Multi-dimensional stationary internal layers for spatially inhomogeneous reaction-diffusion equations with balanced nonlinearity

N. N. Nefedov and K. Sakamoto

Abstract. We deal with reaction-diffusion equations of bistable type in an inhomogeneous medium. When the reaction term is balanced in the sense that a bulk potential energy attains the same global minimum at the two stable equilibria for each spatial point, we derive a free-boundary problem whose solutions determine equilibirum interfaces. We show that a non-degenerate solution of the free-boundary problem gives rise to an equilibrium internal layer solution of the reaction-diffusion equation, and moreover, the stability property of the latter is obtained from a linearization of the free boundary problem.

1. Introduction

1.1. Background. Internal layers, which separate two stable bulk states by a sharp transition near hypersurfaces (called interfaces), are often observed in bistable reaction-diffusion equations when the reaction rate is stronger than the diffusion effect. The motion of such interfaces is considered as the evolution of spatio-temporal patterns generated by the reaction-diffusion equation. Therefore, investigations of interfacial phenomena, such as internal transition layers and interface motions, are of crucial importance in our understanding of pattern-formation mechanisms in nature.

The Allen-Cahn equation with cubic nonlinearity

$$\text{(A-C)} \qquad \frac{\partial u}{\partial t} = \varepsilon^2 \Delta u - (u^3 - u) \qquad x \in \mathbf{R}^N,\ t > 0,$$

is a typical example in which internal layers spontaneously develop and evolve when the diffusion coefficient is rather small, $0 < \varepsilon^2 \ll 1$. By rescaling time, one can write (A-C) as follows:

(A-C-s)
$$\frac{\partial u}{\partial t} = \varepsilon \Delta u - \frac{1}{\varepsilon}(u^3 - u),$$

(A-C-ss)
$$\frac{\partial u}{\partial t} = \Delta u - \frac{1}{\varepsilon^2}(u^3 - u).$$

It should be noted here that no spatial re-scaling has been performed to obtain the latter two equations from (A-C). Slow dynamics in (A-C), say dynamics of order $O(\varepsilon)$, is clearly captured by (A-C-s), while to describe much slower dynamics of order $O(\varepsilon^2)$ the third equation (A-C-ss) may be adequate. It is therefore our advantage to employ three forms of the same equation inter-changeably, according to the dynamical behavior of our attention.

The ordinary differntial equation (ODE), or the reaction kinetics, associated with (A-C)

(ODE)
$$\frac{du}{dt} = u - u^3$$

is of bistable type, in the sense that $u = \pm 1$ are its two stable equilibria. The equilibrium $u = 0$ is located in between the two stable ones, and in fact, it is precisely on the boundary of the basins of attraction for $u = \pm 1$. Since the reaction rate is much stronger than the diffusion rate in (A-C), it is naturally expected that ODE-dynamics will dominate the behavior of solutions, at least in the initial stage. In fact, it is rigorously proven by Chen [2] that the solution of (A-C) with initial condition $u(x, 0) = u_0(x)$ develops transition laryers near $\{x \in \mathbf{R}^N \mid u_0(x) = 0\}$. Namely, for $t \approx 0$

$$u(x, t) \approx -1 \qquad \{x \in \mathbf{R}^N \mid u_0(x) < 0\} =: \mathscr{D}^{(-)}(0),$$

$$u(x, t) \approx 1 \qquad \{x \in \mathbf{R}^N \mid u_0(x) > 0\} =: \mathscr{D}^{(+)}(0).$$

In this sense, $\Gamma_0 = \{x \in \mathbf{R}^N \mid u_0(x) = 0\}$ is called the initial *interface*.

When the transition layer becomes sharp enough, the diffusion effect $\varepsilon^2 \Delta u$ also becomes comparble with the reaction term and can not be neglected anymore. It is also shown by Chen [2] that at this stage the interface $\Gamma(t) = \{x \in \mathbf{R}^N \mid u(x, t) = 0\}$ starts to move according to the mean curvature flow:

(MCF)
$$\mathbf{V}(x; \Gamma(t)) = -\kappa(x; \Gamma(t)) \qquad x \in \Gamma(t), \, t > 0.$$

Here, the scale of time is that of (A-C-ss), and $\mathbf{V}(x; \Gamma(t))$ is the speed of $\Gamma(t)$, measured in the unit normal direction $v(x, \Gamma(t))$ on $\Gamma(t)$ pointing into the interior of $\mathscr{D}^{(+)}(t)$, where $\mathscr{D}^{(\pm)}(t) = \{x \mid \pm u(x, t) > 0\}$. The symbol $\kappa(x; \Gamma)$ stands for the sum of principal curvatures of Γ at $x \in \Gamma$. We agree to call it simply the *mean curvature* of $\Gamma(t)$. The sign of $\kappa(x; \Gamma(t))$ is chosen so that it

is positive if the center of the curvature sphere is in $\mathscr{D}^{(-)}(t)$. More precisely, we define it by

$$\kappa(x; \Gamma(t)) = \mathrm{div}_x\, v(x; \Gamma(t))$$

in which the unit normal vector v is smoothly extended to a neighborhood of $\Gamma(t)$.

The Allen-Cahn equation above is a special case of the following reaction-diffusion equation

(RD) $$\frac{\partial u}{\partial t} = \varepsilon^2 \Delta u - W'(u),$$

in which $W(u)$ is a double-well potential and $W'(u) = \frac{dW(u)}{du}$. $W(u)$ is called a double-well potential when it has two local minima at, say, $u = a_-$ and $u = a_+$ $(a_- < a_+)$. When $W(u) = u^4/4 - u^2/2$, (RD) is nothing but (A-C). In the dynamics of (RD) driven by the double-well potential, the difference of potential values at the two wells $u = a_\pm$ plays an important role. The difference is measured as

(PD) $$W(a_+) - W(a_-) = \int_{a_-}^{a_+} W'(u)du =: [W]_-^+.$$

Note that for a double-well potential, the nonlinearity $-W'(u)$ is of bistable type. When the difference $[W]_-^+$ is zero, the corresponding nonlinearity $-W'(u)$ is called **balanced**. When the nonlinearity is bistable and balanced, the description above for (A-C) is equally valid for (RD).

When the nonlinearity is bistable but not balanced, it is also shown by Chen [2] that the motion of interface $\Gamma(t)$ is described by

(CS) $$\mathbf{V}(x; \Gamma(t)) = c^W \qquad x \in \Gamma(t),\, t > 0,$$

where the time scale is that of (A-C-s), and c^W is a constant which is of the same sign as $[W]_-^+$. The motion law (CS) is in accord with our intuition. If, for example, $c^W > 0$, namely, $[W]_-^+ > 0$ (and hence the well at $u = a_-$ is deeper than that at $u = a_+$, cf. (PD)), then (CS) says that the interface $\Gamma(t)$ invades $\mathscr{D}^{(+)}(t)$, the region occupied by $u = a_+$. Therefore, eventually $\mathscr{D}^{(-)}(t)$, which is occupied by $u = a_-$, will dominate the entire region.

For more detail of the interface dynamics of (RD), we refer to [2] by Chen.

1.2. Statement of Problem. In the previous subsection, interfacial phenomena have been described for a spatially homogeneous reaction-diffusion equation. From a viewpoint of application, it is natural to consider spatially inhomogeneous equations, since environments in which reaction and diffusion take

place is, realistically speaking, non-uniform. For reaction-diffusion equations, there are at least two ways of introducing spatial inhomogeneity: one in the diffusion rate; the other in the reaction term. In this paper, we restrict our consideration strictly to the latter situation.

Let us consider a spatially inhomogeneous reaction-diffusion equation

$$(1.1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon^2 \Delta u - f(u, x, \varepsilon), & (x \in \mathscr{D} \subset \mathbf{R}^N, t > 0), \\[2mm] \dfrac{\partial u}{\partial \mathbf{n}} = 0, & (x \in \partial\mathscr{D}, t > 0), \end{cases}$$

with the no-flux boundary conditions. In (1.1), $\mathscr{D}$ is a smooth bounded domain and $\mathbf{n}$ stands for the *inward* unit normal vector on $\partial\mathscr{D}$. The nonlinear term $f(u, x, \varepsilon)$ is assumed to be smooth and derived from a double-well potential $W(u, x, \varepsilon)$:

$$(1.2) \qquad f(u, x, \varepsilon) = \frac{\partial W(u, x, \varepsilon)}{\partial u}$$

with $u = \phi^{(\pm)}(x, \varepsilon)$ denoting the locations of two wells, satisfying

$$(1.2\text{-a}) \qquad \phi^{(-)}(x, \varepsilon) < \phi^{(+)}(x, \varepsilon) \qquad (x \in \bar{\mathscr{D}}).$$

It is expected, from the discussion on the Allen-Cahn equation, that the difference in the values of potential at the two wells will play an important role in describing the dynamics of (1.1). Let us denote the difference at each $x \in \bar{\mathscr{D}}$ by $[W]_-^+(x)$:

$$(1.3) \quad [W]_-^+(x) = \int_{\phi^{(-)}(x)}^{\phi^{(+)}(x)} f(u, x, 0)\,du = W(\phi^{(+)}(x), x, 0) - W(\phi^{(-)}(x), x, 0)$$

where $\phi^{(\pm)}(x) = \phi^{(\pm)}(x, 0)$ (cf. (1.2-a)).

Along with (1.1), it is convenient to also consider its (time) re-scaled versions:

$$(1.1\text{-s}) \qquad \frac{\partial u}{\partial t} = \varepsilon \Delta u - \frac{1}{\varepsilon} f(u, x, \varepsilon),$$

$$(1.1\text{-ss}) \qquad \frac{\partial u}{\partial t} = \Delta u - \frac{1}{\varepsilon^2} f(u, x, \varepsilon).$$

In this paper, we are mainly concerned with interfacial phenomena for (1.1), and in particular, with equilibrium internal transition layers and their stability properties.

Before we go further, let us make precise the conditions to be imposed on the nonlinearity f.

(A1) The equation $f(u, x, 0) = 0$ has exactly three solutions $u = \phi^{(\pm)}(x), \phi^{(0)}(x)$ with

$$\phi^{(-)}(x) < \phi^{(0)}(x) < \phi^{(+)}(x) \qquad x \in \bar{\mathscr{D}}.$$

Moreover,

$$f_u^{\pm}(x) \equiv f_u(\phi^{(\pm)}(x), x, 0) > 0 \qquad x \in \bar{\mathscr{D}}.$$

Under the assumption **(A1)**, it is known that the following problem

$$(1.4) \quad \begin{cases} \dfrac{d^2 \tilde{Q}_0}{dz^2} + c \dfrac{d\tilde{Q}_0}{dz} - f(\tilde{Q}_0, x, 0) = 0 & z \in \mathbf{R}, \\[2mm] \lim_{z \to \pm\infty} \tilde{Q}_0(z) = \phi^{(\pm)}(x), \quad \tilde{Q}_0(0) = \dfrac{1}{2}(\phi^{(+)}(x) + \phi^{(-)}(x)) \end{cases}$$

has a unique solution $(\tilde{Q}_0(z; x), c(x))$, where $x \in \bar{\mathscr{D}}$ is regarded as a parameter. The solution pair $(\tilde{Q}_0(z; x), c(x))$ satisfy the following properties: There exist constants $C > 0$ and $\delta > 0$, independent of x, such that

$$(1.4\text{-a}) \qquad |\tilde{Q}_0(z; x) - \phi^{(\pm)}(x)| \leq Ce^{-\delta|z|} \qquad \text{as } z \to \pm\infty,$$

$$(1.4\text{-b}) \qquad \frac{\partial \tilde{Q}_0(z; x)}{\partial z} > 0, \qquad \left|\frac{\partial \tilde{Q}_0(z; x)}{\partial z}\right|, \left|\frac{\partial^2 \tilde{Q}_0(z; x)}{\partial z^2}\right| \leq Ce^{-\delta|z|},$$

$$(1.4\text{-c}) \qquad [W]_-^+(x) = c(x) \int_{-\infty}^{\infty} \left(\frac{\partial \tilde{Q}_0(z; x)}{\partial z}\right)^2 dz.$$

The solution $u(x, t)$ of (1.1) starting from an initial function $u_0(x)$, similar to the case of (RD), will develop internal transition layers near

$$\left\{x \in \mathscr{D} \mid u_0(x) = \frac{1}{2}(\phi^{(+)}(x) + \phi^{(-)}(x))\right\} = \Gamma_0,$$

and the interface $\Gamma(t)$ evolves according to

$$(1.5) \qquad \qquad \mathbf{V}(x; \Gamma(t)) = c(x) \qquad (x \in \Gamma(t), t > 0).$$

The time scale of (1.5) is the same as (1.1-s). Here and in what follows, we always treat the cases where interfaces are staying uniformly away from the boundary of doamin $\partial\mathscr{D}$.

From our standpoint of investigating the existence of equilibrium internal layer solutions, it is natural to ask the next question:

If the interface equation (1.5) has a smooth equilibrium solution Γ, then does (1.1) have a family of equilibrium solutions with transition layers on Γ for small $\varepsilon > 0$?

It turns out that the answer to this question is rather delicate. In [4], Fife and Greenlee prove that the answer is affirmative if the condition

(F-G) $$\nabla_x c(x)|_\Gamma \cdot v(x,\Gamma) < 0, \qquad x \in \Gamma$$

is fulfilled. Here Γ is a smooth equilibrium solution of (1.5). Namely, $\Gamma = \{x \in \mathscr{D} \mid c(x) = 0\}$ is a closed manifold of codimension 1. Moreover, the solution thus obtained is a stable equilibrium of (1.1). It is of crucial importance to note that the normal vector v above is pointing into the *region where the solution u assumes values close to $\phi^{(+)}$*. Since $c(x) \equiv 0$ on Γ, the condition above says that in the two regions away from the interface Γ the solution takes values close to the absolute minimum of the potential $W(u,x,0)$.

On the other hand, it is also pointed out in [9], in the context of the same question for a system of reaction-diffusion equations, that if the condition

(Reverse F-G) $$\nabla_x c(x)|_\Gamma \cdot v(x,\Gamma) > 0 \qquad x \in \Gamma$$

is the case, then there may exist infinitely many internal laryer solutions which exhibit sharp transitions near Γ. By examining the proof in [9] and interpreting it in our situation, we can state the following criterion on the existence of equilibrium internal layer solutions.

THEOREM 1.1. *Let Γ be a smooth equilibrium solution of* (1.5). *If it is non-degenerate in the sense that the spectrum of the linearized operator*

(1.6) $$\varepsilon \Delta^\Gamma + (\nabla_x c(x)|_\Gamma \cdot v(x,\Gamma)),$$

defined on Γ (Δ^Γ is the Laplace-Beltrami operator on Γ), is bounded away from zero uniformly in $\varepsilon \in (0,\varepsilon_0]$ for some $\varepsilon_0 > 0$, then (1.1) *has a family of solutions with sharp transitions along Γ.*

Since Δ^Γ is a non-positive operator, it is evident that if $\nabla_x c(x)|_\Gamma \cdot v(x,\Gamma) < 0$, then the spectrum of the operator (1.6) is uniformly bounded away from zero. Hence the criterion above is compatible with the result by Fife and Greenlee [4]. On the other hand, if the sign is opposite, $\nabla_x c(x)|_\Gamma \cdot v(x,\Gamma) > 0$, then *the spectrum of* (1.6) *hits zero infinitely often* as $\varepsilon \to 0$.

The purpose of this paper is to investigate the existence and stability properties of internal layer solutions of (1.1) when the nonlinearity is of **balanced type** at each point $x \in \bar{\mathscr{D}}$. Namely, we impose the following condition on f:

(A2) $[W]_-^+(x) \equiv 0$ on $\bar{\mathscr{D}}$, or equivalently, $c(x) \equiv 0$ on $\bar{\mathscr{D}}$.

Under this condition, we may normalize W so that $W(\phi^{(\pm)}(x),x,0) \equiv 0$.

If **(A2)** is the case, two kinds of degeneracies occur;

- any closed manifold $\Gamma \subset \mathscr{D}$ of codimension one is an equilibrium of (1.5);
- the corresponding linear operator (1.6) reduces to $\varepsilon \Delta^{\Gamma}$ which has the 0-eigenvalue as well as infinitely many eigenvalues converging to 0 as $\varepsilon \to 0$,

making Theorem 1.1 invalid. Therefore, we need to establish a selection principle to identify possible equilibrium interfaces, and to develop a method to study stability properties of corresponding transition layer solutions.

Under the conditions **(A1)** and **(A2)**, one can show, along the line of arguments employed in Nakamura et al. [7], that the interface equation for (1.1-ss) is given by

$$(1.7) \qquad \mathbf{V}(x; \Gamma(t)) = -\kappa(x; \Gamma(t)) + \frac{J(x; \Gamma(t))}{m(x)}, \qquad (x \in \Gamma(t), t > 0),$$

where

$$(1.7\text{-a}) \qquad m(x) = \int_{-\infty}^{\infty} \left(\frac{\partial \tilde{Q}_0(z; x)}{\partial z} \right)^2 dz, \qquad (x \in \bar{\mathscr{D}}),$$

$$(1.7\text{-b}) \qquad J(x; \Gamma) = \int_{-\infty}^{\infty} [z(\nabla_x f(u, x, 0)|_{u=\tilde{Q}_0(z; x)} \cdot v(x; \Gamma))$$

$$+ f_\varepsilon(\tilde{Q}_0(z; x), x, 0)] \frac{\partial \tilde{Q}_0(z; x)}{\partial z} \, dz, \qquad (x \in \Gamma).$$

There is an important relation between $m(x)$ and $J(x; \Gamma)$:

$$(1.7\text{-c}) \quad J(x; \Gamma) = -\nabla_x m(x) \cdot v(x, \Gamma) + \int_{-\infty}^{\infty} f_\varepsilon(\tilde{Q}_0(z; x), x, 0) \frac{\partial \tilde{Q}_0(z; x)}{\partial z} \, dz$$

$$= -\nabla_x m(x) \cdot v(x, \Gamma) + \frac{\partial}{\partial \varepsilon} [W(\phi^{(+)}(x), x, \varepsilon) - W(\phi^{(-)}(x), x, \varepsilon)]|_{\varepsilon=0}.$$

In terms of the potential W, the quantity $m(x)$ is expressed as

$$(1.7\text{-d}) \qquad m(x) = \int_{\phi^{(-)}(x)}^{\phi^{(+)}(x)} \sqrt{2W(u, x, 0)} \, du$$

which may be interpreted as the *unit energy neccessary for the system to make transition* in the v-direction from $\phi^{(-)}(x)$ to $\phi^{(+)}(x)$. Therefore, when the nonlinearity is independent of ε, (1.7-c) implies that the interface equation (1.7) is written as

$$\mathbf{V}(x; \Gamma(t)) = -\kappa(x; \Gamma(t)) - \nabla_v(\log m(x)), \qquad (x \in \Gamma(t), t > 0),$$

which is a slight generalization of the interface equation (2.20) in Nakamura et al. [7]. The latter equation shows that the interface tends to move in the direction to decrease the interfacial energy $m(x)$ if the curvature-effect is neglected.

The main objective of this paper is to answer the following question:

Under the conditions (A1) and (A2), if the interface equation (1.7) has a smooth equilibrium solution Γ, then does (1.1) have an equilibrium solution with internal transition layers on Γ? If such a solution exists, is it stable or unstable?

In order to state our problem succinctly, let us define a class $\mathscr{F}$ of interfaces.

$$(1.8) \qquad \mathscr{F} = \{\Gamma \subset \mathscr{D} \mid \Gamma \text{ is an } (N-1)\text{-dimensional, smooth,} \\ \text{and closed manifold}\}.$$

For a given $\Gamma \in \mathscr{F}$ and in its sufficiently small neighborhood

$$(1.9) \qquad \Gamma^{(\delta)} \equiv \{x \in \mathscr{D} \mid \mathrm{dist}(x, \Gamma) < \delta\}$$

for some $\delta > 0$, we introduce a local coordinate system via

$$(1.10) \quad \Gamma^{(\delta)} \ni x \mapsto (y, r) \in \Gamma \times (-\delta, \delta) \qquad \text{with } r = \begin{cases} \mathrm{dist}(x, y), & \text{if } x \in \mathscr{D}_\Gamma^{(+)}, \\ -\mathrm{dist}(x, y), & \text{if } x \in \mathscr{D}_\Gamma^{(-)}, \end{cases}$$

where $y \in \Gamma$ is such that $\mathrm{dist}(x, \Gamma) = \mathrm{dist}(x, y)$, and $\mathscr{D}_\Gamma^{(\pm)}$ are sub-domains of $\mathscr{D}$ divided by Γ. Later in this paper, $\mathscr{D}_\Gamma^{(\pm)}$ are, respectively, the domains where $u(x) \approx \phi^{(\pm)}(x)$ (cf. (2.1), (2.2) below). Denoting by $\nu(y, \Gamma)$ the unit normal vector of Γ at $y \in \Gamma$ pointing into $\mathscr{D}_\Gamma^{(+)}$, one can write the coordinate system in (1.10) as

$$(1.11) \qquad x = y + r\nu(y, \Gamma), \qquad (x \in \Gamma^{(\delta)}, y \in \Gamma, |r| \leq \delta).$$

For an element $\Gamma \in \mathscr{F}$, we also define its *r-shift*, Γ_r, by

$$(1.12) \qquad \Gamma_r = \{x = y + r\nu(y, \Gamma) \mid y \in \Gamma\}$$

for $|r| < \delta$.

In accordance with the right-hand side of (1.7), let us define a function $V_1(x, \Gamma)$ for $\Gamma \in \mathscr{F}$ by

$$(1.13) \qquad V_1(x, \Gamma) \equiv -\kappa(x, \Gamma)m(x) + J(x; \Gamma), \qquad x \in \mathscr{D}.$$

As our selection principle for equilibrium interfaces, we now impose the condition that (1.7) has a smooth equilibrium solution.

(A3)   There exists a $\Gamma \in \mathscr{F}$ such that

$$V_1(x, \Gamma) \equiv 0 \qquad \text{on } \Gamma.$$

It is not so easy to find such a $\Gamma \in \mathscr{F}$ as in **(A3)**. It is a *free-interface problem* to be investigated in its own right (cf. §2.2 below).

In order to answer the question above, we need an extra *non-degeneracy condition*. Let us define an elliptic operator $\mathscr{A}^\Gamma$ by

$$(1.14) \qquad \mathscr{A}^\Gamma R(x) := m(x)\left(\Delta^\Gamma + \sum_{j=1}^{N-1} \kappa_j(x)^2 \right) R(x) + \nabla_\Gamma m(x) \cdot \nabla_\Gamma R(x)$$

$$- \kappa(x; \Gamma) \frac{\partial m(x)}{\partial v(x; \Gamma)} R(x) + J_r(x; \Gamma) R(x) \qquad x \in \Gamma,$$

$$= \mathrm{div}_\Gamma(m(x)\nabla_\Gamma R(x)) + m(x)\left(\sum_{j=1}^{N-1} \kappa_j(x)^2 \right) R(x)$$

$$- \kappa(x; \Gamma) \frac{\partial m(x)}{\partial v(x; \Gamma)} R(x) + J_r(x; \Gamma) R(x) \qquad x \in \Gamma,$$

where $\Delta^\Gamma, \mathrm{div}_\Gamma$, and ∇_Γ stand, respectively, for the *Laplace-Beltrami*, *divergence* and *gradient* operators on the manifold Γ with respect to the metric induced on Γ from the Eulidean metric in $\mathscr{D}$, and $\kappa_j(x)$ $(j = 1, \ldots, N-1)$ are principal curvatures of Γ at x. The function $J_r(x; \Gamma)$ is defined by

$$(1.15) \qquad J_r(x; \Gamma) \equiv \frac{\partial}{\partial r} J(x + rv(x; \Gamma); \Gamma_r)|_{r=0}.$$

The operator $\mathscr{A}^\Gamma R$ in (1.14) is the *linearization* of $V_1(x, \Gamma)$ in the direction of

$$\{ x + R(x)v(x, \Gamma) \,|\, x \in \Gamma \}.$$

This follows from the identity

$$\frac{\partial}{\partial \tau} J(x + \tau R(x)v(x; \Gamma); \Gamma_{\tau R(x)})|_{\tau=0} = \nabla_\Gamma m(x) \cdot \nabla_\Gamma R(x) + J_r(x; \Gamma) R(x),$$

which is verified via direct computations. The operator $\mathscr{A}^\Gamma$ also emerges naturally from C^1-matching conditions in §4.2, below.

Let us consider an eigenvalue problem for $\mathscr{A}^\Gamma$

$$(1.16) \qquad \mathscr{A}^\Gamma \Theta(x) = \lambda m(x)\Theta(x), \qquad x \in \Gamma.$$

Since (1.16) is a self-adjoint eigenvalue problem, its eigenvalues are all real. We denote them as

(1.16′) $\sigma_\Gamma = \{\lambda_j^\Gamma\}_{j=0}^\infty, \qquad \lambda_0^\Gamma > \lambda_1^\Gamma > \cdots > \lambda_j^\Gamma \to -\infty,$

where only *distinct eigenvalues* are listed. The *multiplicity* of λ_j^Γ is denoted by $m_j^\Gamma \geq 1$. The non-degeneracy condition on Γ is:

(A4)   0 is not an eigenvalue of (1.16), i.e., $0 \notin \sigma_\Gamma$.

Let us now outline the contents of the paper.

In the next section, we will state our main theorem and present some examples to which it applies. These examples naturally give rise to interesting geometric variational problems. We then prove the main theorem in §3, while §§4 and 5 are devoted to the proof of technical results used in §3.

2.   Main results

2.1.   Existence and Stability of Layers.
The following is our main theorem.

THEOREM 2.1.   *Assume that the conditions* **(A1)**, **(A2)**, **(A3)**, *and* **(A4)** *are satisfied.*

(i)   *There exist* $\varepsilon_0 > 0$ *and a family of equilibrium solutions* $u(x, \varepsilon)$ *of* (1.1) *such that for each fixed* $d_0 > 0$

(2.1) $\lim_{\varepsilon \to 0} u(x, \varepsilon) = \begin{cases} \phi^{(-)}(x), & x \in \mathscr{D}_\Gamma^{(-)} \backslash \Gamma^{(d_0)}, \\ \phi^{(+)}(x), & x \in \mathscr{D}_\Gamma^{(+)} \backslash \Gamma^{(d_0)}, \end{cases}$

uniformly.

(ii)   *If* $\lambda_0^\Gamma < 0$, *then the solution* $u(x, \varepsilon)$ *is asymptotically stable. If there exists* $k \geq 0$ *such that* $\lambda_k^\Gamma > 0 > \lambda_{k+1}^\Gamma$, *then the solution* $u(x, \varepsilon)$ *is unstable with instability index equal to* $\sum_{j=0}^k m_j$.

Nakashima [8] established results similar to Theorem 2.1. She proves the existence of *stable* internal transition layers in one-dimensional situation. Her method, however, is based on comparison principles and hence unable to prove *unstable* solutions. Our theorem is a generalization of the results in [8] to multi-dimensional domains, including unstable situations.

Theorem 2.1 *justifies* the interface equation (1.7), in the context of equilibrium solutions. A dynamic version of such a justification may be established by the method similar to the one presented below (cf. §§3, 4, and 5).

2.2.   Examples.
In this subsection, we deal with simple examples to which Theorem 2.1 applies.

EXAMPLE 2.1.   Let us consider the situation where the nonlinearity f in (1.1) is given by

(2.2) $$f(u, x, \varepsilon) \equiv (u - \varepsilon\alpha(x))(u^2 - 1).$$

In this case, the corresponding $\tilde{Q}_0$ of (1.4) is easily found to be

(2.3) $$\tilde{Q}_0(\tau; x) \equiv \hat{Q}_0(\tau) := \tanh\left(\frac{\tau}{\sqrt{2}}\right).$$

In the sequel, the function $\hat{Q}_0(\tau)$ will be used frequently. The functions $m(x)$ and $J(x; \Gamma)$ of (1.7-a) and (1.7-b), respectively, are easily computed:

(2.4) $$m(x) \equiv \frac{2\sqrt{2}}{3} \quad (x \in \bar{\mathscr{D}}), \qquad J(x; \Gamma) \equiv \frac{4}{3}\alpha(x) \quad (x \in \Gamma).$$

Therefore the condition **(A3)** demands that there should exist a $\Gamma \in \mathscr{F}$ such that

(2.5) $$\kappa(x; \Gamma) = \sqrt{2}\alpha(x), \qquad x \in \Gamma.$$

This is a problem of highly geometric nature. Such a hypersurface is called a *surface with prescribed mean curvature* in differential geometry literature. One can verify that (2.5) is the first variational equation for the functional $F_\alpha(\Gamma)$ defined by

(2.6) $$F_\alpha(\Gamma) := \int_\Gamma dS_x^\Gamma - \int_{\mathscr{D}_\Gamma^{(-)}} \alpha(x)dx \qquad (\Gamma \in \mathscr{F}),$$

where dS_x^Γ is the surface element of Γ.

Let us assume that (2.5) does have a smooth solution Γ, or equivalently, that (2.6) has a critical point Γ which is regular enough (we will later treat a special case where we can easily find a solution). The linear operator $\mathscr{A}^\Gamma$ in (1.14) is given by

(2.7) $$\frac{3}{2\sqrt{2}}\mathscr{A}^\Gamma R(x) = \left(\Delta^\Gamma + \sum_{j=1}^{N-1} \kappa_j(x)^2\right)R(x) + \frac{\partial\alpha(x)}{\partial v(x; \Gamma)}R(x) \qquad x \in \Gamma.$$

The first two terms on the right-hand side of (2.7), $\Delta^\Gamma + \sum_{j=1}^{N-1} \kappa_j(x)^2$, is called the Jacobi operator on Γ, which describes the first variation of $-\kappa(x; \Gamma)$. In fact $-\frac{3}{2\sqrt{2}}\mathscr{A}^\Gamma R(x)$ is the second variation of F_α at Γ in the direction of R.

EXAMPLE 2.2. In this example, we consider the case where the non-linearity f is of the form

(2.8) $$f(u, x, \varepsilon) \equiv h(x)^2(u^3 - u).$$

The function $h(x)$ is smooth and strictly positive on $\bar{\mathscr{D}}$. One can easily identify $\tilde{Q}_0(\tau; x), m(x)$ and $J(x; \Gamma)$ as

$$(2.9) \qquad \tilde{Q}_0(\tau; x) = \hat{Q}_0(h(x)\tau), \qquad m(x) = \frac{2\sqrt{2}}{3}h(x),$$

$$J(x; \Gamma) = -\frac{2\sqrt{2}}{3}\frac{\partial h(x)}{\partial v(x; \Gamma)} \qquad x \in \Gamma.$$

Therefore the free-interface problem in the present case is given by

$$(2.10) \qquad h(x)\kappa(x; \Gamma) + \frac{\partial h(x)}{\partial v(x; \Gamma)} = 0, \qquad x \in \Gamma.$$

This is the first variational equation for the functional $E_h(\Gamma)$ defined by

$$(2.11) \qquad E_h(\Gamma) := \int_\Gamma h(x)\,dS_x^\Gamma.$$

The free-interface problem (2.10) may look similar to (2.5), but it is substantially different, since in (2.10) the value of mean curvature at $x \in \Gamma$ depends not only on the the prescribed value $h(x)$, but also on the normal direction $v(x; \Gamma)$ of the free interface. To the best of our knowledge, there seems to be no general condition on h that ensures the existence of solutions to (2.10), except for a special case to be treated in §2.3 below. It is a natural strategy to find minimizers (or critical points) of E_h in order to obtain solutions of (2.10). This problem deserves a separate treatment.

Assuming that (2.10) has a smooth solution Γ, the linear operator in (1.14) is given by

$$(2.12) \qquad \frac{3}{2\sqrt{2}}\mathscr{A}^\Gamma R(x) \equiv h(x)\left(\Delta^\Gamma + \sum_{j=1}^{N-1}\kappa_j(x)^2\right)R(x) + \nabla_\Gamma h(x) \cdot \nabla_\Gamma R(x)$$

$$- \frac{\partial h(x)}{\partial v(x; \Gamma)}\kappa(x; \Gamma)R(x) - \frac{\partial^2 h(x)}{\partial v(x; \Gamma)^2}R(x),$$

which is the second variation of (2.11) at Γ.

EXAMPLE 2.3. Let the nonlinearity f in (1.1) be given by

$$(2.13) \qquad f(u, x, \varepsilon) \equiv u(u^2 - \phi(x)^2),$$

in which $\phi(x) > 0$ is a smooth function defined on $\bar{\mathscr{D}}$. Then we have

$$(2.14) \qquad \tilde{Q}_0(\tau; x) = \phi(x)\hat{Q}_0(\phi(x)\tau), \qquad m(x) = \frac{2\sqrt{2}}{3}\phi(x)^3,$$

$$J(x; \Gamma) = -c_*\phi(x)^2\frac{\partial \phi(x)}{\partial v(x; \Gamma)},$$

where

$$c_* = 2 \int_{\mathbf{R}} \hat{Q}_0(\tau) \frac{d\hat{Q}_0(\tau)}{d\tau} \tau \, d\tau = 2\sqrt{2}.$$

The free-interface problem is therefore written as

$$(2.15) \qquad \kappa(x; \Gamma)\phi(x)^3 + 3\phi(x)^2 \frac{\partial \phi(x)}{\partial \nu(x; \Gamma)} = 0 \qquad x \in \Gamma.$$

This is almost identical to (2.10). In fact, (2.15) is the first variational equation of the functional $\hat{E}_\phi(\Gamma)$ defined by

$$(2.16) \qquad \hat{E}_\phi(\Gamma) \equiv \int_{\Gamma} \phi(x)^3 dS_x^\Gamma.$$

Assuming that (2.15) has a smooth solution Γ, the operator $\mathscr{A}^\Gamma$ is given by

$$(2.17) \qquad \frac{3}{2\sqrt{2}} \mathscr{A}^\Gamma R(x) \equiv \phi(x)^3 \left(\Delta^\Gamma + \sum_{j=1}^{N-1} \kappa_j(x)^2 \right) R(x) + \nabla_\Gamma(\phi(x)^3) \cdot \nabla_\Gamma R(x)$$

$$- \frac{\partial \phi(x)^3}{\partial \nu(x; \Gamma)} \kappa(x; \Gamma) R(x) - \frac{\partial^2 \phi(x)^3}{\partial \nu(x; \Gamma)^2} R(x).$$

In the examples above, we have been naturally lead to geometric variational problems associated with the functionals F_α, E_h and $\hat{E}_\phi$, defined respectively by (2.6), (2.11) and (2.16). Critical points of these functionals correspond to the solutions of the free-interface problems (2.5), (2.10) and (2.15), respectively. The solutions of the latter problems in turn give rise to stationary solutions of (1.1) with internal transition layers on the free-interface. Also, the stability of the transition layer solutions is read off from the index of the critical points.

2.3. Analysis of Examples. In this subsection, the examples above will be analyzed when the inhomogeneity is radially symmetric. Let $\mathscr{D}$ contain the origin of $\mathbf{R}^N$ and α, h and ϕ depend only on $r = |x|$. We do not, however, assume that $\mathscr{D}$ is radially symmetric. In this situation, we look for the free-interface Γ among spheres with center at the origin. For the sake of definiteness, we assume that interiors of free surfaces correspond to $\mathscr{D}_\Gamma^{(-)}$. The other case where $\mathscr{D}_\Gamma^{(+)}$ corresponds to the interior of Γ can be treated similarly.

The free-interface problem is nothing but the equilibrium problem of the interface equation (1.7). It is therefore more illuminating to cast it in a dynamic version. When the initial interface is a sphere with center at the origin, the solution of (1.7) remains to be a concentric sphere, since the spatial

inhomogeneity is radially symmetric. Therefore, the dynamic versions of the free-interface problems (2.5), (2.10) and (2.15) are respectively written as:

(2.18)   (Example 2.1): $\dfrac{dr}{dt} = -\dfrac{d}{dr}\left(\log r^{N-1} - \int_0^r \alpha(s)ds\right),$

(Example 2.2): $\dfrac{dr}{dt} = -\dfrac{d}{dr}\log(r^{N-1}h(r)),$

(Example 2.3): $\dfrac{dr}{dt} = -\dfrac{d}{dr}\log(r^{N-1}\phi(r)^3).$

Therefore radial dynamics of (1.7) in these examples are determined by the profile of the corresponding potentials.

Let $r = r_0$ be an equilibrium point of (2.18). The stability of the corresponding interface Γ with respect to radial pertubations is determined by the sign of the second derivative of the potintials at $r = r_0$. The stability property with respect to non-radial perturbations is encoded in the eigenvalues of the linear operator in (2.7), (2.12), or (2.17).

One can easily verify that eigenfunctions of the operator are all spherical harmonics. The eigenvalue λ_j^Γ corresponding to spherical harmonics of degree $j \geq 0$ is given by

(2.19)   (Example 2.1): $\lambda_j^\Gamma = -\dfrac{j(j+N-2)}{r_0^2} - \dfrac{d^2}{dr^2}\left(\log r^{N-1} - \int_0^r \alpha(s)ds\right)\Big|_{r=r_0},$

(Example 2.2): $\lambda_j^\Gamma = -\dfrac{j(j+N-2)}{r_0^2} - \dfrac{d^2}{dr^2}\log(r^{N-1}h(r))\Big|_{r=r_0},$

(Example 2.3): $\lambda_j^\Gamma = -\dfrac{j(j+N-2)}{r_0^2} - \dfrac{d^2}{dr^2}\log(r^{N-1}\phi(r)^3)\Big|_{r=r_0}.$

Based upon the information provided by (2.19), Theorems 2.1 and 2.2 now apply to produce the existence and stability properties of the corresponding stationary solutions of (1.1) with internal transition layers.

Remark 2.2. Our theorem naturally applies to one-dimensional cases, where we can reproduce the results in [8].

Let $\mathscr{D} = (0,1) \subset \mathbf{R}$. The interface equation for Example 2.1 is given by

$$\frac{dr}{dt} = \alpha(r).$$

A simple zero $r_0 \in (0,1)$ of $\alpha(r)$ therefore gives rise to an equilibrium solution of (1.1) for small $\varepsilon > 0$, which is asymptotically stable if $\alpha'(r_0) < 0$ and unstable if $\alpha'(r_0) > 0$.

On the other hand, the interface equation for Example 2.4 is given by

$$\frac{dr}{dt} = -\frac{h'(r)}{h(r)}.$$

Therefore, a nondegenerate critical point $r_0 \in (0,1)$ of the function $h(r)$; $h'(r_0) = 0$, $h''(r_0) \neq 0$, gives rise to a family of internal transition layer solutions of (1.1) near $r = r_0$. The solutions are asymptotically stable (resp. unstable) if $h(r)$ attains a local minimum (resp. local maximum).

3. Proof of Theorem 2.1

We will prove Theorem 2.1 in this section. Since our proof is rather lengthy, we will delegate the computational details to §§4 and 5.

Let $u(x, \varepsilon)$ be the desired solution of Theorem 2.1. We define the *equilibrium interface* Γ_ε by

$$\Gamma_\varepsilon = \left\{ x \in \mathscr{D} \,|\, u(x, \varepsilon) = \frac{1}{2} (\phi^{(+)}(x) + \phi^{(-)}(x)) \right\}.$$

Without loss of generality, we may assume that $\phi^{(+)}(x) + \phi^{(-)}(x) \equiv 0$ on $\bar{\mathscr{D}}$. For, if not, we change the unknown u by $u \mapsto u - \frac{1}{2}(\phi^{(+)}(x) + \phi^{(-)}(x))$. Then the new unknown satisfies

$$\varepsilon^2 \Delta u = \bar{f}(u, x, \varepsilon),$$

where

$$\bar{f}(u, x, \varepsilon) = f\left(u + \frac{1}{2}(\phi^{(+)}(x) + \phi^{(-)}(x)), x, \varepsilon \right) + \frac{\varepsilon^2}{2} \Delta(\phi^{(+)}(x) + \phi^{(-)}(x)).$$

For this new nonlinearity $\bar{f}$, the conditions **(A1)** and **(A2)** are fulfilled. We therefore define the interface Γ_ε by

(3.1) $$\Gamma_\varepsilon = \{ x \in \mathscr{D} \,|\, u(x, \varepsilon) = 0 \} \in \mathscr{F},$$

which is unknown a priori. The condition **(A3)** demands that $\lim_{\varepsilon \to 0} \Gamma_\varepsilon$ exists. Therefore we expect that Γ_ε is expressed as the graph of a function $R^\varepsilon(y)$ over $\Gamma := \lim_{\varepsilon \to 0} \Gamma_\varepsilon$ in terms of the local coordinate system (y, r) in (1.10) and (1.11):

(3.2) $$\Gamma_\varepsilon = \{ x \in \mathscr{D} \,|\, x = y + \varepsilon R^\varepsilon(y) \nu(y), \, y \in \Gamma \}.$$

We will first establish the existence of approximate solutions with an arbitrarily high accuracy.

Proposition 3.1.   *For any integer $k \geq 2$, one can choose a smooth function $R_k^\varepsilon(y)$ $(y \in \Gamma)$ for which there exists a family of smooth approximate solutions $u_k^\varepsilon(x)$ of the problem*

$$(3.3) \qquad \begin{cases} \varepsilon^2 \Delta u - f(u, x, \varepsilon) = 0 & (x \in \mathcal{D}), \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & (x \in \partial \mathcal{D}), \end{cases}$$

such that

$$\{x \in \mathcal{D} \mid u_k^\varepsilon(x) = 0\} = \{x \in \mathcal{D} \mid x = y + \varepsilon R_k^\varepsilon(y)v(y), y \in \Gamma\}$$

and

$$\frac{\partial u_k^\varepsilon}{\partial \mathbf{n}} = 0 \quad (x \in \partial \mathcal{D}), \qquad \|\varepsilon^2 \Delta u_k^\varepsilon - f(u_k^\varepsilon, x, \varepsilon)\|_{C_\varepsilon^\alpha(\bar{\mathcal{D}})} = O(\varepsilon^{k+1-\alpha-\alpha'})$$

for $0 < \alpha < 1$ and $0 < \alpha' \leq 1$. Here $C_\varepsilon^\alpha(\bar{\mathcal{D}})$ is the usual Hölder space $C^\alpha(\bar{\mathcal{D}})$ with the weighted norm

$$\|p\|_{C_\varepsilon^\alpha(\bar{\mathcal{D}})} = \sup_{x \in \bar{\mathcal{D}}} |p(x)| + \varepsilon^\alpha \sup_{x \neq x' \in \bar{\mathcal{D}}} \frac{|p(x) - p(x')|}{|x - x'|^\alpha}.$$

We give here some comments on Proposition 3.1, although it will be proven in §4.

The function $R_k^\varepsilon(y)$ is given by a finite sum

$$R_k^\varepsilon(y) = R_1(y) + \varepsilon R_2(y) + \cdots + \varepsilon^{k-2} R_{k-1}(y),$$

in which the functions $R_j(y)$ $(0 \leq j \leq k-1)$ are determined by solving linear elliptic equations on Γ:

$$(3.4) \qquad \mathscr{A}^\Gamma R_1 = h_1(y) \text{ (a known function)} \qquad (j = 1),$$

$$(3.5) \qquad \mathscr{A}^\Gamma R_j = h_j(y, R_1, \ldots, R_{j-1})$$

$$\text{(a function determined by } R_1, \ldots, R_{j-1}) \qquad (j \geq 2).$$

Thanks to the condition **(A4)**, the operator $\mathscr{A}^\Gamma$ is invertible and hence (3.4) and (3.5) are uniquely solvable. The elliptic equations (3.4) and (3.5) will appear as C^1-matching conditions in the asymptotic expansions in §4.

We now linearize (3.3) around the approximate solution u_k^ε:

$$L_k^\varepsilon \varphi := \varepsilon^2 \Delta \varphi - f_u(u_k^\varepsilon, x, \varepsilon)\varphi,$$

and consider the associated eigenvalue problem:

$$(3.6) \qquad \begin{cases} L_k^\varepsilon \varphi = \lambda \varphi & (x \in \mathscr{D}), \\ \dfrac{\partial \varphi}{\partial \mathbf{n}} = 0 & (x \in \partial \mathscr{D}). \end{cases}$$

DEFINITION 3.2. An eigenvalue λ^ε of (3.6) is called a *non-critical eigenvalue* if there exist constants $\delta > 0$, $\varepsilon_* > 0$ such that

$$|\lambda^\varepsilon| \geq \delta \qquad (\forall \varepsilon \in (0, \varepsilon_*]).$$

An eigenvalue of (3.6) is called a *critical eigenvalue* if it is not non-critical.

The following result plays a decisive role in our proof of Theorem 2.1.

PROPOSITION 3.3. (i) *There exists $\delta_* > 0$ and $\varepsilon_* > 0$ so that non-critical eigenvalues λ^ε of* (3.6) *satisfy*

$$\lambda^\varepsilon \leq -\delta_*, \qquad \forall \varepsilon \in (0, \varepsilon_*].$$

(ii) *Let λ^ε be a critical eigenvalue of* (3.6) *with $k \geq 2$. Then*

$$\lim_{\varepsilon \to 0} \lambda^\varepsilon = 0 \qquad and \qquad \lim_{\varepsilon \to 0} \frac{\lambda^\varepsilon}{\varepsilon^2} = \lambda^* \in \sigma_\Gamma.$$

We will prove this proposition in §5.

From this proposition and the condition **(A4)**, we immediately conclude that the linear operator L_k^ε is invertible. This also allows us to establish the following.

PROPOSITION 3.4. *There exist constants $C > 0$ and $\varepsilon_* > 0$ so that the estimate*

$$\|(L_k^\varepsilon)^{-1}\|_{C_\varepsilon^{1/2}(\mathscr{D}) \to C_\varepsilon^{1/2}(\mathscr{D})} \leq \frac{C}{\varepsilon^{2[N/4]+4}} \qquad for \ \varepsilon \in (0, \varepsilon_*]$$

is valid, where $\left[\frac{N}{4}\right]$ stands for the integer-part of one-quater of the dimension N.

PROOF. Proposition 3.3 and the condition **(A4)** imply that

$$L_k^\varepsilon : W^{2,2}(\mathscr{D}) \to L^2(\mathscr{D})$$

is invertible for $\varepsilon \in (0, \varepsilon_*]$ (where $\varepsilon_* > 0$ is adequately small), and that there exists $C > 0$ such that

$$(3.7) \qquad \|u\|_{L^2} \leq \frac{C}{\varepsilon^2} \|v\|_{L^2}, \qquad \varepsilon \in (0, \varepsilon_*],$$

where $v \in C^{1/2}(\bar{\mathscr{D}})$ and $u = (L_k^\varepsilon)^{-1} v$.

In the sequel, we simply use the same symbol C to denote positive constants, independent of ε, which may differ from line to line.

By using the L^p-estimates for the Laplacian in the equation

$$\Delta u = \frac{1}{\varepsilon^2} \{ f_u(u_k^\varepsilon, x, \varepsilon) u + v \},$$

we have

(3.8) $\|u\|_{W^{2,p}(\mathscr{D})} \le \dfrac{C}{\varepsilon^2} (\|u\|_{L^p} + \|v\|_{L^p})$ $(1 < \forall p < \infty).$

Our strategy of proof is to use (3.8), combined with the Sobolev embeddings

(3.9) $\|u\|_{L^p} \le C\|u\|_{W^{2,p'}},$ $1 \le p \le \dfrac{p'N}{N-2p'},$ if $N \ge 2p',$

and

(3.10) $\|u\|_{C^\alpha} \le C\|u\|_{W^{2,p}},$ with $\alpha = \min\left\{1, 2 - \dfrac{N}{p}\right\},$ if $2p > N.$

1. For $N = 1, 2, 3$, we first use (3.10) with $p = 2$. Taking $\alpha = 1/2$, and using (3.8) (with $p = 2$) and (3.7), we have

$$\|u\|_{C_\varepsilon^{1/2}} \le \|u\|_{C^{1/2}} \le C\|u\|_{W^{2,2}} \le \frac{C}{\varepsilon^2} (\|u\|_{L^2} + \|v\|_{L^2})$$

$$\le \frac{C}{\varepsilon^2} \left(\frac{C}{\varepsilon^2} \|v\|_{L^2} + \|v\|_{L^2} \right) \le \frac{C}{\varepsilon^4} \|v\|_{C_\varepsilon^{1/2}}.$$

This completes the proof for $N = 1, 2, 3$.

2. For $N \ge 4$, we argue as follows. The estimate

(3.11) $\|u\|_{W^{2,2}} \le \dfrac{C}{\varepsilon^4} \|v\|_{L^2},$ $\forall N \ge 1$

has been already established above by using (3.8) and (3.7). For each integer $j \ge 0$, we will establish the esimate

(3.12) $\|u\|_{W^{2,2N/(N-4j)}} \le \dfrac{C}{\varepsilon^{2j+4}} \|v\|_{L^{2N/(N-4j)}},$ $\forall N \ge 4j.$

Let us prove (3.12) by induction. For $j = 0$, (3.12) reduces to (3.11). Assume now that (3.12) is true for some $j \ge 0$. By using (3.9) with $p' = \frac{2N}{N-4j}$, we obtain, for $N \ge 4(j+1)$,

$$\|u\|_{L^p} \le C\|u\|_{W^{2,2N/(N-4j)}},   1 \le \forall p \le \frac{2N}{N-4(j+1)}.$$

Using this with $p = \frac{2N}{N-4(j+1)}$ in (3.8), and then using (3.12) in the resulting estimate, we obtain

$$\|u\|_{W^{2,2N/(N-4(j+1))}} \leq \frac{C}{\varepsilon^{2(j+1)+4}} \|v\|_{L^{2N/(N-4(j+1))}}, \qquad \forall N \geq 4(j+1).$$

Therefore, we have established (3.12) for all nonnegative integer j.

On the other hand, if $N = 4j, 4j+1, 4j+2, 4j+3$, then $2 \times \frac{2N}{N-4j} > N$, and hence (3.10) applies with $\alpha = 1/2$, giving rise to

$$\|u\|_{C^{1/2}} \leq C\|u\|_{W^{2,2N/(N-4j)}}.$$

Since $\|u\|_{C_\varepsilon^{1/2}} \leq \|u\|_{C^{1/2}}$ and $\|v\|_{L^p} \leq C\|v\|_{C_\varepsilon^{1/2}}$ for any $p \geq 1$, the last estimate and (3.12) establish for each $j \geq 0$ that

$$\|u\|_{C_\varepsilon^{1/2}} \leq \frac{C}{\varepsilon^{2j+4}} \|v\|_{C_\varepsilon^{1/2}}, \qquad \text{for } N = 4j, 4j+1, 4j+2, 4j+3.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

We are now ready to prove Theorem 2.1.

PROOF OF THEOREM 2.1. The method of proof presented below is a generalization of an idea first employed in [5].

We look for a true solution $u(x, \varepsilon)$ of (3.3) near the approximate solution u_k^ε:

$$u(x, \varepsilon) = u_k^\varepsilon(x) + \varphi(x).$$

Problem (3.3) is now recast, in terms of φ, as

$$(3.13) \qquad \begin{cases} L_k^\varepsilon \varphi = \mathcal{N}^\varepsilon(\varphi) + \mathcal{R}^\varepsilon & \text{in } \mathcal{D}, \\ \dfrac{\partial \varphi}{\partial \mathbf{n}} = 0 & \text{on } \partial\mathcal{D}, \end{cases}$$

where

$$\begin{cases} \mathcal{N}^\varepsilon(\varphi) = f(u_k^\varepsilon + \varphi, x, \varepsilon) - f(u_k^\varepsilon, x, \varepsilon) - f_u(u_k^\varepsilon, x, \varepsilon)\varphi, \\ \mathcal{R}^\varepsilon = -\varepsilon^2 \Delta u_k^\varepsilon + f(u_k^\varepsilon, x, \varepsilon). \end{cases}$$

Replacing φ by $\varepsilon^l \tilde{\varphi}$ with $l = 2\left[\frac{N}{4}\right] + 4$, we rewrite (3.13) as follows:

$$(3.14) \qquad \tilde{\varphi} = \mathcal{S}^\varepsilon \tilde{\varphi} := \varepsilon^{-l}(L_k^\varepsilon)^{-1}\{\mathcal{N}^\varepsilon(\varepsilon^l \tilde{\varphi}) + \mathcal{R}^\varepsilon\}.$$

Applying Proposition 3.1 with $k = 2l$, we have

$$\|\mathcal{R}^\varepsilon\|_{C_\varepsilon^{1/2}(\bar{\mathcal{D}})} = O(\varepsilon^{2l+1/2}).$$

Since $|\mathcal{N}^\varepsilon(\varphi)| = O(|\varphi|^2)$, Proposition 3.4 yields

$$\|\varepsilon^{-l}(L_k^\varepsilon)^{-1}\mathcal{N}^\varepsilon(\varepsilon^l\tilde{\varphi})\|_{C_\varepsilon^{1/2}(\bar{\mathscr{D}})} = O(\|\tilde{\varphi}\|_{C^0(\bar{\mathscr{D}})}\|\tilde{\varphi}\|_{C_\varepsilon^{1/2}(\bar{\mathscr{D}})})$$

$$\|\varepsilon^{-l}(L_k^\varepsilon)^{-1}\mathcal{R}^\varepsilon\|_{C_\varepsilon^{1/2}(\bar{\mathscr{D}})} = O(\varepsilon^{1/2}).$$

These allow us to show that $\mathscr{S}^\varepsilon : C_\varepsilon^{1/2}(\bar{\mathscr{D}}) \to C_\varepsilon^{1/2}(\bar{\mathscr{D}})$ is a contraction on a small ball of radius $O(\varepsilon^{1/2})$ around zero in $C_\varepsilon^{1/2}(\bar{\mathscr{D}})$. Therefore, the fixed-point equation (3.14) has a unique solution $\tilde{\varphi} = \tilde{\varphi}^\varepsilon$ with $\|\tilde{\varphi}^\varepsilon\|_{C_\varepsilon^{1/2}(\bar{\mathscr{D}})} = O(\varepsilon^{1/2})$. The desired solution is then given by

$$u(x,\varepsilon) = u_k^\varepsilon(x) + \varphi^\varepsilon(x) \qquad \text{with } \|\varphi^\varepsilon\|_{C_\varepsilon^{1/2}(\bar{\mathscr{D}})} = O(\varepsilon^{l+1/2}).$$

This completes the proof of the existence-part of Theorem 2.1.

In order to prove the stability-part of the theorem, we consider the critical eigenvalues of

$$L^\varepsilon = \varepsilon^2\Delta - f_u(u(x,\varepsilon),x,\varepsilon).$$

However, we have

$$L^\varepsilon = L_k^\varepsilon - [f_u(u(x,\varepsilon),x,\varepsilon) - f_u(u_k^\varepsilon(x),x,\varepsilon)]$$

and

$$\|f_u(u(\cdot,\varepsilon),\cdot,\varepsilon) - f_u(u_k^\varepsilon,\cdot,\varepsilon)\|_{L^\infty} = O(\varepsilon^{l+1/2}).$$

Therefore, the critical eigenvalues of L^ε are at most $O(\varepsilon^{l+1/2})$ away $(l \geq 4)$ from those of L_k^ε. In other words, Proposition 3.3 is also valid for the critical eigenvalues of L^ε. This fact establishes the stability properties in Theorem 2.1.

4. Asymptotic expansion of approximate solutions

In this section, we will prove Proposition 3.1.

The construction of the approximate solution u_k^ε in Proposition 3.1 consists of three parts; outer expansion, boundary correction, and inner expansion. The outer expansion deals with the approximation in the bulk regions $(\mathscr{D}_\Gamma^{(\pm)})$. The outer approximation in general does not satisfy the boundary conditions in (3.3). The boundary correction then modifies the outer approximation so as to satisfy the boundary conditions. The inner expansion takes care of the sharp transition behavior of u_k^ε near the interface Γ_ε where a stretched spatial scale is to be introduced.

4.1. Outer Expansion. We substitute the formal expression

$$(4.1) \qquad\qquad u(x) = \bar{u}_\varepsilon^{(\pm)}(x) := \sum_{j\geq 0}\varepsilon^j\bar{u}_j^{(\pm)}(x)$$

into the differential equation in (3.3). The superscripts "$(\pm)$" indicate that the relevant functions are defined on the two subdomains $\mathscr{D}_\Gamma^{(\pm)}$, respectively. We then expand the left hand side of the resulting equation in the ε-power series. Equating to zero the coefficient of each power of ε, we obtain an array of equations. The lowest order equation is

(4.2)
$$0 = f(\bar{u}_0^{(\pm)}, x, 0).$$

According to **(A1)**, we choose

(4.3)
$$\begin{cases} \bar{u}_0^{(-)}(x) = \phi^{(-)}(x), & (x \in \mathscr{D}_\Gamma^{(-)}), \\ \bar{u}_0^{(+)}(x) = \phi^{(+)}(x), & (x \in \mathscr{D}_\Gamma^{(+)}). \end{cases}$$

The equations for $\bar{u}_j^{(\pm)}(x)$ $(j \geq 1)$ are given by

(4.4)
$$0 = f_u(\phi^{(\pm)}(x), x, 0)\bar{u}_1^{(\pm)}(x) + f_\varepsilon(\phi^{(\pm)}(x), x, 0), \qquad (x \in \mathscr{D}_\Gamma^{(\pm)}),$$

for $j = 1$ and

(4.5)
$$f_u(\phi^{(\pm)}(x), x, 0)\bar{u}_j^{(\pm)}(x)$$
$$= \Delta\bar{u}_{j-2}^{(\pm)}(x) + \text{function depending on } \bar{u}_0^{(\pm)}(x), \ldots, \bar{u}_{j-1}^{(\pm)}(x) \quad (x \in \mathscr{D}_\Gamma^{(\pm)})$$

for $j \geq 2$. Thanks to **(A1)**, we have $f_u(\phi^{(\pm)}(x), x, 0) > 0$ on $\bar{\mathscr{D}}_\Gamma^{(\pm)}$, and hence $\bar{u}_j^{(\pm)}(x)$ is uniquely determined successively for $j = 1, 2, \ldots$.

4.2. Inner Expansion. There is a jump between the outer expansions $\sum_{j \geq 0} \varepsilon^j \bar{u}_j^{(-)}(x)$ and $\sum_{j \geq 0} \varepsilon^j \bar{u}_j^{(+)}(x)$ on Γ. The inner expansion bridges the jump by introducing sharp transition layers along Γ. In order to describe the transiton layers, it is adequate to work with the local coordinate system (y, r) near Γ, defined in (1.10). A function $p(x)$ of $x \in \Gamma^{(\delta)}$ is also expressed as $p(r, y)$, with the relation $x = y + rv(y)$ being understood. The symbol $p_r(r, y)$, for example, means

$$p_r(r, y) = \frac{\partial}{\partial r} p(r, y) = \nabla p(x) \cdot v(x).$$

We now introduce a stretched spatial scale in the r-direction. However, note that the set $\{r = 0\}$ simply corresponds to the reduced interface Γ, while our interface Γ_ε is given by (3.1), (3.2). Therefore, we define the stretched variable z by

(4.6)
$$z = \frac{r - \varepsilon R^\varepsilon(y)}{\varepsilon}, \qquad \text{or} \qquad r = \varepsilon z + \varepsilon R^\varepsilon(y),$$

with

$$R^\varepsilon(y) = \sum_{j\geq 0} \varepsilon^j R_{j+1}(y).$$

We will substitute into (3.3) the formal expression

$$(4.7) \qquad u^\varepsilon(x) = u^\varepsilon(r, y) := \sum_{j\geq 0} \varepsilon^j \bar{u}_j^{(\pm)}(\varepsilon z + \varepsilon R^\varepsilon(y), y) + \sum_{j\geq 0} \varepsilon^j Q_j^{(\pm)}(z, y)$$

$$= \sum_{j\geq 0} \varepsilon^j q_j^{(\pm)}(z, y),$$

where

$$(4.7\text{-}0) \qquad q_0^{(\pm)}(z, y) = \bar{u}_0^{(\pm)}(0, y) + Q_0^{(\pm)}(z, y) = \phi^{(\pm)}(0, y) + Q_0^{(\pm)}(z, y),$$

$$(4.7\text{-}1) \qquad q_1^{(\pm)}(z, y) = \bar{u}_1^{(\pm)}(0, y) + (z + R_1)\bar{u}_{0r}^{(\pm)}(0, y) + Q_1^{(\pm)}(z, y)$$

$$(4.7\text{-}2) \qquad q_2^{(\pm)}(z, y) = \bar{u}_2^{(\pm)}(0, y) + (z + R_1)\bar{u}_{1r}^{(\pm)}(0, y)$$

$$+ \frac{1}{2}(z + R_1)^2 \bar{u}_{0rr}^{(\pm)}(0, y) + R_2 \bar{u}_{0r}^{(\pm)}(0, y) + Q_2^{(\pm)}(z, y)$$

$$(4.7\text{-}k) \qquad q_k^{(\pm)}(z, y) = Q_k^{(\pm)}(z, y) + \sum_{j=0}^{k} \frac{1}{(k-j)!} \left(\frac{d^{k-j}}{d\varepsilon^{k-j}} \bar{u}_j^{(\pm)}(\varepsilon z + \varepsilon R^\varepsilon, y) \bigg|_{\varepsilon=0} \right)$$

for general $k \geq 2$.

In order to write down equations for $q_k^{(\pm)}$, we need to express the Laplacian Δ in terms of the local coordinates (r, y) and (z, y).

The Laplacian Δ is expressed as

$$(4.8) \qquad\qquad \Delta = \frac{\partial^2}{\partial r^2} + \kappa(r, y)\frac{\partial}{\partial r} + \Delta^\Gamma(r, y),$$

in terms of the coordinate system (r, y) in (3.1) and (3.2), where

$\kappa(r, y)$:   the sum of principal curvatures (the mean curvature, for short) of $\Gamma(r)$,

$$\Gamma(r) := \{x \in \mathcal{D} \,|\, x = y + rv(y), y \in \Gamma\}$$

at $x = y + rv(y) \in \Gamma(r)$,

$\Delta^\Gamma(r, y)$:   the Laplace-Beltrami operator on $\Gamma(r)$ acting on functions of y. In order to obtain equations for $q_k^{(\pm)}$, we further need to express (4.8) in terms of (z, y).

LEMMA 4.1.   *The Laplacian in* (4.8) *is expressed as follows.*

(4.9) $$\varepsilon^2 \varDelta = \frac{\partial^2}{\partial z^2} + \varepsilon \kappa(0, y) \frac{\partial}{\partial z} + \sum_{j \geq 2} \varepsilon^j [-M_{j-1} + P_{j-2}],$$

where

(4.10)   (a) $$M_1 = (\varDelta^\Gamma R_1) \frac{\partial}{\partial z} + 2 \nabla_\Gamma R_1 \cdot \nabla_\Gamma \left(\frac{\partial}{\partial z} \right) - \kappa_r(0, y) R_1 \frac{\partial}{\partial z} - |\nabla_\Gamma R_1|^2 \frac{\partial^2}{\partial z^2},$$

         (b) $$M_j = (\varDelta^\Gamma R_j) \frac{\partial}{\partial z} + 2 \nabla_\Gamma R_j \cdot \nabla_\Gamma \left(\frac{\partial}{\partial z} \right)$$

$$- \kappa_r(0, y) R_j \frac{\partial}{\partial z} - 2 \nabla_\Gamma R_1 \cdot \nabla_\Gamma R_j \frac{\partial^2}{\partial z^2} \qquad (j \geq 2),$$

         (c) $$P_0 = \varDelta^\Gamma + z \kappa_r(0, y) \frac{\partial}{\partial z},$$

         (d) P_j: *dierential operator depending only on* $R_1, \ldots, R_j$.

PROOF.   Let us denote by $(g_{ij}) = (g_{ij}(r, y))$ $(i, j = 1, \ldots, N-1)$ the cova-riant metric tensor on $\Gamma(r)$, induced from the Euclidean metric in $\mathbf{R}^N$, at $x = y + r\nu(y)$. We also use the symbols $g = \det(g_{ij})$ and $(g^{ij}) = (g_{ij})^{-1}$. Under the change of variables in (4.6), we have

$$\frac{\partial}{\partial r} \mapsto \frac{1}{\varepsilon} \frac{\partial}{\partial z}, \qquad \frac{\partial^2}{\partial r^2} \mapsto \frac{1}{\varepsilon^2} \frac{\partial^2}{\partial z^2},$$

and

$$\frac{\partial}{\partial y_i} \mapsto \frac{\partial}{\partial y_i} - \frac{\partial R^\varepsilon}{\partial y_i} \frac{\partial}{\partial z}.$$

Using (4.8), $\varepsilon^2 \varDelta$ is therefore written as

(4.8-ε) $$\varepsilon^2 \varDelta = \frac{\partial^2}{\partial z^2} + \varepsilon \kappa(\varepsilon z + \varepsilon R^\varepsilon, y) \frac{\partial}{\partial z} + \varepsilon^2 \tilde{\varDelta}_\varepsilon^\Gamma(z, y),$$

where

$$\tilde{\varDelta}_\varepsilon^\Gamma(z, y) = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{N-1} \left(\frac{\partial}{\partial y_i} - \frac{\partial R^\varepsilon}{\partial y_i} \frac{\partial}{\partial z} \right) \left\{ \sqrt{g} g^{ij} \left(\frac{\partial}{\partial y_j} - \frac{\partial R^\varepsilon}{\partial y_j} \frac{\partial}{\partial z} \right) \right\}.$$

One can compute this explicitly as follows.

(4.11) $\tilde{\Delta}_{\varepsilon}^{\Gamma}(z, y) = \Delta^{\Gamma}(\varepsilon z + \varepsilon R^{\varepsilon}, y) - [\Delta^{\Gamma}(\varepsilon z + \varepsilon R^{\varepsilon}, y)R^{\varepsilon}]\dfrac{\partial}{\partial z}$

$$- 2\nabla_{\Gamma} R^{\varepsilon} \cdot \nabla_{\Gamma}\left(\dfrac{\partial}{\partial z}\right) + |\nabla_{\Gamma} R^{\varepsilon}|^2 \dfrac{\partial^2}{\partial z^2}$$

$$- \dfrac{\varepsilon}{\sqrt{g}} \sum_{i,j=1}^{N-1} \dfrac{\partial}{\partial r}(\sqrt{g}g^{ij})|_{r=\varepsilon z+\varepsilon R^{\varepsilon}} \left(\dfrac{\partial R^{\varepsilon}}{\partial y_i}\right)\dfrac{\partial}{\partial y_j}$$

$$+ \dfrac{\varepsilon}{\sqrt{g}} \sum_{i,j=1}^{N-1} \dfrac{\partial}{\partial r}(\sqrt{g}g^{ij})|_{r=\varepsilon z+\varepsilon R^{\varepsilon}} \left(\dfrac{\partial R^{\varepsilon}}{\partial y_i}\right)\left(\dfrac{\partial R^{\varepsilon}}{\partial y_j}\right)\dfrac{\partial}{\partial z}.$$

Substituting $R^{\varepsilon} = \sum_{k\geq 1} \varepsilon^{k-1} R_k$ into (4.8-ε) and (4.11), and expanding the resulting equation in the ε-power series, we obtain (4.9) and (4.10).   □

We are now ready to write down the equations for $q_k^{(\pm)}(z, y)$. In the sequel, partial differentiation with respect to z will be denoted by "dot", $\dot{q}$. We also use the short hand notation $f(u, r, y, \varepsilon)$ in place of $f(u, x, \varepsilon)$ with the relation $x = y + rv(y)$ being understood.

The equations for $q_k^{(\pm)}$ are given as follows.

(4.12) $\ddot{q}_0^{(\pm)} = f(q_0^{(\pm)}, 0, y, 0),$

(4.12-1) $\ddot{q}_1^{(\pm)} - f_u(q_0^{(\pm)}, 0, y, 0)q_1^{(\pm)} = F_1^{(\pm)}(z)$

$$:= -\kappa\dot{q}_0^{(\pm)} + f_r(q_0^{(\pm)}, 0, y, 0)(z + R_1) + f_{\varepsilon}(q_0^{(\pm)}, 0, y, 0),$$

where $\kappa = \kappa(0, y)$, $f_r = \frac{\partial}{\partial r}f$, and $f_{\varepsilon} = \frac{\partial}{\partial \varepsilon}f$. In the sequel, $f(*)$ means $f(q_0^{(\pm)}, 0, y, 0)$.

(4.12-2) $\ddot{q}_2^{(\pm)} - f_u(*)q_2^{(\pm)} = F_2^{(\pm)}(z)$

$$:= -\kappa\dot{q}_1^{(\pm)} + M_1 q_0^{(\pm)} - P_0 q_0^{(\pm)} + f_r(*)R_2$$

$$+ \dfrac{1}{2}f_{uu}(*)(q_1^{(\pm)})^2 + f_{ur}(*)(z + R_1)q_1^{(\pm)} + f_{u\varepsilon}(*)q_1^{(\pm)}$$

$$+ f_{r\varepsilon}(*)(z + R_1) + \dfrac{1}{2}f_{rr}(*)(z + R_1)^2 + \dfrac{1}{2}f_{\varepsilon\varepsilon}(*),$$

(4.12-k) $\ddot{q}_k^{(\pm)} - f_u(*)q_k^{(\pm)} = F_k^{(\pm)}(z)$

$$:= -\kappa\dot{q}_{k-1}^{(\pm)} + M_{k-1}q_0^{(\pm)} + f_r(*)R_k$$

$$+ f_{uu}(*)q_1^{(\pm)}q_{k-1}^{(\pm)} + f_{ur}(*)R_{k-1}q_1^{(\pm)} + f_{u\varepsilon}(*)q_{k-1}^{(\pm)}$$

$$+ f_{ur}(*)(z + R_1)q_{k-1}^{(\pm)} + f_{r\varepsilon}(*)R_{k-1} + f_{rr}(*)(z + R_1)R_{k-1} + \bar{f}_k^{(\pm)}.$$

The symbol $\bar{f}_k^{(\pm)}$ in the last of (4.12-k) represents terms which depend only on $q_0^{(\pm)}, \ldots, q_{k-2}^{(\pm)}$ and $R_1, \ldots, R_{k-2}$ ($k \geq 3$). We consider the equations in (4.12), (4.12-k) as defined on $(-\infty, 0)$ (for superscript $(-)$) and $(0, \infty)$ (for superscript $(+)$), respectively. These equations are supplemented by the following conditions.

(4.13) $$q_k^{(\pm)}(0, y) \equiv 0 \qquad \text{(interface condition)},$$

(4.14) $$Q_k^{(\pm)}(z, y) = O(e^{-\beta|z|})$$

$$\text{as } |z| \to \infty \text{ for some } \beta > 0 \quad \text{(inner-outer matching)},$$

(4.15) $$\dot{q}_k^{(-)}(0, y) = \dot{q}_k^{(+)}(0, y) \qquad (C^1\text{-mactching condition}).$$

The conditions (4.13) come from the definition of the interface $\Gamma_\varepsilon = \{x \mid u(x, \varepsilon) = 0\}$. The conditions (4.14) are called the inner-outer matching conditions. The exponential decay in these conditions guarantees that the inner corrections ($Q_k^{(\pm)}(z, y)$) do not disrupt the outer approximation in the region away from the interface. The conditions (4.15) are the C^1-matching conditions. Once the conditions in (4.13) and (4.15) are satisfied, the solutions $q_k^{(-)}(z, y)$ ($z \in (-\infty, 0]$) and $q_k^{(+)}(z, y)$ ($z \in [0, \infty)$) are joined smoothly across $z = 0$, giving rise to a smooth function $q_k(z, y)$ defined for $z \in \mathbf{R}$:

$$q_k(z, y) = \begin{cases} q_k^{(-)}(z, y), & (z \in (-\infty, 0]), \\ q_k^{(+)}(z, y), & (z \in [0, \infty)). \end{cases}$$

It will be shown that the C^1-matching conditions are equivalent to (3.4) and (3.5).

The equations (4.12) have trival solutions $q_0^{(\pm)}$ which also satisfy the conditions (4.13), (4.14), and (4.15) with $k = 0$:

$$q_0^{(-)} = \tilde{Q}_0(z; y) \quad (z \in (-\infty, 0]), \qquad q_0^{(+)} = \tilde{Q}_0(z; y) \quad (z \in [0, \infty)),$$

where $\tilde{Q}_0$ is the function defined as the unique solution of (1.4). Note that we have normalized $\tilde{Q}_0$ so that $\tilde{Q}_0(0; y) \equiv 0$. From now on, we do not distinguish $q_0^{(\pm)}$ and simply denote them as q_0. *Note also that q_0 is defined for all $x \in \bar{\mathscr{D}}$ via $q_0(z, x) := \tilde{Q}_0(z; x)$. We therefore denote by $q_0(z, r, y)$ the extended function $\tilde{Q}_0(z; y + rv(y))$.*

The equations (4.12-k) ($k \geq 1$) are inhomogeneous linear ordinary differential equations for $q_k^{(\pm)}$ with $y \in \Gamma$ being a parameter:

(4.16) $$\left[\frac{d^2}{dz^2} - f_u(q_0(z, y), 0, y, 0) \right] q_k^{(\pm)} = F_k^{(\pm)}(z) \qquad (k \geq 1).$$

It is easily shown that (4.16) have unique solutions $q_k^{(\pm)}(z, y)$ ($\pm z \in [0, \infty)$) that satisfy the conditions (4.13) and (4.14):

(4.17)

$$q_k^{(\pm)}(z, y) = \dot{q}_0(z) \int_0^z \frac{1}{[\dot{q}_0(z')]^2} \int_{\pm\infty}^{z'} \dot{q}_0(z'', y) F_k^{(\pm)}(z'') dz'' dz', \qquad \pm z \in [0, \infty).$$

From (4.17), one finds that the C^1-matching condition (4.15) is equivalent to

(4.18) $$\int_{\mathbf{R}} \dot{q}_0(z, y) F_k(z) dz = 0 \qquad (k \geq 1),$$

where $F_k(z) = F_k^{(\pm)}(z)$ for $\pm z \in [0, \infty)$.

Let us apply (4.18) to (4.12-1). We need not distinguish $F_1^{(\pm)}$, since they constitute a smooth function $F_1(z)$ defined on $\mathbf{R}$, thanks to the C^1-matching condition for $q_0^{(\pm)}$. One can also see that two functions $F_k^{(\pm)}(z)$ give rise to the smooth function $F_k(z)$ defined on $\mathbf{R}$ as soon as the C^1-matching conditions for $q_j^{(\pm)}$ $(0 \leq j \leq k - 1)$ are satisfied.

The condition (4.18) for $k = 1$ is

$$0 = -\kappa \int_{\mathbf{R}} [\dot{q}_0(z)]^2 dz + \int_{\mathbf{R}} z f_r(q_0(z), 0, y, 0) \dot{q}_0(z) dz$$

$$+ R_1 \int_{\mathbf{R}} f_r(q_0(z), 0, y, 0) \dot{q}_0(z) dz + \int_{\mathbf{R}} f_\varepsilon(q_0(z), 0, y, 0) \dot{q}_0(z) dz.$$

Recall that $q_0(z, y) = \tilde{Q}_0(z; y)$, and hence $\int_{\mathbf{R}} [\dot{q}_0(z)]^2 dz = m(0, y)$. We also have

$$\int_{\mathbf{R}} f_r(q_0(z), 0, y, 0) \dot{q}_0(z) dz = \int_{\phi^{(-)}(0, y)}^{\phi^{(+)}(0, y)} f_r(u, 0, y, 0) du \equiv \frac{\partial}{\partial r} c(r, y)|_{r=0} = 0,$$

because $c(r, y) \equiv 0$ according to **(A2)**. Therefore the C^1-matching condition for $q_1^{(\pm)}$ is written as

(4.19) $$0 = -\kappa(x) m(x) + J(x; \Gamma), \qquad x \in \Gamma.$$

The condition **(A3)** says that the reduced interface Γ satisfies this equation. Note that the solutions $q_1^{(\pm)}$ depend on R_1, but the C^1-matching condition on them does not.

PROPOSITION 4.2. *The C^1-matching condition (4.15) (or (4.18)) for $k \geq 2$ is equivalent to*

(4.20) $$\begin{cases} \mathscr{A}^\Gamma R_1 = h_1(y), & y \in \Gamma \text{ (for } k = 2\text{)}, \\ \mathscr{A}^\Gamma R_{k-1} = h_{k-1}(y, R_1, \ldots, R_{k-2}), & y \in \Gamma \text{ (for } k > 2\text{)}, \end{cases}$$

where $\mathscr{A}^\Gamma$ is the linear elliptic operator on Γ, defined in (1.14), and $h_j(y, R_1, \ldots, R_{j-1})$ is a smooth function on Γ which is determined by $R_1, \ldots, R_{j-1}$.

PROOF. We first note that the coefficient of R_k in the C^1-matching condition (4.18) is zero:

$$\int_{\mathbf{R}} \dot{q}_0(z) f_r(q_0, 0, y, 0) dz = 0,$$

thanks to the condition (A3); $c(r, y) \equiv 0$.

We divide the proof into two cases; case (1) $k = 2$, and case (2) $k > 2$.

Case (1): The solution $q_1(z)$ is expressed as $q_1(z) = \bar{q}_1(z) + R_1 p_0(z)$, where

$$p_0(z) = \frac{\partial}{\partial R_1} q_1(z) = \frac{\partial}{\partial r} q_0(z, r, y)|_{r=0}$$

is the unique solution of

$$\begin{cases} \ddot{p}_0 - f_u(*)p_0 = f_r(*) & z \in \mathbf{R}, \\ p_0(0) = 0, \end{cases}$$

which decays exponentially to zero as $z \to \pm\infty$, and $\bar{q}_1$ is the unique solution of

$$\begin{cases} \ddot{\bar{q}}_1 - f_u(*)\bar{q}_1 = -\kappa \dot{q}_0 + z f_r(*) + f_\varepsilon(*), & z \in \mathbf{R}, \\ \bar{q}_1(0) = 0, \end{cases}$$

which grows linearly in z as $z \to \pm\infty$.

By using $q_1 = \bar{q}_1 + R_1 p_0$, one finds that in the C^1-matching condition

$$(4.21) \qquad \int_{\mathbf{R}} \dot{q}_0(z) F_2(z) dz = 0,$$

the terms involving $R_1(y)$ are expressed as

$$\left\{ I_1(y) - \kappa \int_{\mathbf{R}} \dot{p}_0 \dot{q}_0 \, dz \right\} R_1(y) + \int_{\mathbf{R}} (M_1 q_0) \dot{q}_0 \, dz + I_2(y)[R_1(y)]^2,$$

where

$$I_1(y) = \int_{\mathbf{R}} [f_{uu}(*)\bar{q}_1 p_0 + f_{ur}(*)\bar{q}_1] \dot{q}_0 \, dz$$

$$+ \int_{\mathbf{R}} [f_{ur}(*)z p_0 + z f_{rr}(*) + f_{u\varepsilon}(*)p_0 + f_{r\varepsilon}(*)] \dot{q}_0 \, dz,$$

$$I_2(y) = \int_{\mathbf{R}} \left[\frac{1}{2} f_{uu}(*)[p_0]^2 + f_{ur}(*)p_0 + \frac{1}{2} f_{rr}(*) \right] \dot{q}_0 \, dz.$$

Recall here from (4.10) that M_1 is a differential operator involving R_1.

We will establish the following:

- Claim 1: $I_2(y) \equiv 0$.
- Claim 2: We have $\int_{\mathbf{R}} \dot{p}_0 \dot{q}_0 \, dz = \frac{1}{2} m_r(0, y)$, where $m_r = \frac{\partial}{\partial r} m$.
- Claim 3: The integral involving the operator M_1 is reduced to

$$\int_{\mathbf{R}} (M_1 q_0) \dot{q}_0 \, dz = m(y) \Delta^{\Gamma} R_1 + \nabla_{\Gamma} R_1 \cdot \nabla_{\Gamma} m(y) - \kappa_r(0, y) m(y) R_1(y)$$

where $\kappa_r = \frac{\partial}{\partial r} \kappa(r, y)|_{r=0}$.
- Claim 4: $I_1(y) = -\frac{\kappa}{2} m_r(0, y) + J_r(y; \Gamma)$.

Note that $\kappa_r(0, y) = -\sum_{j=1}^{N-1} \kappa_j(y)^2$. Therefore, if the claims are proven, the C^1-matching condition for $q_2^{(\pm)}$ is shown to be the same as the elliptic equation $\mathscr{A}^{\Gamma} R_1 = h_1(y)$ on Γ.

Proof of Claims:

Claim 1. Integrating by parts and using the fact $\lim_{z \to \pm\infty} p_0(z) = \phi^{(\pm)}(0, y)$, we have

$$I_2(y) = \int_{\mathbf{R}} \left[\frac{1}{2} (f_u(*))_z (p_0)^2 + (f_r(*))_z p_0 \right] dz + \frac{1}{2} \int_{\phi^{(-)}}^{\phi^{(+)}} f_{rr}(u, 0, y, 0) du$$

$$= \frac{1}{2} f_u(\phi^{(+)}, 0, y, 0)(\phi_r^{(+)})^2 - \frac{1}{2} f_u(\phi^{(-)}, 0, y, 0)(\phi_r^{(-)})^2$$

$$+ f_r(\phi^{(+)}, 0, y, 0)\phi_r^{(+)} - f_u(\phi^{(-)}, 0, y, 0)\phi_r^{(-)}$$

$$- \int_{\mathbf{R}} \{ f_u(*) p_0 + f_r(*) \} \dot{p}_0 \, dz + \frac{1}{2} \int_{\phi^{(-)}}^{\phi^{(+)}} f_{rr}(u, 0, y, 0) du$$

$$(4.22) \qquad = \frac{1}{2} \{ f_u(\star^+) \phi_r^{(+)} + f_r(\star^+) \} \phi_r^{(+)} - \frac{1}{2} \{ f_u(\star^-) \phi_r^{(-)} + f_r(\star^-) \} \phi_r^{(-)}$$

$$(4.23) \qquad + \frac{1}{2} \left\{ f_r(\star^+) \phi_r^{(+)} - f_r(\star^-) \phi_r^{(-)} + \int_{\phi^{(-)}}^{\phi^{(+)}} f_{rr}(u, 0, y, 0) du \right\}$$

$$(4.24) \qquad - \int_{\mathbf{R}} \{ f_u(*) p_0 + f_r(*) \} \dot{p}_0 \, dz,$$

in which

$$f(\star^{\pm}) = f(\phi^{(\pm)}(0, y), 0, y, 0).$$

Since $f(\phi^{(\pm)}(r, y), r, y, 0) \equiv 0$ implies

$$f_u(\phi^{(\pm)}(r, y), r, y, 0)\phi_r^{(\pm)}(r, y) + f_r(\phi^{(\pm)}(r, y), r, y, 0) \equiv 0,$$

by setting $r = 0$, we find that (4.22) is identically equal to zero. Differentiating the identity

$$0 \equiv \int_{\phi^{(-)}(r,y)}^{\phi^{(+)}(r,y)} f(u, r, y, 0) du$$

twice with respect to r and setting $r = 0$, we also find that (4.23) is zero. The fact $\dot{p}_0(\pm\infty) = 0$, together with the identity

$$\int_{\mathbf{R}} \{f_u(*)p_0 + f_r(*)\}\dot{p}_0 \, dz = \int_{\mathbf{R}} \ddot{p}_0\dot{p}_0 \, dz = \frac{1}{2}(\dot{p}_0(\infty)^2 - \dot{p}_0(-\infty)^2),$$

implies that (4.24) vanishes.

Claim 2. Since $p_0(z) = \frac{\partial}{\partial r}q_0(z, r, y)|_{r=0}$, we have

$$\int_{\mathbf{R}} \dot{p}_0\dot{q}_0 \, dz = \frac{1}{2}\frac{\partial}{\partial r}\int_{\mathbf{R}} [\dot{q}_0(z, r, y)]^2 dz|_{r=0} = \frac{1}{2}m_r(0, y).$$

Claim 3. From the definition of the operator M_1 in (4.10) and $m(x) = \int_{\mathbf{R}} [\dot{q}_0(z; x)]^2 dz$, we have

$$\int_{\mathbf{R}} (M_1 q_0)\dot{q}_0 \, dz = (\Delta^{\Gamma} R_1) \int_{\mathbf{R}} (\dot{q}_0)^2 dz + 2\nabla_{\Gamma} R_1 \cdot \int_{\mathbf{R}} (\nabla_{\Gamma}\dot{q}_0)\dot{q}_0 \, dz$$

$$- \kappa_r(0, y)R_1 \int_{\mathbf{R}} (\dot{q}_0)^2 dz - \int_{\mathbf{R}} \ddot{q}_0\dot{q}_0 \, dz$$

$$= m(y)\Delta^{\Gamma} R_1 + \nabla_{\Gamma} R_1 \cdot \nabla_{\Gamma} m(y) - \kappa_r(y)m(y)R_1.$$

Claim 4. We use the fact $p_0 = \frac{\partial}{\partial r}q_0$. From the definition of $J(x; \Gamma)$

(1.7-b) $$J(x; \Gamma) = \int_{\mathbf{R}} \{zf_r(q_0, x, 0) + f_{\varepsilon}(q_0, x, 0)\}\dot{q}_0 \, dz,$$

we obtain

$$J_r(y; \Gamma) = \int_{\mathbf{R}} z\{f_{ur}(q_0, x, 0)p_0 + f_{rr}(q_0, x, 0)\}\dot{q}_0 \, dz + \int_{\mathbf{R}} zf_r(q_0, x, 0)\dot{p}_0 \, dz$$

$$+ \int_{\mathbf{R}} \{f_{u\varepsilon}(q_0, x, 0)p_0 + f_{r\varepsilon}(q_0, x, 0)\}\dot{q}_0 \, dz + \int_{\mathbf{R}} f_{\varepsilon}(q_0, x, 0)\dot{p}_0 \, dz.$$

By using the relation

$$\ddot{p}_0 - f_u(*)\dot{p}_0 = [f_{uu}(*)p_0 + f_{ur}(*)]\dot{q}_0,$$

which is obtained by differentiating the equation for p_0 with respect to z, we have

$$I_1(y) - J_r(y; \Gamma) = \int_{\mathbf{R}} \{f_{uu}(*)p_0\dot{q}_0 + f_{ur}(*)\dot{q}_0\}\bar{q}_1 \, dz - \int_{\mathbf{R}} \{zf_r(*) + f_\varepsilon(*)\}\dot{p}_0 \, dz$$

$$= \int_{\mathbf{R}} \{\ddot{p}_0 - f_u(*)\dot{p}_0\}\bar{q}_1 \, dz - \int_{\mathbf{R}} \{zf_r(*) + f_\varepsilon(*)\}\dot{p}_0 \, dz$$

$$= \int_{\mathbf{R}} \dot{p}_0\{\ddot{\bar{q}}_1 - f_u(*)\bar{q}_1 - zf_r(*) + f_\varepsilon(*)\}dz$$

$$= \int_{\mathbf{R}} \dot{p}_0(-\kappa\dot{q}_0)dz \qquad \text{(via the equation for } \bar{q}_1\text{)}$$

$$= -\frac{\kappa}{2}m_r(0, y) \qquad \text{(as in the proof of Claim 2).}$$

This completes the proof of Claim 4.

Case (2): Since $q_{k-1} = R_{k-1}p_0 + \bar{q}_{k-1}$ $(k \geq 2)$, with $\bar{q}_{k-1}$ being independent of R_{k-1}, in the C^1-matching condition $\int_{\mathbf{R}} \dot{q}_0(z)F_k(z)dz = 0$ for $k > 2$, the terms involving R_{k-1} are expressed as (cf. (4.12-k))

$$(4.25) \qquad \left(-\kappa \int_{\mathbf{R}} \dot{p}_0\dot{q}_0 \, dz\right)R_{k-1} + \int_{\mathbf{R}} (M_{k-1}q_0)\dot{q}_0 \, dz$$

$$+ \left(\int_{\mathbf{R}} \{f_{uu}(*)q_1p_0 + f_{ur}(*)q_1\}\dot{q}_0 \, dz\right.$$

$$+ \int_{\mathbf{R}} \{zf_{ur}(*)p_0 + zf_{rr}(*) + f_{u\varepsilon}(*)p_0 + f_{r\varepsilon}(*)\}\dot{q}_0 \, dz\bigg)R_{k-1}$$

$$+ \left(\int_{\mathbf{R}} \{f_{ur}(*)p_0 + f_{rr}(*)\}\dot{q}_0 \, dz\right)R_1R_{k-1}.$$

The first line of (4.25) is computed as in the proof of Claims 2 and 3 with R_1 being replaced by R_{k-1}. The second and third lines of (4.25) are the same as $I_1(y)R_{k-1}$, except that q_1 is replacing $\bar{q}_1$. Therefore, we have

$$(4.26) \qquad \text{(the second and third lines of (4.25))}$$

$$= J_r(y; \Gamma)R_{k-1} + R_{k-1}\int_{\mathbf{R}} \dot{p}_0\{\ddot{q}_1 - f_u(*)q_1 - zf_r(*) - f_\varepsilon(*)\}dz$$

$$= J_r(y; \Gamma) + R_{k-1}\int_{\mathbf{R}} \dot{p}_0(-\kappa\dot{q}_0)dz + R_1R_{k-1}\int_{\mathbf{R}} f_r(*)\dot{p}_0 \, dz.$$

On the other hand, integrating the first integrand by parts, the fourth line of (4.25) is computed as

(4.27)   (the fourth line of (4.25))

$$= R_1 R_{k-1} \left[f_r(\star^+) \phi_r^{(+)} - f_r(\star^-) \phi_r^{(-)} + \int_{\mathbf{R}} f_{rr}(*) \dot{q}_0 \, dz \right]$$

$$- R_1 R_{k-1} \int_{\mathbf{R}} f_r(*) \dot{p}_0 \, dz = -R_1 R_{k-1} \int_{\mathbf{R}} f_r(*) \dot{p}_0 \, dz.$$

The third term in line (4.26) and the last term in (4.27) cancel. Therefore we conclude that the C^1-matching condition for $q_k^{(\pm)}$ is

$$\mathscr{A}^{\Gamma} R_{k-1} = h_{k-1}(y, R_1, \dots, R_{k-2}). \qquad \square$$

4.3. Boundary Correction. The outer approximation $\bar{u}_\varepsilon^{(\pm)}$ in (4.1) does not necessarily satisfy the boundary conditions $\frac{\partial u}{\partial \mathbf{n}} = 0$ on $\partial \mathscr{D}$. In order to modify the outer approximation, we introduce a stretched coordinate system near $\partial \mathscr{D}$. Let us express a point $x \in \bar{\mathscr{D}}$ near $\partial \mathscr{D}$ as

(4.28) $x = s + \rho \mathbf{n}(s) \qquad (s \in \partial \mathscr{D}, \rho \geq 0).$

The Laplacian Δ in the coordinate system (ρ, s) is given by

(4.29) $\Delta = \dfrac{\partial^2}{\partial \rho^2} + \kappa^{\partial}(\rho, s) \dfrac{\partial}{\partial \rho} + \Delta^{\partial}(\rho, s),$

where $\kappa^{\partial}(\rho, s)$ is the mean curvature of $S(\rho)$,

$$S(\rho) := \{ x \in \bar{\mathscr{D}} \mid x = s + \rho \mathbf{n}(s), s \in \partial \mathscr{D} \},$$

at $x = s + \rho \mathbf{n}(s)$, and $\Delta^{\partial}(\rho, s)$ is the Laplace-Beltrami operator on $S(\rho)$.

The stretched variable in the $\mathbf{n}(s)$-direction is introduced by

(4.30) $z = \dfrac{\rho}{\varepsilon}, \qquad \text{or} \qquad \rho = \varepsilon z.$

In terms of the stretched coordinate system (z, s), the Laplacian is expressed as

(4.29-ε) $\varepsilon^2 \Delta = \dfrac{\partial^2}{\partial z^2} + \varepsilon \kappa^{\partial}(0, s) \dfrac{\partial}{\partial z}$

$$+ \varepsilon^2 \sum_{k \geq 0} \varepsilon^k \left[\dfrac{z^k}{k!} \dfrac{\partial^k}{\partial \rho^k} \Delta^{\partial}(\rho, s)|_{\rho=0} + \dfrac{z^{k+1}}{(k+1)!} \dfrac{\partial^{k+1}}{\partial \rho^{k+1}} \kappa^{\partial}(\rho, s)|_{\rho=0} \dfrac{\partial}{\partial z} \right].$$

We denote by $\bar{u}_\varepsilon^b(\rho, s)$ the outer approximation $\bar{u}_\varepsilon^{(\pm)}(x)$ in the coordinate (ρ, s):

$$\bar{u}_\varepsilon^b(\rho, s) = \bar{u}_\varepsilon^{(\pm)}(s + \rho \mathbf{n}(s)) \qquad (s \in \partial \mathscr{D}).$$

In the boundary correction, we modify the outer approximation in the form

$$(4.31) \qquad u_\varepsilon^b(x) = u_\varepsilon^b(\rho, s) = \sum_{j\geq 0} \varepsilon^j \bar{u}_j^b(\varepsilon z, s) + \sum_{j\geq 0} \varepsilon^j B_j(z, s)$$

$$= \sum_{j\geq 0} \varepsilon^j b_j(z, s),$$

where

$$(4.31\text{-}0) \qquad b_0(z, s) = \bar{u}_0^b(0, s) + B_0(z, s)$$

$$(4.31\text{-}k) \qquad b_k(z, s) = B_k(z, s) + \sum_{j=0}^{k} \frac{1}{(k-j)!} \frac{\partial^{k-j}}{\partial \varepsilon^{k-j}} \bar{u}_j^b(\varepsilon z, s)|_{\varepsilon=0} \qquad (k \geq 1).$$

The boundary conditions in (3.3) now read

$$\frac{\partial u^\varepsilon(x)}{\partial \mathbf{n}} = \frac{\partial}{\partial \rho} u_\varepsilon^b(\rho, s)|_{\rho=0} = \frac{1}{\varepsilon} \sum_{j\geq 0} \varepsilon^j \dot{b}_j(0, s) = 0,$$

where we used *dot* to indicate the differentiation with respect to ρ. Therefore, we require the conditions

$$(4.32) \qquad\qquad\qquad \dot{b}_k(0, s) = 0 \qquad (k \geq 0),$$

and

$$(4.33) \qquad\qquad \lim_{z\to\infty} B_k(z, s) = 0 \qquad \text{exponentially } (k \geq 0).$$

Substituting (4.31) into (3.3) and using (4.29-ε), we obtain an equation for $b_k(z, s)$ $(k \geq 0)$. The equation for b_0 is

$$(4.34) \qquad\qquad \ddot{b}_0 - f(b_0, s, 0) = 0, \qquad z \in [0, \infty) \, (s \in \partial\mathscr{D}).$$

This has the trivial solution $b_0(z, s) \equiv \bar{u}_0^b(0, s)$ (i.e., $B_0(z, s) \equiv 0$), which satisfies the conditions (4.32) and (4.33) with $k = 0$. The equation for b_k for $k \geq 1$ is of the following form

$$(4.34\text{-}k) \qquad\qquad \ddot{b}_k - K(s)^2 b_k = g_k(z, s), \qquad z \in [0, \infty),$$

where $K(s) = \sqrt{f_u(\bar{u}_0^b(0, s), s, 0)} > 0$ $(s \in \partial\mathscr{D})$ and g_k is a function depending only on $b_0, \ldots, b_{k-1}$. Solutions of (4.34-k) satisfying (4.33) are uniquely given by

$$(4.35) \quad b_k(z, s) = a_k e^{-Kz} - \frac{1}{2K} \int_0^z e^{-K(z-z')} g_k(z', s) dz' + \frac{1}{2K} \int_\infty^z e^{K(z-z')} g_k(z', s) dz'$$

with $K = K(s)$, where a_k is an arbitrary constant. From (4.35), we find that b_k satisfies the condition (4.32) if a_k is given by

$$a_k = \frac{1}{2K(s)} \int_\infty^0 e^{-K(s)z} g_k(z,s) dz.$$

This completes the construction of the boundary correction.

Putting the above ingredients together, we obtain the desired approximation $u_k^\varepsilon(x)$ in Proposition 3.1. We now choose smooth cut-off functions $\theta^0(r), \theta^+(r)$ and $\theta^-(r)$ such that

$$0 \le \theta^i(r) \le 1 \quad (i = 0, +, -), \qquad \theta^0(r) + \theta^+(r) + \theta^-(r) \equiv 1 \quad (r \in \mathbf{R})$$

and

$$\theta^0(r) = \begin{cases} 1, & |r| \le 1, \\ 0, & |r| \ge 2, \end{cases} \qquad \theta^+(r) = \begin{cases} 1, & r \ge 2, \\ 0, & r \le 1, \end{cases} \qquad \theta^-(r) = \begin{cases} 0, & r \ge -1, \\ 1, & r \le -2. \end{cases}$$

We also use symbols $d(x, \Gamma), x_\Gamma, d(x, \partial\mathscr{D})$, and x_∂ defined by

$$d(x, \Gamma) = \begin{cases} \text{dist}(x, \Gamma), & x \in \mathscr{D}^{(+)} \\ -\text{dist}(x, \Gamma), & x \in \mathscr{D}^{(-)} \end{cases}$$

and $x_\Gamma \in \Gamma$ is defined by $|d(x, \Gamma)| = \text{dist}(x, x_\Gamma)$ when $|d(x, \Gamma)| \le 2d^*$, while

$$d(x, \partial\mathscr{D}) = \text{dist}(x, \partial\mathscr{D}), \qquad x \in \bar{\mathscr{D}},$$

and $x_\partial \in \partial\mathscr{D}$ is defined by $d(x, \partial\mathscr{D}) = \text{dist}(x, x_\partial)$ when $d(x, \partial\mathscr{D}) \le 2d^*$.

Let $\beta > 0$ be the same constants as in (4.14) and $\bar{K} = \min\{K(s) \mid s \in \partial\mathscr{D}\}$. The desired approximation is defined for $x \in \mathscr{D}_\Gamma^{(\pm)}$ by

$$(4.36) \qquad u_k^\varepsilon(x) = \theta^- \left(\frac{\beta d(x, \Gamma)}{(k+1)\varepsilon|\log\varepsilon|} \right) \sum_{j=0}^k \varepsilon^j \bar{u}_j^{(-)}(x)$$

$$+ \theta^+ \left(\frac{\beta d(x, \Gamma)}{(k+1)\varepsilon|\log\varepsilon|} \right) \sum_{j=0}^k \varepsilon^j \bar{u}_j^{(+)}(x)$$

$$+ \theta^0 \left(\frac{\beta d(x, \Gamma)}{(k+1)\varepsilon|\log\varepsilon|} \right) \sum_{j=0}^k \varepsilon^j q_j \left(\frac{d(x, \Gamma) - \varepsilon R_k^\varepsilon(x_\Gamma)}{\varepsilon}, x_\Gamma \right)$$

$$+ \theta^0 \left(\frac{\bar{K} d(x, \partial\mathscr{D})}{(k+1)\varepsilon|\log\varepsilon|} \right) \sum_{j=0}^k \varepsilon^j B_j \left(\frac{d(x, \partial\mathscr{D})}{\varepsilon}, x_\partial \right)$$

$$+ \theta^+ \left(\frac{\bar{K} d(x, \partial\mathscr{D})}{(k+1)\varepsilon|\log\varepsilon|} \right) \sum_{j=0}^k \varepsilon^j \bar{u}_j^b(x),$$

where we simply write q_j in place of $q_j^{(\pm)}$, since the latter $(q_j^{(\pm)})$ have been smoothly joined by the C^1-matching conditions. We can easily verify

$$\frac{\partial u_k^\varepsilon(x)}{\partial \mathbf{n}} = 0, \qquad x \in \partial \mathscr{D}.$$

It is also routine computations to verify

$$\|\varepsilon^2 \varDelta u_k^\varepsilon(\cdot) - f(u_k^\varepsilon(\cdot), \cdot, \varepsilon)\|_{C_\varepsilon^\alpha(\bar{\mathscr{D}})} = O(\varepsilon^{k+1-\alpha-\alpha'}) \qquad \text{as } \varepsilon \to 0.$$

This completes the proof of Proposition 3.1.

5. Asymptotic expansion of eigenvalue

In this section, we shall prove Proposition 3.3, by using the results in [3] and [1].

We first use the estimate established in [3]: There exists a constant $C > 0$, independent of small $\varepsilon > 0$, so that any eigenvalue λ^ε of (3.6) satisfies

$$\lambda^\varepsilon \leq C\varepsilon^2.$$

This estimate, together with the definition of non-critical eigenvalues, proves Proposition 3.3 (i).

We now proceed to the proof of Proposition 3.3 (ii). Let us denote by $(\varphi^\varepsilon, \lambda^\varepsilon)$ a critical eigenpair of (3.6):

$$(3.6) \qquad \begin{cases} L_k^\varepsilon \varphi^\varepsilon := \varepsilon^2 \varDelta \varphi^\varepsilon - f_u(u_k^\varepsilon, x, \varepsilon)\varphi^\varepsilon = \lambda^\varepsilon \varphi^\varepsilon, & x \in \mathscr{D}, \\ \partial \varphi^\varepsilon / \partial \mathbf{n} = 0, & x \in \partial \mathscr{D}. \end{cases}$$

The theory developed in [1] (cf. Lemmas 4.1 and 4.2, therein) says: In order to approximate the critical eigenpair $(\varphi^\varepsilon, \lambda^\varepsilon)$, it suffices to find asymptotic expansions

$$(5.1) \qquad \begin{cases} \varphi^\varepsilon = \varphi_0 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \cdots, \\ \lambda^\varepsilon = \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots, \end{cases}$$

so that the right hand side of (5.1) satisfies (3.6) *approximately*, namely, for some $k > 1$

$$\begin{cases} L_k^\varepsilon \varphi^\varepsilon - \lambda^\varepsilon \varphi^\varepsilon = O(\varepsilon^k), & x \in \mathscr{D}, \\ \partial \varphi^\varepsilon / \partial \mathbf{n} = 0, & x \in \partial \mathscr{D}, \end{cases} \qquad \text{as } \varepsilon \to 0.$$

We therefore need to show:
   1. The coefficient $\lambda_1 = 0$, which essentially says that the interface evolves according to the time scale of (1.1-ss);
   2. The coefficient λ_2 is an element of σ_Γ.
   By using Lemma 2.1 in [1], one can prove that there exist constants $C > 0$,

$\delta > 0$, and $b > 0$, indepedent of $\varepsilon \in (0, \varepsilon_*]$, so that any critical eigenfunction of (3.6) satisfies

(5.2) $$|\varphi^\varepsilon(x)| \leq C \exp\left\{-\frac{\delta}{\varepsilon} \operatorname{dist}(x, \Gamma)\right\}, \qquad \operatorname{dist}(x, \Gamma) \geq \varepsilon b.$$

It is appropriate, therefore, to consider (3.6) in terms of the coordinate system (z, y) introduced in (4.6). We therefore express the critical eigenfucntion φ^ε as

(5.3) $$\varphi^\varepsilon(x) = \varphi^\varepsilon(z, y) = \varphi_0(z, y) + \varepsilon\varphi_1(z, y) + \varepsilon^2\varphi_2(z, y) + \cdots$$

with respect to the stretched coordinate system near Γ_ε. We impose the boundary conditions

(5.4) $$\varphi_j(z, y) = O(e^{-\delta|z|}) \qquad \text{as } |z| \to \infty \text{ for some } \delta > 0 \ (j \geq 0),$$

according to (5.2). The potential term of the differential equation in (3.6) is expressed as

(5.5) $$f_u(q^\varepsilon(z, y), \varepsilon z, y, \varepsilon) = f_u(q_0(z, y), 0, y, \varepsilon) + \sum_{j \geq 1} \varepsilon^j f_u^{(j)}(z),$$

where $q^\varepsilon(z, y) = \sum_{j \geq 0} \varepsilon^j q_j(z, y)$ and

$$f_u^{(j)}(z) := \frac{1}{j!} \frac{\partial^j}{\partial \varepsilon^j} f_u(q^\varepsilon(z, y), \varepsilon z, y, \varepsilon)\big|_{\varepsilon=0}.$$

Let us substitute (5.3), (5.1) into (3.6). By using (4.9) and (5.5), and equating like powers of ε, we obtain equations of φ_j $(j \geq 0)$. They read as follows.

(5.6) $$\ddot{\varphi}_0 - f_u(*)\varphi_0 = 0,$$

(5.6-1) $$\ddot{\varphi}_1 - f_u(*)\varphi_1 = -\kappa\dot{\varphi}_0 + f_u^{(1)}(z)\varphi_0 + \lambda_1\varphi_0 =: l_1(z),$$

(5.6-2) $$\ddot{\varphi}_2 - f_u(*)\varphi_2 = -\kappa\dot{\varphi}_1 + M_1\varphi_0 - P_0\varphi_0$$
$$+ f_u^{(1)}(z)\varphi_1 + f_u^{(2)}\varphi_0 + \lambda_2\varphi_0 + \lambda_1\varphi_1 =: l_2(z),$$

(5.6-k) $$\ddot{\varphi}_k - f_u(*)\varphi_k = -\kappa\dot{\varphi}_{k-1} + M_1\varphi_{k-2} - P_0\varphi_{k-2} + \sum_{j=2}^{k-1}(M_j - P_{j-1})\varphi_{k-j-1}$$
$$+ f_u^{(1)}(z)\varphi_{k-1} + f_u^{(2)}(z)\varphi_{k-2} + \sum_{j=3}^{k} f_u^{(j)}\varphi_{k-j}$$
$$+ \lambda_k\varphi_0 + \sum_{j=1}^{k-3} \lambda_{k-j}\varphi_j + \lambda_2\varphi_{k-2} + \lambda_1\varphi_{k-1} =: l_k(z).$$

We show the solvability of these equations.

The equation (5.6) has a unique solution satisfying (5.4) with $j = 0$:

(5.7) $$\varphi_0(z, y) = \Theta_0(y)\dot{q}_0(z, y),$$

where Θ_0 is an arbitrary smooth function defined on Γ. The function Θ_0 will be determined as an eigenfucntion of (1.16).

The equations (5.6-k) with $k \geq 1$ are all linear inhomogeneous equations. These equations have a family of solutions which satisfy the condition (5.4) if and only if the solvability condition

(5.8) $$\int_{\mathbf{R}} \dot{q}_0(z, y)l_k(z)dz = 0$$

is satisfied. It turns out that the solvability condition determines λ_k and φ_{k-2} in terms of λ_j and φ_{j-2} with $2 \leq j \leq k - 1$.

1. We will show $\lambda_1 = 0$. Let us apply the solvability condition (5.8) to (5.6-1). It is given by

(5.9) $$0 = \lambda_1\Theta_0\int_{\mathbf{R}}(\dot{q}_0)^2dz - \kappa\Theta_0\int_{\mathbf{R}}\ddot{q}_0\dot{q}_0\,dz + \Theta_0\int_{\mathbf{R}}f_u^{(1)}(z)(\dot{q}_0)^2dz.$$

Note that $\int_{\mathbf{R}}\ddot{q}_0\dot{q}_0\,dz = 0$ and $\int_{\mathbf{R}}f_r(*)\dot{q}_0\,dz = 0$. Integrating by parts and using the equation (4.12-1) for q_1, we have

$$\int_{\mathbf{R}}f_u^{(1)}(z)(\dot{q}_0)^2dz = \int_{\mathbf{R}}\{f_{uu}(*)q_1 + f_{ur}(*)(z + R_1) + f_{u\varepsilon}(*)\}(\dot{q}_0)^2dz$$

$$= \int_{\mathbf{R}}\{(f_u(*))_zq_1 + (f_r(*))_z(z + R_1) + (f_\varepsilon(*))_z\}\dot{q}_0\,dz$$

$$= -\int_{\mathbf{R}}\{f_u(*)q_1 + f_r(*)(z + R_1) + f_\varepsilon(*)\}\ddot{q}_0\,dz$$

$$\quad - \int_{\mathbf{R}}\{f_u(*)\dot{q}_1 + f_r(*)\}\dot{q}_0\,dz$$

$$= -\int_{\mathbf{R}}\{\ddot{q}_1 + \kappa\dot{q}_0\}\ddot{q}_0\,dz - \int_{\mathbf{R}}\ddot{q}_0\dot{q}_1\,dz - \int_{\mathbf{R}}f_r(*)\dot{q}_0\,dz$$

$$= -\int_{\mathbf{R}}\ddot{q}_1\ddot{q}_0\,dz + \int_{\mathbf{R}}\ddot{q}_0\ddot{q}_1\,dz = 0.$$

Therefore (5.9) implies $0 = m(y)\lambda_1\Theta_0$. On the other hand, the normalization $\int_{\mathscr{D}}(\varphi^\varepsilon(x))^2dx = 1$ gives rise to $1 = \int_\Gamma(y)\Theta_0(y)dS_y^\Gamma$, and hence to $\Theta_0(y) \not\equiv 0$. We conclude that

(5.10) $$\lambda_1 = 0$$

and that $\varphi_1(z, y)$ is given by

(5.11) $$\varphi_1(z, y) = \Theta_1(y)\dot{q}_0(z, y) + \bar{\varphi}_1(z, y)\Theta_0(y),$$

where Θ_1 is an arbitrary smooth function and $\bar{\varphi}_1$ is the unique solution of

(5.12) $$\begin{cases} \ddot{\bar{\varphi}}_1 - f_u(*)\bar{\varphi}_1 = -\kappa\ddot{q}_0 + f_u^{(1)}(z)\dot{q}_0, & z \in \mathbf{R}, \\ \bar{\varphi}_1(0, y) = 0, \end{cases}$$

decaying to zero as $z \to \pm\infty$. Comparing (5.12) and the equation for $\dot{q}_1$, which is obtained by differentiating (4.12-1) with respect to z, we find that

(5.13) $$\dot{q}_1(z, y) - \bar{\varphi}_1(z, y) = \frac{\dot{q}_1(0, y)}{\dot{q}_0(0, y)}\dot{q}_0(z, y) + p_0(z, y),$$

where p_0 is the same as appeared in §4; $p_0 = \partial q_0/\partial r|_{r=0}$.

2. Let us now apply the solvability condition (5.8) to (5.6-2). By using (5.7), we have

(5.14) $$0 = \lambda_2 m(y)\Theta_0 + \int_{\mathbf{R}} \{-\kappa\dot{\varphi}_1 + M_1\varphi_0 - P_0\varphi_0\}\dot{q}_0\, dz$$

$$+ \int_{\mathbf{R}} f_u^{(1)}(z)\varphi_1\dot{q}_0\, dz + \int_{\mathbf{R}} f_u^{(2)}(z)\varphi_0\dot{q}_0\, dz.$$

Thanks to (5.11) and the fact $\int_{\mathbf{R}} f_u^{(1)}(z)(\dot{q}_0)^2 dz = 0$, we find that the second line of (5.14) is expressed as

(5.15) (the second line of (5.14))

$$= \Theta_0\left(\int_{\mathbf{R}} f_u^{(1)}(z)\bar{\varphi}_1\dot{q}_0\, dz + \int_{\mathbf{R}} f_u^{(2)}(z)(\dot{q}_0)^2 dz\right).$$

We first establish some identities.

PROPOSITION 5.1. *We have the following relations:*

(i) $$\int_{\mathbf{R}} f_u^{(1)}(z)\bar{\varphi}_1\dot{q}_0\, dz + \int_{\mathbf{R}} f_u^{(2)}(z)(\dot{q}_0)^2 dz$$

$$= \frac{1}{2}\kappa m_r(y) - J_r(y; \Gamma) + \int_{\mathbf{R}} \{-\kappa\dot{q}_1 + M_1 q_0 - P_0 q_0\}\ddot{q}_0\, dz;$$

(ii) $$\Theta_0\int_{\mathbf{R}} (M_1 q_0)\ddot{q}_0\, dz + \int_{\mathbf{R}} (M_1\varphi_0)\dot{q}_0\, dz = 0;$$

(iii) $$\Theta_0\int_{\mathbf{R}} (-\kappa\dot{q}_1\ddot{q}_0)dz + \int_{\mathbf{R}} (-\kappa\dot{\varphi}_1\dot{q}_0)dz = \frac{1}{2}\kappa m_r\Theta_0;$$

(iv) $$\Theta_0\int_{\mathbf{R}} (P_0 q_0)\ddot{q}_0\, dz + \int_{\mathbf{R}} (P_0\varphi_0)\dot{q}_0\, dz$$

$$= m(y)\Delta^{\Gamma}\Theta_0 + \nabla_{\Gamma} m(y) \cdot \nabla_{\Gamma}\Theta_0 - \kappa_r m(y)\Theta_0.$$

By using the proposition and employing the relation $\kappa_r = -\sum_{j=1}^{N-1} \kappa_j(y)^2$, we arrive at:

THEOREM 5.2. (i) *The solvability condition for* (5.6-2), *namely,* (5.14) *is rewritten as*

$$0 = \lambda_2 m(y)\Theta_0 - \mathscr{A}^\Gamma \Theta_0,$$

i.e., λ_2 *is an eigenvalue of* (1.16).
   (ii) *For* $k > 2$, *the solvability condition for* (5.6-k) *is written as*

(5.16) $(\lambda_2 m(y) - \mathscr{A}^\Gamma)\Theta_{k-2} + \lambda_k m(y)\Theta_0 = H_k(y, \lambda_2, \ldots, \lambda_{k-1}; \Theta_0, \ldots, \Theta_{k-3}),$

where the right hand side is a known function of the variables indicated.

Theorem 5.2 (i) and (5.10) establish the proof of Proposition 3.3 (ii).
   As for proof, Theorem 5.2 (i) follows immediately from Proposition 5.1.
   The proof of Theorem 5.2 (ii) is reduced to that of (i) as follows.   In the solvability condition

$$\int_{\mathbf{R}} l_k(z)\dot{q}_0(z)dz = 0$$

for (5.6-k), we leave on the left side those terms involving $\varphi_{k-1}, \varphi_{k-2}$, and λ_k, and move the rest to the right hand side.   Since we have shown $\lambda_1 = 0$, the left hand side is written as

(5.17) $\int_{\mathbf{R}} \{-\kappa \dot{\varphi}_{k-1} + M_1 \varphi_{k-2} - P_0 \varphi_{k-2}$

$$+ f_u^{(1)}(z)\varphi_{k-1} + f_u^{(2)}(z)\varphi_{k-2} + \lambda_k \varphi_0 + \lambda_2 \varphi_{k-2}\}\dot{q}_0 \; dz.$$

Note that φ_{k-1} and φ_{k-2} are expressed as

$$\varphi_{k-2}(z, y) = \Theta_{k-2}(y)\dot{q}_0(z, y) + \bar{\varphi}_{k-2}(z, y),$$

$$\varphi_{k-1}(z, y) = \Theta_{k-1}(y)\dot{q}_0(z, y) + \bar{\varphi}_1(z, y)\Theta_{k-2}(y) + \bar{\varphi}_{k-1}(z, y),$$

where $\bar{\varphi}_1$ is as defined in (5.12), and the functions $\bar{\varphi}_{k-1}$ and $\bar{\varphi}_{k-2}$ are indepenent of λ_k, Θ_{k-2}, and Θ_{k-1}.   Moving the terms involving these latter functions from (5.17) to the right hand side of the above solvability condition, (5.17) is similar to the solvability condition for φ_2.   The only differences are; Θ_0 is replaced by Θ_{k-2}; and the extra term $\lambda_k m(y)\Theta_0$ is added.   Therefore, applying Proposition 5.1, the statement (ii) of Theorem 5.2 follows.

Theorem 5.2 (ii) may be used to show that the critical eigenvalues of (3.6) are approximated as accurate as one wishes, in the ε-power series.

PROOF OF PROPOSITION 5.1.   The proof is computational and integration by parts will be used frequently.

(i)   We first integrate by parts to obtain

$$\int_{\mathbf{R}} f_u^{(2)}(z)(\dot{q}_0)^2 dz = \int_{\mathbf{R}} \{(f_u(*))_z q_2 + (f_r(*))_z R_2\} \dot{q}_0 \, dz$$

$$+ \int_{\mathbf{R}} \left\{ \frac{1}{2}(f_{uu}(*))_z (q_1)^2 + (f_{ur}(*))_z (z + R_1) q_1 \right.$$

$$+ (f_{u\varepsilon}(*))_z q_1 + (f_{r\varepsilon}(*))_z (z + R_1)$$

$$\left. + \frac{1}{2}(f_{rr}(*))_z (z + R_1)^2 + \frac{1}{2}(f_{\varepsilon\varepsilon}(*))_z \right\} \dot{q}_0 \, dz$$

$$(5.18) \quad = -\int_{\mathbf{R}} \{f_u(*)q_2 + f_r(*)R_2\} \ddot{q}_0 \, dz - \int_{\mathbf{R}} f_u(*)\dot{q}_0 \dot{q}_2 \, dz$$

$$- \int_{\mathbf{R}} \left\{ \frac{1}{2} f_{uu}(*)(q_1)^2 + f_{ur}(*)(z + R_1)q_1 + f_{u\varepsilon}(*)q_1 + f_{r\varepsilon}(*)(z + R_1) \right.$$

$$\left. + \frac{1}{2} f_{rr}(*)(z + R_1)^2 + \frac{1}{2} f_{\varepsilon\varepsilon}(*) \right\} \ddot{q}_0 \, dz$$

$$(5.19) \quad - \int_{\mathbf{R}} f_u^{(1)}(z)\dot{q}_1 \dot{q}_0 \, dz - \int_{\mathbf{R}} f_{ur}(*)q_1 \dot{q}_0 \, dz - \int_{\mathbf{R}} \{f_{r\varepsilon}(*) + f_{rr}(*)(z + R_1)\} \dot{q}_0 \, dz.$$

By using the identity

$$-\int_{\mathbf{R}} f_u(*)\dot{q}_0 \dot{q}_2 \, dz = -\int_{\mathbf{R}} \ddot{q}_0 \dot{q}_2 \, dz = \int_{\mathbf{R}} \ddot{q}_2 \dot{q}_0 \, dz$$

in the second term of line (5.18), and using (5.13) together with the relation

$$\int_{\mathbf{R}} f_u^{(1)}(z)(\dot{q}_0)^2 dz = 0$$

in the first term of line (5.19), we continue the computation above as follows.

$$\int_{\mathbf{R}} f_u^{(2)}(z)(\dot{q}_0)^2 dz$$

$$= \int_{\mathbf{R}} \ddot{q}_0 \left\{ \ddot{q}_2 - f_u(*)q_2 - f_r(*)R_2 - \frac{1}{2} f_{uu}(*)(q_1)^2 - f_{ur}(*)(z + R_1)q_1 \right.$$

$$\left. - f_{u\varepsilon}(*)q_1 - f_{r\varepsilon}(*)(z + R_1) - \frac{1}{2} f_{rr}(*)(z + R_1)^2 - \frac{1}{2} f_{\varepsilon\varepsilon}(*) \right\} dz$$

$$- \int_{\mathbf{R}} \{ f_{ur}(*)q_1 + f_u^{(1)}(z)(p_0 + \bar{\varphi}_1) \} \dot{q}_0 \, dz$$

$$- \int_{\mathbf{R}} \{ f_{r\varepsilon}(*) + f_{rr}(*)(z + R_1) \} \dot{q}_0 \, dz.$$

Thanks to the equation (4.12-2) for q_2, we have

$$(5.20) \qquad \int_{\mathbf{R}} f_u^{(2)}(z)(\dot{q}_0)^2 dz + \int_{\mathbf{R}} f_u^{(1)}(z)\bar{\varphi}_1 \dot{q}_0 \, dz$$

$$= \int_{\mathbf{R}} \{ -\kappa \dot{q}_1 + M_1 q_0 - P_0 q_0 \} \ddot{q}_0$$

$$- \int_{\mathbf{R}} \{ f_u^{(1)}(z)p_0 + f_{ur}(*)q_1 + f_{r\varepsilon}(*) + f_{rr}(*)(z + R_1) \} \dot{q}_0$$

$$= \int_{\mathbf{R}} \{ -\kappa \dot{q}_1 + M_1 q_0 - P_0 q_0 \} \ddot{q}_0 - I_1(y) - 2R_1 I_2(y)$$

$$= \int_{\mathbf{R}} \{ -\kappa \dot{q}_1 + M_1 q_0 - P_0 q_0 \} \ddot{q}_0 - J_r + \frac{\kappa}{2} m_r$$

(cf. Claims 1 and 4 in §4).

This completes the proof of (i).

(ii) From the definition (4.10-(a)) of M_1, we have

$$\int_{\mathbf{R}} (M_1 q_0)\ddot{q}_0 \, dz = (\varDelta^\Gamma R_1) \int_{\mathbf{R}} \dot{q}_0 \ddot{q}_0 \, dz + 2\nabla_\Gamma R_1 \cdot \int_{\mathbf{R}} \nabla_\Gamma(\dot{q}_0)\ddot{q}_0 \, dz$$

$$- \kappa_r R_1 \int_{\mathbf{R}} \dot{q}_0 \ddot{q}_0 - |\nabla_\Gamma R_1|^2 \int_{\mathbf{R}} (\ddot{q}_0)^2 dz$$

$$= 2\nabla_\Gamma R_1 \cdot \int_{\mathbf{R}} \nabla_\Gamma(\dot{q}_0)\ddot{q}_0 \, dz - |\nabla_\Gamma R_1|^2 \int_{\mathbf{R}} (\ddot{q}_0)^2 dz.$$

On the other hand, by using (5.7) and (4.10-(a)), we have

$$\int_{\mathbf{R}} (M_1\varphi_0)\dot{q}_0 \, dz = (\Delta^\Gamma R_1) \int_{\mathbf{R}} \ddot{q}_0\dot{q}_0 \, dz\Theta_0 + 2\nabla_\Gamma R_1 \cdot \int_{\mathbf{R}} \nabla_\Gamma(\ddot{q}_0)\dot{q}_0 \, dz\Theta_0$$

$$+ 2(\nabla_\Gamma R_1 \cdot \nabla_\Gamma \Theta_0) \int_{\mathbf{R}} \ddot{q}_0\dot{q}_0 \, dz - |\nabla_\Gamma R_1|^2 \int_{\mathbf{R}} \ddot{q}_0\dot{q}_0 \, dz\Theta_0$$

$$= -2\nabla_\Gamma R_1 \cdot \int_{\mathbf{R}} \nabla_\Gamma(\dot{q}_0)\ddot{q}_0 \, dz\Theta_0 + |\nabla_\Gamma R_1|^2 \int_{\mathbf{R}} (\ddot{q}_0)^2 dz\Theta_0.$$

Therefore we get

$$\Theta_0 \int_{\mathbf{R}} (M_1 q_0)\ddot{q}_0 \, dz + \int_{\mathbf{R}} (M_1\varphi_0)\dot{q}_0 \, dz = 0.$$

(iii)   Since $-\int \dot{q}_1\ddot{q}_0 \, dz = \int \ddot{q}_1\dot{q}_0 \, dz$, by using (5.11) and (5.13), we have

$$\Theta_0 \int_{\mathbf{R}} -\kappa\dot{q}_1\ddot{q}_0 \, dz + \int_{\mathbf{R}} -\kappa\dot{q}_0\varphi_1 \, dz = \kappa \int_{\mathbf{R}} (\ddot{q}_1\Theta_0 - \varphi_1)\dot{q}_0$$

$$= \kappa\Theta_0 \int_{\mathbf{R}} \dot{p}_0\dot{q}_0 \, dz + \left(\frac{\dot{q}_1(0)}{\dot{q}_0(0)}\Theta_0 - \Theta_1\right)\kappa \int_{\mathbf{R}} \ddot{q}_0\dot{q}_0 \, dz = \frac{1}{2}\kappa m_r\Theta_0.$$

(iv)   Recall the definition (4.10-(c)) of P_0.   We have

$$\int_{\mathbf{R}} (P_0 q_0)\ddot{q}_0 \, dz = \int_{\mathbf{R}} (\Delta^\Gamma q_0)\ddot{q}_0 \, dz + \kappa_r \int_{\mathbf{R}} z\dot{q}_0\ddot{q}_0 \, dz$$

$$= -\int_{\mathbf{R}} (\Delta^\Gamma \dot{q}_0)\dot{q}_0 \, dz - \frac{1}{2}\kappa_r m(y).$$

On the other hand, we also have,

$$\int_{\mathbf{R}} (P_0\varphi_0)\dot{q}_0 \, dz = \int_{\mathbf{R}} (\Delta^\Gamma \dot{q}_0\Theta_0)\dot{q}_0 \, dz + \kappa_r \int_{\mathbf{R}} z\dot{q}_0\ddot{q}_0 \, dz\Theta_0$$

$$= \Theta_0 \int_{\mathbf{R}} (\Delta^\Gamma \dot{q}_0)\dot{q}_0 \, dz + 2\nabla_\Gamma\Theta_0 \cdot \int_{\mathbf{R}} (\nabla_\Gamma \dot{q}_0)\dot{q}_0 \, dz$$

$$+ (\Delta^\Gamma \Theta_0) \int_{\mathbf{R}} (\dot{q}_0)^2 dz - \frac{1}{2}\kappa_r m(y)\Theta_0$$

$$= m\Delta^\Gamma \Theta_0 + \nabla_\Gamma\Theta_0 \cdot \nabla_\Gamma m - \frac{1}{2}\kappa_r m\Theta_0 + \Theta_0 \int_{\mathbf{R}} (\Delta^\Gamma \dot{q}_0)\dot{q}_0 \, dz.$$

Therefore, we finally obtain

$$\Theta_0 \int_{\mathbf{R}} (P_0 q_0)\ddot{q}_0 \, dz + \int_{\mathbf{R}} (P_0\varphi_0)\dot{q}_0 \, dz = m\Delta^\Gamma \Theta_0 + \nabla_\Gamma m \cdot \nabla_\Gamma\Theta_0 - \kappa_r m\Theta_0.$$

This concludes the proof of Proposition 5.1.   □

REMARK 5.3. By using Theorem 5.2 (ii), it is possible to obtain higher order approximations of the critical eigenvalues. Although, this is of interest in its own right, we do not exhibit the details in this paper.

References

[1] N. D. Alikakos, G. Fusco and V. Stefanopoulos, Critical spectrum and stability of interfaces for a class of reaction-diffusion equations. J. Differential Equations **126** (1996), no. 1, 106–167.

[2] X.-F. Chen, Generation and propagation of interfaces in reaction-diffusion equations, J. Differential Equations **96** (1992), 116–141.

[3] X.-F. Chen, Spectrum for the Allen-Cahn, Cahn-Hilliard, and Phase-Field Equations for Generic Interfaces, Comm. in Part. Diff. Equat. **19** (1994), 1371–1395.

[4] P. C. Fife and W. M. Greenlee, Interior transition layers from elliptic boundary value problems with a small parameter, Russian Math. Surveys **29** (1974), no. 4, 103–131.

[5] J. K. Hale and K. Sakamoto, Existence and stability of transition layers, Japan J. Appl. Math. **5** (1988), 367–405.

[6] H. Ikeda, On the asymptotic solutions for a weakly coupled elliptic boundary value problem with a small parameter, Hiroshima Math. J. **16** (1986), 227–250.

[7] K.-I. Nakamura, H. Matano, D. Hilhorst, and R. Schätzle, Singular Limit of a Reaction-Diffusion Equation with a Spatially Inhomogeneous Reaction Term, J. Stat. Phys. **95** (1999), 1165–1184.

[8] K. Nakashima, Stable transition layers in a balanced bistable equation, Differential Integral Equations **13** (2000), no. 7–9, 1025–1038.

[9] K. Sakamoto, Internal layers in high-dimensional domains, Proc. Royal Soc. of Edinburgh **128A** (1998), 359–401.

N. N. Nefedov
Department of Physics, Moscow State University
Vorobjovi Gori, 119899 *Moscow, Russia*
E-mail: nefedov@phys.msu.ru

K. Sakamoto
Department of Mathematical and Life Sciences
Graduate School of Science, Hiroshima University
Kagamiyama 1-3-1, 739-8526 *Higashi-Hiroshima, Japan*
E-mail: kuni@math.sci.hiroshima-u.ac.jp