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IRREDUCIBLE CHARACTERS OF SEMISIMPLE LIE GROUPS I

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1. Introduction. Let G be a connected semisimple Lie group. In [16], a collection of problems in the representation theory of G was set forth; one of the major ones was the determination of the irreducible characters of G. (This problem is not solved in the present paper.) The main theorem of [16] (Theorem 3.7 and Lemma 3.11 below) is a simple description of how these characters behave under "coherent continuation" (defined by 2.6 below). From this theorem, reducibility conditions for some standard induced representations were obtained. Unfortunately, the proof given in [16] for Theorem 3.7 is extremely complicated, and not very enlightening. The first purpose of this paper is to present a greatly simplified proof of this theorem, using Duflo's realization of the primitive ideals in the enveloping algebra of the complexified Lie algebra $\mathfrak{G}$ of G (cf. [4]).

Our study of irreducible characters is along the lines sketched in [16]. As indicated there (in Theorem 6.18) it suffices to determine the irreducible characters with a given nonsingular infinitesimal character. These form a finite set A, which has been parametrized by Langlands in [11]. Thus we may write $A = \{\Theta_1, \ldots, \Theta_N\}$; here each Θ_i is characterized in a certain way, but not explicitly known. To each $\Theta \in A$, a finite set $\{U_\alpha(\Theta)\}$ of invariant eigendistributions was associated in [16]; this will be described in section 3 (Definition 3.8). Each $U_\alpha(\Theta)$ is a sum of elements of A, with non-negative integral multiplicities. Once these multiplicities are known (for all α and Θ) the Θ_i can be computed explicitly by a finite algorithm. (This algorithm will be given in Section 5.) So the problem we consider is the determination of these multiplicities. (For reasons discussed in Section 2, this seems to be the simplest way of describing the irreducible characters.) Some of the multiplicities were found in [16]. Here we determine some more of them (Theorems 4.12 and 4.14). More importantly, we give a result relating the multiplicities to the dimensions of certain Ext groups (Theorem 3.9). This turns out to be an extremely powerful computational tool. In Section 6 we illustrate these results with a computation in $SP(3, 1)$.

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2. Notation and preliminary results. Throughout this paper, G will be a connected semisimple Lie group. The arguments of [16] relied heavily on the theory of parabolic induction, so it was convenient there to work in Harish-Chandra's category of reductive groups. Certain technical problems arose in that category, however, which were avoided by assuming that the Cartan subgroups of G were abelian; and for simplicity G was assumed to be linear. These assumptions eliminate many extremely interesting phenomena associated with the double covers of split linear groups; and since the arguments used here do not require the assumptions, it seems worthwhile to drop them. Allowing disconnected groups, on the other hand, does not seem to be worth the complications involved (notably in the theory of finite dimensional representations). We will tacitly assume that G has finite center; but the results all hold for the general case, with occasional trivial modifications of the proofs.

Fix a maximal compact subgroup K of G, and let θ be the associated Cartan involution. We write $\mathfrak{G}_0$, $\mathfrak{k}_0$ for the Lie algebras of G and K, $\mathfrak{G}$, $\mathfrak{k}$ for their complexifications, and $U(\mathfrak{G})$, $U(\mathfrak{k})$ for the enveloping algebras; analogous notation is used for other subgroups. The Killing form and its various restrictions and dualizations are all written $\langle\ ,\ \rangle$. The -1 eigenspace of θ in $\mathfrak{G}_0$ is written $\mathfrak{p}_0$.

We will make heavy use of the category $\mathfrak{M}(\mathfrak{G}, K)$ of *compatible* $(\mathfrak{G}, K)$ *modules* introduced by Lepowsky in [12]. An object X of $\mathfrak{M}(\mathfrak{G}, K)$ is a module for $\mathfrak{G}$ and K simultaneously; if $x \in X$, then $\langle K \cdot x\rangle$ (the span of the vectors $k \cdot x$ for $k \in K$) is finite dimensional, and the differential of the action of K is the action of $\mathfrak{k}$. If $\delta \in \hat{K}$ (the set of equivalence classes of irreducible representations of K) then the δ-primary subspace $X(\delta) \subseteq X$ is well defined, and $X = \bigoplus_{\delta \in \hat{K}} X(\delta)$. If V is a locally finite representation of K (i.e., $\langle K \cdot v\rangle$ is finite dimensional for every $v \in V$), then $U(\mathfrak{G}) \otimes \mathfrak{k}\ V$ can be given the structure of a compatible $(\mathfrak{G}, K)$ module; and it is projective in $\mathfrak{M}(\mathfrak{G}, K)$. It follows easily that every compatible $(\mathfrak{G}, K)$ module has a projective resolution, and therefore that the functor $\mathrm{Hom}_{\mathfrak{G}, K}$ has derived functors $\mathrm{Ext}^i_{\mathfrak{G}, K}$. The basic theory of these functors is developed in [1]. We will often omit the subscript $(\mathfrak{G}, K)$, since we will not consider any other kind of Ext group.

Let $\mathfrak{F}(\mathfrak{G}, K)$ denote the subcategory of $\mathfrak{M}(\mathfrak{G}, K)$ consisting of compatible $(\mathfrak{G}, K)$ modules with finite composition series. Such a module X has a *global character* $\Theta(X)$, which is a distribution on G; this is most easily defined as the sum of the characters of the irreducible composition factors of X. (By a theorem of Casselman, X is the Harish-Chandra module of a certain representation of G on a Hilbert space; and $\Theta(X)$ is the character of this representation. We will make no use of this fact, however.) The characters of inequivalent irreducible $(\mathfrak{G}, K)$ modules are linearly independent. It follows that the lattice $\mathcal{V}(\mathfrak{G}, K)$ of virtual $(\mathfrak{G}, K)$ modules (i.e., formal finite combinations of irreducible $(\mathfrak{G}, K)$ modules with integer coefficients) may be identified with a lattice of distributions on G, which has as a basis the set of irreducible characters. This lattice of distributions is called the lattice of *virtual characters*. In particular every virtual $(\mathfrak{G}, K)$ module V has a character $\Theta(V)$. We may of course identify $\mathcal{V}(\mathfrak{G}, K)$ with the Grothendieck group of $\mathcal{F}(\mathfrak{G}, K)$.

Write $\hat{G}$ for the set of equivalence classes of irreducible $(\mathfrak{G}, K)$ modules. Langlands has parametrized $\hat{G}$ in the following way. Let P be a parabolic subgroup of G, with Langlands decomposition $P = MAN$. (Thus M is θ-stable, and $\mathfrak{a}_0 \subseteq \mathfrak{p}_0$.) Fix a tempered representation δ of M, and a character $\nu \in \hat{A} \cong \mathfrak{a}^*$. We say that ν is *positive* (respectively *strictly positive*) *with respect to* P if $\langle \mathrm{Re}\, \nu, \alpha \rangle \geqslant 0$ (respectively $\langle \mathrm{Re}\, \nu, \alpha \rangle > 0$) for every root α of $\mathfrak{a}$ in $\mathfrak{N}$. Since N is normal in P, δ and ν define a representation of P which is trivial on N; we write this as $\delta \otimes \nu \otimes 1$ or $\delta \otimes \nu$. Put

$$I^P_{\delta\otimes\nu} = I_{\delta\otimes\nu} = \mathrm{Ind}^G_P\, \delta \otimes \nu \otimes 1.$$

Here induction means normalized induction. The K-finite vectors in $I_{\delta\otimes\nu}$ form a $(\mathfrak{G}, K)$ module with a finite composition series, which we may also write as $I_{\delta\otimes\nu}$. If ν is strictly positive with respect to P, then Langlands has constructed in [11] a canonical irreducible quotient $J_{\delta\otimes\nu}$ of $I_{\delta\otimes\nu}$. (D. Miličić has observed that the proof of Lemma 3.13 of [11] actually shows that $J_{\delta\otimes\nu}$ is the *unique* irreducible quotient of $I_{\delta\otimes\nu}$.) Again we will often consider $J_{\delta\otimes\nu}$ as a $(\mathfrak{G}, K)$ module.

THEOREM 2.1 (*Langlands* [11]). *Let X be an irreducible $(\mathfrak{G}, K)$ module. Then there is a parabolic subgroup $P = MAN$ of G, a tempered representation $\delta \in \hat{M}$, and a character $\nu \in \hat{A}$ strictly positive with respect to P, such that X is equivalent to $J^P_{\delta\otimes\nu}$. Furthermore the pair $(P, \delta \otimes \nu)$ is unique up to conjugation in G.*

We call a pair $(P, \delta \otimes \nu)$ as in the theorem a set of *Langlands data*. The tempered representations of the groups M arising in this theorem have been completely determined. (This depends on the connectedness of G or something similar; at any rate the tempered representations of arbitrary reductive groups are not known.) For G linear this is announced in [9]; the general case is treated in [17]. In particular the characters $\Theta(\delta \otimes \nu) = \Theta(I_{\delta\otimes\nu})$ occurring in the theorem, which we will call *standard*, can in principle be computed: the problem comes down to computing the characters of discrete series representations, which can be done usıng Harish-Chandra's proof of their uniqueness. This is certainly not a completely satisfactory description of the standard characters; but it is nonetheless reasonable to try to describe the irreducible characters by expressing them in terms of standard characters. This will be our goal. First we show that this is possible in principle.

PROPOSITION 2.2. *$J_{\delta\otimes\nu}$ occurs exactly once as a composition factor of $I^P_{\delta\otimes\nu}$. Suppose $J^{P'}_{\delta'\otimes\nu'}$ is a composition factor of $I^P_{\delta\otimes\nu}$. (Here $(P, \delta \otimes \nu)$ and $(P', \delta' \otimes \nu')$ are sets of Langlands data.) Then either $J^{P'}_{\delta'\otimes\nu'} = J^P_{\delta\otimes\nu}$, or $\langle \mathrm{Re}\, \nu', \mathrm{Re}\, \nu' \rangle < \langle \mathrm{Re}\, \nu, \mathrm{Re}\, \nu \rangle$.*

This result was discovered independently by several people. A proof may be found in [16] (Proposition 2.10).

COROLLARY 2.3. *The standard characters form a basis for the lattice of virtual characters of G. More precisely, there is an equation*

$$\Theta(J_{\delta\otimes\nu}) = \Theta(\delta\otimes\nu) + \sum_{i=1}^{N} n_i\Theta(\delta_i\otimes\nu_i), \qquad (*)$$

for some integers n_i; and $n_i \neq 0$ implies $\langle \mathrm{Re}\,\nu_i, \mathrm{Re}\,\nu_i\rangle < \langle \mathrm{Re}\,\nu, \mathrm{Re}\,\nu\rangle$.

This follows from Proposition 2.2 by an obvious induction. The first statement of the Corollary was first proved by Zuckerman (unpublished); the rest is again due to several people independently. One might at first suspect that $(*)$ is as bad as possible when $J_{\delta\otimes\nu}$ is finite dimensional. In that case Zuckerman has computed it explicitly. It has several nice features, notably that the n_i are all ± 1 (or zero). Unfortunately this is not true in general, and it seems unlikely that one can compute $(*)$ in a nice closed form. Our goal will be to construct a nice algorithm for computing $(*)$, with the hope that the complications in the result arise only from the fact that the algorithm has many steps. (As indicated in the introduction, even this program is not completely carried out here.) There is an obvious analogy with the algorithm mentioned above for computing discrete series characters. That algorithm is based on Harish-Chandra's "matching conditions" relating the behavior of a character on two Cartan subgroups, which are extremely simple in form; but the discrete series characters themselves are quite complicated.

Let $\mathfrak{Z}(\mathfrak{G})$ denote the center of $U(\mathfrak{G})$, and let $\mathfrak{h} \subseteq \mathfrak{G}$ be a Cartan subalgebra. Write $W = W(\mathfrak{G}/\mathfrak{h})$ for the Weyl group of $\mathfrak{h}$ in $\mathfrak{G}$. If $S(\mathfrak{h})$ is the symmetric algebra of $\mathfrak{h}$, then Harish-Chandra has defined an algebra isomorphism $\xi, : \mathfrak{Z}\cdot(\mathfrak{G}) \to S(\mathfrak{h})^W$. ($\xi$ is constructed using a system of positive roots, but is independent of the choice of that system.) If $\lambda \in \mathfrak{h}^*$, we define $\chi_\lambda : \mathfrak{Z}(\mathfrak{G}) \to \mathbf{C}$ by $\chi_\lambda(z) = \xi(z)(\lambda)$. In this way Spec $\mathfrak{Z}(\mathfrak{G})$ is identified with the set of W orbits in $\mathfrak{h}^*$. If X is an irreducible $(\mathfrak{G}, K)$ module, then $\mathfrak{Z}(\mathfrak{G})$ acts by scalars on X; if it acts through the homomorphism χ_λ, we say X has *infinitesimal character* λ. If X is a $(\mathfrak{G}, K)$ module of finite length, we define

$$P_\lambda(X) = \{x \in X \mid \text{for all } z \in \mathfrak{Z}(\mathfrak{G}) \text{ there is a positive integer } n \text{ such that } (z - \chi_\lambda(z))^n x = 0\}.$$

Then

$$X = \sum_{\lambda\in\mathfrak{h}^*/W} P_\lambda(X),$$

a finite direct sum; this is obvious. The functor P_λ on $\mathcal{F}(\mathfrak{G}, K)$ lifts to a homomorphism of $\mathcal{V}(\mathfrak{G}, K)$ to itself. In particular, if Θ is a virtual character, then $P_\lambda(\Theta)$ is well defined. In fact if Θ is known explicitly (as a function) then it is trivial to compute $P_\lambda(\Theta)$ explicitly.

Finally, suppose F is a finite dimensional $(\mathfrak{G}, K)$ module. If X is a $(\mathfrak{G}, K)$ module, so is $X \otimes F$; and if X has finite composition series, so does $X \otimes F$. In that case $\Theta(X \otimes F) = \Theta(X) \cdot \Theta(F)$, which is explicitly computable if $\Theta(X)$ is known and F is specified. (Notice that $\Theta(F)$ is a smooth function, so that the product is well defined.) There is a natural isomorphism

$$\mathrm{Hom}_{\mathfrak{G}, K}(X \otimes F, Y) \cong \mathrm{Hom}_{\mathfrak{G}, K}(X, Y \otimes F^*),$$

which easily implies a natural isomorphism

$$\mathrm{Ext}^i(X \otimes F, Y) \cong \mathrm{Ext}^i(X, Y \otimes F^*). \tag{2.4}$$

If X and Y have finite composition series, then

$$\mathrm{Hom}_{(\mathfrak{G}, K)}(P_\lambda(X), Y) \cong \mathrm{Hom}_{(\mathfrak{G}, K)}(X, P_\lambda(Y)).$$

Since Ext is defined in the larger category $\mathfrak{M}(\mathfrak{G}, K)$, it is not obvious that the corresponding result holds for Ext, but in fact it is proved in [1]:

$$\mathrm{Ext}^i(P_\lambda(X), Y) \cong \mathrm{Ext}^i(X, P_\lambda(Y)). \tag{2.5}$$

Write $\Delta = \Delta(\mathfrak{G}, \mathfrak{h})$ for the set of roots of $\mathfrak{h}$ in $\mathfrak{G}$. The infinitesimal character λ is called *nonsingular* if $\langle \alpha, \lambda \rangle \neq 0$ for all $\alpha \in \Delta(\mathfrak{G}, \mathfrak{h})$. (Clearly this depends only on the W orbit of λ in $\mathfrak{h}^*$.) Fix a system $\Delta^+ \subseteq \Delta$ of positive roots. We associate to Δ^+ a positive Weyl chamber $C \subseteq \mathfrak{h}^*$:

$$\begin{aligned} C = \{\lambda \in \mathfrak{h}^* \mid \mathrm{Re}\langle \alpha, \lambda \rangle > 0 \quad &\text{or} \quad \mathrm{Re}\langle \alpha, \lambda \rangle = 0 \\ \text{and} \quad \mathrm{Im}\langle \alpha, \lambda \rangle > 0, \quad &\text{for all} \quad \alpha \in \Delta^+ \}. \end{aligned}$$

Then $\bar{C}$ is a fundamental domain for the action of W on $\mathfrak{h}^*$. An element of $\bar{C}$ is called *dominant*; an element of C is called *strictly dominant*.

Recall from [16], section 5 Schmid's theory of coherent continuation of characters. If Θ is a virtual character with strictly dominant infinitesimal character λ, and $\mu \in \mathfrak{h}^*$ is a weight of a finite dimensional representation of G, then $S_\mu(\Theta)$ is a virtual character with infinitesimal character $\lambda + \mu$. (In [16] we had to consider μ as a character of a special Cartan subgroup of G; the connectedness of G allows us to be more careless here.) If F is a finite dimensional representation of G, and $\Delta(F) \subseteq \mathfrak{h}^*$ is the set of weights of F with multiplicities, then

$$\Theta \cdot \Theta(F) = \sum_{\mu \in \Delta(F)} S_\mu(\Theta); \tag{2.6}$$

this fact and $P_{\lambda+\mu}(S_\mu(\Theta)) = S_\mu(\Theta)$ determine $S_\mu(\Theta)$ uniquely, and the only problem is to define it. This is done in [16], using Harish-Chandra's theorem that a character is a locally L^1 function; it is perhaps worth remarking that use of that theorem can be avoided by combining recent work on Zuckerman on constructing the discrete series algebraically with the results of [17].

If X is a $\mathfrak{G}$ module, we write $\operatorname{Ann} X \subseteq U(\mathfrak{G})$ for the annihilator of X. We need some information about the relationship of Ann to P_λ and $\otimes F$; the precise form of the result is not very important, but the fact that a result exists will be crucial. Let $h : U(\mathfrak{G}) \to U(\mathfrak{G}) \otimes U(\mathfrak{G})$ be the Hopf map: h is an algebra map, and for $x \in \mathfrak{G}$, $h(x) = x \otimes 1 + 1 \otimes x$. If X and Y are $U(\mathfrak{G})$ modules, then $X \otimes Y$ is a $U(\mathfrak{G}) \otimes U(\mathfrak{G})$ module in an obvious way. The usual $U(\mathfrak{G})$ module structure of $X \otimes Y$ is obtained via the map h: if $v \in X \otimes Y$ and $u \in U(\mathfrak{G})$, then $u \cdot v = h(u) \cdot v$. The kernel of the map

$$U(\mathfrak{G}) \otimes U(\mathfrak{G}) \to (U(\mathfrak{G})/\operatorname{Ann} X) \otimes (U(\mathfrak{G})/\operatorname{Ann} Y)$$

is easily seen to be $\operatorname{Ann} X \otimes U(\mathfrak{G}) + U(\mathfrak{G}) \otimes \operatorname{Ann} Y$. We deduce

LEMMA 2.7. *If X and Y are $U(\mathfrak{G})$ modules, then the annihilator of $X \otimes Y$ as a $U(\mathfrak{G})$ module is*

$$h^{-1}(\operatorname{Ann} X \otimes U(\mathfrak{G}) + U(\mathfrak{G}) \otimes \operatorname{Ann} Y).$$

LEMMA 2.8. *Let X be a $(\mathfrak{G}, K)$ module of finite length, say with at most n irreducible composition factors. Suppose that these factors have infinitesimal characters contained in*

$$\{\lambda_0, \lambda_1, \ldots, \lambda_N\} \subseteq \overline{C}.$$

Then

$$\operatorname{Ann}(P_{\lambda_0}(X)) = \left\{ u \in U(\mathfrak{G}) \mid \text{for all } z \in \mathfrak{Z}(\mathfrak{G}), \left[u \cdot \prod_{j=1}^{N} (z - \chi_{\lambda_j}(z))^n \right] \in \operatorname{Ann} X \right\}.$$

Proof. Choose a filtration $\{0\} = F^0 \subseteq F^1 \subseteq \cdots \subseteq F^n = X$, such that F^i/F^{i-1} is irreducible or zero for $1 \leqslant i \leqslant n$. Put $X_j = P_{\lambda_j}(X)$. Since the λ_i lie in $\overline{C}$, they are distinct modulo W; so $X = \bigoplus_{j=0}^N X_j$. Each X_j may be regarded as a quotient of X, and hence inherits a filtration $\{X_j^i\}$ with the same properties as $\{F^i\}$. Of course X_j^i/X_j^{i-1} has infinitesimal character λ_j; so if $z \in \mathfrak{Z}(\mathfrak{G})$, one sees by induction that $(z - \chi_{\lambda_j}(z))^i$ annihilates X_j^i. In particular $(z - \chi_{\lambda_j}(z))^n$ annihilates X_j. If we write A for the subalgebra of $U(\mathfrak{G})$ defined in the lemma, then it follows that $\operatorname{Ann}(P_{\lambda_0}(X)) \subseteq A$.

Conversely, suppose $u \in A$; we must show u annihilates $P_{\lambda_0}(X)$. Choose $z \in \mathfrak{Z}(\mathfrak{G})$ such that $\chi_{\lambda_0}(z) \neq \chi_{\lambda_j}(z)$ for all $j > 0$. Now $z - \chi_{\lambda_0}(z)$ defines a nilpotent operator T on $P_{\lambda_0}(X) : T^n = 0$. So (computing in $\operatorname{End} P_{\lambda_0}(X)$), $z - \chi_{\lambda_j}(z) = T + (\chi_{\lambda_0}(z) - \chi_{\lambda_j}(z))$, which is invertible for $j > 0$. So

$$\prod_{j=1}^{N} (z - \chi_{\lambda_j}(z))^n = S$$

is invertible as an operator on $P_{\lambda_0}(X)$. Since $S \cdot u$ annihilates all of X, it annihilates $P_{\lambda_0}(X)$; so u itself must annihilate $P_{\lambda_0}(X)$. Q.E.D.

The main idea of the proofs given in section 3 for the results of [16] can now be described very simply. By a theorem of Duflo, the annihilator of any irreducible $U(\mathfrak{G})$ module is the annihilator of some irreducible highest weight module. (This will be described more carefully in the next section.) We will translate statements about irreducible $(\mathfrak{G}, K)$ modules into statements about their annihilators, and then back into statements about highest weight modules. In this last form they are often much easier to prove.

3. Coherent continuation across walls. Recall our fixed choice of a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{G}$, positive root system $\Delta^+ \subseteq \Delta(\mathfrak{G}, \mathfrak{h})$, and Weyl chamber $C \subseteq \mathfrak{h}^*$. Throughout most of this paper we will be concerned with $(\mathfrak{G}, K)$ modules X of finite length having a fixed (generalized) infinitesimal character λ. This means that $P_\lambda(X) = X$, and thus that for all $z \in \mathfrak{Z}(\mathfrak{G})$, $z - \chi_\lambda(z)$ acts nilpotently (not necessarily trivially) on X. Generally λ will be nonsingular, so we can and do assume $\lambda \in C$. We define

$$\Delta_\lambda = \left\{ \alpha \in \Delta(\mathfrak{G}, \mathfrak{h}) \,\middle|\, \frac{2\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} \in \mathbf{Z} \right\}$$

$$\Delta_\lambda^+ = \Delta_\lambda \cap \Delta^+$$

$$W_\lambda = W(\Delta_\lambda) \subseteq W.$$

It is a standard result that Δ_λ is a root system, so that the Weyl group W_λ is well defined. It should be emphasized that Δ_λ need not correspond to a subalgebra of $\mathfrak{G}$. Let Π_λ denote the set of simple roots of Δ_λ^+.

LEMMA 3.1. *For each $\alpha \in \Pi_\lambda$ there exists a positive root system $\Psi_\alpha \subseteq \Delta(\mathfrak{G}, \mathfrak{h})$ such that $\Psi_\alpha \supseteq \Delta_\lambda^+$, and α is simple for Ψ_α.*

This generalizes the standard result that every root is conjugate under W to a simple root, and can be proved in the same way; details are left to the reader.

Fix $\alpha \in \Pi_\lambda$, and a positive system Ψ_α as in Lemma 3.1. Choose an integral weight μ_α^1, dominant for Ψ_α, and so large that $\lambda + \mu_\alpha^1$ is dominant and nonsingular for Ψ_α. (If λ is already dominant for Ψ_α—for example if λ is integral—we could take $\mu_\alpha^1 = 0$.) Put $m = 2\langle \alpha, \lambda + \mu_\alpha^1 \rangle / \langle \alpha, \alpha \rangle$, and define μ_α^2 to be m times the fundamental weight of Ψ_α corresponding to α: thus if β is a simple root of Ψ_α, $\langle \beta, \mu_\alpha^2 \rangle = 0$ if $\alpha \neq \beta$, and $2\langle \alpha, \mu_\alpha^2 \rangle / \langle \alpha, \alpha \rangle = m$. It follows that if $\delta \in \Pi_\lambda$, then $\langle \delta, \lambda + \mu_\alpha^1 - \mu_\alpha^2 \rangle \geqslant 0$, with equality if and only if $\delta = \alpha$.

Following Zuckerman [19], we now introduce several functors on $\mathfrak{F}(\mathfrak{G}, K)$. Let F_α^i denote the finite dimensional irreducible $\mathfrak{G}$ module of highest weight μ_α^i with respect to Ψ_α, and $(F_\alpha^i)^*$ its contragredient (which has lowest weight $-\mu_\alpha^i$). Possibly passing to a finite covering group, we may assume that F_α^i exponentiates to a representation of G. (We may often do this without further comment; the F_α^i appear only as a technical device for constructing certain representations which will be defined on our original group.) If X is a $(\mathfrak{G}, K)$ module of finite length, we define

$$\psi_\alpha^1(X) = P_{\lambda+\mu_\alpha^1}\big((P_\lambda(X)) \otimes F_\alpha^1\big)$$

$$\psi_\alpha^2(X) = P_{\lambda+\mu_\alpha^1-\mu_\alpha^2}\big((P_{\lambda+\mu_\alpha^1}(X)) \otimes (F_\alpha^2)^*\big)$$

$$\varphi_\alpha^2(X) = P_{\lambda+\mu_\alpha^1}\big((P_{\lambda+\mu_\alpha^1-\mu_\alpha^2}(X)) \otimes F_\alpha^2\big)$$

$$\varphi_\alpha^1(X) = P_\lambda\big((P_{\lambda+\mu_\alpha^1}(X)) \otimes (F_\alpha^1)^*\big).$$

Put $\psi_\alpha = \psi_\alpha^2 \circ \psi_\alpha^1$, $\varphi_\alpha = \varphi_\alpha^1 \circ \varphi_\alpha^2$. The functors φ_α^2 and ψ_α^2 are precisely of the type considered in Definition 1.1 of [19]. Since λ may not be dominant with respect to Ψ_α, the other two are not precisely of this kind; but since only the integral roots really matter, the proofs of [19] go over almost without change. (Compare [8], and Theorem 5.20(a) of [16].) For $\gamma \in \mathfrak{h}^*$, we write $\mathcal{F}_\gamma(\mathfrak{G}, K)$ for the subcategory of $\mathcal{F}(\mathfrak{G}, K)$ consisting of modules with generalized infinitesimal character γ. Then we have

PROPOSITION 3.2 (cf. [19], Theorem 1.2 and 1.3, and section 3). *The functor ψ_α^1 restricts to an isomorphism of $\mathcal{F}_\lambda(\mathfrak{G}, K)$ with $\mathcal{F}_{\lambda+\mu_\alpha^1}(\mathfrak{G}, K)$, with natural inverse φ_α^1. If $n = 2\langle\alpha, \lambda\rangle/\langle\alpha, \alpha\rangle$, $\mu_\alpha = \mu_\alpha^2 - \mu_\alpha^1$, and X has generalized infinitesimal character λ, then*

(a) $\Theta(\psi_\alpha(X)) = S_{-\mu_\alpha}(\Theta(X))$

(b) $\Theta(\varphi_\alpha\psi_\alpha(X)) = \Theta(X) + S_{-n\alpha}(\Theta(X))$.

If X is irreducible, then $\psi_\alpha(X)$ is primary, of length at most two.

Proof. The first statement is proved in the same way as Theorem 1.2 of [19]. Formulas (a) and (b) follow from the definition of S_μ and the computations of section 3 of [19] (compare also the proof of Theorem 5.20 of [16]). The last statement follows from Zuckerman's proof of Theorem 1.3 of [19]; the number 2 is the order of the stabilizer of $\lambda - \mu_\alpha$ in W. Q.E.D.

The first thing to notice about this result is that (at least on the level of characters) $\varphi_\alpha\psi_\alpha$ is independent of all choices. By Theorem 1.2 of [19], $\varphi_\alpha\psi_\alpha(X) = 0$ if and only if $\psi_\alpha(X) = 0$. Accordingly we make

Definition 3.3. Suppose X is a $(\mathfrak{G}, K)$ module of finite length with nonsingular infinitesimal character λ. The *Borho-Jantzen-Duflo τ-invariant* $\tau(X)$ is the subset of Π_λ defined by $\alpha \in \tau(X)$ if and only if $\psi_\alpha(X) = 0$ (which is equivalent to $S_{-n\alpha}(\Theta(X)) = -\Theta(X)$ by the remarks above and Proposition 3.2(b)).

The τ invariant of Borho-Jantzen and Duflo (cf. [2], [4]) is defined for certain ideals in $U(\mathfrak{G})$; in case X is irreducible, what is defined above is their $\tau(\text{Ann } X)$.

Now fix an irreducible $(\mathfrak{G}, K)$ module X with nonsingular infinitesimal character λ. For each $\alpha \in \Pi_\lambda - \tau(X)$, we will define a new $(\mathfrak{G}, K)$ module $U_\alpha(X)$. The existence of $U_\alpha(X)$ is the main theorem of [16]; we give now a new proof.

LEMMA 3.4. *Suppose A and B are $(\mathfrak{G}, K)$ modules of finite length. Then there are natural isomorphisms*

$$\operatorname{Ext}^*(\varphi_\alpha A, B) \cong \operatorname{Ext}^*(A, \psi_\alpha B)$$

$$\operatorname{Ext}^*(A, \varphi_\alpha B) \cong \operatorname{Ext}^*(\psi_\alpha A, B).$$

Proof. These follow immediately from (2.4) and (2.5). Q.E.D.

The preceding result was first proved by G. Zuckerman (unpublished). Suppose now that X is irreducible as above, and that $\alpha \in \Pi_\lambda - \tau(X)$. By Definition 3.3 and Proposition 3.2, $\psi_\alpha X$ is non-zero and primary, say with unique irreducible constituent Z. By Lemma 3.4,

$$\begin{aligned}\operatorname{Hom}_{\mathfrak{G}, K}(X, \varphi_\alpha\psi_\alpha X) &\cong \operatorname{Hom}_{\mathfrak{G}, K}(\psi_\alpha X, \psi_\alpha X)\\ &\cong \operatorname{Hom}_{\mathfrak{G}, K}(\varphi_\alpha\psi_\alpha X, X).\end{aligned} \tag{3.5}$$

The middle term here has a canonical non-zero element, namely the identity. This gives rise to two maps

$$0 \to X \to \varphi_\alpha\psi_\alpha X \to X \to 0. \tag{3.6}$$

The first map is injective and the second surjective, since X is irreducible.

THEOREM 3.7 ([16], Theorem 5.15 and Corollary 6.17). *Let X be an irreducible $(\mathfrak{G}, K)$ module with non-singular infinitesimal character λ. With notation as above, fix a root $\alpha \in \Pi_\lambda - \tau(X)$. Then*

(a) *(3.6) is a chain complex, i.e., the composition of the maps is zero.*

(b) *$\psi_\alpha X$ is irreducible.*

Proof. We will show that if (a) or (b) fails, then $\varphi_\alpha\psi_\alpha X \cong X \oplus Y$, with Y irreducible and $\psi_\alpha Y \cong \psi_\alpha X$. Thus $\mathfrak{Z}(\mathfrak{G})$ acts by scalars on $\varphi_\alpha\psi_\alpha X$. By a separate argument, we will show that the Casimir operator of $\mathfrak{G}$ cannot act by scalars on $\varphi_\alpha\psi_\alpha X$. This contradiction will prove the theorem.

Notice first of all that $\Theta(\psi_\alpha\varphi_\alpha\psi_\alpha X) = 2\Theta(\psi_\alpha X)$; this is essentially Lemma 3.3 of [19]. (As remarked earlier, the functors ψ_α and φ_α are not precisely those considered by Zuckerman, but the same argument works.) Suppose first that (b) fails. In this case Proposition 3.2 implies that $\psi_\alpha X$ has exactly two irreducible, isomorphic composition factors (which we call Z). It follows trivially that $\operatorname{Hom}_{\mathfrak{G}, K}(\psi_\alpha X, \psi_\alpha X)$ has dimension two or four, depending on whether $\psi_\alpha X$ is completely reducible or not. In either case we have from 3.5 an injection

$$0 \to X \oplus X \to \varphi_\alpha\psi_\alpha X.$$

Let Q be the quotient module $\varphi_\alpha\psi_\alpha X/X \oplus X$. To prove the first claim in this case it suffices to show that $Q = 0$, or equivalently that $\operatorname{Hom}_{\mathfrak{G}, K}(\varphi_\alpha\psi_\alpha X, Q) = 0$. By Lemma 3.4, this is $\operatorname{Hom}_{\mathfrak{G}, K}(\psi_\alpha X, \psi_\alpha Q)$. But

$$\Theta(\psi_\alpha Q) = \Theta(\psi_\alpha\varphi_\alpha\psi_\alpha X) - 2\Theta(\psi_\alpha X) = 0$$

by the remark above; so $\psi_\alpha Q = 0$, proving the claim.

Suppose next that (b) holds but (a) fails. Let Y_0 be the kernel of $\varphi_\alpha\psi_\alpha X \to X$. Then (since (a) fails) 3.6 provides an isomorphism $\varphi_\alpha\psi_\alpha X \cong X \oplus Y_0$. Let Y be an irreducible subrepresentation of Y_0. Then

$$0 \neq \operatorname{Hom}_{\mathfrak{G}, K}(Y, \varphi_\alpha\psi_\alpha X) \cong \operatorname{Hom}_{\mathfrak{G}, K}(\psi_\alpha Y, \psi_\alpha X).$$

Since $\psi_\alpha X$ is assumed irreducible, it follows that $\Theta(\psi_\alpha Y) = \Theta(\psi_\alpha X) + \Theta_1$, with Θ_1 the character of a representation. Hence

$$\Theta(\psi_\alpha(Y_0/Y)) = \Theta(\psi_\alpha\varphi_\alpha\psi_\alpha X) - \Theta(\psi_\alpha X) - \Theta(\psi_\alpha Y) = -\Theta_1$$

since $\Theta(\psi_\alpha\varphi_\alpha\psi_\alpha X) = 2\Theta(\psi_\alpha X)$. Now the left side is the character of a representation, and the right side is minus the character of a representation. Hence $\Theta_1 = 0$, and $\psi_\alpha(Y_0/Y) = 0$. It follows that

$$\operatorname{Hom}_{\mathfrak{G}, K}(\varphi_\alpha\psi_\alpha X, Y_0/Y) \cong \operatorname{Hom}_{\mathfrak{G}, K}(\psi_\alpha X, \psi_\alpha(Y_0/Y)) = 0,$$

and hence that $Y_0/Y = 0$, i.e., $Y_0 = Y$. This proves the claim in general.

Let Ω be the Casimir operator of $\mathfrak{G}$. We want to show that Ω does not act by scalars on $\varphi_\alpha\psi_\alpha X$, or equivalently that $\Omega - \chi_\lambda(\Omega) \notin \operatorname{Ann}(\varphi_\alpha\psi_\alpha X)$. By Proposition 3.2, it is enough to show that Ω does not act by scalars on $\varphi_\alpha^2(\psi_\alpha(X))$; in other words we may replace X by $\psi_\alpha^1(X)$, λ by $\lambda + \mu_\alpha^1$, and μ_α by μ_α^2. Thus we may assume that μ_α is dominant for Ψ_α. (Similar reductions will be made hereafter without much comment.) Put $F_\alpha = F_\alpha^2$. Let Z be the unique irreducible composition factor of $\psi_\alpha X$. Since Z is a subrepresentation of $\psi_\alpha X$, $\operatorname{Ann}(Z) \supseteq \operatorname{Ann}(\psi_\alpha X)$. Let L be an irreducible $\mathfrak{G}$-module (not necessarily a $(\mathfrak{G}, K)$ module), and suppose $\operatorname{Ann}(L) \supseteq \operatorname{Ann}(Z)$. By Lemmas 2.7 and 2.8,

$$\operatorname{Ann}(\varphi_\alpha L) \supseteq \operatorname{Ann}(\varphi_\alpha\psi_\alpha X).$$

(Here $\varphi_\alpha L = P_\lambda(L \otimes F_\alpha)$. The functor P_λ makes sense because $L \otimes F_\alpha$ is annihilated by an ideal of finite codimension in $\mathfrak{Z}(\mathfrak{G})$. This will be obvious for the L we consider; or one could appeal to a general theorem of Kostant ([10].) So it suffices to construct an L with $\operatorname{Ann}(L) \supseteq \operatorname{Ann}(Z)$, and show that Ω does not act by scalars on $P_\lambda(L \otimes F_\alpha)$. As the annihilator of an irreducible $\mathfrak{G}$-module, $\operatorname{Ann}(Z)$ is by definition a *primitive ideal* in $U(\mathfrak{G})$. The set of primitive ideals containing $\ker(\chi_{\lambda - \mu_\alpha}) \subseteq \mathfrak{Z}(\mathfrak{G})$ has a unique maximal element (as was proved by Dixmier); and by a deep theorem of Duflo, this element is precisely the annihilator of the irreducible Ψ_α-highest weight module L, with highest weight

$\lambda - \mu_\alpha - \rho(\Psi_\alpha)$. (For all this see [4]; the theory of highest weight modules is described in [3].) Here $\rho(\Psi_\alpha)$ is just half the sum of the roots of Ψ_α; we will write this simply as ρ below. Recall that the $\mathfrak{G}$-module L is completely reducible for $\mathfrak{h}$; and every weight occurring is of the form $\lambda - \mu_\alpha - \rho - Q$, with Q a sum of roots of Ψ_α. The highest weight $\lambda - \mu_\alpha - \rho$ occurs with multiplicity one, as do all the weights of the form $\lambda - \mu_\alpha - \rho - m\alpha$, for m a non-negative integer.

Since $\mathrm{Ann}(L) \supseteq \mathrm{Ann}(Z)$, it remains to show that Ω does not act by scalars on $P_\lambda(L \otimes F_\alpha)$. This we prove by a reduction to $\mathfrak{sl}(2)$. Since every weight of F_α is of the form $\mu_\alpha - Q$, with Q a sum of roots of Ψ_α, we see that every weight of $L \otimes F_\alpha$ is of the form $\lambda - \rho - Q$. Let V denote the subspace of weights of the form $\lambda - \rho - m\alpha$, and let $\mathfrak{G}_1 \cong \mathfrak{sl}(2, \mathbf{C})$ be the subalgebra of $\mathfrak{G}$ generated by the root vectors for the roots $\pm\alpha$. Then V is invariant under $\mathfrak{h}$ and $\mathfrak{G}_1$, and is annihilated by the root vectors for roots $\beta \in \Psi_\alpha - \{\alpha\}$. By a standard computation, we can write

$$\Omega = \Omega_1 + \sum_{\beta \in \Psi_\alpha - \{\alpha\}} X_{-\beta} X_\beta + r(h).$$

Here X_β is a root vector for the root β; $r(h)$ is a polynomial of degree two in $\mathfrak{h}$, invariant under translation by α; and Ω_1 is the Casimir operator for $\mathfrak{G}_1$. Hence the second term annihilates V, and $r(h)$ acts by a fixed scalar c_1 on V. In particular Ω preserves V, and acts by $\Omega_1 + c_1$ there.

Let $0 \neq v_0 \in V$ be a vector of weight $\lambda - \rho$. By the definition of χ_λ, $\Omega \cdot v_0 = \chi_\lambda(\Omega) v_0$. Let $Q_\lambda(V)$ denote the generalized eigenspace of Ω in V with eigenvalue $\chi_\lambda(\Omega)$. By the preceding paragraph, this is just the generalized eigenspace of Ω_1 in V with eigenvalue $\chi_\lambda(\Omega) - c_1$. Let F_1 denote the subspace of F_α corresponding to weights of the form $\mu_\alpha - m\alpha$, and L_1 the subspace of L corresponding to weights of the form $\lambda - \mu_\alpha - \rho - m\alpha$. It is standard that F_1 is the irreducible $\mathfrak{G}_1$ module of dimension $2\langle \alpha, \lambda\rangle / \langle \alpha, \alpha\rangle = n$; and it follows from an earlier remark on the weights of L_1 that the weights of L_1 as a $\mathfrak{G}_1$ module are $0, -\alpha, -2\alpha, \ldots$, each occurring with multiplicity one. Finally observe that, as a $\mathfrak{G}_1$ module, $V \cong L_1 \otimes F_1$. With all of this explicit information, it is straightforward to compute that Ω_1 does not act by scalars on $Q_\lambda(V)$: one writes down the action of $\mathfrak{G}_1$ on bases of weight vectors in F_1 and L_1. The rather tedious details are left to the reader (compare for example section 7.2 of [7]). Hence Ω does not act by scalars on $Q_\lambda(V)$. To complete the proof of Theorem 3.7, it remains to show that $Q_\lambda(V) \subseteq P_\lambda(L \otimes F_\alpha)$. Let $\mathfrak{h}_\alpha$ be the kernel of α in $\mathfrak{h}$, and suppose $z \in \mathfrak{Z}(\mathfrak{G})$. By the Harish-Chandra homomorphism, we can write

$$z = p_z(\Omega_1, h) + \sum_{\beta \in \Psi_\alpha - \{\alpha\}} T_\beta X_\beta . \qquad (*)$$

Here p_z is a polynomial in Ω_1 and $\mathfrak{h}_\alpha$, and $T_\beta \in U(\mathfrak{G})$. Just as in the case $z = \Omega$, the second term annihilates V, and $\mathfrak{h}_\alpha$ acts by evaluation at $\lambda - \rho$. Now by definition of χ_λ, $z \cdot v_0 = \chi_\lambda(z) \cdot v_0$. Then $(*)$ implies that z has generalized eigenvalue $\chi_\lambda(z)$ on the entire generalized eigenspace of Ω_1 in V containing v_0. But this says precisely that $Q_\lambda(V) \subseteq P_\lambda(L \otimes F_\alpha)$. Q.E.D.

Definition 3.8. Let X be an irreducible $(\mathfrak{G}, K)$ module with infinitesimal character λ, and suppose $\alpha \in \Pi_\lambda - \tau(X)$. In the chain complex 3.6, let $K_\alpha = K_\alpha(X)$ be the kernel of $\varphi_\alpha\psi_\alpha X \to X$, $Q_\alpha = Q_\alpha(X)$ the quotient $\varphi_\alpha\psi_\alpha X/X$, and $U_\alpha = U_\alpha(X)$ the cohomology $K_\alpha/X \cong \ker(Q_\alpha \to X)$.

By Proposition 3.2, $\Theta(Q_\alpha) = \Theta(K_\alpha) = S_{-n\alpha}(\Theta)$.

THEOREM 3.9. *Let X be an irreducible $(\mathfrak{G}, K)$ module with nonsingular infinitesimal character λ, and suppose $\alpha \in \Pi_\lambda - \tau(X)$. Then*

(a) *$\varphi_\alpha\psi_\alpha X$ has X as its unique irreducible subrepresentation and as its unique irreducible quotient.*

(b) *$\alpha \in \tau(U_\alpha(X))$, and $\mathfrak{Z}(\mathfrak{G})$ acts by scalars on $U_\alpha(X)$.*

(c) *Suppose B is a $(\mathfrak{G}, K)$ module of finite length with (generalized) infinitesimal character λ, and $\alpha \in \tau(B)$. Then*

$$\operatorname{Hom}_{\mathfrak{G}, K}(B, U_\alpha(X)) \cong \operatorname{Ext}^1(B, X).$$

More generally, if $\operatorname{Ext}^i(B, X) = 0$, *then*

$$\operatorname{Ext}^i(B, U_\alpha(X)) \cong \operatorname{Ext}^{i-1}(B, X) \oplus \operatorname{Ext}^{i+1}(B, X).$$

Dually, if $\operatorname{Ext}^i(X, B) = 0$, *then*

$$\operatorname{Ext}^i(U_\alpha(X), B) \cong \operatorname{Ext}^{i-1}(X, B) \oplus \operatorname{Ext}^{i+1}(X, B).$$

(This theorem was essentially proved in the course of joint work with B. Speh, except for the higher Ext formulas which are joint work with G. Zuckerman.)

Proof. We begin with the first part of (b). As we saw in the proof of Theorem 3.7, $\Theta(\psi_\alpha\varphi_\alpha\psi_\alpha X) = 2\Theta(\psi_\alpha X)$; so

$$\Theta(\psi_\alpha U_\alpha) = \Theta(\psi_\alpha\varphi_\alpha\psi_\alpha X) - 2\Theta(\psi_\alpha X) = 0.$$

This proves the first part of (b). By Lemma 3.4 and Theorem 3.7(b),

$$\operatorname{Hom}_{\mathfrak{G}, K}(X, \varphi_\alpha\psi_\alpha X) \cong \operatorname{Hom}_{\mathfrak{G}, K}(\psi_\alpha X, \psi_\alpha X) \cong \mathbf{C};$$

so X occurs only once as a subrepresentation of $\varphi_\alpha\psi_\alpha X$. Similarly it occurs once as a quotient. To prove (a), suppose $Y \neq X$ is an irreducible subrepresentation of $\varphi_\alpha\psi_\alpha X$. Then Y occurs in U_α, so $\psi_\alpha Y = 0$ (since $\alpha \in \tau(U_\alpha)$); so

$$0 \neq \operatorname{Hom}_{\mathfrak{G}, K}(Y, \varphi_\alpha\psi_\alpha X) \cong \operatorname{Hom}_{\mathfrak{G}, K}(\psi_\alpha Y, \psi_\alpha X) = 0,$$

a contradiction. For the second part of (b), suppose $z \in \mathfrak{Z}(\mathfrak{G})$. Then $z - \chi_\lambda(z)$ defines a self intertwining operator $I : K_\alpha \to K_\alpha$. Since $z - \chi_\lambda(z)$ annihilates X, I annihilates X. Since X occurs only once in K_α, $I(K_\alpha) \subseteq K_\alpha$ is a subrepresentation not containing X. By (a), $I(K_\alpha) = 0$. In particular, $z - \chi_\lambda(z)$ annihilates $U_\alpha(X)$. For (c), notice that $\alpha \in \tau(B)$ but $\alpha \notin \tau(X)$; so X does not occur in B, so $\mathrm{Hom}_{\mathfrak{G},K}(B, X) = \mathrm{Ext}^0(B, X) = 0$, so the second formula really generalizes the first; so we consider the second. We have two short exact sequences

$$0 \to X \to \varphi_\alpha \psi_\alpha X \to Q_\alpha \to 0 \tag{$*$}$$

$$0 \to U_\alpha \to Q_\alpha \to X \to 0. \tag{$**$}$$

Now $\psi_\alpha B = 0$ since $\alpha \in \tau(B)$; so

$$\mathrm{Ext}^j(B, \varphi_\alpha \psi_\alpha X) \cong \mathrm{Ext}^j(\psi_\alpha B, \psi_\alpha X) = 0$$

by Lemma 3.4. The long exact sequence for $\mathrm{Ext}^j(B, *)$ attached to $(*)$ gives

$$\mathrm{Ext}^j(B, X) \cong \mathrm{Ext}^{j-1}(B, Q_\alpha).$$

By the long exact sequence for $(**)$,

$$\cdots \to \mathrm{Ext}^{i-1}(B, Q_\alpha) \to \mathrm{Ext}^{i-1}(B, X) \to \mathrm{Ext}^i(B, U_\alpha)$$
$$\to \mathrm{Ext}^i(B, Q_\alpha) \to \mathrm{Ext}^i(B, X) \to \cdots .$$

Now replacing $\mathrm{Ext}^{i-1}(B, Q_\alpha) \cong \mathrm{Ext}^i(B, X)$ by zero, and $\mathrm{Ext}^i(B, Q_\alpha)$ by Ext^{i+1} (B, X), we find a short exact sequence

$$0 \to \mathrm{Ext}^{i-1}(B, X) \to \mathrm{Ext}^i(B, U_\alpha) \to \mathrm{Ext}^{i+1}(B, X) \to 0.$$

The dual assertion is proved similarly. Q.E.D.

THEOREM 3.10. *Let X be an irreducible $(\mathfrak{G}, K)$ module with nonsingular infinitesimal character λ, and suppose $\alpha \in \Pi_\lambda - \tau(X)$, $\beta \in \tau(X)$.*

(a) *If β is orthogonal to α, then $\beta \in \tau(U_\alpha)$.*

(b) *If α and β span a root system of type A_2, then U_α has exactly one irreducible constituent Y with $\beta \notin \tau(Y)$.*

(c) *If α and β span a root system of type B_2, then U_α has one or two irreducible constituents Y_i with $\beta \notin \tau(Y_i)$.*

Proof. We will make repeated use of

LEMMA 3.11. *Let X be an irreducible $(\mathfrak{G}, K)$ module with nonsingular infinitesimal character λ, and suppose $\alpha \in \Pi_\lambda$. Let Θ be the character of X.*

(a) *If $\alpha \in \tau(X)$, then $S_{-n\alpha}(\Theta) = -\Theta$.*

(b) *If $\alpha \notin \tau(X)$, then $S_{-n\alpha}(\Theta) = \Theta + \Theta_1$, with Θ_1 the character of a representation, and $\alpha \in \tau(Y)$ for every irreducible constituent Y of Θ_1.*

(c) *If $\alpha \notin \tau(X)$, and Θ' is an arbitrary virtual character with infinitesimal character λ, then the multiplicity of X in Θ' is the same as the multiplicity of X in $S_{-n\alpha}(\Theta')$.*

(These results were all formulated in [16]).

Proof. (a) is contained in Definition 3.3; (b) is a reformulation of 3.9(b). For (c), (a) and (b) imply that every irreducible constituent Y of $\Theta' - S_{-n\alpha}(\Theta')$ has $\alpha \in \tau(Y)$. Q.E.D.

Definition 3.12. A virtual character Θ with nonsingular infinitesimal character λ is called *α-singular* ($\alpha \in \Pi_\lambda$) if $S_{-n\alpha}(\Theta) = -\Theta$, or equivalently (by 3.11(c)) if every irreducible constituent Y of Θ has $\alpha \in \tau(Y)$. A $(\mathfrak{G}, K)$ module of finite length is called α-singular if its character is (equivalently if α is in its τ-invariant).

Finally we need a simple composition law for coherent continuation.

LEMMA 3.13. *Let Θ be a virtual character with nonsingular infinitesimal character $\lambda \in C$, and let μ_1 and μ_2 be weights of finite dimensional representations. If $\lambda + \mu_1 \in w \cdot C$ for some $w \in W$ (the Weyl group of $\mathfrak{h}$ in $\mathfrak{G}$) then*

$$S_{\mu_2}(S_{\mu_1}(\Theta)) = S_{\mu_1 + w \cdot \mu_2}(\Theta).$$

Proof. This is an immediate consequence of the definition of $S_\mu(\Theta)$ (cf. [16], section 5). Q.E.D.

We proceed now with the proof of Theorem 3.10. Let Θ be the character of X. For $w \in W_\lambda$, $w\lambda - \lambda$ is a sum of roots with integral coefficients, and is therefore a weight of a finite dimensional representation. Accordingly we can define $w \cdot \Theta = S_{w\lambda - \lambda}(\Theta)$. This of course makes sense for any character Θ' with infinitesimal character λ; and by Lemma 3.13 one finds immediately that for $w_1, w_2 \in W_\lambda$,

$$w_1 \cdot (w_2 \cdot \Theta') = (w_1 w_2) \cdot \Theta'.$$

Let s_β denote reflection about the root β. With this notation, Θ' is α-singular if and only if $s_\alpha \cdot \Theta' = -\Theta'$. For 3.10(a), we have $\Theta(U_\alpha) = s_\alpha(\Theta) - \Theta$. By hypothesis Θ is β-singular, and s_α and s_β commute. Hence

$$\begin{aligned} s_\beta \cdot \Theta(U_\alpha) &= s_\beta s_\alpha(\Theta) - s_\beta \Theta \\ &= s_\alpha(s_\beta \Theta) - s_\beta \Theta \\ &= -s_\alpha(\Theta) + \Theta = -(\Theta(U_\alpha)). \end{aligned}$$

Consider now 3.10(b). By hypothesis $s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta$, and $s_\beta \cdot \Theta = -\Theta$. Hence $s_\alpha s_\beta s_\alpha \Theta = -s_\beta s_\alpha \Theta$, which means that $s_\beta s_\alpha \Theta$ is α-singular. Using Lemma 3.11(b), we write

$$s_\alpha \Theta = \Theta + \Theta^\alpha + \Theta^{\alpha\beta};$$

here Θ^α and $\Theta^{\alpha\beta}$ are characters of representations, $\Theta^{\alpha\beta}$ is α-singular and β-singular, and no constituent of Θ^α is β-singular. Hence, by Lemma 3.11 again

$$s_\beta s_\alpha \Theta = -\Theta + \left(\Theta^\alpha + \Theta_1^\beta + \Theta_1^{\alpha\beta}\right) - \Theta^{\alpha\beta};$$

here Θ_1^β and $\Theta_1^{\alpha\beta}$ are characters of representations, $\Theta_1^{\alpha\beta}$ is α-singular and β-singular, and no constituent of Θ_1^β is α-singular. Since $s_\beta s_\alpha \Theta$ is α-singular, this implies that $\Theta_1^\beta = \Theta$. In particular $\Theta^\alpha \neq 0$. Suppose Θ^α is reducible; say $\Theta^\alpha = (\Theta^\alpha)' + (\Theta^\alpha)''$, with both terms non-zero characters of representations. Then $\Theta_1^\beta = (\Theta_1^\beta)' + (\Theta_1^\beta)''$ with obvious notation; and both terms are characters of representations. Furthermore, applying the previous observation that $\Theta^\alpha \neq 0$ to $(\Theta^\alpha)'$ and $(\Theta^\alpha)''$ with α replaced by β implies that $(\Theta_1^\beta)'$ and $(\Theta_1^\beta)''$ are both non-zero. This contradicts the irreducibility of $\Theta = \Theta_1^\beta$. So Θ^α is in fact irreducible, proving 3.10(b).

3.10(c) requires a little more effort. We can begin in the same way: by hypothesis $s_\beta \Theta = -\Theta$ and $s_\beta s_\alpha s_\beta s_\alpha = s_\alpha s_\beta s_\alpha s_\beta$, so we deduce that $s_\alpha s_\beta s_\alpha \Theta$ is β-singular. We can use the notation of the preceding case for $s_\alpha \cdot \Theta$ and $s_\beta s_\alpha \cdot \Theta$. By Lemma 3.11,

$$s_\alpha s_\beta s_\alpha \cdot \Theta = -(\Theta + \Theta^\alpha + \Theta^{\alpha\beta}) - (\Theta^\alpha) + (\Theta_1^\beta + \Theta_2^\alpha + \Theta_2^{\alpha\beta}) - \Theta_1^{\alpha\beta} + \Theta^{\alpha\beta}.$$

Here Θ_2^α and $\Theta_2^{\alpha\beta}$ are characters of representations; $\Theta_2^{\alpha\beta}$ is α- and β-singular, and no constituent of Θ_2^α is β-singular. Since $s_\alpha s_\beta s_\alpha \cdot \Theta$ is β-singular, it follows that $\Theta_2^\alpha = 2\Theta^\alpha$. It is no longer obvious that $\Theta^\alpha \neq 0$ (i.e., that $\beta \notin \tau(U_\alpha)$). This will be proved later; but assume it for a moment. To prove (c), it is enough to show that Θ^α cannot have as many as three irreducible constituents. Suppose that it does. Choose a chain $0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_N = U_\alpha$ of submodules of U_α, such that $F_i/F_{i-1} = Y_i$ is an irreducible $(\mathfrak{G}, K)$ module; then $\Theta(U_\alpha) = \sum_{i=1}^N \Theta(Y_i)$. Since $\Theta^\alpha \neq 0$, we can define integers i_1 and i_2 so that $\beta \in \tau(Y_i)$ if $i < i_1$ or $i > i_2$, but $\beta \notin \tau(Y_{i_1})$ and $\beta \notin \tau(Y_{i_2})$. Since Θ^α has at least three irreducible constituents, there is an integer i_3 with $i_1 < i_3 < i_2$, and $\beta \notin \tau(Y_{i_3})$. We will show that X is a constituent of $U_\beta(Y_{i_1})$ and of $U_\beta(Y_{i_2})$, and hence that X occurs at least twice in Θ_1^β. Assume this for a moment. Then $\Theta_1^\beta = 2\Theta + \Theta'$; here Θ' is the character of an α-singular representation with no β-singular constituents, and $\Theta' \neq 0$ because of the contribution of Y_{i_3}. (Here we are using our earlier assertion that $\Theta^\alpha \neq 0$, applied to Y_{i_3} instead of X, with α and β reversed.) Thus $\Theta_2^\alpha = 2\Theta^\alpha + (\Theta')^\alpha$, and $(\Theta')^\alpha \neq 0$ by our earlier assertion again. This contradicts $\Theta_2^\alpha = 2\Theta^\alpha$, proving that in fact Θ^α cannot have as many as three constituents. It remains to establish the two assertions made in the course of the proof.

First we show that X is a constituent of $U_\beta(Y_{i_1})$. Let G_1 be the preimage of F_{i_1} in K_α under the natural projection $U_\alpha \cong K_\alpha / X$, and let G_0 be the preimage of F_{i-1}. We have an exact sequence

$$0 \to G_0 \to G_1 \to Y_{i_1} \to 0.$$

By Theorem 3.9(a), Y_{i_1} is not a submodule of $G_1 \subseteq \varphi_\alpha \psi_\alpha X$. So the sequence does not split, and $\mathrm{Ext}^1(Y_{i_1}, G_0) \neq 0$. (The description of Ext in terms of exact sequences in this category may be found in [1].) By Theorem 3.9(c),

$$\mathrm{Hom}_{\mathfrak{G}, K}\big(U_\beta(Y_{i_1}), G_0\big) \neq 0.$$

As a submodule of $\varphi_\alpha \psi_\alpha X$, G_0 has X as its unique irreducible submodule (Theorem 3.9(a)). It follows that X is a constituent of $U_\beta(Y_{i_1})$. In exactly the same way one shows that X is a constituent of $U_\beta(Y_{i_2})$.

Finally, we must show that $\Theta^\alpha \neq 0$. Since X is β-singular, this is equivalent to showing that $\varphi_\alpha \psi_\alpha X$ is not β-singular, or equivalently that $\operatorname{Ann}(\psi_\beta \varphi_\alpha \psi_\alpha X) \neq U(\mathfrak{G})$. By Lemmas 2.7 and 2.8, it is enough to replace $\psi_\alpha X$ by some other module L with $\operatorname{Ann}(L) \supseteq \operatorname{Ann}(\psi_\alpha X)$, and show that $\varphi_\alpha L$ is not β-singular. Just as in the proof of Theorem 3.7, we will let L be an irreducible highest weight module. This time we will argue by a reduction to $\mathfrak{sp}(2)$, however; so we need

LEMMA 3.14. *With notation as at the beginning of this section, suppose that* $\{\alpha, \beta\} \subseteq \Pi_\lambda$ *are adjacent simple roots whose span in* $\Delta(\mathfrak{G}, \mathfrak{h})$ *is not of type* G_2. *Then we can choose* $\Psi_\alpha = \Psi_\beta$: *that is, there is a positive root system* Ψ *for* $\mathfrak{h}$ *in* $\mathfrak{G}$, *containing* Δ_λ^+, *such that* α *and* β *are simple for* Ψ.

Proof. Choose an element $\gamma \in \mathfrak{h}^*$ such that

(a) $\langle \gamma, \alpha \rangle = \langle \gamma, \beta \rangle = 0$;

(b) if $\delta \in \Delta(\mathfrak{G}, \mathfrak{h})$, then $\langle \gamma, \delta \rangle$ is real and non-zero unless δ is in the span of α and β;

(c) γ is dominant for Δ_λ^+.

Such γ clearly exist. Let Δ_1 be the span of α and β in $\Delta(\mathfrak{G}, \mathfrak{h})$. Then Δ_1 is a root system of rank two, containing two adjacent roots, and not of type G_2; so Δ_1 is of type A_2 or B_2. On the other hand, $\Delta_\lambda \cap \Delta_1$ has the same properties, and so is also of type A_2 or B_2. Since there is no containment $B_2 \supseteq A_2$, it follows that $\Delta_\lambda \cap \Delta_1 = \Delta_1$. Hence α and β are the simple roots of a positive system Δ_1^+ for Δ_1. Set

$$\Psi = \{\delta \in \Delta(\mathfrak{G}, \mathfrak{h}) \mid \langle \delta, \gamma \rangle > 0 \quad \text{or} \quad \delta \in \Delta_1^+\};$$

then one easily checks that Ψ has the required properties. Q.E.D.

We turn now to the construction of L. Choose $\Psi = \Psi_\alpha = \Psi_\beta$ as in Lemma 3.14, and assume that this choice has been used to construct μ_α^i, μ_β^i, and so on as at the beginning of this section. Obviously we may assume $\mu_\alpha^1 = \mu_\beta^1$. Just as in the proof of Theorem 3.7, this allows us to reduce to the case $\mu_\alpha^1 = \mu_\beta^1 = 0$, so that μ_α and μ_β are dominant for Ψ. Put $F_\alpha = F_\alpha^2$, and let L be the irreducible highest weight module of highest weight $\lambda - \mu_\alpha - \rho$ with respect to Ψ. By the theorem of Duflo already mentioned ([4]), $\operatorname{Ann}(L) \supseteq \operatorname{Ann}(\psi_\alpha X)$. Let $\mathfrak{G}_1 \subseteq \mathfrak{G}$ be the subalgebra generated by $\mathfrak{h}$ and the root vectors for the roots $\pm\alpha$ and $\pm\beta$: $\mathfrak{G}_1 \cong \mathfrak{sp}(2) +$ center. Let L_1 be the submodule of L spanned by weight vectors for weights $\lambda - \mu_\alpha - \rho - m\alpha - n\beta$. It is easy to see that L_1 is the irreducible highest weight module for $\mathfrak{G}_1$ of highest weight $\lambda - \rho$. Similarly, we let F_1 be the submodule of F_α consisting of the weights $\mu_\alpha - m\alpha - n\beta$; then F_1 is irreducible, with highest weight μ_α. Let $V \subseteq L \otimes F_\alpha$ be the subspace of weights of the form $\lambda - \rho - m\alpha - n\beta$; then $V \cong L_1 \otimes F_1$ as a $\mathfrak{G}_1$ module. Furthermore, V is annihilated by the root vectors X_γ for $\gamma \in \Psi$ but γ not in the span of α and β. Let ρ_1 be half the sum of the positive roots spanned by α and β. It follows from known results that V contains as a composition factor the irreducible highest weight module for $\mathfrak{G}_1$ of highest weight $s_\alpha s_\beta(\lambda - \rho + \rho_1) - \rho_1 = s_\alpha s_\beta \lambda - \rho$. (We will say more about this in a moment.) It is easy to deduce that $L \otimes F_\alpha$ contains as a composition factor the irreducible highest weight module for $\mathfrak{G}$ of highest weight $s_\alpha s_\beta \lambda - \rho$, which we call L_2. By definition of χ_λ, L_2 has infinitesimal character $\chi_{s_\alpha s_\beta \lambda} = \chi_\lambda$, and hence occurs in $\varphi_\alpha L$. Our goal is to prove that $\psi_\beta \varphi_\alpha L \neq 0$; it suffices to show that $\psi_\beta L_2 \neq 0$. This is essentially obvious, since $s_\alpha s_\beta \cdot \beta \in -\Psi$; a proof may be found in [8], Theorem 2.11.

We have now proved Theorem 3.10, except for the assertion made above that $L_1 \otimes F_1$ contains as a composition factor the irreducible highest weight module (for $\mathfrak{G}$) of highest weight $s_\alpha s_\beta \lambda - \rho$. There are many ways to see this. The simplest to explain (although certainly not the simplest to carry out) is the following. There is a nice theory of formal characters for highest weight modules (cf. [3]); and the assertion could certainly be verified if one knew the characters of all irreducible highest weight modules for $\mathfrak{sp}(2)$. This is equivalent to knowing the composition series of the Verma modules for $\mathfrak{sp}(2)$, and these are known: all composition factors occur with multiplicity one, and they are precisely those given by Verma's theorem. For details see [8]. The required computation is left to the reader: as hinted above, we would actually suggest that the reader look for one of the subtler but easier proofs. Q.E.D.

Theorem 3.9 suggests that it would be very nice if $U_\alpha(X)$ were completely reducible. This is true in many examples; so we make

CONJECTURE 3.15. *If X is an irreducible $(\mathfrak{G}, K)$ module with nonsingular infinitesimal character λ, and $\alpha \in \Pi_\lambda - \tau(X)$, then $U_\alpha(X)$ is completely reducible.*

We can at least prove a related result. By [1], Corollary 7.5, there is a unique automorphism $\mu : G \to G$, preserving K, such that if X is an irreducible $(\mathfrak{G}, K)$ module, then $X^\mu \cong X^*$; here X^μ is the obvious twisting of X by μ, and X^* is the (K-finite) contragredient module to X. The functors $(\ )^\mu$ and $(\ )^*$ are well defined on all of $\mathfrak{F}(\mathfrak{G}, K)$.

LEMMA 3.16. *If X and Y are $(\mathfrak{G}, K)$ modules of finite length, then*

$$\mathrm{Ext}^*(X, Y) \cong \mathrm{Ext}^*(X^\mu, Y^\mu)$$

$$\mathrm{Ext}^*(X, Y) \cong \mathrm{Ext}^*(Y^*, X^*).$$

Proof. This is obvious if we interpret Ext in terms of long exact sequences (cf. [1]). Q.E.D.

If X is a $(\mathfrak{G}, K)$ module of finite length, define $\tilde{X} \cong (X^\mu)^*$; $X \to \tilde{X}$ is a contravariant functor, and if X is irreducible then $X \cong \tilde{X}$. It is clear that this functor commutes with φ_α and ψ_α; so we deduce

PROPOSITION 3.17. *If X is an irreducible $(\mathfrak{G}, K)$ module with nonsingular infinitesimal character λ, and $\alpha \in \Pi_\lambda - \tau(X)$, then $\tilde{U}_\alpha(X) \cong U_\alpha(X)$.*

LEMMA 3.18. *If X and Y are $(\mathfrak{G}, K)$ modules of finite length, then*

$$\mathrm{Ext}^*(X, Y) \cong \mathrm{Ext}^*(\tilde{Y}, \tilde{X}).$$

COROLLARY 3.19. *Conjecture 6.21 of [16] (namely that the irreducible constituents of $U_\alpha(X)$ occur with multiplicity one) implies Conjecture 3.15. More precisely, suppose Y is an irreducible subrepresentation of $U_\alpha(X)$ which occurs exactly once as a composition factor. Then Y is a direct summand of $U_\alpha(X)$.*

Proof. Clearly it suffices to establish the last statement. So suppose $\mathrm{Hom}_{\mathfrak{G}, K}(Y, U_\alpha(X)) \neq 0$. By Lemma 3.18, $\mathrm{Hom}_{\mathfrak{G}, K}(U_\alpha(X), Y) \neq 0$. Choose a nontrivial map $U_\alpha(X) \to Y$, and let W be its kernel. Since Y occurs only once as a composition factor of $U_\alpha(X)$, Y does not occur in W; so if we regard Y as a submodule of $U_\alpha(X)$, $Y \cap W = 0$. Hence we have an injection $Y \oplus W \hookrightarrow U_\alpha(X)$. Since these modules have the same character, the map is an isomorphism. Q.E.D.

This completes our "general" results on U_α. In the next section we will collect more "specific" results, describing U_α in terms of the Langlands classification of $\hat{G}$; these are to some extent taken from [16]. There is one more result which we will state here, however.

THEOREM 3.20. *Let X be an irreducible $(\mathfrak{G}, K)$ module with nonsingular infinitesimal character λ. Suppose μ is a weight of a finite dimensional representation of G, and $\lambda + \mu$ is dominant with respect to Δ_λ^+.*

(a) *If $\lambda + \mu$ is nonsingular, then $S_\mu \cdot \Theta(X)$ is an irreducible character.*

(b) *In general, $S_\mu \cdot \Theta(X)$ is irreducible or zero.*

Proof. These results are essentially proved in [16], Theorem 5.15 and Theorem 5.20(a). The hypotheses on G are different, but this is irrelevant for the proofs. Q.E.D.

3.20(a) is a fairly harmless perturbation of Zuckerman's results in [19], and the proof is formal. The proof given for (b) in [16] is rather complicated, and uses the Langlands classification. If $\lambda + \mu$ is singular with respect to exactly one root α, then (b) is easily deduced from Theorem 3.7(b), which of course was proved without using the Langlands classification. It would be interesting to generalize that argument to obtain 3.20(b).

4. Partial computation of $U_\alpha(X)$. Fix a nonsingular infinitesimal character λ. The set $\hat{G}_\lambda$ of equivalence classes of irreducible $(\mathfrak{G}, K)$ modules with infinitesimal character λ is finite. Suppose that X is an irreducible $(\mathfrak{G}, K)$ module with infinitesimal character λ; say $X \cong J_{\delta \otimes \nu}$. Our first goal in the present section is to compute $\tau(X)$ in terms of $\delta \otimes \nu$; this has essentially been done in [16]. At the same time, if $\alpha \in \Pi_\lambda - \tau(X)$, we will determine certain special composition factors Y of $U_\alpha(X)$. There are zero, one, or two such composition factors, and they occur with multiplicity one; essentially they are the ones given by Theorem 6.16 of [16]. ("Essentially" here and above refers to problems arising from the non-linearity of G.) With these results in hand, we can prove a sort of duality theorem (Theorem 4.14) which is one of our most powerful general results on U_α. We begin with a slight reformulation of Langlands' classification of $\hat{G}_\lambda$.

LEMMA 4.1 (Harish-Chandra). *Let $P = MAN$ be a parabolic subgroup of G, and $\delta \in \hat{M}$ a tempered representation. If δ has non-singular infinitesimal character (in particular if $J_{\delta\otimes\nu} \in \hat{G}_\lambda$ for some $\nu \in \hat{A}$) then there is a cuspidal parabolic subgroup $P_1 = M_1A_1N_1$ of G, a discrete series representation $\delta_1 \in \hat{M}_1$, and a unitary character $\nu_1 \in (A_1 \cap M)\hat{}$, such that $P_1 \subseteq P$ (so that $P_1 \cap M = M_1(A_1 \cap M)(N_1 \cap M)$ is a cuspidal parabolic subgroup of M) and*

$$\delta = \operatorname{Ind}_{P_1\cap M}^{M}(\delta_1 \otimes \nu_1 \otimes 1).$$

Proof. Without the hypothesis on δ, we can find P_1, δ_1, and ν_1 so that δ is a constituent of $\operatorname{Ind}_{P_1\cap M}^{M}\delta_1 \otimes \nu_1 \otimes 1$. (This result of Harish-Chandra may be found for example in [11], Lemma 4.10.) It follows from the hypothesis that ν_1 is not orthogonal to any real root of $A_1 \cap M$ in M; so by [5], pp. 195–197, the induced representation is irreducible. Q.E.D.

Now let $P = MAN$ be a cuspidal parabolic subgroup of G, $\delta \in \hat{M}$ a discrete series representation, and $\nu \in \hat{A}$. If ν is positive with respect to P, and $I_{\delta\otimes\nu}$ has nonsingular infinitesimal character λ, then an induction by stages argument allows us to identify $I_{\delta\otimes\nu}$ with an induced representation of the sort considered by Langlands (cf. section 2). In particular $I_{\delta\otimes\nu}$ has a unique irreducible quotient $J_{\delta\otimes\nu}$. One can show that $J^P_{\delta\otimes\nu} \cong J^{P'}_{\delta'\otimes\nu'}$ (with δ and δ' discrete series representations) if and only if $(MA, \delta\otimes\nu)$ is conjugate under G to $(M'A', \delta'\otimes\nu')$; this follows from Theorem 2.1 and a comparison of characters of unitarily induced representations. Harish-Chandra's parametrization of the discrete series of M now suggests a way to parametrize $\hat{G}_\lambda$ by certain characters of Cartan subgroups of G. This is described carefully in [16]; we recall the definitions briefly. Let $B = T^+A$ be a θ-stable Cartan subgroup of G, with $T^+ = B \cap K$, and $A = \exp(\mathfrak{b}_0 \cap \mathfrak{p}_0)$. Let $MA = G^A$ be the Langlands decomposition of the centralizer of A in M. An *M-regular pseudocharacter* (or *regular character*) γ of B is an ordered pair $(\Gamma, \bar{\gamma})$, with Γ an irreducible representation of B_0 and $\bar{\gamma} \in \mathfrak{h}^*$ nonsingular with respect to $\Delta(\mathfrak{M}; \mathfrak{t}^+)$. (Since H may be non-abelian, Γ need not be a character.) Furthermore the following compatibility condition is required: $\bar{\gamma}|_{\mathfrak{t}^+}$ should be purely imaginary, so that it defines a positive root system $\Delta^+(\mathfrak{M}, \mathfrak{t}^+)$. Write $\rho_{\mathfrak{M}}$, $\rho_{\mathfrak{M}\cap\mathfrak{k}}$ for the obvious half sums of positive roots. Then we want $d\Gamma = \bar{\gamma} + \rho_{\mathfrak{M}} - 2\rho_{\mathfrak{M}\cap\mathfrak{k}}$. This notation will be heavily abused: we may write γ to mean Γ or $\bar{\gamma}$, but the meaning should be clear from context. (Thus, if $h \in B$ and $\alpha \in \Delta(\mathfrak{G}, \mathfrak{h})$, $\gamma(h)$ means $\Gamma(h)$, and $\langle\gamma, \alpha\rangle$ means $\langle\bar{\gamma}, \alpha\rangle$.) The Weyl group $W(G/B)$ of B in G acts on $\hat{B}'$ in an obvious way. Associated to $\gamma|_{T^+}$

there is a representation $\pi_M(\gamma|_{T^+})$ in the discrete series of M; and of course $\gamma|_A$ defines a one dimensional character of A. Fix a parabolic subgroup P associated to B; thus $P = MAN$. Set

$$\pi_G(P, \gamma) = \pi(P, \gamma) = \operatorname{Ind}_P^G \pi_M(\gamma|_{T^+}) \otimes (\gamma|_A) \otimes 1.$$

Such a representation is called a *generalized principal series representation*. The character of this representation depends only on γ (and not on our choice of P); we write it as $\Theta(\gamma)$. If γ is nonsingular (i.e., $\gamma = (\Gamma, \bar{\gamma})$, and $\bar{\gamma} \in \mathfrak{b}^*$ is nonsingular with respect to $\Delta(\mathfrak{G}, \mathfrak{b})$) and we choose P so that $\gamma|_A$ is positive with respect to P, then $\pi(P, \gamma)$ is the sort of representation considered above, and hence has a Langlands quotient $\bar{\pi}(\gamma)$ (which, as the notation indicates, does not depend on P). Here we should perhaps point out that $\pi(P, \gamma)$ has infinitesimal character $\bar{\gamma}$ if $\gamma = (\Gamma, \bar{\gamma})$. Thus $\bar{\pi}(\gamma) = J_{\pi_M(\gamma|_{T^+}) \otimes (\gamma|_A)}$. (An attempt to formulate the results below in the $J_{\delta \otimes \nu}$ notation should convince the reader that this abandonment of traditional notation is justified.) We may sometimes write $\bar{\Theta}(\gamma)$ for $\Theta(\bar{\pi}(\gamma))$. The Langlands classification theorem now reads

THEOREM 4.2. *Let X be an irreducible $(\mathfrak{G}, K)$ module with nonsingular infinitesimal character. Then there is a θ-stable Cartan subgroup B of G and an M-regular pseudocharacter $\gamma \in \hat{B}'$ such that $X \cong \bar{\pi}(\gamma)$. Furthermore (B, γ) is unique up to conjugation by G.*

This can be extended to singular infinitesimal characters (cf. [16], Theorem 2.9).

To study coherent continuation, we need to introduce some further notation. Let B be as above, and fix a positive system $\Psi \subseteq \Delta(\mathfrak{M}, \mathfrak{t}^+)$; define $\rho_{\mathfrak{M}}$ and $\rho_{\mathfrak{M} \cap \mathfrak{k}}$ using Ψ. A *Ψ-pseudocharacter* (or *Ψ-character*) γ of B is an ordered pair $(\Gamma, \bar{\gamma})$, with Γ an irreducible representation of B and $\bar{\gamma} \in \mathfrak{b}^*$. The compatibility condition is simply $d\Gamma = \bar{\gamma} + \rho_{\mathfrak{M}} - 2\rho_{\mathfrak{M} \cap \mathfrak{k}}$; thus $\bar{\gamma}$ is not required to be dominant or nonsingular. The set of Ψ-pseudocharacters of B is written $\hat{B}_\Psi$. Following the procedure of Hecht and Schmid in the linear case ([6]), one can associate to $\gamma|_{T^+}$ a virtual character $\Theta_M(\Psi, \gamma|_{T^+})$ for M. Thus we can define

$$\Theta(\Psi, \gamma) = \Theta_G(\Psi, \gamma) = \operatorname{Ind}_P^G \Theta_M(\Psi, \gamma|_{T^+}) \otimes (\gamma|_A) \otimes 1.$$

Recall from section 2 our fixed Cartan subalgebra $\mathfrak{h}$ and positive root system Δ^+, with respect to which coherent continuation was defined. If $\gamma \in \mathfrak{b}^*$ is nonsingular, we define $\Delta_\gamma^+ = \Delta_\gamma^+(\mathfrak{G}, \mathfrak{b})$ to be the unique positive system with respect to which γ is (strictly) dominant. Then there is a unique inner automorphism c of $\mathfrak{G}$ such that $c \cdot \mathfrak{h} = \mathfrak{b}$ and $c \cdot \Delta^+ = \Delta_\gamma^+$. If $\mu \in \mathfrak{h}^*$, we define $\mu_\gamma = \mu_{\Delta_\gamma^+} = c \cdot \mu \in \mathfrak{b}^*$.

THEOREM 4.3 (cf. [16], Corollary 5.12). *Suppose $B = T^+A$ is a θ-stable Cartan subgroup of G, Ψ is a positive system for $\Delta(\mathfrak{M}, \mathfrak{t}^+)$, and $\gamma \in \hat{B}_\Psi$ is nonsingular. If $\mu \in \mathfrak{h}^*$ is a weight of a finite dimensional representation of G, then*

$$S_\mu \cdot \Theta(\Psi, \gamma) = \Theta(\Psi, \gamma + \mu_\gamma).$$

Proof. The argument given in [16] for linear groups works in the present situation as well. (Notice that μ_γ lifts to a character of B which is a weight of a finite dimensional representation; $\gamma + \mu_\gamma$ means $(\Gamma \otimes \mu_\gamma, \bar{\gamma} + \mu_\gamma)$.) Q.E.D.

We need the Hecht-Schmid character identities in the present context. Since disconnected linear groups are treated in Proposition 5.14 of [16], and connected non-linear groups in [15], we will merely state the necessary definitions. So let $B = T^+A$ be a θ-stable Cartan subgroup of G, and Ψ a positive root system for $\mathfrak{t}^+$ in $\mathfrak{M}$. (As usual $MA = G^A$.) Fix a simple noncompact root $\beta \in \Psi$. Let $(\mathfrak{a}_0^\beta)^+$ be a one dimensional subalgebra of $\mathfrak{p}_0$ contained in the sum of the β and $-\beta$ root spaces. Then $(\mathfrak{a}_0^\beta)^+$ is unique up to conjugation by T^+. Set $\mathfrak{a}^\beta = (\mathfrak{a}^\beta)^+ + \mathfrak{a}$, $(\mathfrak{t}^+)^\beta = \{t \in \mathfrak{t} \mid \beta(t) = 0\}$. Then $\mathfrak{b}^\beta = (\mathfrak{t}^+)^\beta + \mathfrak{a}^\beta$ is a θ-stable Cartan subalgebra of $\mathfrak{G}$; let $B^\beta = (T^+)^\beta A^\beta$ be the associated Cartan subgroup. Let $M^\beta A^\beta$ be the Langlands decomposition of G^{A_β}. The roots of $(\mathfrak{t}^+)^\beta$ in $\mathfrak{M}^\beta$ may be identified with the roots of $\mathfrak{t}^+$ in $\mathfrak{M}$ orthogonal to β; so the intersection of Ψ with this set defines a positive system Ψ^β for $(\mathfrak{t}^+)^\beta$ in $\mathfrak{M}^\beta$. Fix $\gamma \in \hat{B}_\Psi$; we want to define $\gamma^\beta \in \hat{B}^\beta_{\Psi^\beta}$. Put $(T_1^+)^\beta = (T^+)^\beta \cap T^+$, $B_1^\beta = (T_1^+)^\beta A^\beta$. If $\gamma = (\Gamma, \bar{\gamma})$, we define $\bar{\gamma}^\beta$ so that $\bar{\gamma}^\beta|_{\mathfrak{a}} = \bar{\gamma}|_{\mathfrak{a}}$, $\bar{\gamma}^\beta|_{(\mathfrak{t}^+)^\beta} = \bar{\gamma}|_{(\mathfrak{t}^+)^\beta}$, and if $\tilde{\beta}$ is a real root of $\mathfrak{b}^\beta$ supported on $\mathfrak{a}_0^\beta$, then $\langle \bar{\gamma}^\beta, \tilde{\beta} \rangle = \langle \bar{\gamma}, \beta \rangle$. (There is an ambiguity of sign in the choice of $\tilde{\beta}$, so $\bar{\gamma}^\beta$ is not quite determined; but either choice will do.) We define Γ_1^β as a representation of B_1^β to be $\Gamma|_{(T_1^+)^\beta}$ on $(T_1^+)^\beta$, and $\exp(\bar{\gamma}^\beta|_{\mathfrak{a}_0^\beta})$ on A^β. One easily checks that $(T_1^+)^\beta$ meets every component of T^+, so that Γ_1^β is irreducible. Just as in the linear case, $B_1^\beta = B^\beta$ if and only if the simple reflection $s_\beta \notin W(G/B)$. In that case we put $\Gamma^\beta = \Gamma_1^\beta$, and γ^β is defined. If $s_\beta \in W(G/B)$, then B_1^β has index two in B^β, and we put

$$\Gamma^\beta = \mathrm{Ind}_{B_1^\beta}^{B^\beta} \Gamma_1^\beta .$$

If B^β is abelian, Γ^β obviously splits as a direct sum of two irreducible components Γ_+^β and Γ_-^β; we can define $\gamma_\pm^\beta$ accordingly. In general Γ^β is either irreducible or splits into two components $\Gamma_\pm^\beta$, according as s_β does or does not fix $\Gamma_1^\beta|_{(T_1^+)^\beta}$ in its action on $(\hat{T}_1^+)^\beta$. So we define either γ^β or $\gamma_\pm^\beta$. The character identities are

THEOREM 4.4. *Suppose $B \subseteq G$ is a θ-stable Cartan subgroup, $\Psi \subseteq \Delta(\mathfrak{M}, \mathfrak{t}^+)$ is a positive root system, and $\beta \in \Psi$ is a simple root. Suppose $\gamma \in \hat{B}_\Psi$.*

(a) *If β is compact, and $2\langle \beta, \bar{\gamma} \rangle = n$, then $\Theta(\Psi, \gamma) + \Theta(\Psi, \gamma - n\beta) = 0$.*

(b) *If β is noncompact, define B^β and γ^β (or $\gamma_\pm^\beta$) as above. Then if $\tilde{\gamma} = (\Gamma - \beta, \bar{\gamma}) \in \hat{B}_{s_\beta \Psi}$,*

$$\Theta(\Psi, \gamma) + \Theta(s_\beta \Psi, \tilde{\gamma}) = \Theta(\Psi^\beta, \gamma^\beta) \quad \text{or} \quad \Theta(\Psi^\beta, \gamma_+^\beta) + \Theta(\Psi^\beta, \gamma_-^\beta).$$

Proof. See [15] and [16], Proposition 5.14; the few additional details needed are left to the reader. Q.E.D.

We will need to know the condition for a character to appear on the right side of the identity of 4.4(b). So fix a θ-stable Cartan subgroup $B = T^+A$, a positive system Ψ for $\mathfrak{t}^+$ in $\mathfrak{M}$, and a *real* root $\tilde{\beta}$ of $\mathfrak{b}$ in $\mathfrak{G}$. Choose a three-dimensional subalgebra $\varphi_{\tilde{\beta}} : \mathfrak{sl}(2, \mathbf{R}) \to \mathfrak{G}_0$ through the root $\tilde{\beta}$ in such a way that $\varphi_{\tilde{\beta}}\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right) \in \mathfrak{a}_0$, and $\varphi_{\tilde{\beta}}(-{}^tX) = \theta(\varphi_{\tilde{\beta}}(X))$. Set $m_{\tilde{\beta}} = \exp(\varphi_{\tilde{\beta}}\left(\begin{smallmatrix}0 & -\pi\\ \pi & 0\end{smallmatrix}\right)) \in T^+$. (There is ambiguity in the definition of $\varphi_{\tilde{\beta}}$, so $m_{\tilde{\beta}}$ is only defined up to inverse.) Define $\epsilon_{\tilde{\beta}} = \pm 1$ as in the remarks after Proposition 5.14 of [16]. By a slight extension of the argument there, we find

PROPOSITION 4.5. *With notation as above, $\Theta(\Psi, \gamma)$ occurs on the right side of a character identity corresponding to the root $\tilde{\beta}$ if and only if the eigenvalues of $\gamma(m_{\tilde{\beta}})$ are of the form $\epsilon_{\tilde{\beta}}\exp(\pm 2\pi i\langle \gamma, \tilde{\beta}\rangle / \langle \tilde{\beta}, \tilde{\beta}\rangle)$.*

Notice that if G is not linear, we need not have $m_{\tilde{\beta}}{}^2 = 1$; so there can be a character identity even if $2\langle \gamma, \tilde{\beta}\rangle / \langle \tilde{\beta}, \tilde{\beta}\rangle \notin \mathbf{Z}$. In that case, however, we can at least rule out the second possibility of 4.4(b): the element $m_{\tilde{\beta}}$ lies in the identity component of the more compact Cartan subgroup on which the character identity is based, and so acts by scalars in the representation Γ_1^{β} defined earlier. The reflection s_{β} (if it exists) takes $m_{\tilde{\beta}}$ to $m_{\tilde{\beta}}^{-1}$, and so does not preserve Γ_1^{β} unless $m_{\tilde{\beta}}$ acts by ± 1. We summarize this as

Remark 4.6. In the setting of 4.4(b), the second case can arise only if $2\langle \gamma, \beta\rangle / \langle \beta, \beta\rangle \in \mathbf{Z}$.

We can now begin to study coherent continuation of irreducible representations. The next few results are implicit in [16]. By Theorem 3.20, we can coherently continue across non-integral walls without affecting irreducibility. Our goal is to see how such continuation affects the parameters in the Langlands classification.

LEMMA 4.7. *Let $B = T^+A$ be a θ-stable Cartan subgroup of G, $P = MAN$ an associated parabolic subgroup, and $\gamma \in \hat{B}'$. Suppose γ is nonsingular; recall the associated positive root system $\Delta_\gamma^+(\mathfrak{G}, \mathfrak{b})$. Suppose that for every root $\alpha \in \Delta_\gamma^+$, either*

(a) *$\theta\alpha \in \Delta_\gamma^+$, or*

(b) *α is complex, and $2\langle \alpha, \gamma\rangle / \langle \alpha, \alpha\rangle \notin \mathbf{Z}$, or*

(c) *α is real, and the eigenvalues of $\gamma(m_\alpha)$ are not of the form $\epsilon_\alpha\exp(\pm 2\pi i\langle \gamma, \alpha\rangle / \langle \alpha, \alpha\rangle)$ (cf. Proposition 4.5), or*

(d) *$\alpha|\mathfrak{a}$ is a root of $\mathfrak{a}$ in $\mathfrak{N}$.*

Then $\pi(P, \gamma)$ has a unique irreducible quotient, namely $\bar{\pi}(\gamma)$.

Proof. Choose P' so that $\gamma|\mathfrak{a}$ is positive with respect to P'. Then $\pi(P', \gamma)$ has a unique irreducible quotient, namely $\bar{\pi}(\gamma)$; and there is an integral intertwining operator $I(P', P, \gamma)$ from $\pi(P', \gamma)$ to $\pi(P, \gamma)$ (cf. [16], Section 3). Clearly it is enough to show that $I(P', P, \gamma)$ is an isomorphism. This is proved in the same way as Theorem 5.15 of [16] (cf. [16], Lemma 5.16 and Proposition 6.1). Q.E.D.

COROLLARY 4.8 (cf. [16], Corollary 5.17). *In the setting of Lemma 4.7, suppose $\mu \in \mathfrak{h}^*$ is a weight of a finite dimensional representation of G. Suppose $\gamma + \mu_\gamma$ is nonsingular, and dominant with respect to those roots $\alpha \in \Delta_\gamma^+$ such that either*

(a) α *is imaginary, or*
(b) $2\langle\alpha, \gamma\rangle/\langle\alpha, \alpha\rangle \in \mathbf{Z}$, *or*
(c) α *is real, and the eigenvalues of* $\gamma(m_\alpha)$ *are of the form* $\epsilon_\alpha \exp(\pm 2\pi i\langle\gamma, \alpha\rangle/\langle\alpha, \alpha\rangle)$.
Then $S_\mu \cdot \overline{\Theta}(\gamma) = \overline{\Theta}(\gamma + \mu_\gamma)$.

Proof. Fix a parabolic subgroup $P = MAN$ associated to B, so that $\gamma|_{\mathfrak{a}}$ is positive with respect to P. Then $\bar{\pi}(\gamma)$ is the unique irreducible quotient of $\pi(P, \gamma)$. By condition (a), $\gamma + \mu_\gamma \in \hat{B}'$. Let F be the finite dimensional irreducible representation of G of extremal weight μ. Then it is easy to see that $P_{\gamma+\mu_\gamma}(\pi(P, \gamma) \otimes F)$ has a unique irreducible quotient, whose character is $S_\mu(\Theta(\bar{\pi}(\gamma)))$. (Compare the proof of Theorem 5.20(a) in [16].) On the other hand, an argument identical to that given for Corollary 5.9 of [16] shows that $P_{\gamma+\mu_\gamma}(\pi(P, \gamma) \otimes F) \cong \pi(P, \gamma + \mu_\gamma)$. By Lemma 4.7 and the hypotheses on $\gamma + \mu_\gamma$, the unique irreducible quotient of $\pi(P, \gamma + \mu_\gamma)$ is $\bar{\pi}(\gamma + \mu_\gamma)$. Q.E.D.

This result tells us how to cross non-integral complex walls, and non-integral real walls where the parity condition of 4.5 doesn't hold. We are left with crossing noncompact imaginary non-integral walls, and real walls where the parity condition holds.

LEMMA 4.9. *In the setting of Corollary* 4.8, *fix a simple noncompact imaginary root* $\beta \in \Delta_\gamma^+$ *such that* $2\langle\gamma, \beta\rangle/\langle\beta, \beta\rangle \notin \mathbf{Z}$. *Suppose* $\gamma + \mu_\gamma$ *is strictly dominant with respect to* $s_\beta(\Delta_\gamma^+)$. *Then* (*with notation as in Theorem* 4.4) $S_\mu(\overline{\Theta}(\gamma)) = \overline{\Theta}((\gamma + \mu_\gamma)^\beta)$.

Proof. Let $\Psi = \Delta_\gamma^+ \cap \Delta(\mathfrak{t}^+, \mathfrak{M})$; then we consider $\gamma + \mu_\gamma$ as an element of $\hat{B}_\Psi$, and define B^β and $(\gamma + \mu_\gamma)^\beta \in \hat{B}^\beta_{\Psi^\beta}$ as for Theorem 4.4. By remark 4.6, $(\gamma + \mu_\gamma)^\beta$ (as opposed to $(\gamma + \mu_\gamma)^\beta_\pm$) exists; and the condition that $\gamma + \mu_\gamma$ be dominant with respect to $s_\beta(\Delta_\gamma^+)$ clearly implies that $(\gamma + \mu_\gamma)^\beta$ is dominant with respect to Ψ^β. So $(\gamma + \mu_\gamma)^\beta$ may be regarded as an element of $(\hat{B}^\beta)'$, so the statement makes sense. The proof is quite similar to that given for Theorem 6.16(g) of [16]; details are left to the reader. Q.E.D.

This result immediately implies one about crossing a non-integral real wall with the parity condition; we leave its formulation to the reader.

We need some notational conventions to state the next result. Suppose $\gamma \in \hat{B}'$, and $\bar{\pi}(\gamma)$ has nonsingular infinitesimal character $\lambda \in \mathfrak{h}^*$, with λ dominant with respect to our fixed positive system $\Delta^+(\mathfrak{G}, \mathfrak{h})$. There is a unique inner automorphism of $\mathfrak{G}$ which takes $(\mathfrak{h}, \Delta^+(\mathfrak{G}, \mathfrak{h}), \lambda)$ to $(\mathfrak{b}, \Delta_\gamma^+, \gamma)$; we denote this by bar. Thus $\bar{\gamma} = \bar{\lambda}$ (a notation for which we apologize), and each root $\alpha \in \Delta(\mathfrak{G}, \mathfrak{h})$ determines a root $\bar{\alpha} \in \Delta(\mathfrak{G}, \mathfrak{b})$. Thus for example, to describe $\tau(\bar{\pi}(\gamma))$, we need only describe $\bar{\tau}(\bar{\pi}(\gamma)) \subseteq \Delta_\gamma^+$; this is a subset of $\overline{\Pi}_\lambda \subseteq \overline{\Delta_\lambda^+} \subseteq \Delta_\gamma^+$. This we now do.

THEOREM 4.12 (cf. [16], Theorem 6.16). *With notation as above, suppose* $X \cong \bar{\pi}(\gamma)$ *has nonsingular infinitesimal character* λ, *and* $\bar{\alpha} \in \overline{\Pi}_\lambda$ *is actually simple with respect to* Δ_γ^+. *Put* $n = 2\langle\bar{\alpha}, \gamma\rangle/\langle\bar{\alpha}, \bar{\alpha}\rangle$.

(a) *If* $\bar{\alpha}$ *is real and* $\gamma(m_{\bar{\alpha}}) = -(-1)^n \epsilon_{\bar{\alpha}}$ (*cf.* 4.5) *then* $\alpha \notin \tau(X)$, *and* $\Theta(U_\alpha(X)) = \Theta_0$.

(b) *If* $\bar{\alpha}$ *is real and* $\gamma(m_{\bar{\alpha}}) = (-1)^n \epsilon_{\bar{\alpha}}$, *then* $\alpha \in \tau(X)$.

(c) *If* $\bar{\alpha}$ *is complex and* $\theta\bar{\alpha} \in \Delta_\gamma^+$, *then* $\alpha \notin \tau(X)$, *and*

$$\Theta(U_\alpha(X)) = \overline{\Theta}(\gamma - n\bar{\alpha}) + \Theta_0.$$

(d) *If* $\bar{\alpha}$ *is complex and* $\theta\bar{\alpha} \notin \Delta_\gamma^+$, *then* $\alpha \in \tau(X)$.

(e) *If* $\bar{\alpha}$ *is compact imaginary, then* $\alpha \in \tau(X)$.

(f) *If* $\bar{\alpha}$ *is noncompact imaginary, then* $\alpha \notin \tau(X)$. *Define* $\gamma^{\bar{\alpha}}$ *or* $\gamma_\pm^{\bar{\alpha}}$ *as in Theorem* 4.4; *then*

$$\Theta(U_\alpha(X)) = \overline{\Theta}(\gamma^{\bar{\alpha}}) + \Theta_0 \quad \textit{or} \quad \overline{\Theta}(\gamma_+^{\bar{\alpha}}) + \overline{\Theta}(\gamma_-^{\bar{\alpha}}) + \Theta_0 \ \textit{respectively}.$$

In each case, Θ_0 *is the character of a representation, and every constituent of* Θ_0 *occurs in* $\pi(\gamma)$.

Proof. Except for the very last assertion, this is proved for linear groups in [16] (Theorem 6.16). The extension to the non-linear case is quite trivial—all the difficulties were contained in Theorem 6.9 of [16], which is an immediate consequence of Lemma 6.5 of [16] and Lemma 3.11 of the present paper. (As mentioned in the Introduction, the proof of these results without Theorem 3.7 is a highly non-trivial matter.) So we confine our attention to the last statement. Consider first (a); we must show that every constituent of Θ_0 occurs in $\pi(\gamma)$. As is shown in [16] in the proof of Theorem 6.16, $S_{-n\alpha}(\Theta(\gamma)) = \Theta(\gamma)$. Write

$$\Theta(\gamma) = \overline{\Theta}(\gamma) + \Theta_1 + \cdots + \Theta_r,$$

with Θ_i an irreducible character. Then

$$\begin{aligned}\Theta_0 &= \Theta(U_\alpha(\bar{\pi}(\gamma)))\\ &= S_{-n\alpha}(\overline{\Theta}(\gamma)) - \overline{\Theta}(\gamma)\\ &= S_{-n\alpha}(\Theta(\gamma) - \Theta_1 - \cdots - \Theta_r) - (\Theta(\gamma) - \Theta_1 - \cdots - \Theta_r)\\ &= \sum_{i=1}^r \Theta_i - S_{-n\alpha}(\Theta_i).\end{aligned}$$

We know that Θ_0 is the character of a representation. But by Lemma 3.11, every term on the right is the negative of the character of a representation, except those terms for which $\alpha \in \tau(\Theta_i)$; in that case $\Theta_i - S_{-n\alpha}(\Theta_i) = 2\Theta_i$. Thus every constituent of Θ_0 is such a Θ_i, proving the last assertion in case (a).

Consider next (c). Choose a cuspidal parabolic subgroup $P = MAN$ associated to B so that $\gamma - n\bar{\alpha}|_{\mathfrak{a}}$ is positive with respect to P. Then $\bar{\pi}(\gamma - n\bar{\alpha})$ is the *unique* irreducible quotient of $\pi(P, \gamma - n\bar{\alpha})$; and Lemma 4.7 implies that $\bar{\pi}(\gamma)$ is the unique irreducible quotient of $\pi(P, \gamma)$. Recall now the functors φ_α and ψ_α of section 3, which were defined using certain choices Ψ_α and μ_α. We may as well assume that $\overline{\Psi}_\alpha = \Delta_\gamma^+$, since α is simple for Δ_γ^+. In that case $\gamma - \bar{\mu}_\alpha$ is dominant with respect to Δ_γ^+, and singular only with respect to the complex root α. Hence $\gamma - \bar{\mu}_\alpha \in \hat{B}'$; by Corollary 5.9 of [16],

$$\psi_\alpha(\pi(P, \gamma)) \cong \pi(P, \gamma - \bar{\mu}_\alpha)$$

(cf. Theorem 4.3 above. The point is that $(\operatorname{Ind} \pi) \otimes F$ is naturally isomorphic to $\operatorname{Ind}(\pi \otimes F)$; the results we are using follow easily.) The surjection $\pi(P, \gamma) \to \bar{\pi}(\gamma)$ therefore gives rise to a surjection

$$\pi(P, \gamma - \bar{\mu}_\alpha) \to \psi_\alpha(\bar{\pi}(\gamma)).$$

(In fact $\psi_\alpha(\bar{\pi}(\gamma))$ is the Langlands quotient of $\pi(P, \gamma - \bar{\mu}_\alpha)$, as is proved in [16], Corollary 5.17; but we do not need this.) Accordingly we obtain a surjection

$$\varphi_\alpha(\pi(P, \gamma - \bar{\mu}_\alpha)) \to \varphi_\alpha\psi_\alpha(\bar{\pi}(\gamma)).$$

Now Lemma 5.8 of [16] provides a short exact sequence

$$0 \to \pi(P, \gamma - n\bar{\alpha}) \to \varphi_\alpha(\pi(P, \gamma - \bar{\mu}_\alpha)) \to \pi(P, \gamma) \to 0.$$

A simple diagram chase gives rise to a short exact sequence

$$0 \to K_1 \to \varphi_\alpha\psi_\alpha(\bar{\pi}(\gamma)) \to Q_1 \to 0,$$

with K_1 a quotient of $\pi(P, \gamma - n\bar{\alpha})$ and Q_1 a quotient of $\pi(P, \gamma)$. Now $\Theta(\varphi_\alpha\psi_\alpha(\bar{\pi}(\gamma))) = 2\overline{\Theta}(\gamma) + \Theta(U_\alpha(\bar{\pi}(\gamma))$. By Lemma 3.11(c), $\bar{\pi}(\gamma)$ occurs exactly as often in $\pi(P, \gamma)$ as in $\pi(P, \gamma - n\alpha)$; and the first multiplicity is one by Proposition 2.2. So K_1 and Q_1 both contain $\bar{\pi}(\gamma)$ exactly once. By Theorem 3.9(a), $\bar{\pi}(\gamma)$ is a subrepresentation of K_1 and a quotient of Q_1. Set $K_2 = K_1/\bar{\pi}(\gamma)$, $Q_2 = \ker(Q_1 \to \bar{\pi}(\gamma))$. Then we have an exact sequence

$$0 \to K_2 \to U_\alpha(\bar{\pi}(\gamma)) \to Q_2 \to 0,$$

with K_2 a quotient of $\pi(P, \gamma - n\bar{\alpha})$ and Q_2 a subquotient of $\bar{\pi}(\gamma)$. To prove the last assertion, it is enough to show that each irreducible constituent Y of $U_\alpha(\bar{\pi}(\gamma))$, $Y \not\cong \bar{\pi}(\gamma - n\bar{\alpha})$, occurs in Q_2. So choose a submodule Z of $U_\alpha(\bar{\pi}(\gamma))$ which contains Y as a quotient, and is minimal with respect to this property. Then clearly Y is the *unique* irreducible quotient of Z. If $Z \cap K_2 \neq Z$, then there is a non-trivial quotient map $Z \to Q_2$, and it follows that Y occurs in Q_2. So we may as well assume $Z \subseteq K_2$. Since $Y \not\cong \bar{\pi}(\gamma - n\bar{\alpha})$ is the unique irreducible quotient of Z, the quotient map $K_2 \to \bar{\pi}(\gamma - n\bar{\alpha})$ (recall that K_2 is a quotient of $\pi(P, \gamma - n\bar{\alpha})$) is necessarily trivial on Z. So $\bar{\pi}(\gamma - n\bar{\alpha})$ is not a constituent of Z. Define $\tilde{Z}$ as at the end of section 3. Then $\tilde{Z}$ is a quotient of $\tilde{U}_\alpha(\bar{\pi}(\gamma)) \cong U_\alpha(\bar{\pi}(\gamma))$, and $\tilde{Z}$ has Y as its unique irreducible submodule. Since the unique irreducible quotient $\bar{\pi}(\gamma - n\bar{\alpha})$ of K_2 does not occur in $\tilde{Z}$, the restriction of the quotient map $U_\alpha(\bar{\pi}(\gamma)) \to \tilde{Z}$ to K_2 is necessarily trivial. So this map lifts to a surjection $Q_2 \to \tilde{Z}$; in particular Y occurs in Q_2. This proves the last claim in case (c). (It should be remarked that we have *not* established that the multiplicity of Y in Θ_0 is bounded by the multiplicity of Y in $\pi(\gamma)$.)

Consider next case (f); suppose for definiteness that we are in the case when $\gamma^{\bar{\alpha}}$ (instead of $\gamma^{\bar{\alpha}}_{\pm}$) exists; the other case is similar. Choose a parabolic subgroup $P^{\bar{\alpha}}$ associated to $B^{\bar{\alpha}}$ so that $\gamma^{\bar{\alpha}}|_{\mathfrak{a}^{\bar{\alpha}}}$ is positive with respect to P, and the (restricted) real root $\tilde{\bar{\alpha}}$ of $\mathfrak{a}^{\bar{\alpha}}$ in $\mathfrak{N}^{\bar{\alpha}}$ is simple as a restricted root. Then there is a unique parabolic subgroup $P \supseteq P^{\bar{\alpha}}$ associated to B, and Lemma 4.7 easily implies that $\bar{\pi}(\gamma)$ is the unique irreducible quotient of $\pi(P, \gamma)$. We claim that there are exact sequences

$$0 \to X^{\bar{\alpha}} \to \varphi_\alpha \psi_\alpha(\pi(P, \gamma)) \to \pi(P, \gamma) \to 0$$

$$0 \to \pi(P, \gamma_{\bar{\alpha}}) \to \pi(P^{\bar{\alpha}}, \gamma^{\bar{\alpha}}) \to X^{\bar{\alpha}} \to 0.$$

Here $\gamma_{\bar{\alpha}} = (\Gamma - (n+1)\alpha, \bar{\gamma} - n\alpha) \in \hat{B}'$. A little thought will convince the reader that this is what happens in $SL(2, \mathbf{R})$. The general case will essentially be left to the reader, but the idea is the following. By a straightforward induction by stages argument, we may assume $P = G$, i.e. that $\pi(\gamma) = \bar{\pi}(\gamma)$ is a discrete series representation. In this situation $\pi(\gamma^{\bar{\alpha}})$ has exactly three composition factors, namely $\pi(\gamma_{\bar{\alpha}})$, $\pi(\gamma)$, and $\bar{\pi}(\gamma^{\bar{\alpha}})$ (cf. [16], Proposition 5.22 and the remarks after it). The required exact sequences are then obvious on the level of characters because of Theorem 4.4; one only has to check that the composition factors fit together properly, and this is easy. For by Langlands' classification theorem, $\pi(P^{\bar{\alpha}}, \gamma^{\bar{\alpha}})$ has $\bar{\pi}(\gamma^{\bar{\alpha}})$ as its unique irreducible quotient. The kernel of the quotient map has composition factors $\pi(\gamma)$ and $\pi(\gamma_{\bar{\alpha}})$, and therefore splits as a direct sum by Theorem 1.6 of [14]. The structure of the composition series of $\varphi_\alpha\psi_\alpha(\pi(\gamma))$ can be deduced from Theorem 3.9(a), since $U_\alpha(\pi(\gamma)) \cong \bar{\pi}(\gamma^\alpha)$ in this case.

By the second exact sequence, $X^{\bar{\alpha}}$ has $\bar{\pi}(\gamma^{\bar{\alpha}})$ as its unique irreducible quotient. The last claim of the theorem now follows from the first exact sequence exactly as in case (c). Q.E.D.

The constituents of $U_\alpha(X)$ described by Theorem 4.12 are called *special*. Using Corollary 4.8, Lemma 4.9, and Theorem 4.12, it is possible to determine $\tau(\bar{\pi}(\gamma))$ for any $\gamma \in \hat{B}'$, with nonsingular infinitesimal character λ: if $\alpha \in \Pi_\lambda$, the first two results allow us to modify γ so that $\bar{\alpha}$ is simple in Δ^+_γ. Then the third result determines whether or not $\alpha \in \tau(\bar{\pi}(\gamma))$. If G is linear, then the situation of Lemma 4.9 never arises, and we have the following simple result.

COROLLARY 4.13. *Suppose G is linear, $\gamma \in \hat{B}'$, and $\bar{\pi}(\gamma)$ has infinitesimal character λ. Then $\bar{\tau}(\bar{\pi}(\gamma)) \subseteq \overline{\Pi}(\lambda)$ consists of those roots $\alpha \in \Delta^+_\gamma$ which are simple with respect to $\overline{\Delta^+_\lambda}$, and such that either*

(a) *α is real, and $\gamma(m_\alpha) = (-1)^n \epsilon_\alpha$ ($n = 2\langle \alpha, \gamma \rangle / \langle \alpha, \alpha \rangle$), or*

(b) *α is complex, and $\theta\alpha \notin \Delta_\gamma^+$, or*
(c) *α is compact imaginary.*

It should be emphasized that this result is false for non-linear groups: $\bar{\tau}(\bar{\pi}(\gamma))$ contains the set described above, but (as one sees for the double cover of $SU(2, 1)$) it may be strictly larger. We have not obtained a "closed form" for $\bar{\tau}(\bar{\pi}(\gamma))$ in general.

To conclude this section, we give the duality theorem mentioned earlier.

THEOREM 4.14. *Suppose X and Y are irreducible $(\mathfrak{G}, K)$ modules with nonsingular infinitesimal character λ. Fix α and $\beta \in \Pi_\lambda$, and suppose $\alpha \notin \tau(X)$, $\beta \notin \tau(Y)$, $\alpha \in \tau(Y)$, and $\beta \in \tau(X)$. Suppose also that α and β do not span a subsystem of $\Delta(\mathfrak{G}, \mathfrak{h})$ of type G_2. Then the multiplicity of Y in $U_\alpha(X)$ equals the multiplicity of X in $U_\beta(Y)$. Their common value is computable (in terms of the Langlands classification) and is zero or one.*

(The hypothesis that α and β not span a G_2 can be removed by calculations in G_2.)

Proof. By way of motivation, suppose that Conjecture 3.15 were available. Then the multiplicity of Y in $U_\alpha(X)$ would be the dimension of

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{G}, K}(Y, U_\alpha(X)) &\cong \mathrm{Ext}^1(Y, X)\\ &\cong \mathrm{Hom}_{\mathfrak{G}, K}(U_\beta(Y), X)\end{aligned}$$

by Theorem 3.9(c), which would be the multiplicity of X in $U_\beta(Y)$. The point of the theorem is that it doesn't depend on Conjecture 3.15, and that one gets a multiplicity one result.

If α is orthogonal to β, both multiplicities are zero by Theorem 3.10. So we may assume α and β are adjacent. Choose $\Psi \subseteq \Delta(\mathfrak{G}, \mathfrak{h})$ as in Lemma 3.14. By the reduction argument used in Theorems 3.7 and 3.10, we may as well assume λ is dominant for Ψ. Write $X \cong \bar{\pi}(\gamma_1)$, $Y \cong \bar{\pi}(\gamma_2)$; say $|\mathrm{Re}\, \gamma_1|_{\mathfrak{a}_1}| \leqslant |\mathrm{Re}\, \gamma_2|_{\mathfrak{a}_2}|$. The hypotheses obviously imply $Y \not\cong X$; so by Proposition 2.2, Y is not a constituent of $\pi(\gamma_1)$. By Theorem 4.12, the multiplicity of Y in $U_\alpha(X)$ is zero or one, and is computable. (For Y must be one of the special constituents of $U_\alpha(X)$ described there.) Suppose Y does occur in $U_\alpha(X)$. Let Z be a submodule of $K_\alpha(X)$ in which Y occurs, and minimal with respect to this property. Then X is the unique irreducible subrepresentation of Z, and Y is the unique irreducible quotient. Put $Z_0 = Z/X \subseteq U_\alpha(X)$, and $Z_1 = \ker(Z \to Y)$. Since Y is the unique irreducible quotient of Z, the sequence

$$0 \to Z_1 \to Z \to Y \to 0$$

does not split, and hence $\mathrm{Ext}^1(Y, Z_1) \neq 0$. We claim that $\beta \in \tau(Z_1)$. If α and β span an A_2, this follows from Theorem 3.10(b). Suppose then that α and β span a B_2, and that Z_1 has some constituent W with $\beta \notin \tau(W)$. Put $Z_2 = Z_1/X$; then

W occurs in Z_2. The injection $Z_0 \hookrightarrow U_\alpha(X)$ gives rise to a surjection $U_\alpha(X) \to \tilde{Z}_0$ (Proposition 3.17); write K for the kernel of this map. Then we have an injection $Z_0/Z_0 \cap K \to \tilde{Z}_0$. Since Z_0 has Y as its unique irreducible quotient, Y occurs only once in Z_0, and Y is the unique irreducible subrepresentation of $\tilde{Z}_0$, it follows that $Z_0/Z_0 \cap K$ is isomorphic to Y or 0; i.e., $K \supseteq Z_2$. The exact sequence

$$0 \to K \to U_\alpha(X) \to \tilde{Z}_0 \to 0$$

now says that $U_\alpha(X)$ has at least three constituents which are not β-singular: Y, W in K, and W in $\tilde{Z}_0$. This contradicts Theorem 3.10(c), and proves that in fact $\beta \in \tau(Z_1)$. Recall that $\mathrm{Ext}^1(Y, Z_1) \neq 0$; by Theorem 3.9(c), $\mathrm{Hom}_{\mathfrak{G}, K}(U_\beta(Y), Z_1) \neq 0$. Since $Z_1 \subseteq Z \subseteq K_\alpha(X)$, X is the unique irreducible subrepresentation of Z_1. Thus X occurs in $U_\beta(Y)$. This argument did not use the assumption $|\mathrm{Re}\, \gamma_1|_{\mathfrak{a}_1}| \leqslant |\mathrm{Re}\, \gamma_2|_{\mathfrak{a}_2}|$, and is therefore symmetric in X and Y. So X occurs in $U_\beta(Y)$ if and only if Y occurs in $U_\alpha(X)$. If α and β span an A_2, Theorem 3.10(b) completes the proof. Suppose then that α and β span a B_2. We know that Y occurs in $U_\alpha(X)$ if and only if X occurs in $U_\beta(Y)$, and that Y has multiplicity one or zero in $U_\alpha(X)$. By Theorem 3.10(c), the desired result can fail only if Y occurs in $U_\alpha(X)$ (exactly once) and X occurs twice in $U_\beta(Y)$, which we therefore assume to get a contradiction. First note that $U_\beta(Y)$ can have no other α-nonsingular constituents by Theorem 3.10(c). In particular the special constituents of $U_\beta(Y)$ given by Theorem 4.12 (which have longer "$\mathfrak{a}$-parameter" than Y and therefore cannot be isomorphic to X) are α-singular. Next, an examination of the proof of Theorem 3.10(c) shows that since $U_\beta(Y)$ has two α-nonsingular constituents, Y must be the *unique* β-nonsingular constituent of $U_\alpha(X)$. Recall also that we saw that Y was one of the special constituents of $U_\alpha(X)$. If we let α_1 and β_1 correspond to α and β in $\Delta^+_{\gamma_1}$, the existence of such constituents implies that $\theta\alpha_1 \in \Delta^+_{\gamma_1}$. We now examine several cases separately. Let α_2 and β_2 correspond to α and β in $\Delta^+_{\gamma_2}$.

Case I. α_1 *and* β_1 *imaginary.* Thus α_1 is noncompact and β_1 is compact. By Theorem 4.12(f), α_2 is real, β_2 is complex, and $\theta\beta_2 = s_{\alpha_2}\beta_2$, which is obviously positive. The special constituent of $U_\beta(Y)$ is easily computed from 4.12(c), and (essentially computing inside $\mathfrak{sp}(2, \mathbb{R})$ or $\mathfrak{so}(4, 1)$) one finds that it is α-nonsingular. This contradicts the observations made above.

Case II. α_1 *imaginary*, β_1 *complex.* Here α_1 is noncompact and $\theta\beta_1 \notin \Delta^+_{\gamma_1}$ (since $\beta \in \tau(X)$). Put $m = 2\langle \beta, \lambda\rangle / \langle \beta, \beta\rangle$. By Theorem 4.12, X is the special constituent of $U_\beta(\bar{\pi}(\gamma_1 - m\beta_1))$. Set $\gamma_3 = \gamma_1 - m\beta_1$, and define α_3 and β_3 in the obvious way; then $\beta_3 = -\beta_1$, and $\alpha_3 = s_{\beta_1}(\alpha_1) = \alpha_1 + r\beta_1$, with $r = 1$ or 2. Hence $\theta\alpha_3 = \alpha_1 + r\theta\beta_1 = \alpha_3 + r\beta_3 + r\theta\beta_1$. We consider two possibilities separately.

Subcase (a). $\theta\beta_1$ *does not lie in the span of* β_1 *and* α_1. In this case, since $\theta\beta_1$ is negative for $\Delta^+_{\gamma_3}$ and does not lie in the span of the simple roots β_3 and α_3, $\alpha_3 + r\beta_3 + r\theta\beta_1 \notin \Delta^+_{\gamma_3}$. Thus $\theta\alpha_3$ is negative. If $\theta\alpha_3 = -\alpha_3$, then we could write $\theta\beta_1$ in terms of β_1 and α_1, which we cannot; so α_3 is complex. By 4.12(d), $\bar{\pi}(\gamma_3)$

is α-singular. By what we have already proved, it follows that $\bar{\pi}(\gamma_3)$ occurs in $U_\alpha(X)$, contradicting our previous observation that Y is the unique β-nonsingular constituent of $U_\alpha(X)$.

Subcase (b). *$\theta\beta_1$ lies in the span of β_1 and α_1.* Since α_1 is imaginary, we deduce at once that $\theta\beta_1 = -s_{\alpha_1}(\beta_1)$. Thus Theorem 4.12(f) implies that $\theta\beta_2 = -\beta_2$; so this case is included in Case III.

Case III. *β_2 is real.* Choose P associated to B_2 so that $\gamma_2|\mathfrak{a}_2$ is positive with respect to P; then $\pi(P, \gamma_2)$ has a unique irreducible quotient, namely $\bar{\pi}(\gamma_2) \cong Y$. Arguing as in the proof of Theorem 4.12(c), say, we find an exact sequence.

$$0 \to \pi(P, \gamma_2) \to \varphi_\beta\psi_\beta(\pi(P, \gamma_2)) \to \pi(P, \gamma_2) \to 0$$

and a surjection $\varphi_\beta\psi_\beta(\pi(P, \gamma_2)) \to \varphi_\beta\psi_\beta(Y)$. (Here the main point is that $\gamma_2 - m\beta_2$ is conjugate to γ_2 ([16], proof of 6.16(a)) and the corresponding intertwining operator is an isomorphism (Lemma 4.7)).) So we have an exact sequence

$$0 \to K_1 \to \varphi_\beta\psi_\beta(Y) \to Q_1 \to 0,$$

with K_1 and Q_1 quotients of $\pi(P, \gamma_2)$. So Y occurs once in K_1 and Q_1. Furthermore Y is both the unique quotient and the unique subrepresentation of K_1, so $Y \cong K_1$. Hence $Q_1 \cong Q_\beta(Y)$, proving that $Q_\beta(Y)$ is a quotient of $\pi(P, \gamma_2)$. In particular X occurs in $U_\beta(Y)$ at most as often as in $\pi(P, \gamma_2)$. But $\Theta(\gamma_2) = \Theta(\gamma_1) + \Theta' + \Theta_0$, with $\alpha \in \tau(\Theta_0)$ and $\Theta' = 0$ or $\Theta(\gamma')$ (some $\gamma' \in \hat{B}_1'$, with $|\mathrm{Re}\, \gamma'|_{\mathfrak{a}_1}| = |\mathrm{Re}\, \gamma_1|_{\mathfrak{a}_1}|$ and γ' not conjugate to γ_1); this follows from the character identity of Theorem 4.4. X occurs once in $\Theta(\gamma_1)$ and not at all in Θ' and Θ_0; so X occurs exactly once in $\pi(P, \gamma_2)$, and hence at most once in $U_\beta(Y)$, contradicting our assumptions.

Case IV. *α_1 imaginary, β_1 real.* This cannot happen, since then α_1 and β_1 would be orthogonal.

Case V. *α_1 complex, β_1 imaginary.* We have $\theta\alpha_1 \in \Delta_{\gamma_1}^+$, and β_1 compact. Then by Theorem 4.12(c) $\gamma_2 = \gamma_1 - n\alpha_1$ ($n = 2\langle\alpha, \lambda\rangle/\langle\alpha, \alpha\rangle$ as usual) so that

$$\beta_2 = s_{\alpha_1}(\beta_1) = \beta_1 + r\alpha_1,$$

with $r = 1$ or 2. Hence $\theta\beta_2 = \beta_1 + r(\theta\alpha_1)$. Now $\beta_1 \in \Delta_{\gamma_2}^+$, and $\theta\alpha_1 \in \Delta_{\gamma_1}^+ - \{\alpha_1\}$ $\subseteq \Delta_{\gamma_2}^+$, so $\theta\beta_2 > 0$. Furthermore β_2 is not imaginary, since β_1 is and α_1 is not. So β_2 is complex, and $U_\beta(Y)$ contains $\bar{\pi}(\gamma_3)$, with $\gamma_3 = \gamma_2 - m\beta_2$. Furthermore (with obvious notation)

$$\begin{aligned} \alpha_3 &= s_{\beta_2}(\alpha_2) \\ &= s_{\beta_2}(-\alpha_1) \\ &= -\alpha_1 + s\beta_2 \\ &= (rs-1)\alpha_1 + s\beta_1. \end{aligned}$$

Hence $\theta\alpha_3 = (rs - 1)\theta\alpha_1 + s\beta_1$. Computing in B_2 shows $rs = 2$, so α_3 is complex. As remarked above, the special constituent of $U_\beta(Y)$ must be α-singular, so $\theta\alpha_3 \notin \Delta^+_{\gamma_3}$. Since $\Delta^+_{\gamma_3} = (\Delta^+_{\gamma_1} - \{\alpha_1, \beta_1 + r\alpha_1\}) \cup \{-\alpha_1, -\beta_1 - r\alpha_1\}$, it follows that $\theta\alpha_1 = \alpha_1$ or $\beta_1 + r\alpha_1$. The first is impossible since α_1 is complex, and the second since it implies $r = -1$.

Case VI. *α_1 complex, β_1 complex.* Here $\theta\alpha_1 \in \Delta^+_{\gamma_1}$, and $\theta\beta_1 \notin \Delta^+_{\gamma_1}$; $\alpha_2 = -\alpha_1$, and $\beta_2 = s_{\alpha_1}(\beta_1) = \beta_1 + r\alpha_1$. By Case III we may assume β_2 is not real; so since $\beta \notin \tau(Y)$, we have $\theta\beta_2 \in \Delta^+_{\gamma_2}$. If $\theta\beta_2 \notin \Delta^+_{\gamma_1}$, this implies $\theta\beta_2 = -\alpha_1$, and hence $\theta\alpha_1 = -\beta_2 \notin \Delta^+_{\gamma_1}$, a contradiction. So $\theta\beta_2 \in \Delta^+_{\gamma_1}$. Put $\gamma_3 = \gamma_1 - m\beta_1$; then X is a special constituent of $U_\beta(\overline{\pi}(\gamma_3))$. If $\alpha \in \tau(\overline{\pi}(\gamma_3))$, then this implies that $\overline{\pi}(\gamma_3)$ is a (β-nonsingular) constituent of $U_\alpha(X)$, a contradiction. So $\alpha \notin \tau(\overline{\pi}(\gamma_3))$. Now $\alpha_3 = s_{\beta_1}(\alpha_1) = \alpha_1 + s\beta_1$. We separate several cases.

(a) *β_2 is complex.* Then $U_\beta(Y)$ contains $\overline{\pi}(\gamma_4)$ as a special constituent, with $\gamma_4 = \gamma_2 - m\beta_2$. So by the remarks earlier, $\alpha \in \tau(\overline{\pi}(\gamma_4))$. But $\alpha_4 = s_{\beta_2}(\alpha_2) = \alpha_2 + s\beta_2 = (rs - 1)\alpha_1 + s\beta_1$. As usual $rs = 2$, so $\alpha_4 = \alpha_3$. Now $\alpha \in \tau(\overline{\pi}(\gamma_4))$, and $\alpha \notin \tau(\overline{\pi}(\gamma_3))$; by Theorem 4.12, this is possible only if α_3 is complex, and $\theta\alpha_3 \in \Delta^+_{\gamma_3} - \Delta^+_{\gamma_4}$. This difference consists of the roots $\{-\beta_1, \alpha_1, \beta_2\}$, of which only α_1 has the right length; so

$$\theta(\alpha_1 + s\beta_1) = \alpha_1 .$$

This gives $s\theta\beta_1 = \alpha_1 - \theta\alpha_1$, which implies β_1 is real, a contradiction.

(b) *β_2 is imaginary and α_3 is complex.* Since $\beta \notin \tau(Y)$, β_2 is noncompact; and since $\alpha \notin \tau(\overline{\pi}(\gamma_3))$, $\theta\alpha_3 \in \Delta^+_{\gamma_3}$. It follows that $\theta\alpha_3 \in \Delta^+_{\gamma_1}$. But as we saw in case (a), $\alpha_3 = s_{\beta_2}(\alpha_2)$. In this case θ and s_{β_2} commute so $\theta\alpha_3 = s_{\beta_2}(\theta\alpha_2) = s_{\beta_2}(-\theta\alpha_1)$. We know $\theta\alpha_1 \in \Delta^+_{\gamma_1}$, so this is possible only if $\theta\alpha_1$ lies in the span of α_1 and β_1. The only roots of the right length in this span are α_1, which is ruled out since $\alpha_1 \neq \theta\alpha_1$, and $\alpha_1 + s\beta_1$, which is ruled out as in case (a).

(c) *β_2 is imaginary and α_3 is imaginary.* Since α_3 and β_2 have the same span as α_1 and β_1, this is impossible.

(d) *β_2 is imaginary and α_3 is real.* Computing in B_2, one finds that β_2 and α_3 are not orthogonal, so this is impossible.

Since $\theta\beta_2 \in \Delta^+_{\gamma_2}$, this exhausts the possibilities for Case VI.

Case VII. *α_1 complex, β_1 real.* As usual $\theta\alpha_1 \in \Delta^+_{\gamma_1}$; and $\beta_1 \in \tau(X) \Rightarrow \gamma_1(m_{\beta_1}) = (-1)^m\epsilon_{\beta_1}$. We have $\gamma_2 = \gamma_1 - n\alpha_1$, $\alpha_2 = -\alpha_1$, and $\beta_2 = \beta_1 + r\alpha_1$. Since β_1 is real and α_1 is complex, β_2 has non-zero imaginary part; so β_2 is imaginary or complex. Since $\beta \notin \tau(Y)$, this implies that $\theta\beta_2 \in \Delta^+_{\gamma_2}$. We separate two subcases.

(a) *β_2 is complex.* Then $U_\beta(Y)$ has as a special constituent $\overline{\pi}(\gamma_4)$, with $\gamma_4 = \gamma_2 - m\beta_2$ as usual. Furthermore $\alpha_4 = s_{\beta_1}(\alpha_1) = \alpha_1 + s\beta_1$ just as in case VI. Hence α_4 is complex or imaginary. Since $\alpha \in \tau(\overline{\pi}(\gamma_4))$ as usual, this implies that either α_4 is compact imaginary, or $\theta\alpha_4 \notin \Delta^+_{\gamma_4}$. Since $\theta\alpha_4 = \theta\alpha_1 - s\beta_1$, and $\theta\alpha_1 \in \Delta^+_{\gamma_1}$, the second possibility would force $\theta\alpha_4$ to be in the span of α_1 and β_1; combined with $\theta\alpha_4 \neq -\alpha_4$ and $\theta\alpha_4 \notin \Delta^+_{\gamma_4}$, this implies $\theta\alpha_4 = \alpha_1$. Thus $\alpha_4 + \theta\alpha_4 = 2\alpha_1 + s\beta_1$ is imaginary; but $\beta_2 = \beta_1 + r\alpha_1 = \beta_1 + 2\alpha/s = (2\alpha_1 + s\beta_1)/s$, so β_2 is imaginary, a contradiction. For the first possibility (α_4 compact imaginary), computing in B_2 shows that α_4 and β_1 are not orthogonal. Since β_1 is real, this is impossible.

(b) β_2 *is imaginary*. Since $\beta \notin \tau(Y)$, β_2 is noncompact. Notice that the span of α and β is stable under θ; this allows us to reduce to computations to the *real* subalgebra of $\mathfrak{G}_0$ corresponding to the roots α_1 and β_1. Since β_2 is noncompact imaginary, this subalgebra is isomorphic to $\mathfrak{sp}(2, \mathbf{R})$. There are two essentially different possibilities. Suppose first that β is long. Define a more compact Cartan subgroup B_3 via Cayley transform through the real root β_1, and choose $\gamma_3 \in \hat{B}_3'$ so that $\pi(\gamma_1)$ occurs on the right side of the character identity for $\pi(\gamma_3)$ and the noncompact simple root β_3 (cf. Proposition 4.5). Note that α_3 is an imaginary root. There is ambiguity in the choice of γ_3; it could be replaced by $\gamma_3' = (\Gamma_3 - (m+1)\beta_3, \bar{\gamma}_3 - m\beta_3)$, which replaces α_3 by $s_{\beta_3}(\alpha_3) = \alpha_3 + \beta_3$ (because β is long). If α_3 is noncompact, then $\alpha_3 + \beta_3$ is compact; so possibly replacing γ_3 by γ_3', we may assume α_3 is compact. Then $\alpha \in \tau(\bar{\pi}(\gamma_4))$, and X occurs in $U_\beta(\bar{\pi}(\gamma_4))$; so $\bar{\pi}(\gamma_4)$ occurs in $U_\alpha(X)$, a contradiction. So suppose finally that β is short. Define B_4 via Cayley transform through the noncompact imaginary root β_2. Then we can fix simple (real) roots α_4 and β_4 of $\mathfrak{b}_4$ in $\mathfrak{G}$ corresponding to α_2 and β_2 via the Cayley transform. Thus we obtain an element $m_{\alpha_4} \in B_4$. Since β is short, it is easy to see that m_{α_4} acts by -1 on the β_4 root space of $\mathfrak{b}_4$ in $\mathfrak{G}$, and hence that m_{α_4} acts by -1 in the β_2 direction in $\mathfrak{b}$. In particular m_{α_4} does not fix γ_2 in its action on $\hat{B}'$. The proof of the character identities (Theorem 4.4) now shows that the right side of the character identity for γ_2 and β_2 has two terms corresponding to $(\gamma_4)_\pm \in \hat{B}_4'$, and therefore that $U_\beta(Y)$ has two special constituents $\bar{\pi}((\gamma_4)_\pm)$. Since m_{α_4} is easily seen to have different eigenvalues in $(\gamma_4)_+$ and $(\gamma_4)_-$ (from their construction), Theorem 4.12(a) shows that α cannot lie in the τ invariant of both of these constituents. This contradicts the hypothesis that X is the unique α-nonsingular constituent of Y, and completes case VII. Since we saw that $\theta\alpha_1 \in \Delta^+_{\gamma_1}$, α_1 cannot be real; so we have treated all possible cases, and proved the theorem. Q.E.D.

It hardly needs to be remarked that the preceding proof is extremely unsatisfactory. It is not a complete accident, however: it is easy to see from the first part of the proof that, if the result is true, then it can be verified along the lines adopted here. From the computational point of view, therefore, the case-by-case arguments are unnecessary: the methods of the first part allow one to determine completely the β-nonsingular constituents of $U_\alpha(X)$, and the fact that the answer happens to have a nice general form is not too important.

5. Computation of characters. In this section we describe irreducible characters in terms of the $U_\alpha(X)$. More precisely, suppose that for each irreducible $(\mathfrak{G}, K)$ module X with nonsingular infinitesimal character λ, and $\alpha \in \Pi_\lambda - \tau(X)$ the irreducible constituents of $U_\alpha(X)$ and their multiplicities are known (in terms of the Langlands classification). Then we will describe how to compute the irreducible characters as integer combinations of the standard characters $\Theta(\delta \otimes \nu)$ described in section 2. (Recall that the standard characters are computable via an algorithm, although they have not been expressed in closed form.) By Theorem 6.18 of [16] (which can now be generalized to non-linear groups without difficulty) we may confine our attention to characters with nonsingular infinitesimal character. It is convenient to assume that the characters of groups of lower dimension than G are known (although everything could be formulated, a bit more clumsily, in terms of G itself). Fix $\lambda_0 \in \mathfrak{h}^*$ nonsingular, and let $C_1, \ldots, C_N$ be the set of Weyl chambers in $\mathfrak{h}^*$ dominant with respect to $\Delta^+_{\lambda_0}$. Choose weights μ_i of finite dimensional representations of G such that $\lambda_i = \lambda_0 + \mu_i \in C_i$. We confine our attention to the finite set of irreducible characters with infinitesimal character equal to some λ_i. By the proof of Corollary 2.3, it is enough to compute the irreducible composition factors (and their multiplicities) for the various $\pi(\gamma)$ having infinitesimal character $\lambda \in \{\lambda_i\}$. This we do by induction on

$$l(\gamma) = \tfrac{1}{2}|\{\alpha \in \Delta^+_\gamma | \theta\alpha \notin \Delta^+_\gamma\}| + \tfrac{1}{2}\dim \mathfrak{a}$$

(Here $\gamma \in \hat{B}'$, and $\mathfrak{a}$ is the split component of $\mathfrak{b}$.) This notion of "length" has the following nice properties, which are trivial to verify. Suppose $\alpha \in \Delta^+_\gamma$ is simple. If α is complex, $\theta\alpha \in \Delta^+_\gamma$, and γ' is dominant for $s_\alpha(\Delta^+_\gamma)$, then $l(\gamma') = l(\gamma) + 1$. If α is noncompact imaginary, and γ' occurs on the right side of the corresponding character identity (Theorem 4.4), then $l(\gamma') = l(\gamma) + 1$. Suppose then that the composition series of $\pi(\gamma')$ is known when $\pi(\gamma')$ has infinitesimal character $\lambda' \in \{\lambda_i\}$ and $l(\gamma') < l(\gamma)$, for some $\gamma \in \hat{B}$, such that $\pi(\gamma)$ has infinitesimal character $\lambda \in \{\lambda_i\}$. Choose a positive root system $\Delta^+_1 \subseteq \Delta(\mathfrak{G}, \mathfrak{b})$ such that if $\alpha \in \Delta^+_\gamma$ is real, or $\theta\alpha \in \Delta^+_\gamma$, then $\alpha \in \Delta^+_1$, and such that $\beta \in \Delta^+_1$ and $\theta\beta \notin \Delta^+_1$ implies β is real; this is certainly possible. Suppose first that $\Delta^+_1 \neq \Delta^+_\gamma$. Then there is a simple root $\alpha \in \Delta^+_\gamma$ such that $\alpha \notin \Delta^+_1$. Clearly α is complex, and $\theta\alpha \notin \Delta^+_\gamma$. Choose a weight μ of a finite dimensional representation of G such that $\lambda + \mu \in W(\mathfrak{G}/\mathfrak{h}) \cdot \{\lambda_i\}$, and $\gamma + \mu_\gamma$ is dominant for $s_\alpha(\Delta^+_\gamma)$: if α is integral for γ, we take $\mu_\gamma = -n\alpha$; otherwise μ can be taken to be the difference of two of the μ_i chosen above. Now $l(\gamma + \mu_\gamma) = l(\gamma) - 1$, so the composition series of $\pi(\gamma + \mu_\gamma)$ is known. Furthermore

$$\Theta(\gamma) = S_{-\tilde{\mu}}(\Theta(\gamma + \mu_\gamma))$$

here $\tilde{\mu} = s_{\tilde{\alpha}}(\mu)$, with $\tilde{\alpha} \in \Delta(\mathfrak{G}, \mathfrak{h})$ corresponding to α. If α is not integral for γ, the effect of $S_{-\tilde{\mu}}$ on the irreducible constituents of $\Theta(\gamma + \mu_\gamma)$ is known from Corollary 4.8 and Lemma 4.9, so the composition series of $\Theta(\gamma)$ is known. If α is integral with respect to γ, then $\tilde{\mu} = n\tilde{\alpha}$. Write

$$\Theta(\gamma + \mu_\gamma) = \Sigma n_i \Theta_i + \Sigma m_j \Theta'_j,$$

with Θ_i and Θ'_j irreducible, $\tilde{\alpha} \notin \tau(\Theta_i)$, and $\tilde{\alpha} \in \tau(\Theta'_j)$; the Θ_i and Θ'_j are specified in terms of the Langlands classification, and the n_i and m_j are explicitly known integers by induction. Then

$$\Theta(\gamma) = \Sigma n_i \Theta_i + \Sigma n_i U_\alpha(\Theta_i) - \Sigma m_j \Theta'_j;$$

so if the decomposition of each $U_\alpha(\Theta_i)$ into irreducible constituents is known, so is that for $\Theta(\gamma)$.

So we may suppose that $\Delta_\gamma^+ = \Delta_1^+$, i.e., that $\alpha \in \Delta_\gamma^+$ and $\theta\alpha \notin \Delta_\gamma^+$ implies α is real. We define a parabolic subalgebra $\mathfrak{S} = l + \mathfrak{N}$ of $\mathfrak{G}$ so that $l \supseteq \mathfrak{b}$, $\Delta(l, \mathfrak{b})$ is the set of real roots of $\mathfrak{b}$ in $\mathfrak{G}$, and $\Delta(\mathfrak{N}, \mathfrak{b}) = \{\alpha \in \Delta_\gamma^+ | \theta\alpha \in \Delta_\gamma^+\}$. Clearly $\mathfrak{S}$ is θ-stable, and l corresponds to a connected reductive subgroup L of G whose semisimple part is split over $\mathbf{R}$. In this setting the composition series of $\pi(\gamma)$ can be computed in terms of the composition series for a certain representation of L (cf. [16], Theorem 4.15, Corollary 4.18, and Proposition 4.19. Recent joint work with G. Zuckerman provides a more conceptual proof of these results, which will appear in a future paper.) Since we are assuming that the result is known for groups of lower dimension, we are reduced to the case $L = G$, i.e., to $B = T^+A$, with T^+ discrete; and the inductive hypothesis means that the composition series of $\pi(\gamma')$ is known whenever B' is not split and $\pi(\gamma')$ has infinitesimal character $\lambda' \in \{\lambda_i\}$. If $\pi(\gamma)$ is irreducible, we are of course done; so suppose it is not. By intertwining operator considerations (cf. [16], section 3), there is a root $\alpha \in \Delta(\mathfrak{G}, \mathfrak{b})$ such that the eigenvalues of $\gamma(m_\alpha)$ are of the form $\epsilon_\alpha \cdot \exp(\pm 2\pi i\langle\gamma, \alpha\rangle/\langle\alpha, \alpha\rangle)$. Suppose first that we can find such an α so that the corresponding character identity of Theorem 4.4 has only one term on the right. (This happens for example if α is not integral. Cf. Remark 4.6). Thus if B_α is the Cartan subgroup with one dimensional compact factor obtained from B via Cayley transform, and Ψ_α is a positive root system (i.e., choice of positive root) for $\mathfrak{t}_\alpha^+$ in $\mathfrak{M}_\alpha$, then there is $\gamma_\alpha \in \hat{B}'$, with $\gamma_\alpha|_{\mathfrak{t}_\alpha^+}$ dominant for Ψ_α, such that

$$\Theta(\gamma) = \Theta(\Psi_\alpha, \gamma_\alpha) + \Theta(-\Psi_\alpha, \tilde{\gamma}_\alpha)$$

(cf. Theorem 4.4). The first term on the right has a composition series which is known by induction. The second is not a standard character; but it is obtained from a standard character by a sequence of wall crossings of the sort used in the first part of the algorithm. We know how these wall crossings affect composition series, so the composition series of $\Theta(-\Psi_\alpha, \tilde{\gamma}_\alpha)$ is computable; details are left to the reader. So in this case we can determine the constituents of $\Theta(\gamma)$ and their multiplicities. So we may assume that *no non-integral roots* α *satisfying the parity condition exist*. We may also assume that G is simple. First we claim that $\tau(\overline{\Theta}(\gamma)) \neq \emptyset$. Now the element m_α^2 of B depends (up to inverse) only on the length of α. Furthermore if $b \in B$, then $bm_\alpha b^{-1} = m_\alpha^{\pm 1}$. It follows that if $\gamma(m_\alpha)$ has eigenvalues ± 1, then in fact $\gamma(m_\alpha) = +1$ or -1, and $\gamma(m_\beta) = \pm 1$ whenever α and β have the same length.

Suppose first that all roots of G have the same length, or that $\mathfrak{G}$ is of type G_2. We know that for some α, $\gamma(m_\alpha) = \pm 1$ (since $\pi(\gamma)$ is reducible); so by the remarks above (or a computation in the second case) $\gamma(m_\beta) = \pm 1$ for *all* β. Put

$$R = \left\{ \alpha \in \Delta(\mathfrak{G}, \mathfrak{b}) \mid \frac{2\langle\alpha, \gamma\rangle}{\langle\alpha, \alpha\rangle} = n \in \mathbf{Z}, \quad \text{and} \quad \gamma(m_\alpha) = -\epsilon_\alpha(-1)^n \right\}.$$

In the present situation $\epsilon_\alpha = -(-1)^{2\langle\alpha,\rho\rangle/\langle\alpha,\alpha\rangle}$ (cf. [16], before 5.15), so

$$R = \left\{\alpha \in \bar{\Delta}_\lambda \mid (-1)^{2\langle\alpha,\gamma+\rho\rangle/\langle\alpha,\alpha\rangle} = \gamma(m_\alpha)\right\}$$

(recall that $\bar{\Delta}_\lambda = \{\alpha \in \Delta(\mathfrak{G}, \mathfrak{b}) \mid 2\langle\alpha,\gamma\rangle/\langle\alpha,\alpha\rangle \in \mathbf{Z}\}$). We claim that R is a root system. Clearly $\alpha \in R$ if and only if $-\alpha \in R$. Suppose $\alpha, \beta \in R$; we must show that $s_\beta\alpha \in R$. Now $s_\beta\alpha = \alpha + r\beta$, with $r = 2\langle\alpha,\beta\rangle/\langle\beta,\beta\rangle$, and $m_{s_\beta\alpha} = m_\alpha$ or $m_\alpha m_\beta$ according as $2\langle\alpha,\beta\rangle/\langle\alpha,\alpha\rangle$ is even or odd (for this and similar computations, compare section 6 of [17].) Hence

$$\frac{2\langle s_\beta\alpha, \gamma+\rho\rangle}{\langle\alpha,\alpha\rangle} = \frac{2\langle\alpha,\gamma+\rho\rangle}{\langle\alpha,\alpha\rangle} + r\cdot\frac{2\langle\beta,\gamma+\rho\rangle}{\langle\alpha,\alpha\rangle},$$

and

$$\gamma(m_{s_\beta\alpha}) = \gamma(m_\alpha) \quad \text{or} \quad \gamma(m_\alpha m_\beta)$$

according as $2\langle\alpha,\beta\rangle/\langle\alpha,\alpha\rangle$ is even or odd. Recalling that $r = 2\langle\alpha,\beta\rangle/\langle\alpha,\alpha\rangle$, we obtain

$$\frac{2\langle s_\beta\alpha, \gamma+\rho\rangle}{\langle\alpha,\alpha\rangle} = \frac{2\langle\alpha,\gamma+\rho\rangle}{\langle\alpha,\alpha\rangle} + \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\cdot\frac{2\langle\beta,\gamma+\rho\rangle}{\langle\beta,\beta\rangle}.$$

The desired equality $(-1)^{2\langle s_\beta\alpha,\gamma+\rho\rangle/\langle\alpha,\alpha\rangle} = \gamma(m_{s_\beta\alpha})$ is now obvious. So R is a root system. On the other hand, Theorem 4.12 implies that $\bar{\tau}(\bar{\pi}(\gamma)) = \bar{\Pi}_\lambda - (\bar{\Pi}_\lambda \cap R)$. So if $\tau(\bar{\pi}(\gamma))$ is empty, then $R \supset \bar{\Pi}_\lambda$, so $R \supseteq \bar{\Delta}_\lambda$ since R is a root system. But this contradicts the assumed existence of a root $\alpha \in \Delta_\gamma^+$ with $2\langle\alpha,\gamma\rangle/\langle\alpha,\alpha\rangle = n \in \mathbf{Z}$, and $\gamma(m_\alpha) = \epsilon_\alpha(-1)^n$. So $\tau(\bar{\pi}(\gamma))$ is nonempty in this case.

Suppose next that there are two root lengths, differing by a factor of two. A computation for $\mathfrak{sp}(2, \mathbf{R})$ shows that if α is short, then $m_\alpha^2 = 1$; this result therefore holds for G as well. In particular $\gamma(m_\alpha) = \pm 1$ when α is short. If this is true of long roots as well, the argument for the other case applies. So we may assume that for some long root β, $\gamma(m_\beta) \neq \pm 1$. Thus $\gamma(m_\beta^2) \neq 1$; since m_β^2 depends only on the length of β, $\gamma(m_{\beta'}) \neq \pm 1$ whenever β' is long. Define R exactly as before; now R consists of short roots. The computations of the previous case easily imply that R is invariant under reflections about long roots in Δ_λ. If $\tau(\bar{\pi}(\gamma))$ is empty, then R contains all the short roots in Π_λ. Now a subsystem of a root system containing all the short simple roots and invariant under reflections about long roots contains all the short roots. So R contains all the short roots. But this contradicts the reducibility of $\pi(\gamma)$ just as before. So in all cases $\tau(\bar{\pi}(\gamma))$ is nonempty. (We have actually proved more, namely that $\tau(\bar{\pi}(\gamma))$ meets each simple factor of Δ_λ containing some root α with $\gamma(\bar{m}_\alpha) = \epsilon_\alpha\cdot(-1)^n$.)

Suppose then that $\alpha \in \bar{\tau}(\bar{\pi}(\gamma))$, and that the corresponding character identity has two terms on the right. (Recall that if there is only one term, then the composition series of $\pi(\gamma)$ is completely known.) So there is some $\gamma_1 \in \hat{B}'$ and a character identity

$$\Theta(\Psi_\alpha, \gamma_\alpha) + \Theta(-\Psi_\alpha, \tilde{\gamma}_\alpha) = \Theta(\gamma) + \Theta(\gamma_1).$$

Just as before, the composition series of the left side is assumed to be known by induction. We may as well assume $\gamma_1|_{\mathfrak{a}} = \gamma|_{\mathfrak{a}}$. We will call γ and γ_1 *adjacent*. Let $\tilde{\alpha} \in \Pi_\lambda$ correspond to α.

By induction by stages, one sees that in fact $\Theta(\gamma) - \Theta(\Psi_\alpha, \gamma_\alpha) = \Theta_0$ is the character of a representation, and $\tilde{\alpha} \in \tau(\Theta_0)$. Hence $\Theta(\gamma)$ and $\Theta(\Psi_\alpha, \gamma_\alpha)$ have the same $\tilde{\alpha}$-nonsingular constituents.

Definition 5.1. Suppose $D \subseteq \Pi_\lambda$. A character Θ with infinitesimal character λ is said to be *known up to D* if the multiplicity of X in Θ is known whenever $\tau(X)$ does not contain D.

Thus we have just seen that $\Theta(\gamma)$ is known up to $\tau(\bar{\pi}(\gamma))$. If Θ_1 is known up to D_1, Θ_2 is known up to D_2, and $\Theta_1 + \Theta_2$ is known up to $D_1 \cup D_2$, then obviously one can compute Θ_i up to $D_1 \cup D_2$. Let $\sim$ denote the equivalence relation on $\hat{B}'$ generated by adjacency as defined above. We have proved

LEMMA 5.2. *With notation as above, set $D_\gamma = \bigcup_{\gamma' \sim \gamma} \tau(\bar{\pi}(\gamma'))$. Then $\Theta(\gamma)$ is known up to D_γ.*

On the other hand, the sets D_γ are fairly large:

LEMMA 5.3. *Suppose α and β are adjacent roots in $\bar{\Pi}_\lambda$, with $\gamma(m_\alpha)$ and $\gamma(m_\beta)$ both ± 1, $\alpha \in \bar{\tau}(\bar{\pi}(\gamma))$, and $|\beta|/|\alpha| \neq 1/2$. Then $\tilde{\beta} \in D_\gamma$.*

Proof. Let $\gamma_1 \in \hat{B}'$ be adjacent to γ via the character identity corresponding to α. The hypotheses imply $2\langle \alpha, \beta \rangle / \langle \beta, \beta \rangle$ is odd, and hence that m_β acts by -1 on $\mathfrak{t}_\alpha^+$; it follows from the construction of the character identity that $\gamma(m_\beta) = -\gamma_1(m_\beta)$. So if $\beta \notin \bar{\tau}(\bar{\pi}(\gamma))$, then $\beta \in \bar{\tau}(\bar{\pi}(\gamma_1))$. Q.E.D.

Because of these lemmas, the main difficulty in determining the composition series of $\Theta(\gamma)$ comes from simple factors of Δ_λ whose simple roots do not meet $\tau(\bar{\pi}(\gamma))$. So we need to rule out certain composition factors *a priori*. Let Δ_λ^1 be the product of the simple factors of Δ_λ which meet $\tau(\bar{\pi}(\gamma))$; put $\Pi_\lambda^1 = \Delta_\lambda^1 \cap \Pi_\lambda$.

LEMMA 5.4. *With assumptions as above, suppose $\bar{\pi}(\gamma_1)$ is a composition factor of $\pi(\gamma)$, and suppose $\alpha \in \Delta_{\gamma_1}^+$. If α is imaginary, or real with $\gamma_1(m_\alpha)$ satisfying the parity condition, then α is integral. Furthermore the root $\tilde{\alpha} \in \Delta(\mathfrak{G}, \mathfrak{h})$ corresponding to α lies in Δ_λ^1.*

Proof. We proceed by *downward* induction on $l(\gamma_1)$, treating all the infinitesimal characters $\{\lambda_i\}$ at once. Notice that if $\gamma_1 = \gamma$, the result is true: this follows from the remark at the end of the proof that $\tau(\bar{\pi}(\gamma)) \neq \emptyset$, together with our assumption that Δ_γ^+ has no non-integral roots satisfying the parity condition. In general, choose a positive root system $\Delta_1^+ \subseteq \Delta(\mathfrak{G}, \mathfrak{b}_1)$ (with $\gamma_1 \in \hat{B}_1'$) such that if $\beta \in \Delta_{\gamma_1}^+$ is imaginary, or $-\theta\beta \in \Delta_{\gamma_1}^+$, then $\beta \in \Delta_1^+$; and such that $\beta \in \Delta_1^+$ and $-\theta\beta \notin \Delta_1^+$ implies β is imaginary. Suppose first that $\Delta_{\gamma_1}^+ \neq \Delta_1^+$. Choose a simple root $\beta \in \Delta_{\gamma_1}^+$ such that $\beta \notin \Delta_1^+$; necessarily β is complex, and $\theta\beta \in \Delta_{\gamma_1}^+$. Choose a weight μ of a finite dimensional representation of G such that $\lambda + \mu \in W(\mathfrak{G}/\mathfrak{h}) \cdot \{\lambda_i\}$, and $\gamma_1 + \mu_{\gamma_1}$ is dominant for $s_\beta(\Delta_{\gamma_1}^+)$; if β is integral, we take $\mu_{\gamma_1} = -m\beta$ ($m = 2\langle \gamma_1, \beta\rangle / \langle \beta, \beta\rangle$); otherwise μ can be taken to be the difference of two of the μ_i chosen at the beginning of this section. Then $l(\gamma_1 + \mu_{\gamma_1}) = l(\gamma_1) + 1$. Furthermore $\gamma + \mu_\gamma$ has no non-integral roots satisfying the parity condition (since γ doesn't), and the set Δ_λ^1 attached to $\gamma + \mu_\gamma$ is the same as that for γ. We claim that $\bar{\pi}(\gamma_1 + \mu_{\gamma_1})$ is a composition factor of $\pi(\gamma + \mu_\gamma)$ or of $\pi(\gamma)$. If β is non-integral this is clear, and if β is integral it is easily deduced from Theorem 4.12(c). So by induction there are no roots $\alpha \in \Delta(\mathfrak{G}, \mathfrak{b})$ which are "bad" with respect to $\gamma_1 + \mu_{\gamma_1}$ and the conditions of the lemma; so there are none for γ_1 either.

So we may assume $\Delta_{\gamma_1}^+ = \Delta_1^+$, i.e., that $\beta \in \Delta_{\gamma_1}^+$ and $\theta\beta \in \Delta_{\gamma_1}^+$ implies β is imaginary. Put $M_1A_1 = G^{A_1}$ ($B_1 = T_1^+A_1$). Then M_1 is split over $\mathbb{R}$ since G is, so all its simple factors are noncompact. Furthermore every simple root of $\mathfrak{t}_1^+$ in $\mathfrak{M}_1$ is simple as a root of $\mathfrak{b}_1$ in $\mathfrak{G}$, by the special nature of $\Delta_{\gamma_1}^+$. Suppose that Lemma 5.4 fails for γ_1. Suppose first that there is an imaginary root α with $\tilde{\alpha} \notin \Delta_\lambda^1$; clearly we may assume α is simple and noncompact. Define B^α as for the character identities. Choose a weight μ of a finite dimensional representation of G as before, so that $\gamma_1 + \mu_{\gamma_1}$ is dominant for $s_\alpha(\Delta_{\gamma_1}^+)$. Using either Lemma 4.9 or Theorem 4.12(f) (according as α is or is not integral) we obtain $\gamma_1^\alpha \in (\hat{B}^\alpha)'$, with $l(\gamma_1^\alpha) = l(\gamma_1) + 1$, so that $\bar{\pi}(\gamma_1^\alpha)$ occurs in $\pi(\gamma)$ or in $\pi(\gamma + \mu_\gamma)$. Furthermore the real root of B^α corresponding to α satisfies the parity condition, but $\tilde{\alpha} \notin \Delta_\lambda$; and this contradicts the inductive hypothesis. So (still assuming the failure of Lemma 5.4) all imaginary roots correspond to roots in Δ_λ^1, but there is a real root α satisfying the parity condition, with $\tilde{\alpha} \notin \Delta_\lambda^1$. Since we know that Lemma 5.4 holds for $\gamma_1 = \gamma$, M_1 is noncompact; so there is a noncompact simple imaginary root β, which is necessarily integral; say $2\langle \gamma_1, \beta\rangle / \langle \beta, \beta\rangle = m$. Just as above we can define B^β and γ_1^β; the only problem is to see that γ_1^β fails to satisfy Lemma 5.4 (to get a contradiction). Let α' be the root of $\mathfrak{b}^\beta$ in $\mathfrak{G}$ corresponding to α under the Cayley transform defining $\mathfrak{b}^\beta$; obviously α' is real, and $\tilde{\alpha}' = \tilde{\alpha} \notin \Delta_\lambda^1$. The only question is whether $\gamma_1^\beta(m_{\alpha'})$ has eigenvalues $\epsilon_{\alpha'} \cdot \exp(\pm 2\pi i \langle \gamma_1^\beta, \alpha'\rangle / \langle \alpha', \alpha'\rangle)$. Now $m_{\alpha'} = m_\alpha$, and $\langle \gamma_1^\beta, \alpha'\rangle = \langle \gamma_1, \alpha\rangle$. By the construction of the character identities, $\gamma_1^\beta(m_{\alpha'}) = \gamma_1(m_\alpha)$; so we need only show that $\epsilon_\alpha = \epsilon_{\alpha'}$. This requires a little work. Define a Cartan subgroup B_α by a Cayley transform through α. Since everything is defined within the attached equal rank group M_α, we might as well assume for the moment that $B_\alpha = T$ is compact. Let α_0 and β_0 be the roots of $\mathfrak{t}$ in $\mathfrak{G}$ corresponding to α and β via the Cayley transform defining T. Recall that the roots of $\mathfrak{t}_1$ in $\mathfrak{M}_1$ may be identified with the roots of $\mathfrak{t}$ in $\mathfrak{G}$ orthogonal to α_0; if $\delta \in \Delta(\mathfrak{t}_1, \mathfrak{M}_1)$ is identified with δ_0, and δ_0 is (say) noncompact, then δ is noncompact if and only if δ_0 is strongly orthogonal to α_0. Let $\Psi_1 \subseteq \Delta(\mathfrak{t}_1, \mathfrak{M}_1)$ be the positive root system defined by γ_1. Choose a positive system Ψ for $\mathfrak{t}$ in $\mathfrak{G}$ so that α_0 is simple, the span of α_0 and β_0 is spanned by simple roots, and Ψ restricts to Ψ_1 on the orthogonal complement of α_0. Write $\rho(\Psi)$ for half the sum of the roots of Ψ, and use other notation analogously. Put $n_\alpha = 2\langle \alpha_0, \rho(\Psi) - 2\rho(\Psi \cap \Delta(\mathfrak{k}))\rangle / \langle \alpha_0, \alpha_0\rangle$ (cf. [16], after 5.14); then $\epsilon_\alpha = (-1)^{n_\alpha}$. $\epsilon_{\alpha'}$ is computed analogously on the Cartan subgroup $B^\beta_{\alpha'}$ obtained via Cayley transform of B^β through α'. We know that β is integral. If α is *not* integral, then inspection of the rank two split groups shows that either β is simple, or α and β are not strongly orthogonal. Suppose first that α and β are strongly orthogonal. Then β_0 is noncompact, and the various Cayley transforms commute with each other: if we define Ψ_β to be the subset of Ψ orthogonal to β_0, and $\Psi_{\mathfrak{k}\beta}$ to be the roots in Ψ_β which are compact and strongly orthogonal to β_0, or noncompact and not strongly orthogonal, then

$$n_{\alpha'} = 2\langle \alpha_0, \rho(\Psi_\beta) - 2\rho(\Psi_{\mathfrak{k}\beta})\rangle / \langle \alpha, \alpha\rangle,$$

and $\epsilon_{\alpha'} = (-1)^{n_{\alpha'}}$. If β is actually simple, then $n_\alpha = n_{\alpha'}$ by (4.3) of [17] (which is due to Schmid), and we are done. If β is not simple, then it is integral by the remark above. If $\tilde{\alpha}$ and $\tilde{\beta}$ span a B_2, then λ is necessarily integral with respect to all of the roots in this B_2 (since it is integral with respect to the long roots $\tilde{\alpha}$ and $\tilde{\beta}$). So $\tilde{\alpha}$ and $\tilde{\beta}$ lie in the same simple factor of Δ_λ. Since we are assuming $\tilde{\alpha} \notin \Delta^1_\lambda$ and $\tilde{\beta} \in \Delta^1_\lambda$, this is a contradiction. If $\tilde{\alpha}$ and $\tilde{\beta}$ span a G_2, then (again because they lie in distinct factors of Δ_λ) they must be the only positive integral roots; so α_0 and β_0 must be compact, a contradiction. So we may assume α and β are not strongly orthogonal; thus they are the short roots of a B_2. Since β is noncompact β_0 is compact, and (computing in $\mathfrak{sp}(2, \mathbf{R})$) one finds that $B_1 \cong B^\beta_{\alpha'}$ by an isomorphism fixing the orthogonal complement of α and β. So let Ψ_α be the subset of Ψ orthogonal to α_0, with $\Psi_{\mathfrak{k}\alpha}$ defined as in the previous case. Then

$$n_{\alpha'} = 2\langle \beta_0, \rho(\Psi_\alpha) - 2\rho(\Psi_{\mathfrak{k}\alpha})\rangle / \langle \beta_0, \beta_0\rangle,$$

and $\epsilon_{\alpha'} = (-1)^{n_{\alpha'}}$. By 4.3 of [17] again, we may replace Ψ_α by Ψ without changing $n_{\alpha'}$. But now β_0 is compact, so $2\langle \beta_0, \rho(\Psi \cap \Delta(\mathfrak{k}))\rangle / \langle \beta_0, \beta_0\rangle \in \mathbf{Z}$. So we need only show that in the present case,

$$2\langle \alpha_0, \rho(\Psi \cap \Delta(\mathfrak{k}))\rangle / \langle \alpha_0, \alpha_0\rangle \in \mathbf{Z}.$$

Set $p = \rho(\Psi \cap \Delta(\mathfrak{k}))$. Then

$$2\langle \alpha_0, p\rangle / \langle \alpha_0, \alpha_0\rangle = 2\langle \beta_0, p\rangle / \langle \beta_0, \beta_0\rangle + 2\langle (\alpha_0 - \beta_0), p\rangle / \langle \alpha_0, \alpha_0\rangle.$$

The first term is an integer; and the second may be written as

$$2\langle (\alpha_0 - \beta_0), 2p\rangle / \langle \alpha_0 - \beta_0, \alpha_0 - \beta_0\rangle.$$

Since $\alpha_0 - \beta_0$ is a root, and $2p$ is integral, this is an integer. So $\epsilon_\alpha = \epsilon_{\alpha'}$. Q.E.D.

We turn at last to the algorithm for computing the multiplicity of $\bar{\pi}(\gamma_1)$ in $\pi(\gamma)$, assuming as always that no non-integral roots for γ satisfy the parity condition. We may as well assume in addition that $\pi(\gamma_1)$ and $\pi(\gamma)$ have the same restriction to the center of G; otherwise the multiplicity is zero. We proceed by downward induction on $l(\gamma_1)$. Suppose first that there is a simple root $\alpha \in \Delta^+_{\gamma_1}$ with $\theta\alpha \in \Delta^+_{\gamma_1}$. Choose a weight μ of a finite dimensional representation of G such that $\lambda + \mu \in W(\mathfrak{G}/\mathfrak{h}) \cdot \{\lambda_i\}$, and $\gamma_1 + \mu_{\gamma_1}$ is dominant for $s_\alpha(\Delta^+_{\gamma_1})$: if α is integral, with $2\langle \alpha, \gamma_1\rangle / \langle \alpha, \alpha\rangle = n$, we take $\mu_{\gamma_1} = -n\alpha$; otherwise μ can be taken to be the difference of two of the μ_i chosen at the beginning of this section. Now $l(\gamma_1 + \mu_{\gamma_1}) = l(\gamma_1) + 1$, so the multiplicity of $\bar{\pi}(\gamma_1 + \mu_{\gamma_1})$ in $\pi(\gamma + \mu_\gamma)$ is known. If α is not integral, this is of course equal to the multiplicity of $\bar{\pi}(\gamma_1)$ in $\bar{\pi}(\gamma)$; so suppose α is integral. If $\tilde{\alpha} \in \tau(\bar{\pi}(\gamma))$ ($\tilde{\alpha}$ the root in Π_λ corresponding to α) then the multiplicity of $\bar{\pi}(\gamma_1)$ in $\pi(\gamma)$ is known by Lemma 5.2. So suppose $\tilde{\alpha} \notin \tau(\bar{\pi}(\gamma))$. Then as we have remarked before (proof of 4.12(a)) $S_{-n\tilde{\alpha}}(\Theta(\gamma)) = \Theta(\gamma)$. Write

$$\Theta(\gamma) = x_0\overline{\Theta}(\gamma_1) + \Sigma n_i\Theta_i^1 + \Sigma x_j\Theta_j^2$$
$$+ m_0\overline{\Theta}(\gamma_1 - n\alpha) + \Sigma m_i\Theta_i^3 + \Sigma y_j\Theta_j^4 .$$

Here the $\overline{\Theta}_i^l$ denote irreducible characters, distinct from each other and from $\overline{\Theta}(\gamma_1)$ and $\overline{\Theta}(\gamma_1 - n\alpha)$; the Θ_i^1 and Θ_j^2 are α-nonsingular, and the Θ_i^3 and Θ_j^4 are α-singular; the Θ_i^1 and Θ_i^3 correspond to γ' with $l(\gamma') > l(\gamma_1)$, and the Θ_j^2 and Θ_j^4 correspond to γ' with $l(\gamma') \leqslant l(\gamma_1)$. Thus the m_i and n_i are known, and the x_j and y_j are (possibly) unknown. If we apply $S_{-n\tilde{\alpha}}$ to this equation, we obtain

$$\Theta(\gamma) = \Theta(\gamma) + x_0 U_{\tilde{\alpha}}\big(\overline{\Theta}(\gamma_1)\big) + \Sigma n_i U_{\tilde{\alpha}}\big(\Theta_i^1\big) + \Sigma x_j U_{\tilde{\alpha}}\big(\Theta_j^2\big)$$
$$- 2m_0\overline{\Theta}(\gamma_1 - n\alpha) - 2\Sigma m_i\Theta_i^3 - 2\Sigma y_j\Theta_j^4.$$

Now look at the occurrences of $\overline{\Theta}(\gamma_1 - n\alpha)$ in this formula. It occurs exactly once in $U_{\tilde{\alpha}}(\overline{\Theta}(\gamma_1))$, a known number of times in the first sum, and $-2m_0$ times on the second line. If we can show that it does *not* occur in any $U_{\tilde{\alpha}}(\Theta_j^2)$, then obviously we can solve for x_0 (which is the multiplicity we want). So suppose $\overline{\Theta}(\gamma_1 - n\alpha)$ occurs in some $U_{\tilde{\alpha}}(\Theta_j^2)$; say $\Theta_j^2 = \overline{\Theta}(\gamma_2)$. Since $\overline{\Theta}(\gamma_2) \neq \overline{\Theta}(\gamma_1)$, $\overline{\Theta}(\gamma_1 - n\alpha)$ is *not* a special constituent of $U_{\tilde{\alpha}}(\overline{\Theta}(\gamma_2))$. So by Theorem 4.12, $\overline{\Theta}(\gamma_1 - n\alpha)$ occurs in $\Theta(\gamma_2)$. Now $l(\gamma_1 - n\alpha) = l(\gamma_1) + 1$, and $l(\gamma_2) \leqslant l(\gamma_1)$; so $l(\gamma_2) < l(\gamma_1 - n\alpha)$. But we have

LEMMA 5.5. *If $\overline{\Theta}(\gamma_1)$ is a composition factor of $\Theta(\gamma_2)$, then $l(\gamma_2) \geqslant l(\gamma_1)$. Equality holds if and only if $\overline{\Theta}(\gamma_1) = \overline{\Theta}(\gamma_2)$.*

This will be proved later. This contradiction shows that $\overline{\Theta}(\gamma_1 - n\alpha)$ in fact cannot occur in $U_{\tilde{\alpha}}(\Theta_j^2)$, and allows us to compute the desired multiplicity in the present case.

So we may assume that no such root α exists. As we saw in the proof of Lemma 5.4, this implies that if $\alpha \in \Delta^+_{\gamma_1}$ and $\theta\alpha \in \Delta^+_{\gamma_1}$, then α is imaginary. Write $B_1 = T_1^+A_1$, $M_1A_1 = G^{A_1}$. M_1 is split over R, and its simple roots may be regarded as a subset of those of G. The roots in $\Delta(\mathfrak{G}, \mathfrak{h})$ corresponding to $\Delta(\mathfrak{t}_1^+,$

$\mathfrak{M}_1$) are contained in Δ_λ^1 by Lemma 5.4; in particular they are all integral. Recall the subset D_γ of Π_λ^1 defined by Lemma 5.2; D_γ meets every simple factor of Π_λ^1 (by the definition of Π_λ^1), and $\Pi_\lambda^1 - D_\gamma$ consists precisely of the short simple roots in some simple components of Π_λ^1 of type B_n, C_n, or F_4 (by Lemma 5.3. To see this, we need only show that if α is a simple root of $\mathfrak{b}$ in $\mathfrak{M}_1$, then $\gamma(m_\alpha) = \pm 1$. This is true because m_α^2 lies in the center of G and in the identity component of M_1; so since $\pi(\gamma_1)$ and $\pi(\gamma)$ have the same restrictions to the center of G,

$$\begin{aligned} \gamma(m_\alpha^2) &= \pi(\gamma)(m_\alpha^2) \\ &= \pi(\gamma_1)(m_\alpha^2) \\ &= \pi_{(M_1)_0}(\gamma_1)(m_\alpha^2). \end{aligned}$$

Here $\pi_{(M_1)_0}(\gamma_1)$ is the discrete series representation of the identity component of M_1 with Harish-Chandra parameter $\bar{\gamma}_1|_{\mathfrak{t}_1^+}$, and the equalities mean that the operators are the same multiple of the appropriate identity operators. Since $\bar{\gamma}_1$ is integral for $\mathfrak{M}_1$, this representation actually lives on a linear quotient of $(M_1)_0$; and clearly $m_\alpha^2 = 1$ in such a quotient. So $\gamma(m_\alpha) = \pm 1$.) Suppose first that there is a noncompact simple root α such that the corresponding root $\tilde{\alpha} \in \Pi_\lambda^1$ lies in D_γ. Then $\tau(\bar{\pi}(\gamma_1))$ does not contain D_γ (since it omits $\tilde{\alpha}$), so the multiplicity of $\bar{\pi}(\gamma_1)$ in $\pi(\gamma)$ is known by Lemma 5.2. So suppose that there are no such roots. We claim that then $\mathfrak{M}_1$ is a product of copies of $\mathfrak{sl}(2)$. For let $\mathfrak{M}_2$ be a simple factor of $\mathfrak{M}_1$. Then the simple roots of $\mathfrak{M}_2$ correspond to some connected subset of a simple factor Π_λ^2 of Π_λ^1, and the noncompact roots correspond to elements of $\Pi_\lambda^2 - D_\lambda$ (and are therefore short). In particular this last set is non-empty, so Π_λ^2 is of type B_n, C_n, or F_4. In particular it has no branches, so neither does the Dynkin diagram of $\mathfrak{M}_2$. If the roots of $\mathfrak{M}_2$ all have the same length, then $(\mathfrak{M}_2)_0 \cong \mathfrak{sl}(n, \mathsf{R})$, which is not equal rank unless $n = 2$. So we may assume that $\mathfrak{M}_2$ has two root lengths. But then it is easy to see that there always exists a long noncompact simple root, contradicting our hypothesis that the noncompact simple roots correspond to $\Pi_\lambda^2 - D_\gamma$. So $\mathfrak{M}_1$ is in fact a product of copies of $\mathfrak{sl}(2)$, corresponding to simple roots *not* in $\tau(\bar{\pi}(\gamma))$. But we claim that $\pi(\gamma)$ cannot have such a composition factor (unless in fact $\mathfrak{M}_1 = 0$, in which case $B_1 = B$ and we are done). We proceed by induction on the number of copies of $\mathfrak{sl}(2)$. Choose some simple noncompact root α, corresponding to $\tilde{\alpha} \in \Pi_\lambda - \tau(\bar{\pi}(\gamma))$. Then $S_{-n\tilde{\alpha}}(\bar{\pi}(\gamma_1))$ contains a composition factor $\bar{\pi}(\gamma_2)$ which is again of the same type, but with $\tilde{\alpha} \in \tau(\bar{\pi}(\gamma_2))$, and one less copy of $\mathfrak{sl}(2)$ in the associated Levi factor (cf. 4.12(f)). Since $S_{-n\tilde{\alpha}}(\pi(\gamma)) = \pi(\gamma)$, $\bar{\pi}(\gamma_2)$ occurs in $\pi(\gamma)$. If $\mathfrak{M}_1$ had more than one $\mathfrak{sl}(2)$, this is impossible by induction. If it has exactly one, then γ_2 is associated to the split Cartan subgroup B, so we must have $\bar{\pi}(\gamma_2) \cong \bar{\pi}(\gamma)$. But $\alpha \in \tau(\bar{\pi}(\gamma_2))$, and $\alpha \notin \tau(\bar{\pi}(\gamma))$, a contradiction.

This completes the algorithm for computing composition series in terms of the U_α.

Proof of Lemma 5.5. We proceed by (upward) induction on $l(\gamma_2)$, proving the result for all (nonsingular) infinitesimal characters simultaneously. Suppose first that there is a simple complex root $\alpha \in \Delta^+_{\gamma_2}$ such that $\theta\alpha \notin \Delta^+_{\gamma_2}$. Choose a weight μ of a finite dimensional representation such that $\gamma_2 + \mu_{\gamma_2}$ is dominant and nonsingular for $s_\alpha(\Delta^+_\gamma)$; if α is integral, we take $\mu_{\gamma_2} = -n\alpha$, with $n = 2\langle\alpha, \gamma_2\rangle/\langle\alpha, \alpha\rangle$. Then $l(\gamma_2 + \mu_{\gamma_2} = l(\gamma_2) - 1$, so the result is known for $\pi(\gamma_2)$. If α is not integral, define γ_3 so that $S_\mu(\overline{\Theta}(\gamma_2)) = \overline{\Theta}(\gamma_3)$. By Corollary 4.8 and Lemma 4.9, $l(\gamma_3) = l(\gamma_1) \pm 1$. By inductive hypothesis, $l(\gamma_3) \leqslant l(\gamma_2 + \mu_{\gamma_2})$, with equality if and only if $\overline{\Theta}(\gamma_3) = \overline{\Theta}(\gamma_2 + \mu_{\gamma_2})$ (which is clearly equivalent to $\overline{\Theta}(\gamma_1) = \overline{\Theta}(\gamma_2)$). So

$$l(\gamma_1) \pm 1 \leqslant l(\gamma_2) - 1,$$

which implies $l(\gamma_1) \leqslant l(\gamma_2)$, with equality only if $l(\gamma_3) = l(\gamma_2 + \mu_{\gamma_2})$. If α is integral, then

$$S_{-n\alpha}(\Theta(\gamma_2 - n\alpha)) = \Theta(\gamma_2).$$

So either $\overline{\Theta}(\gamma_1)$ occurs in $\Theta(\gamma_2 - n\alpha)$ (in which case we are done by induction) or there is an α-nonsingular constituent $\overline{\Theta}(\gamma_3)$ of $\Theta(\gamma_2 - n\alpha)$ such that $\overline{\Theta}(\gamma_1)$ occurs in $U_\alpha(\overline{\Theta}(\gamma_3))$. Now $l(\gamma_3) \leqslant l(\gamma_2 - n\alpha) = l(\gamma_2) - 1$ by induction. So if $\overline{\Theta}(\gamma_1)$ is not a special constituent of $U_\alpha(\overline{\Theta}(\gamma_3))$, then $\overline{\Theta}(\gamma_1)$ occurs in $\Theta(\gamma_3)$, so $l(\gamma_1) \leqslant l(\gamma_3) < l(\gamma_2)$ by induction. If $\overline{\Theta}(\gamma_1)$ is a special constituent, then $l(\gamma_1) = l(\gamma_3) + 1$ by Theorem 4.12; so $l(\gamma_1) \leqslant l(\gamma_2)$. Equality holds only if $l(\gamma_3) = l(\gamma_2 - n\alpha)$, which implies $\overline{\Theta}(\gamma_3) = \overline{\Theta}(\gamma_2 - n\alpha)$ by induction. By Theorem 4.12, the only special constituent of $U_\alpha(\overline{\Theta}(\gamma_2 - n\alpha))$ is $\overline{\Theta}(\gamma_2)$; so $\overline{\Theta}(\gamma_1) = \Theta(\gamma_2)$.

So we may assume that no such root α exists. Arguing as at the beginning of this section, we find that the lemma can now be reduced to the case when G is split, and B_2 is the split Cartan subgroup. In this case $l(\gamma_2) = \frac{1}{2}\dim(G/K) \geqslant l(\gamma_1)$. Equality holds only if B_1 is also split. In this case Proposition 2.2 immediately implies that $\overline{\pi}(\gamma_2) \cong \overline{\pi}(\gamma_1)$. Q.E.D.

A few more remarks about the algorithm as a whole are in order. The first we state as a lemma; all hypotheses on G and γ have been dropped.

LEMMA 5.6. *Suppose $\gamma \in \hat{B}'$, $\overline{\pi}(\gamma)$ has nonsingular infinitesimal character λ, $\alpha \in \Pi_\lambda - \tau(\overline{\pi}(\gamma))$, and $\overline{\alpha} \in \Delta^+_\gamma$ is simple. Suppose that the composition factors and multiplicities are known for $\pi(\gamma)$, as well as for (in the case of 4.12(c)) $\pi(\gamma - n\alpha)$ or (in the case of 4.12(f)) $\pi(\gamma^{\overline{\alpha}}_{\pm})$ and $\pi(\gamma')$ ($\gamma' = (\Gamma - (n+1)\overline{\alpha}, s_\alpha\overline{\gamma})$). Suppose further that for every α-nonsingular constituent X of $\pi(\gamma)$ except $\overline{\pi}(\gamma)$ itself, the composition series of $U_\alpha(X)$ is known (i.e., composition factors and multiplicities.) Then the composition series of $U_\alpha(\overline{\pi}(\gamma))$ is computable.*

The easy proof is left to the reader. The point is that the algorithm we have described for computing composition series on the basis of a complete knowledge of the U_α has an enormous amount of redundancy in it. In practice one can compute U_α's and composition series simultaneously: if some U_α's can be computed using the various general theorems of sections 3 and 4, then some composition series can be computed. Lemma 5.7 then allows one to compute more U_α's, and hence more composition series. It may be that a general description of the U_α's can be given only as an algorithm of this kind.

6. A computational result and an example. It is clear that Theorem 3.9(c) is most useful when $U_\alpha(X)$ is known to be completely reducible. By 3.9(b), $\mathfrak{Z}(\mathfrak{G})$ acts by scalars on $U_\alpha(X)$; so the following result is sometimes helpful.

PROPOSITION 6.1. *Let X be an irreducible $(\mathfrak{G}, K)$ module with nonsingular infinitesimal character. Suppose there is a short exact sequence*

$$0 \to X \to E \to X \to 0$$

of $(\mathfrak{G}, K)$ modules, such that $\mathfrak{Z}(\mathfrak{G})$ acts by scalars on E. Then $E \cong X \oplus X$.

Proof. Say $X \cong \bar{\pi}(\gamma)$, with $\gamma \in \hat{B}'$. Write $B = T^+A$, $\bar{\gamma} = (\bar{\gamma}^+, \bar{\gamma}^-)$, with $\bar{\gamma}^+ \in (\mathfrak{t}^+)^*$ and $\bar{\gamma}^- \in \mathfrak{a}^*$. If μ is a weight of a finite dimensional representation such that $\gamma + \mu_\gamma$ is dominant and nonsingular for Δ_γ^+, then Zuckerman's "periodicity theory" (cf. [19]) allows us to replace γ by $\gamma + \mu_\gamma$. Accordingly we may as well assume that if $\alpha \in \Delta_\gamma^+$ and $\langle \alpha, \bar{\gamma}^+ \rangle = 0$, then α is real. Now we make use of the results of [17]. Define a θ-invariant parabolic subalgebra $\mathfrak{G} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{G}$, with $\mathfrak{l}$ containing $\mathfrak{b}$, by $\Delta(\mathfrak{u}, \mathfrak{b}) = \{\alpha \in \Delta(\mathfrak{G}, \mathfrak{b}) \mid \langle \alpha, \bar{\gamma}^+ \rangle > 0\}$. Choose a Cartan subalgebra $\mathfrak{t}_0$ of $\mathfrak{l}_0 \cap \mathfrak{k}_0$; then $\mathfrak{t}_0$ is also a Cartan subalgebra of $\mathfrak{k}_0$. Fix a positive root system for $\mathfrak{t}$ in $\mathfrak{k}$ compatible with $\mathfrak{u} \cap \mathfrak{k}$. Let $\mu \in \mathfrak{t}^*$ be the highest weight of a lowest K-type of X. Let $2\rho(\mathfrak{u} \cap \mathfrak{p})$ be the sum of the roots of $\mathfrak{t}$ in $\mathfrak{u} \cap \mathfrak{p}$. Section 7 of [17] describes how to compute $\mu - 2\rho(\mathfrak{u} \cap \mathfrak{p})$ from γ. This description involves a representation $\delta \in \hat{T}^+$, which is a certain fixed twist of $\gamma|_{T^+}$ by a one dimensional character of T^+. Possibly passing to a covering group of G and again shifting γ by a weight of a finite dimensional representation, we may obviously assume that $\delta(m_\alpha) \neq -1$, for every simple root α of $\mathfrak{b}$ in $\mathfrak{l}$. In this case, the results of Section 6 of [17] show that the stabilizer W_δ of δ in the Weyl group W_0 generated by reflections about real roots, is W_0 itself. Let E^μ denote the highest weight space of the μ K-type in E, which has dimension two. E^μ is a module for the centralizer $U(\mathfrak{G})^K$ of K in $\mathfrak{G}$. By the results of Lepowsky and McCollum in [13], the result of the proposition is equivalent to the assertion that $U(\mathfrak{G})^K$ acts semisimply on E^μ. This we now prove.

Put $R = \dim(\mathfrak{u} \cap \mathfrak{p})$, and $F = H^R(\mathfrak{u}, E)$; then F is an $(\mathfrak{l}, \mathfrak{l} \cap \mathfrak{k})$ module of finite length (cf. [17], Corollary 3.10). Let $F^{\mu - 2\rho(\mathfrak{u} \cap \mathfrak{p})}$ be the highest weight space of the $\mathfrak{l} \cap \mathfrak{k}$-type $\mu - 2\rho(\mathfrak{u} \cap \mathfrak{p})$ of F. By the results of [17], there is a natural bijection $E^\mu \cong F^{\mu - 2\rho(\mathfrak{u} \cap \mathfrak{p})}$. (Here we are using the fact that μ is strongly $\mathfrak{u}$-minimal in E. This is proved for X in [17], and the result for E follows trivially since E and X have the same set of K-types.) Furthermore there is a map

$\tilde{\xi} : U(\mathfrak{G})^K \rightarrow U(\mathfrak{l})^{\mathfrak{l} \cap \mathfrak{k}}$ which relates the $U(\mathfrak{G})^K$ structure of E^μ to the $U(\mathfrak{l})^{\mathfrak{l}\cap\mathfrak{k}}$ structure of $F^{\mu-2\rho(\mathfrak{u}\cap\mathfrak{p})}$. By Proposition 6.8 of [17], the action of $U(\mathfrak{l})^{\mathfrak{l}\cap\mathfrak{k}}$ on $F^{\mu-2\rho(\mathfrak{u}\cap\mathfrak{p})}$ factors through a certain homomorphism

$$U(\mathfrak{l})^{\mathfrak{l}\cap\mathfrak{k}} \rightarrow S(\mathfrak{b})^{W_\delta};$$

here $W_\delta = W_0$ is the Weyl group generated by reflections about real roots by our assumptions on γ. By Harish-Chandra's theorem, the restriction of this map to $\mathfrak{Z}(\mathfrak{l})$ is already surjective. We write $\bar{\xi} : U(\mathfrak{G})^K \rightarrow S(\mathfrak{b})^{W_0}$ for the composite map. The restriction of $\bar{\xi}$ to $\mathfrak{Z}(\mathfrak{G})$ is nothing but the Harish-Chandra homomorphism $\xi : \mathfrak{Z}(\mathfrak{G}) \rightarrow S(\mathfrak{b})^W$. So we may regard E^μ as a module for $S(\mathfrak{b})^{W_0}$, on which $S(\mathfrak{b})^W$ acts by scalars (namely evaluation at a nonsingular element of $\mathfrak{b}^*$). What we want to show is that $S(\mathfrak{b})^{W_0}$ acts semisimply on E^μ. Clearly it suffices to show that $S(\mathfrak{b})^{W_0}$ acts semisimply on $S(\mathfrak{b})^{W_0}/S(\mathfrak{b})^{W_0}\cdot(S(\mathfrak{b})^W_\gamma)$, where $S(\mathfrak{b})^W_\gamma$ is the kernel of infinitesimal character χ_γ. But this is an immediate consequence of a theorem of Harish-Chandra (cf. [18], Theorem 2.1.3.6). Q.E.D.

Clearly this argument has a lot more mileage in it, and a complete description of the $U_\alpha(X)$ may require a more refined version of Proposition 6.1. Probably an analogous proof exists in terms of the Langlands classification, and this might be a more reasonable point of view.

One general remark on the usefulness of the methods of this paper may be helpful in describing the extent to which the methods are incomplete. Suppose B is a θ-stable Cartan subgroup of G, $\gamma \in \hat{B}'$, $\pi(\gamma)$ has nonsingular infinitesimal character λ, $\alpha \in \Pi_\lambda - \tau(\bar{\pi}(\gamma))$, and the corresponding root $\bar{\alpha} \in \Delta^+_\gamma$ is actually simple. Suppose further that for every irreducible constituent Y of $\pi(\gamma)$, it is *false* that $\tau(Y) \supseteq \tau(\bar{\pi}(\gamma)) \cup \{\alpha\}$. Then $U_\alpha(\bar{\pi}(\gamma))$ is computable—it is completely reducible, and all its constituents have multiplicity one. For let Z be an irreducible $(\mathfrak{G}, K)$ module with infinitesimal character λ; we want to determine the multiplicity of Z in $U_\alpha(\bar{\pi}(\gamma))$. If Z is one of the special constituents of $U_\alpha(\bar{\pi}(\gamma))$, this multiplicity is one (cf. Theorem 4.12). So suppose Z is not a special constituent. If $\alpha \notin \tau(Z)$, then of course Z cannot occur in $U_\alpha(\bar{\pi}(\gamma))$; so assume $\alpha \in \tau(Z)$. If $\tau(Z) \supseteq \tau(\bar{\pi}(\gamma))$, then by hypothesis Z is not a constituent of $\pi(\gamma)$; so Theorem 4.12 implies that Z does not occur in $U_\alpha(\bar{\pi}(\gamma))$. So we may assume $\tau(Z) \not\supseteq \tau(\bar{\pi}(\gamma))$. Pick a root $\beta \in \tau(\pi(\gamma))$ with $\beta \notin \tau(Z)$. By Theorem 4.14, the multiplicity of Z in $U_\alpha(\bar{\pi}(\gamma))$ is one or zero according as $\bar{\pi}(\gamma)$ is or is not a special constituent of $U_\beta(Z)$, which is computable.

So difficulties arise only when $\pi(\gamma)$ has a constituent Y with $\tau(Y) \supseteq \tau(\bar{\pi}(\gamma)) \cup \{\alpha\}$. (Even in this case, the redundancy of the algorithm of section 5 often allows one to make computations.) A little thought will convince the reader that this situation cannot arise very easily. Perhaps the simplest example in which it does occur is $G = SP(3, 1)$, and we conclude this paper with some computations in that example.

So suppose $G = SP(3, 1)$. In this case $\mathfrak{G} = \mathfrak{sp}(4)$. We can take $\mathfrak{h} \cong \mathbf{C}^4$ with the usual basis $\{e_1, e_2, e_3, e_4\}$; then we can take for positive roots $\{2e_i, e_j \pm e_k$

$(j < k)\}$. The simple roots are then $\alpha = e_1 - e_2$, $\beta = e_2 - e_3$, $\gamma = e_3 - e_4$, and $\delta = 2e_4$. We will consider only representations with same infinitesimal character as the trivial representation, namely $\lambda = (4, 3, 2, 1)$. The Killing form (up to a constant) restricts to the usual bilinear form $\langle e_i, e_j \rangle = \delta_{ij}$ on $\mathbb{C}^4$. The Weyl group W is the group of permutations and sign changes of the e_i.

There are two conjugacy classes of Cartan subgroups, namely the compact ones and the noncompact ones; fix θ-stable representatives B_1 and B_2. We may identify $\mathfrak{b}_1$ with $\mathfrak{h}$ in such a way that the roots corresponding to compact roots are $\pm 2e_i$ and $\pm e_i \pm e_j$ $(2 \leqslant i, j \leqslant 4)$. Since B_1 is connected, an element γ of $\hat{B}_1'$ is determined by $\bar{\gamma}$. Those γ corresponding to discrete series with infinitesimal character λ are simply those with $\bar{\gamma} = w \cdot \lambda$ for some $w \in W$; and $w \cdot \lambda$ and $w' \cdot \lambda$ give rise to the same discrete series if and only if $w^{-1}w' \in W(G/B_1)$. Now $W(G/B_1)$ is the Weyl group of the system of compact roots, and therefore consists of sign changes, and permutations of the last three coordinates. Hence it suffices to consider

$$\gamma(1, 1) = (4, 3, 2, 1)$$

$$\gamma(1, 2) = (3, 4, 2, 1)$$

$$\gamma(1, 3) = (2, 4, 3, 1)$$

$$\gamma(1, 4) = (1, 4, 3, 2).$$

Here the first 1 (in $\gamma(1, *)$) refers to the Cartan subgroup B_1. In this case B_2 is connected as well, so the elements of $\hat{B}_2'$ are also identified with their differentials. We identify $\mathfrak{b}_2$ with $\mathfrak{h}$ in such a way that $e_1 - e_2$ corresponds to a real root (and therefore $e_1 + e_2$, $e_3 \pm e_4$, $2e_3$, and $2e_4$ correspond to compact imaginary roots). $W(G/B2)$ contains the reflections about both real and imaginary roots, so we need to consider only the following pseudocharacters:

$$\begin{array}{ll}
\gamma(2, 1) = (4, 3, 2, 1) & \gamma(2, 7) = (4, 1, 3, 2) \\
\gamma(2, 2) = (3, 2, 4, 1) & \gamma(2, 8) = (3, -1, 4, 2) \\
\gamma(2, 3) = (2, 1, 4, 3) & \gamma(2, 9) = (4, -1, 3, 2) \\
\gamma(2, 4) = (4, 2, 3, 1) & \gamma(2, 10) = (3, -2, 4, 1) \\
\gamma(2, 5) = (3, 1, 4, 2) & \gamma(2, 11) = (4, -2, 3, 1) \\
\gamma(2, 6) = (2, -1, 4, 3) & \gamma(2, 12) = (4, -3, 2, 1).
\end{array}$$

Defining the "length" $l(\gamma)$ as in Section 5, one checks easily that the first three have length 1, the next three have length 2, the next two have length 3, the next two have length 4, the next has length 5, and the last has length 6; for if $(x, y, z, w) \in \mathfrak{b}_2^*$, then $\theta(x, y, z, w) = (y, x, z, w)$. Similarly one can calculate τ-invariants, using Corollary 4.13. Using the remarks preceding this example and the algorithm of Section 5, one can begin to compute the various $U_\alpha(X)$, and thus composition series (by induction on $l(\gamma)$). For convenience we write $\pi(\gamma(i, j)) = \pi(i, j)$, $\overline{\Theta}(\gamma(i, j)) = \overline{\Theta}(i, j)$, etc. The results of this computation are given in Table 6.2. There are no difficulties for $l(\gamma) \leqslant 3$; we will do just one case as an example. Suppose that composition series are known for $l(\gamma) \leqslant 1$, and we wish to compute that of $\Theta(2, 4)$. Since $\gamma(2, 4) = (4, 2, 3, 1) = (3, 2, 4, 1) - (e_3 - e_1)$, and $e_3 - e_1$ corresponds to α in $\Delta^+_{\gamma(2,2)}$ ($\gamma(2, 2) = (3, 2, 4, 1)$), we have $\Theta(2, 4) = S_{-\alpha}(\Theta(2, 2))$. Now $l(\gamma(2, 2)) = 1$, so the composition series of $\Theta(2, 2)$ is known—by Table 6.2,

$$\Theta(2, 2) = \overline{\Theta}(2, 2) + \overline{\Theta}(1, 2) + \overline{\Theta}(1, 3). \tag{$*$}$$

Now we compute $S_{-\alpha}$ applied to each summand separately. By Theorem 4.12, $\alpha \notin \tau(\overline{\Theta}(2, 2))$, and the special constituent of $U_\alpha(\overline{\Theta}(2, 2))$ is $\overline{\Theta}(2, 4)$. All other constituents must occur in $\Theta(2, 2)$, and have α in their τ invariant. The only candidate is $\overline{\Theta}(1, 3)$. Now $\beta \notin \tau(\overline{\Theta}(1, 3))$, and $\beta \in \tau(\overline{\Theta}(2, 2))$; so by Theorem 4.14, the multiplicity of $\overline{\Theta}(1, 3)$ in $U_\alpha(\overline{\Theta}(2, 2))$ is the multiplicity of $\overline{\Theta}(2, 2)$ in $U_\beta(\overline{\Theta}(1, 3))$. But one checks easily that $\overline{\Theta}(2, 2)$ is a special constituent of $U_\beta(\overline{\Theta}(1, 3))$, so this multiplicity is one. So

$$U_\alpha\big(\overline{\Theta}(2, 2)\big) = \overline{\Theta}(2, 4) + \overline{\Theta}(1, 3),$$

and

$$S_{-\alpha}\big(\overline{\Theta}(2, 2)\big) = \overline{\Theta}(2, 2) + \overline{\Theta}(2, 4) + \overline{\Theta}(1, 3).$$

By a similar but much easier argument,

$$S_{-\alpha}\big(\overline{\Theta}(1, 2)\big) = \overline{\Theta}(1, 2) + \overline{\Theta}(2, 1).$$

Finally, $\alpha \in \tau(\overline{\Theta}(1, 3))$, so

$$S_{-\alpha}\big(\overline{\Theta}(1, 3)\big) = -\overline{\Theta}(1, 3).$$

Adding the last three equations, and recalling ($*$), we have

$$\Theta(2, 4) = S_{-\alpha}(\Theta(2, 2)) = \overline{\Theta}(2, 4) + \overline{\Theta}(2, 2) + \overline{\Theta}(2, 1) + \overline{\Theta}(1, 2),$$

as claimed in Table 6.2.

Difficulties arise when we try to compute the composition series of $\Theta(2, 9)$; $l(\gamma(2, 9)) = 4$, and we assume Table 6.2 is known for $l(\gamma) < 4$. Arguing as above, we find that $\Theta(2, 9) = S_{-\alpha}(\Theta(2, 8))$. When we try to compute $U_\alpha(\overline{\Theta}(2, 8))$, however, we find that $\Theta(2, 8)$ has a composition factor $\overline{\Theta}(1, 4)$, with $\tau(\overline{\Theta}(1, 4)) \supseteq \tau(\overline{\Theta}(2, 8)) \cup \{\alpha\}$. So Theorem 4.14 does not tell us the multiplicity of $\overline{\Theta}(1, 4)$ in $U_\alpha(\overline{\Theta}(2, 8))$. The right way to deal with this problem is very simple: we simply observe that it is also true that $\Theta(2, 9) = S_{-\delta}(\Theta(2, 7))$, and compute as above; one easily verifies the result given in Table 6.2 for the composition series of $\Theta(2, 9)$. Armed with this information, we could if we wished use Lemma 5.6 to compute $U_\alpha(\overline{\Theta}(2, 8))$. However, there exist still more complicated examples (the simplest I know involves $SP(3, \mathbb{R})$) in which this method does not

TABLE 6.2

Composition series for $SP(3, 1)$

$$\Theta(2, 1) = \overline{\Theta}(2, 1) + \overline{\Theta}(1, 1) + \overline{\Theta}(1, 2)$$

$$\Theta(2, 2) = \overline{\Theta}(2, 2) + \overline{\Theta}(1, 2) + \overline{\Theta}(1, 3)$$

$$\Theta(2, 3) = \overline{\Theta}(2, 3) + \overline{\Theta}(1, 3) + \overline{\Theta}(1, 4)$$

$$\Theta(2, 4) = \overline{\Theta}(2, 4) + \overline{\Theta}(2, 2) + \overline{\Theta}(2, 1) + \overline{\Theta}(1, 2)$$

$$\Theta(2, 5) = \overline{\Theta}(2, 5) + \overline{\Theta}(2, 3) + \overline{\Theta}(2, 2) + \overline{\Theta}(1, 3)$$

$$\Theta(2, 6) = \overline{\Theta}(2, 6) + \overline{\Theta}(2, 3)$$

$$\Theta(2, 7) = \overline{\Theta}(2, 7) + \overline{\Theta}(2, 5) + \overline{\Theta}(2, 4) + \overline{\Theta}(2, 2)$$

$$\Theta(2, 8) = \overline{\Theta}(2, 8) + \overline{\Theta}(2, 6) + \overline{\Theta}(2, 5) + \overline{\Theta}(2, 3) + \overline{\Theta}(1, 4)$$

$$\Theta(2, 9) = \overline{\Theta}(2, 9) + \overline{\Theta}(2, 8) + \overline{\Theta}(2, 7) + \overline{\Theta}(2, 5)$$

$$\Theta(2, 10) = \overline{\Theta}(2, 10) + \overline{\Theta}(2, 8) + \overline{\Theta}(1, 4)$$

$$\Theta(2, 11) = \overline{\Theta}(2, 11) + \overline{\Theta}(2, 10) + \overline{\Theta}(2, 9) + \overline{\Theta}(2, 8) + \overline{\Theta}(2, 6)$$

$$\Theta(2, 12) = \overline{\Theta}(2, 12) + \overline{\Theta}(2, 11) + \overline{\Theta}(2, 6)$$

work; so we describe a direct approach to calculate the multiplicity of $\overline{\Theta}(1, 4)$ in $U_\alpha(\overline{\Theta}(2, 8))$. (The reason for using this example rather than the one for $SP(3, \mathbf{R})$ should be obvious: $SP(3, \mathbf{R})$ has a great many more representations, and would therefore take much longer to describe.)

By Theorems 4.12 and 4.14,

$$U_\alpha\big(\overline{\Theta}(2, 8)\big) = \overline{\Theta}(2, 9) + \overline{\Theta}(2, 6) + X \cdot \Theta(1, 4).$$

We claim that $\bar{\pi}(2, 6)$ is a direct summand of $U_\alpha(\bar{\pi}(2, 8))$. By Corollary 3.19, it suffices to show that

$$\operatorname{Hom}_{\mathfrak{G}, K}(\bar{\pi}(2, 6), U_\alpha(\bar{\pi}(2, 8))) = \mathbf{C}.$$

But by Theorem 3.9(c),

$$\begin{aligned}\operatorname{Hom}_{\mathfrak{G}, K}(\bar{\pi}(2, 6), U_\alpha(\bar{\pi}(2, 8))) &\cong \operatorname{Ext}^1(\bar{\pi}(2, 6), \bar{\pi}(2, 8)) \\ &\cong \operatorname{Hom}_{\mathfrak{G}, K}(U_\beta(\bar{\pi}(2, 6)), \bar{\pi}(2, 8)) = \mathbf{C}\end{aligned}$$

since $U_\beta(\bar{\pi}(2, 6))$ is completely reducible (by the remarks before this example) and contains $\bar{\pi}(2, 8)$ (by Theorem 4.12).

Next, we claim that $\bar{\pi}(1, 4)$ occurs exactly once as a subrepresentation of $U_\alpha(\bar{\pi}(2, 8))$. By Theorem 3.9(c), we must show that $\mathrm{Ext}^1(\bar{\pi}(1, 4), \bar{\pi}(2, 8)) = \mathbf{C}$. Now $\beta \in \tau(\bar{\pi}(1, 4))$; and it follows from Theorems 4.14 and 4.12 that $\bar{\pi}(2, 8) = U_\beta(\bar{\pi}(2, 6))$. Finally

$$\mathrm{Ext}^1(\bar{\pi}(1, 4), \bar{\pi}(2, 6)) \cong \mathrm{Hom}_{\mathfrak{G}, K}(\bar{\pi}(1, 4), U_\beta(\bar{\pi}(2, 6))) = 0$$

by Theorem 3.9(c) again. So by Theorem 3.9(c) with $i = 1$, we obtain

$$\mathrm{Ext}^1(\bar{\pi}(1, 4), \bar{\pi}(2, 8)) \cong \mathrm{Ext}^0(\bar{\pi}(1, 4), \bar{\pi}(2, 6)) \oplus \mathrm{Ext}^2(\bar{\pi}(1, 4), \bar{\pi}(2, 6)).$$

The first summand on the right is zero since $\bar{\pi}(1, 4) \ncong \bar{\pi}(2, 6)$. Using the root γ, which is in $\tau(\bar{\pi}(2, 6))$ but not in $\tau(\bar{\pi}(1, 4))$, a similar argument proves that

$$\mathrm{Ext}^2(\bar{\pi}(1, 4), \bar{\pi}(2, 6)) \cong \mathrm{Ext}^1(\bar{\pi}(2, 3), \bar{\pi}(2, 6)).$$

Now $\delta \notin \tau(\bar{\pi}(2, 3))$, and $\delta \in \tau(\bar{\pi}(2, 6))$; so by Theorem 3.9(c) again,

$$\mathrm{Ext}^1(\bar{\pi}(2, 3), \bar{\pi}(2, 6)) \cong \mathrm{Hom}_{\mathfrak{G}, K}(U_\delta(\bar{\pi}(2, 3)), \bar{\pi}(2, 6)).$$

But by the remarks before this example and Theorems 4.12 and 4.14, $U_\delta(\bar{\pi}(2, 3))$ is completely reducible and contains $\bar{\pi}(2, 6)$ once. So the last Hom has dimension 1. Assembling all this, we find that $\mathrm{Ext}^1(\bar{\pi}(1, 4), \bar{\pi}(2, 8)) = \mathbf{C}$ as claimed.

If we knew that $U_\alpha(\bar{\pi}(2, 8))$ was completely reducible, we would now be done; $\bar{\pi}(1, 4)$ would occur exactly once in $U_\alpha(\bar{\pi}(2, 8))$. Since we do not know this, an additional argument is necessary. First we claim that $\bar{\pi}(2, 9)$ is a direct summand of $U_\alpha(\bar{\pi}(2, 8))$. Just as for $\bar{\pi}(2, 6)$, it is enough to show that $\mathrm{Ext}^1(\bar{\pi}(2, 9), \bar{\pi}(2, 8)) = \mathbf{C}$. Just as for $\bar{\pi}(1, 4)$ one finds that

$$\mathrm{Ext}^1(\bar{\pi}(2, 9), \bar{\pi}(2, 8)) \cong \mathrm{Ext}^2(\bar{\pi}(2, 9), \bar{\pi}(2, 6)).$$

(We need to know that $\mathrm{Ext}^1(\bar{\pi}(2, 9), \bar{\pi}(2, 6)) = 0$; but this is clear since $\beta \notin \tau(\bar{\pi}(2, 6))$, $\beta \in \tau(\bar{\pi}(2, 9))$, and $\bar{\pi}(2, 9)$ does not occur in $U_\beta(\bar{\pi}(2, 6))$.) Since $\gamma \notin \tau(\bar{\pi}(2, 9))$ and $\gamma \in \tau(\bar{\pi}(2, 6))$,

$$\mathrm{Ext}^2(\bar{\pi}(2, 9), \bar{\pi}(2, 6)) \cong \mathrm{Ext}^1(U_\gamma(\bar{\pi}(2, 9)), \bar{\pi}(2, 6)).$$

But by Theorem 4.14, $U_\gamma(\bar{\pi}(2, 9)) \cong \bar{\pi}(2, 11) \oplus \bar{\pi}(2, 7)$. Now $\beta \notin \tau(\bar{\pi}(2, 6))$, $\beta \in \tau(\bar{\pi}(2, 7))$, and $\bar{\pi}(2, 7)$ does not occur in $U_\beta(\bar{\pi}(2, 6))$; so $\mathrm{Ext}^1 \cdot (\bar{\pi}(2, 7), \bar{\pi}(2, 6)) = 0$. So we must show $\mathrm{Ext}^1(\bar{\pi}(2, 11), \bar{\pi}(2, 6)) = \mathbf{C}$. We have $\alpha \in \tau(\bar{\pi}(2, 6))$, $U_\alpha(\bar{\pi}(2, 10)) = \bar{\pi}(2, 11)$, and $\mathrm{Ext}^1(\bar{\pi}(2, 10), \bar{\pi}(2, 6)) = 0$, as is easily verified; so Theorem 3.9(c) implies that

$$\mathrm{Ext}^1(\bar{\pi}(2, 11), \bar{\pi}(2, 6)) \cong \mathrm{Ext}^2(\bar{\pi}(2, 10), \bar{\pi}(2, 6)).$$

A similar argument using β shows that

$$\mathrm{Ext}^2(\bar{\pi}(2, 10), \bar{\pi}(2, 6)) \cong \mathrm{Ext}^1(\bar{\pi}(2, 10), \bar{\pi}(2, 8))$$

(since $\bar{\pi}(2, 8) \cong U_\beta(\bar{\pi}(2, 6))$). Using γ, which is in $\tau(\bar{\pi}(2, 10))$ but not in $\tau(\bar{\pi}(2, 8))$, this becomes

$$\mathrm{Hom}_{\mathfrak{G}, K}(\bar{\pi}(2, 10), U_\gamma(\bar{\pi}(2, 8))).$$

By Theorems 4.12 and 4.14, $U_\gamma(\bar{\pi}(2, 8)) \cong \bar{\pi}(2, 5) \oplus \bar{\pi}(2, 6)$, so the Hom has dimension 1. This proves that $\bar{\pi}(2, 9)$ is a direct summand of $U_\alpha(\bar{\pi}(2, 8))$. We have shown that

$$U_\alpha(\bar{\pi}(2, 8)) \cong \bar{\pi}(2, 6) \oplus \bar{\pi}(2, 9) \oplus X,$$

where $\Theta(X) = x \cdot \Theta(1, 4)$, and X contains $\bar{\pi}(1, 4)$ exactly once as a subrepresentation. But $\mathfrak{Z}(\mathfrak{G})$ acts by scalars on X by 3.9(b); so by Proposition 6.1, X is completely reducible. So $x = 1$. The rest of Table 6.2 is easy to compute.

Table 6.2 (and in fact composition series for all rank one groups) was first obtained by Wallach (unpublished). It is a triviality to invert the formulas of 6.2 to obtain formulas for the $\bar{\Theta}(\gamma)$ in terms of the $\Theta(\gamma)$. It is not obvious that the present derivation of the results is easier than Wallach's; the point is simply to illustrate the results of this paper.

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