d$.

(3) Assume M is c-divisible, $d = 1$ and $t \geq 3$. Then $P \cong \wedge^t P \oplus R[M]^{t-1}$.

(4) Assume R is a unibranched affine algebra over an algebraically closed field and $d = 1$. Then $P \cong \wedge^t P \oplus R[M]^{t-1}$.

1 Introduction

Throughout rings are commutative Noetherian with 1*; projective modules are finitely generated and of constant rank; monoids are commutative cancellative torsion-free;* $\mathbb{Z}_+$ *denote the additive monoid of non-negative integers.*

Let A be a ring and P a projective A-module. An element $p \in P$ is called *unimodular*, if there exists $\phi \in \operatorname{Hom}(P, A)$ such that $\phi(p) = 1$. We say Serre dimension of A (denoted as *Serre dim* A) is $\leq t$, if every projective A-module of rank $\geq t+1$ has a unimodular element. Serre dimension of A measures the surjective stabilization of the Grothendieck group $K_0(A)$. Serre's problem on the freeness of projective $k[X_1, \ldots, X_n]$-modules, k a field, is equivalent to *Serre dim* $k[X_1, \ldots, X_n] = 0$.

After the solution of Serre's problem by Quillen [16] and Suslin [21], many people worked on surjective stabilization of polynomial extension of a ring. Serre [20] proved *Serre dim* $A \leq \dim A$, Plumstead [14] proved *Serre dim* $A[X] \leq \dim A$, Bhatwadekar-Roy [4] proved *Serre dim* $A[X_1, \ldots, X_n] \leq \dim A$ and Bhatwadekar-Lindel-Rao [3] proved *Serre dim* $A[X_1, \ldots, X_n, Y_1^{\pm 1}, \ldots, Y_m^{\pm 1}] \leq \dim A$.

Anderson conjectured an analogue of Quillen-Suslin theorem for monoid algebras over a field which was answered by Gubeladze [8] (see 1.1) as follows.

Theorem 1.1 *Let k be a field and M a monoid. Then M is seminormal if and only if all projective $k[M]$-modules are free.*

Gubeladze [11] asked the following

Question 1.2 *Let $M \subset \mathbb{Z}_+^r$ be a monoid of rank r with $M \subset \mathbb{Z}_+^r$ an integral extension. Let R be a ring of dimension d. Is Serre dim $R[M] \leq d$?*

We answer Question 1.2 for some class of monoids. Recall that a finitely generated monoid M of rank r is called Φ-simplicial if M can be embedded in $\mathbb{Z}_+^r$ and the extension $M \subset \mathbb{Z}_+^r$ is integral (see [10]). A Φ-simplicial monoid is commutative, cancellative and torsion free.

Definition 1.3 Let $\mathcal{C}(\Phi)$ denote the class of seminormal Φ-simplicial monoids $M \subset \mathbb{Z}_+^r = \{t_1^{s_1} \dots t_r^{s_r} \,|\, s_i \geq 0\}$ of rank r such that $M_m = M \cap \{t_1^{s_1} \dots t_m^{s_m} \,|\, s_i \geq 0\}$ for $1 \leq m \leq r$ satisfies the following: Given a positive integer c, there exist integers $c_i > c$ for $i = 1, \dots, m-1$ such that for any ring R, the automorphism $\eta \in Aut_{R[t_m]}(R[t_1, \dots, t_m])$ defined by $\eta(t_i) = t_i + t_m^{c_i}$ for $i = 1, \dots, m-1$, restricts to an R-automorphism of $R[M_m]$.

Note that M_m $(1 \leq m \leq r)$ are in $\mathcal{C}(\Phi)$.

The following result (3.4, 3.8) answers Question 1.2 for monoids in $\mathcal{C}(\Phi)$.

Theorem 1.4 *Let M be a seminormal Φ-simplicial monoid and R a ring of dimension d.*

(1) *If $M \in \mathcal{C}(\Phi)$, then Serre dim $R[M] \leq d$.*

(2) *Assume $rank(M) \leq 3$. Then Serre dim $R[int(M)] \leq d$, where $int(M) = int(\mathbb{R}_+M) \cap \mathbb{Z}_+^3$ and $int(\mathbb{R}_+M)$ is the interior (w.r.t. Euclidean topology) of the cone $\mathbb{R}_+M \subset \mathbb{R}^3$.*

The following result (3.6) follows from (1.4(1)). When R is a field, the result is due to Anderson [1].

Theorem 1.5 *Let R be a ring of dimension d and $M \subset \mathbb{Z}_+^2$ a normal monoid of rank 2. Then Serre dim $R[M] \leq d$.*

The following result answer Question 1.2 partially for 1-dimensional rings (see 3.13, 3.16). The techniques of Kang [12], Roy [17] and Gubeladze's [9] are used to prove the following result. Recall that a monoid M is called *c-divisible* for some integer $c > 1$ if $cX = m$ has a solution in M for all $m \in M$. All c-divisible monoids are seminormal. Further a ring R is called *unibranched* if for any $\mathfrak{p} \in \operatorname{Spec} R$ containing C, there is a unique $\mathfrak{q} \in \operatorname{Spec} \overline{R}$ such that $\mathfrak{q} \cap R = \mathfrak{p}$, where $\overline{R}$ is the integral closure of R and C the conductor ideal of $R \subset \overline{R}$.

Theorem 1.6 *Let R be a ring of dimension 1, M a monoid and P a projective $R[M]$-module of rank r.*

(i) If M is c-divisible and $r \geq 3$, then $P \cong \wedge^r P \oplus R[M]^{r-1}$.

(ii) If R is a unibranched affine algebra over an algebraically closed field, then $P \cong \wedge^r P \oplus R[M]^{r-1}$.

If R is a 1-dimensional anodal ring with finite seminormalization, then (1.6(ii)) is due to Sarwar ([18], Theorem 1.2). If k is an algebraically closed field of characteristic 2, then node $k[X,Y]/(X^2 - Y^2 - Y^3)$ is not anodal but is unibranched by Kang ([12], Example 2).

At the end, we give some applications to minimum number of generators of projective modules.

2 Preliminaries

Let A be a ring and Q an A-module. We say $p \in Q$ is unimodular if the order ideal $O_Q(p) = \{\phi(p) \mid \phi \in Q^*\}$ equals A. The set of all unimodular elements in Q is denoted by $\mathrm{Um}(Q)$. We write $\mathrm{E}_n(A)$ for the group generated by set of all $n \times n$ elementary matrices over A and $\mathrm{Um}_n(A)$ for $\mathrm{Um}(A^n)$. We denote by $\mathrm{Aut}_A(Q)$, the group of all A-automorphisms of Q.

For an ideal J of A, we denote by $\mathrm{E}(A \oplus Q, J)$, the subgroup of $\mathrm{Aut}_A(A \oplus Q)$ generated by all the automorphisms $\Delta_{a\phi} = \left(\begin{smallmatrix} 1 & a\phi \\ 0 & id_Q \end{smallmatrix}\right)$ and $\Gamma_q = \left(\begin{smallmatrix} 1 & 0 \\ q & id_Q \end{smallmatrix}\right)$ with $a \in J$, $\phi \in Q^*$ and $q \in Q$. Further, we shall write $\mathrm{E}(A \oplus Q)$ for $\mathrm{E}(A \oplus Q, A)$. We denote by $\mathrm{Um}(A \oplus Q, J)$ the set of all $(a, q) \in \mathrm{Um}(A \oplus Q)$ with $a \in 1 + J$ and $q \in JQ$.

We state some results for later use.

Proposition 2.1 *(Lindel [13], 1.1) Let A be a ring and Q an A-module. Let Q_s be free of rank r for some $s \in A$. Then there exist $p_1, \ldots, p_r \in Q$, $\phi_1, \ldots, \phi_r \in Q^*$ and $t \geq 1$ such that following holds:*

(i) $0 :_A s'A = 0 :_A {s'}^2 A$, where $s' = s^t$.

(ii) $s'Q \subset F$ and $s'Q^* \subset G$, where $F = \sum_{i=1}^r Ap_i \subset Q$ and $G = \sum_{i=1}^r A\phi_i \subset Q^*$.

(iii) the matrix $(\phi_i(p_j))_{1 \leq i,j \leq r} =$ diagonal $(s', \ldots, s')$. We say F and G are s'-dual submodules of Q and Q^* respectively.

Proposition 2.2 *(Lindel [13], 1.2, 1.3) Let A be a ring and Q an A-module. Assume Q_s is free of rank r for some $s \in A$. Let F and G be s-dual submodules of Q and Q^* respectively. Then*

(i) for $p \in Q$, there exists $q \in F$ such that $\mathrm{ht}\,(O_Q(p + sq)A_s) \geq r$.

(ii) If Q is projective A-module and $\overline{p} \in \mathrm{Um}(Q/sQ)$, then there exists $q \in F$ such that $\mathrm{ht}\,(O_Q(p + sq)) \geq r$.

Proposition 2.3 *(Lindel [13], 1.6) Let Q be a module over a positively graded ring $A = \oplus_{i \geq 0} A_i$ and Q_s be free for some $s \in R = A_0$. Let $T \subseteq A$ be a multiplicatively closed set of homogeneous elements. Let $p \in Q$ be such that $p_{T(1+sR)} \in \mathrm{Um}(Q_{T(1+sR)})$ and $s \in rad(O_Q(p) + A_+)$, where $A_+ = \oplus_{i \geq 1} A_i$. Then there exists $p' \in p + sA_+Q$ such that $p'_T \in \mathrm{Um}(Q_T)$.*

Proposition 2.4 *(Lindel [13], 1.8) Under the assumptions of (2.3), let $p \in Q$ be such that $O_Q(p) + sA_+ = A$ and $A/O_Q(p)$ is an integral extension of $R/(R \cap O_Q(p))$. Then there exists $p' \in \mathrm{Um}(Q)$ with $p' - p \in sA_+Q$.*

The following result is due to Amit Roy ([17], Proposition 3.4).

Proposition 2.5 *Let A, B be two rings with $f : A \to B$ a ring homomorphism. Let $s \in A$ be non-zerodivisor such that $f(s)$ is a non-zerodivisor in B. Assume that we have the following cartesian square.*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A_s & \xrightarrow{f_s} & B_{f(s)} \end{array}$$

Further assume that $\mathrm{SL}_r(B_{f(s)}) = \mathrm{E}_r(B_{f(s)})$ *for some* $r > 0$. *Let* P *and* Q *be two projective* A*-modules of rank* r *such that (i)* $\wedge^r P \cong \wedge^r Q$, *(ii)* P_s *and* Q_s *are free over* A_s, *(iii)* $P \otimes_A B \cong Q \otimes_A B$ *and* $Q \otimes_A B$ *has a unimodular element. Then* $P \cong Q$.

Definition 2.6 (see [10], Section 6) Let R be a ring and M a Φ-simplicial monoid of rank r. Fix an integral extension $M \hookrightarrow \mathbb{Z}_+^r$. Let $\{t_1, \ldots, t_r\}$ be a free basis of $\mathbb{Z}_+^r$. Then M can be thought of as a monoid consisting of monomials in $t_1, \ldots, t_r$.

For $x = t_1^{a_1} \ldots t_r^{a_r}$ and $y = t_1^{b_1} \ldots t_r^{b_r}$ in $\mathbb{Z}_+^r$, define x is *lower* than y if $a_i < b_i$ for some i and $a_j = b_j$ for $j > i$. In particular, t_i is lower than t_j if and only if $i < j$.

For $f \in R[M]$, define the *highest member* $H(f)$ of f as am, where $f = am + a_1 m_1 + \ldots + a_k m_k$ with $m, m_i \in M$, $a \in R \setminus \{0\}, a_i \in R$ and each m_i is strictly lower than m for $1 \leq i \leq k$.

An element $f \in R[\mathbb{Z}_+^r]$ is called *monic* if $H(f) = a t_r^s$, where $a \in U(R)$ (:=units of R) and $s > 0$. An element $f \in R[M]$ is said to be *monic* if f is monic in $R[\mathbb{Z}_+^r]$ via the embedding $R[M] \hookrightarrow R[\mathbb{Z}_+^r]$.

Define M_0 to be the submonoid $\{t_1^{s_1} \ldots t_{r-1}^{s_{r-1}} \,|\, s_i \geq 0\} \cap M$ of M. Clearly M_0 is finitely generated as M is finitely generated. Also $M_0 \hookrightarrow \mathbb{Z}_+^{r-1}$ is integral. Hence M_0 is Φ-simplicial. Further, if M is seminormal, then M_0 is seminormal.

Grade $R[M]$ as $R[M] = R[M_0] \oplus A_1 \oplus A_2 \oplus \ldots$, where A_i is the $R[M_0]$-module generated by the monomials $t_1^{s_1} \ldots t_{r-1}^{s_{r-1}} t_r^i \in M$. For an ideal I in $R[M]$, define its leading coefficient ideal $\lambda(I)$ as $\{a \in R \,|\, \exists f \in I \text{ with } H(f) = am \text{ for some } m \in M\}$. $\blacksquare$

Lemma 2.7 *([10], Lemma 6.5) Let* R *be a ring and* $M \subset \mathbb{Z}_+^r$ *a* Φ*-simplicial monoid. If* $I \subseteq R[M]$ *is an ideal, then* $\mathrm{ht}\,(\lambda(I)) \geq \mathrm{ht}\,(I)$, *where* $\lambda(I)$ *is defined in* (2.6).

3 Main Theorem

This section contains main results stated in the introduction. We also give some examples of monoids in $C(\Phi)$.

3.1 Over $C(\Phi)$ class of monoids

Lemma 3.1 *Let* R *be a ring and* $M \subset \mathbb{Z}_+^r$ *a monoid in* $\mathcal{C}(\Phi)$ *of rank* r. *Let* $f \in R[M] \subset R[\mathbb{Z}_+^r] = R[t_1, \ldots, t_r]$ *with* $H(f) = u t_1^{s_1} \ldots t_r^{s_r}$ *for some unit* $u \in R$. *Then there exist* $\eta \in Aut_R(R[M])$ *such that* $\eta(f)$ *is a monic polynomial in* t_r.

Proof By a property of $C(\Phi)$, choose large $c_1, \ldots, c_{r-1}$ such that $\eta \in Aut_{R[t_r]} R[t_1 \ldots, t_r]$ defined by $\eta(t_i) = t_i + t_r^{c_i}$ for $i = 1, \ldots, r-1$, restricts to an automorphism of $R[M]$. Further, $\eta(f)$ is a monic polynomial in t_r. $\blacksquare$

Lemma 3.2 *Let* R *be a ring of dimension* d *and* $M \subset \mathbb{Z}_+^r$ *a monoid in* $\mathcal{C}(\Phi)$ *of rank* r. *Let* P *be a projective* $R[M]$*-module of rank* $> d$. *Write* $R[M] = R[M_0] \oplus A_1 \oplus A_2 \ldots$, *as defined in* (2.6). *Let*

$A_+ = A_1 \oplus A_2 \oplus \dots$ be an ideal of $R[M]$. Assume that P_s is free for some $s \in R$ and P/sA_+P has a unimodular element. Then the natural map $\mathrm{Um}(P) \to \mathrm{Um}(P/sA_+P)$ is surjective. In particular, P has a unimodular element.

Proof Write $A = R[M]$. Since every unimodular element of P/sA_+P can be lifted to a unimodular element of P_{1+sA_+}, if s is nilpotent, then elements of $1+sA_+$ are units in A and we are done. Therefore, assume that s is not nilpotent.

Let $p \in P$ be such that $\overline{p} \in \mathrm{Um}(P/sA_+P)$. Then $O_P(p) + sA_+ = A$. Hence $O_P(p)$ contains an element of $1+sA_+$. Choose $g \in A_+$ such that $1+sg \in O_P(p)$. Applying (2.2) with sg in place of s, we get $q \in F \subset P$ such that $\mathrm{ht}\,(O_P(p+sgq)) > d$. Since $p+sgq$ is a lift of $\overline{p}$, replacing p by $p+sgq$, we may assume that $\mathrm{ht}\,(O_P(p)) > d$. By (2.7), we get $\mathrm{ht}\,(\lambda(O_P(p))) \geq \mathrm{ht}\,(O_P(p)) > d$. Since $\lambda(O_P(p))$ is an ideal of R, we get $1 \in \lambda(O_P(p))$. Hence there exists $f \in O_P(p)$ such that the coefficient of $H(f)$ (highest member of f) is a unit.

Suppose $H(f) = ut_1^{s_1} \dots t_r^{s_r}$ with u a unit in R. Since $M \in \mathcal{C}(\Phi)$, by (3.1), there exists $\alpha \in \mathrm{Aut}_R(R[M])$ such that $\alpha(f)$ is monic in t_r. Thus we may assume that $O_P(p)$ contains a monic polynomial in t_r. Hence $A/O_P(p)$ is an integral extension of $R[M_0]/(O_P(p) \cap R[M_0])$ and $\overline{p} \in \mathrm{Um}(P/sA_+P)$. By (2.4), there exists $p' \in \mathrm{Um}(P)$ such that $p' - p \in sA_+P$. This means $p' \in \mathrm{Um}(P)$ is a lift of $\overline{p}$. This proves the result. ■

Remark 3.3 In (3.2), we do not need the monoid M to be seminormal. ■

We prove (1.4(1)).

Theorem 3.4 *Let R be a ring of dimension d and M a monoid in $\mathcal{C}(\Phi)$ of rank r. If P is a projective $R[M]$-module of rank $r' \geq d+1$, then P has a unimodular element. In other words, Serre dim $R[M] \leq d$.*

Proof We can assume that the ring is reduced with connected spectrum. If $d = 0$, then R is a field. Since M is seminormal, projective $R[M]$-modules are free, by (1.1). If $r = 0$, then $M = 0$ and we are done by Serre [20]. Assume $d, r \geq 1$ and use induction on d and r simultaneously.

If S is the set of all non-zerodivisor of R, then $\dim S^{-1}R = 0$ and so $S^{-1}P$ is free $S^{-1}R[M]$-module ($d = 0$ case). Choose $s \in S$ such that P_s is free. Consider the ring $R[M]/(sR[M]) = (R/sR)[M]$. Since $\dim R/sR = d-1$, by induction on d, $\mathrm{Um}(P/sP)$ is non-empty.

Write $R[M] = R[M_0] \oplus A_1 \oplus A_2 \dots$, as defined in (2.6). It is easy to see that $M_0 \in \mathcal{C}(\Phi)$. Let $A_+ = A_1 \oplus A_2 \oplus \dots$. Since $R[M]/A_+ = R[M_0]$, by induction on r, $\mathrm{Um}(P/A_+P)$ is non-empty. Write $A = R[M]$ and consider the following fiber product diagram

$$\begin{tikzcd} A/(sA \cap A_+) \arrow[r] \arrow[d] & A/sA \arrow[d] \\ A/A_+ \arrow[r] & A/(s, A_+) \end{tikzcd}$$

If $B = R/sR$, then $A/(s, A_+) = B[M_0]$. Let $u \in \mathrm{Um}(P/A_+P)$ and $v \in \mathrm{Um}(P/sP)$. Let $\overline{u}$ and $\overline{v}$ denote the images of u and v in $P/(s, A_+)P$. Write $P/(s, A_+)P = B[M_0] \oplus P_0$, where P_0 is some projective $B[M_0]$-module of rank $= r' - 1$. Note that $dim(B) = d - 1$ and $\overline{u}, \overline{v}$ are two unimodular elements in $B[M_0] \oplus P_0$.

Case 1. Assume $rank(P_0) \geq \max\{2, d\}$. Then by ([6], Theorem 4.5), there exists $\sigma \in \mathrm{E}(B[M_0] \oplus P_0)$ such that $\sigma(\overline{u}) = \overline{v}$. Lift σ to an element $\sigma_1 \in \mathrm{E}(P/A_+P)$ and write $\sigma_1(u) = u_1 \in \mathrm{Um}(P/A_+P)$. Then images of u_1 and v are same in $P/(s, A_+)P$. Patching u_1 and v over $P/(s, A_+)P$ in the above fiber product diagram, we get an element $p \in \mathrm{Um}(P/(sA \cap A_+)P)$.

Note $sA \cap A_+ = sA_+$. We have P_s is free and P/sA_+P has a unimodular element. Use (3.2), to conclude that P has a unimodular element.

Case 2. Now we consider the remaining case, namely $d = 1$ and $rank(P) = 2$. Since $B = R/sR$ is 0 dimensional, projective modules over $B[M_0]$ and $B[M]$ are free, by (1.1). In particular, P/sP and $P/(s, A_+)P$ are free modules of rank 2 over the rings $B[M]$ and $B[M_0]$ respectively. Consider the same fiber product diagram as above.

Since any two unimodular elements in $\mathrm{Um}_2(B[M_0])$ are connected by an element of $\mathrm{GL}_2(B[M_0])$. Further $B[M_0]$ is a subring of $B[M] = A/sA$. Hence the natural map $\mathrm{GL}_2(B[M]) \to \mathrm{GL}_2(B[M_0])$ is surjective. Hence any automorphism of $P/(s, A_+)P$ can be lifted to an automorphism of P/sP. By same argument as above, patching unimodular elements of P/sP and P/A_+P, we get a unimodular element in $P/(sA \cap A_+)P$. Since $sA \cap A_+ = sA_+$ and P/sA_+P has a unimodular element, by (3.2), P has a unimodular element. This completes the proof. ■

Example 3.5 (1) If M is a Φ-simplicial normal monoid of rank 2, then $M \in \mathcal{C}(\Phi)$. To see this, by ([10], Lemma 1.3), $M \cong (\alpha_1, \alpha_2) \cap \mathbb{Z}_+^2$, where $\alpha_1 = (a, b)$ and $\alpha_2 = (0, c)$ and (α_1, α_2) is the group generated by α_1 and α_2. It is easy to see that $M \cong ((1, a_1), (0, a_2)) \cap \mathbb{Z}_+^2$, where $\gcd(b, c) = g$ and $a_1 = b/g$, $a_2 = c/g$. Hence $M \in \mathcal{C}(\Phi)$.

(2) If $M \subset \mathbb{Z}_+^2$ is a finitely generated rank 2 normal monoid, then it is easy to see that M is Φ-simplicial. Hence $M \in \mathcal{C}(\Phi)$ by (1).

(3) If M is a rank 3 normal quasi-truncated or truncated monoid (see [10], Definition 5.1), then $M \in \mathcal{C}(\Phi)$. To see this, by ([10], Lemma 6.6), M satisfies properties of (1.3). Further, M_0 is a Φ-simplicial normal monoid of rank 2. By (1), $M_0 \in \mathcal{C}(\Phi)$. ■

Corollary 3.6 *Let R be a ring of dimension d and $M \subset \mathbb{Z}_+^2$ a normal monoid of rank 2. Then Serre dim $R[M] \leq d$.*

Proof If M is finitely generated, then result follows from (3.5(2)) and (3.4).

If M is not finitely generated, then write M as a filtered union of finitely generated submonoids, say $M = \cup_{\lambda \in I} M_\lambda$. Since M is normal, the integral closure $\overline{M}_\lambda$ of M_λ is contained in M. Hence $M = \cup_{\lambda \in I} \overline{M}_\lambda$. By ([5], Proposition 2.22), $\overline{M}_\lambda$ is fintely generated. If P is a projective $R[M]$-module, then P is defined over $R[\overline{M}_\lambda]$ for some $\lambda \in I$ as P is finitely generated. Now the result follows from (3.5(2)) and (3.4). ■

The following result follows from (3.5(3)) and (3.4).

Corollary 3.7 *Let R be a ring of dimension d and M a truncated or normal quasi-truncated monoid of rank ≤ 3. Then Serre dim $R[M] \leq d$.*

Now we prove (1.4(2)).

Proposition 3.8 *Let R be a ring of dimension d and M a Φ-simplicial seminormal monoid of rank ≤ 3. Then Serre dim $R[int(M)] \leq d$.*

Proof Recall that $int(M) = int(\mathbb{R}_+ M) \cap \mathbb{Z}_+^3$. Let P be a projective $R[int(M)]$-module of rank $\geq d+1$. Since M is seminormal, by ([5], Proposition 2.40), $int(M) = int(\overline{M})$, where $\overline{M}$ is the normalization of M. Since normalization of a finitely generated monoid is finitely generated (see [5], Proposition 2.22), $\overline{M}$ is a Φ-simplicial normal monoid. By ([10], Theorem 3.1), $int(M) = int(\overline{M})$ is a filtered union of truncated (normal) monoids (see [10], Definition 2.2). Since P is finitely generated, we get P is defined over $R[N]$, where $N \subset int(M)$ is a truncated monoid. By (3.7), *Serre dim* $R[N] \leq d$. Hence P has a unimodular element. Therefore *Serre dim* $R[int(M)] \leq d$. $\blacksquare$

In the following examples, R is a ring of dimension d, Monoid operations are written multiplicatively and $K(M)$ denotes the group of fractions of monoid M.

Example 3.9 For $n > 0$, consider the monoid $M \subset \mathbb{Z}_+^r$ generated by $\{t_1^{i_1} t_2^{i_2} \ldots t_r^{i_r} \,|\, \sum i_j = n\}$. Then M is a Φ-simplicial normal monoid. For integers $c_i = nk_i + 1$, $k_i > 0$ and $i = 1, \ldots, r-1$, consider $\eta \in \text{Aut}_{R[t_r]}(R[t_1, \ldots, t_r])$ defined by $t_i \mapsto t_i + t_r^{c_i}$ for $i = 1, \ldots, r-1$.

A typical monomial in the expansion of $\eta(t_1^{i_1} \ldots t_{r-1}^{i_{r-1}} t_r^{i_r}) = (t_1 + t_r^{c_1})^{i_1} \ldots (t_{r-1} + t_r^{c_{r-1}})^{i_{r-1}} t_r^{i_r}$ will look like $(t_1^{i_1 - l_1} t_r^{c_1 l_1}) \ldots (t_{r-1}^{i_{r-1} - l_{r-1}} t_r^{c_{r-1} l_{r-1}}) t_r^{i_r} = (t_1^{i_1 - l_1} \ldots t_{r-1}^{i_{r-1} - l_{r-1}} t_r^{l_1 + \ldots + l_{r-1} + i_r}) t_r^{n(k_1 l_1 + \ldots + k_{r-1} l_{r-1})}$ which belong to M. So $\eta(R[M]) \subset R[M]$. Similarly, $\eta^{-1}(R[M]) \subset R[M]$. Hence η restricts to an R-automorphism of $R[M]$. Therefore η satisfies the property of (1.3). Hence $M \in \mathcal{C}(\Phi)$. By (3.4), *Serre dim* $R[M] \leq d$. $\blacksquare$

Example 3.10 Let M be a Φ-simplicial monoid generated by monomials $t_1^2, t_2^2, t_3^2, t_1 t_3, t_2 t_3$. For integers $c_j = 2k_j - 1$ with $k_j > 1$, consider the automorphism $\eta \in \text{Aut}_{R[t_3]}(R[t_1, t_2, t_3])$ defined by $t_j \mapsto t_j + t_3^{c_j}$ for $j = 1, 2$. Then it is easy to see that η restricts to an automorphism of $R[M]$.

We claim that M is seminormal but not normal. For this, let

$$z = (t_3^2)^{-1}(t_1 t_3)(t_2 t_3) = t_1 t_2 \in K(M) \setminus M, \text{ but } \mathrm{z}^2 \in \mathrm{M},$$

showing that M is not normal. For seminormality, let

$$z = (t_1^2)^{\alpha_1}(t_2^2)^{\alpha_2}(t_3^2)^{\alpha_3}(t_1 t_3)^{\alpha_4}(t_2 t_3)^{\alpha_5} \in K(M) \text{ with } \alpha_i \in \mathbb{Z} \text{ and } \mathrm{z}^2, \mathrm{z}^3 \in \mathrm{M}.$$

We may assume that $0 \leq \alpha_4, \alpha_5 \leq 1$. Now $z^2 \in M \Rightarrow \alpha_1, \alpha_2 \geq 0$ and $2\alpha_3 + \alpha_4 + \alpha_5 \geq 0$. If $\alpha_3 < 0$, then $\alpha_4 = \alpha_5 = 1$ and $\alpha_3 = -1$. In this case, $z^3 = (t_1^{2\alpha_1+1} t_2^{2\alpha_2+1})^3 \notin M$, a contradiction. Therefore

$\alpha_3 \geq 0$ and $z \in M$. Hence M is seminormal. It is easy to see that $M \in \mathcal{C}(\Phi)$. By (3.4), *Serre dim* $R[t_1^2, t_2^2, t_3^2, t_1t_3, t_2t_3] \leq d$.

$\blacksquare$

Remark 3.11 (1) Let R be a ring and P a projective R-module of rank ≥ 2. Let $\overline{R}$ be the seminormalization of R. It follows from arguments in Bhatwadekar ([2], Lemma 3.1) that $P \otimes_R \overline{R}$ has a unimodular element if and only if P has a unimodular element.

(2) Assume R is a ring of dimension d and $M \in \mathcal{C}(\Phi)$. If $\overline{M}$ the seminormalization of M is in $C(\Phi)$, then *Serre dim* $R[M] \leq max\{1, d\}$ using ([2] and 3.4).

(3) Let $(R, \mathfrak{m}, K)$ be a regular local ring of dimension d containing a field k such that either char $k = 0$ or char $k = p$ and tr-deg $K/\mathbb{F}_p \geq 1$. Let M be a seminormal monoid. Then, using Popescu ([15], Theorem 1) and Swan ([23], Theorem 1.2), we get *Serre dim* $R[M] = 0$. If M is not seminormal, then *Serre dim* $R[M] = 1$ using ([11], [2] and [23]).

Example 3.12 For a monoid M, $\overline{M}$ denotes the seminormalization of M.

1. Let $M \subset \mathbb{Z}_+^2$ be a Φ-simplicial monoid generated by t_1^n, t_1t_2, t_2^n, where $n \in \mathbb{N}$. To see M is normal, let $z = t_1^it_2^j = (t_1^n)^p(t_1t_2)^q(t_2^n)^r \in K(M)$ with $p, q, r \in \mathbb{Z}$ such that $z^t \in M$ for some $t > 0$. Then $i, j \geq 0$. We need to show that $z \in M$. We may assume that $0 \leq q < n$. Since $i, j \geq 0$, we get $p, r \geq 0$. Thus $z \in M$ and M is normal. Hence, by (3.6), *Serre dim* $R[t_1^n, t_1t_2, t_2^n] \leq d$.

2. The monoid $M \subset \mathbb{Z}_+^2$ generated by $t_1^2, t_1t_2^2, t_2^2$ is seminormal but not normal. For this, let $z = (t_1t_2^2)(t_2^2)^{-1} = t_1 \in K(M) \setminus M$. Then $z^2 \in M$ showing that M is not normal. For seminormality, let $z = (t_1^2)^\alpha(t_1t_2^2)^\beta(t_2^2)^\gamma \in K(M)$ with $\alpha, \beta, \gamma \in \mathbb{Z}$ be such that $z^2, z^3 \in M$. We may assume $0 \leq \beta \leq 1$. If $\beta = 0$, then $\alpha, \gamma \geq 0$ and hence $z \in M$. If $\beta = 1$, then $z^2 \in M$ implies $\alpha \geq 0$ and $\gamma + 1 \geq 0$. If $\gamma = -1$, then $z^3 = (t_1)^{6\alpha+3} \notin M$, a contradiction. Hence $\gamma \geq 0$, proving that $z \in M$ and M is seminormal. It is easy to see that $M \in \mathcal{C}(\Phi)$. Therefore, by (3.4), *Serre dim* $R[t_1^2, t_1t_2^2, t_2^2] \leq d$.

3. Let M be a monoid generated by $(t_1^2, t_1t_2^j, t_2^2)$, where $j \geq 3$. Then M is not seminormal. For this, if $z = (t_1t_2^j)(t_2^2)^{-1} = t_1t_2^{j-2} \in K(M) \setminus M$, then $z^2 = t_1^2t_2^{2(j-2)}$ and $z^3 = (t_1^2)(t_1t_2^j)(t_2^{2j-6})$ are in M, showing that M is not seminormal.

   If $j = 3$, then observe that t_1t_2 belongs to $\overline{M}$. Since the monoid genertaed by t_1^2, t_1t_2, t_2^2 is normal, we get that $\overline{M}$ is generated by t_1^2, t_1t_2, t_2^2. Hence *Serre dim* $R[\overline{M}] \leq d$ by (1).

   Observe that if j is odd, then $\overline{M} = (t_1^2, t_1t_2, t_2^2)$ and if j is even, then $\overline{M} = (t_1^2, t_1t_2^2, t_2^2)$. So *Serre dim* $R[\overline{M}] \leq d$ by $(1, 2)$.

   In both cases, applying (3.11(1)), we get *Serre dim* $R[M] \leq$ max $\{1, d\}$.

4. Let M be a monoid generated by $(t_1^3, t_1t_2^2, t_2^3)$ Then M is not seminormal. For this, let $z = (t_1t_2^2)^2t_2^{-3} \in K(M) \setminus M$. Then $z^2 = t_1^3(t_1t_2^2) \in M$ and $z^3 = t_1^6t_2^3 \in M$. Hence seminormalization

of M is $\overline{M} = (t_1^3, t_1^2t_2, t_1t_2^2, t_2^3)$. By (3.9), *Serre dim* $R[\overline{M}] \leq d$. Therefore, applying (3.11(1)), we get *Serre dim* $R[M] \leq \max\{1, d\}$.

■

3.2 Monoid algebras over 1-dimensional rings

The following result proves (1.6(i)).

Theorem 3.13 *Let R be a ring of dimension 1 and M a c-divisible monoid. If P is a projective $R[M]$-module of rank $r \geq 3$, then $P \cong \wedge^r P \oplus R[M]^{r-1}$.*

Proof If R is normal, then we are done by Swan [23]. Assume R is not normal.

Case 1. Assume R has finite normalization. Let $\overline{R}$ be the normalization of R and C the conductor ideal of the extension $R \subset \overline{R}$. Then height of $C = 1$. Hence R/C and $\overline{R}/C$ are zero dimensional rings. Consider the following fiber product diagram

$$\begin{array}{ccc} R[M] & \longrightarrow & \overline{R}[M] \\ \downarrow & & \downarrow \\ (R/C)[M] & \longrightarrow & (\overline{R}/C)[M] \end{array}$$

If $P' = \wedge^r P \oplus R[M]^{r-1}$, then by Swan [23], $P \otimes \overline{R}[M] \cong \wedge^r (P \otimes \overline{R}[M]) \oplus \overline{R}[M]^{r-1} \cong P' \otimes \overline{R}[M]$. By Gubeladze [8], P/CP and P'/CP' are free $(R/C)[M]$-modules. Further, $\mathrm{SL}_r((\overline{R}/C)[M]) = \mathrm{E}_r((\overline{R}/C)[M])$ for $r \geq 3$, by Gubeladze [9]. Now using standard arguments of fiber product diagram, we get $P \cong P'$.

Case 2. Now R need not have finite normalization. We may assume R is a reduced ring with connected spectrum. Let S be the set of all non-zerodivisors of R. By [8], $S^{-1}P$ is a free $S^{-1}R[M]$-module. Choose $s \in S$ such that P_s is a free $R_s[M]$-module.

Now we follow the arguments of Roy ([17],Theorem 4.1). Let $\hat{R}$ denote the s-adic completion R. Then $\hat{R}_{red}$ has a finite normalization. Consider the following fiber product diagram

$$\begin{array}{ccc} R[M] & \longrightarrow & \hat{R}[M] \\ \downarrow & & \downarrow \\ R_s[M] & \longrightarrow & \hat{R}_s[M] \end{array}$$

Since $\hat{R}_s$ is a zero dimensional ring, by [9], $\mathrm{SL}_r(\hat{R}_s[M]) = \mathrm{E}_r(\hat{R}_s[M])$ for $r \geq 3$. If $P' = \wedge^r P \oplus R[M]^{r-1}$, then P_s and P'_s are free $R_s[M]$-modules and by Case 1, $P \otimes \hat{R}[M] \cong P' \otimes \hat{R}[M]$. By (2.5), $P \cong P'$. This completes the proof. ■

The following result is due to Kang ([12], Lemma 7.1 and Remark).

Lemma 3.14 *Let R be a 1-dimensional unibranched affine algebra over an algebraically closed field, $\overline{R}$ the normalization of R and C the conductor ideal of the extension $R \subset \overline{R}$. Then $\overline{R}/C = R/C + a_1R/C + \cdots + a_mR/C$, where $a_i \in \sqrt{C}$ the radical ideal of C in $\overline{R}$.*

Lemma 3.15 *Let R be a 1-dimensional ring, $\overline{R}$ the normalization of R and C the conductor ideal of the extension $R \subset \overline{R}$. Assume $\overline{R}/C = R/C + a_1R/C + \cdots + a_mR/C$, where $a_i \in \sqrt{C}$ the radical ideal of C in $\overline{R}$. Let M be a monoid and write $A = \overline{R}/C$.*

(i) If $\sigma \in \mathrm{SL}_n(A[M])$, then $\sigma = \sigma_1\sigma_2$, where $\sigma_1 \in \mathrm{SL}_n((R/C)[M])$ and $\sigma_2 \in \mathrm{E}_n(A[M])$.

(ii) If P is a projective $R[M]$-module of rank r, then $P \cong \wedge^r P \oplus R[M]^{r-1}$.

Proof (i) Let $\sigma = (b_{ij}) \in \mathrm{SL}_n(A[M])$. Write $b_{ij} = (b_{ij})_0 + (b_{ij})_1 a_1 + \cdots + (b_{ij})_m a_m$, where $(b_{ij})_l \in (R/C)[M]$. If $\alpha = ((b_{ij})_0)$, then $det(\sigma) = 1 = det(\alpha) + c$, where $c \in (\sqrt{C}/C)[M]$. Since $c \in (R/C)[M]$ is nilpotent, $det(\alpha)$ is a unit in $(R/C)[M]$. Let $\beta =$ diagonal $(1/(1-c), 1, \ldots, 1) \in \mathrm{GL}_n((R/C)[M])$ and $\sigma_1 = \alpha\beta \in \mathrm{SL}_n((R/C)[M])$.

Note that $\sigma_1^{-1}\sigma = \beta^{-1}\alpha^{-1}\sigma = \beta^{-1}\, 1/(1-c)\, \overline{\alpha}\sigma$, where $\overline{\alpha} = ((\overline{b}_{ij})_0)$, $(\overline{b}_{ij})_0$ are minors of $(b_{ij})_0$.

$$\sigma_2 := \sigma_1^{-1}\sigma = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \frac{1}{1-c} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{1-c} \end{bmatrix} \begin{bmatrix} 1+c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & 1+c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ c_{n1} & c_{n2} & \cdots & 1+c_{nn} \end{bmatrix},$$

where $c_{ij} \in (\sqrt{C}/C)[M]$.

Note that $\sigma_2 \in \mathrm{SL}_n(A[M])$ and $\sigma_2 = Id$ modulo the nilpotent ideal of $A[M]$. Hence $\sigma_2 \in \mathrm{E}_n(A[M])$. Thus we get $\sigma = \sigma_1\sigma_2$ with the desired properties.

(ii) Follow the proof of (3.13) and use (3.15(i)) to get the result. ∎

Now we prove (1.6(ii)) which follows from (3.14) and (3.15).

Theorem 3.16 *Let R be a 1-dimensional unibranched affine algebra over an algebraically closed field and M a monoid. If P is a projective $R[M]$-module of rank r, then $P \cong \wedge^r P \oplus R[M]^{r-1}$.*

4 Applications

Let R be a ring of dimension d and Q a finitely generated R-module. Let $\mu(Q)$ denote the minimum number of generators of Q. By Forster [7] and Swan [22], $\mu(Q) \leq max\{\mu(Q_\mathfrak{p}) + \dim(R/\mathfrak{p}) | \mathfrak{p} \in \mathrm{Spec}\,(R), Q_\mathfrak{p} \neq 0\}$. In particular, if P is a projective R-module of rank r, then $\mu(P) \leq r + d$.

The following result is well known.

Theorem 4.1 *Let A be a ring such that Serre dim $A \leq d$. Assume A^m is cancellative for $m \geq d+1$. If P is a projective A-module of rank $r \geq d+1$, then $\mu(P) \leq r+d$.*

Proof Assume $\mu(P) = n > r + d$. Consider a surjection $\phi : A^n \twoheadrightarrow P$ with $Q = ker(\phi)$. Then $A^n \cong P \oplus Q$. Since Q is a projective A-module of rank $\geq d + 1$, Q has a unimodular element q. Since $\phi(q) = 0$, ϕ induces a surjection $\overline{\phi} : A^n/qA^n \twoheadrightarrow P$. Since $n - 1 > d$, A^{n-1} is cancellative. Hence $A^{n-1} \cong A^n/qA$ and P is generated by $n - 1$ elements, a contradiction. $\blacksquare$

The following result is immediate from (4.1, 3.4, 3.6 and [6]).

Corollary 4.2 *Let R be a ring of dimension d, M a monoid and P a projective $R[M]$-module of rank $r > d$. Then:*

(i) If $M \in \mathcal{C}(\Phi)$, then $\mu(P) \leq r + d$.

(ii) If $M \subset \mathbb{Z}_+^2$ is a normal monoid of rank 2, then $\mu(P) \leq r + d$.

Schaubhüser [19] proved that *for any ring R of dimension d and $n \geq max\{2, d+1\}$, $E_{n+1}(R[M])$ acts transitively on* $\text{Um}_{n+1}(R[M])$. Using Schaubhüser's result and arguments of Dhorajia-Keshari ([6], Theorem 4.4), we get that if R is a ring of dimension d and P is a projective $R[M]$-module of rank $\geq max\{2, d+1\}$, then $E(R[M] \oplus P)$ acts transitively on $\text{Um}(R[M] \oplus P)$. Therefore the following result is immediate from (4.1 and 3.13).

Corollary 4.3 *Let R be a ring of dimension 1, M a c-divisible monoid and P a projective $R[M]$-module of rank $r \geq 3$. Then $\mu(P) \leq r + 1$.*

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