Families of Nearly Ordinary Eisenstein Series on Unitary Groups (with an Appendix by Kai-Wen Lan)

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Abstract

In this paper we use the doubling method to construct p-adic L-functions and families of nearly ordinary Klingen Eisenstein series from nearly ordinary cusp forms on unitary groups of signature (r, s) and Hecke characters, and prove the constant terms of these Eisenstein series are divisible by the p-adic L-function, following earlier constructions of Eischen-Harris-Li-Skinner and Skinner-Urban. We also make preliminary computations for the Fourier-Jacobi coefficients of the Eisenstein series. This provides a framework to do Iwasawa theory for cusp forms on unitary groups.

1 Introduction

Let p be an odd prime. Let $\mathcal{K}$ be a CM field with the maximal totally real subfield F such that $[F : \mathbb{Q}] = d$. Suppose p is totally split at $\mathcal{K}$. We fix an isomorphic $\iota_p := \mathbb{C}_p \simeq \mathbb{C}$ and a CM type Σ_∞, which means a set of d different embeddings $\mathcal{K} \to \mathbb{C}$ such that $\Sigma_\infty \cup \Sigma_\infty^c$ is the set of all embeddings of $\mathcal{K}$ into $\mathbb{C}$ where c means complex conjugation. This determines a set of embeddings $\mathcal{K} \hookrightarrow \mathbb{C}_p$ using ι_p which we denote as Σ_p. Let $r \geq s \geq 0$ be integers. We often write $a = r - s$ and $b = s$. Let $\mathrm{U}(r, s)$ be the unitary group associated to the skew-Hermitian matrix $\begin{pmatrix} & & 1_b \\ & \zeta & \\ -1_b & & \end{pmatrix}$ where ζ is a diagonal matrix such that $i^{-1}\zeta$ is positive definite.

In [2] the authors constructed the p-adic L-function for an irreducible cuspidal automorphic representation of $\mathrm{U}(r, s)$ that is nearly ordinary at all primes dividing p, which interpolates (the algebraic part of) critical values of the standard L-function of the representation twisted by general CM characters at far from center critical points. The main tool used in *loc.cit* is the doubling method of Piatetski-Shapiro and Rallis. This paper can be thought of as a continuation of their work, but instead using a more general pullback formula of Shimura (which is actually due to Garrett [5], [6] and is called the "Garrett map") to construct p-adic families of Klingen Eisenstein series on $\mathrm{U}(r+1, s+1)$ from the original automorphic representation.

The motivation for doing this is to provide a framework to generalize the important work of Skinner-Urban [28] on Iwasawa main conjectures for GL_2 to forms on general unitary groups. The general strategy is start with a family of cuspforms on the unitary group $\mathrm{U}(r, s)$ and a family of CM characters, we construct a family of Klingen Eisensstein series on the bigger group $\mathrm{U}(r+1, s+1)$. One tries to prove the constant terms of the Klingen Eisenstein family is divisible by the standard p-adic L-function of the cuspforms on $\mathrm{U}(r, s)$ and therefore, the Eisenstein family is congruent to cuspidal families modulo this p-adic L-function. Passing to the Galois side such congruences enable us to construct elements in the Selmer groups, proving one divisibility of the corresponding Iwasawa main conjecture. We have been able to use it to prove one divisibility of the Iwasawa main conjectures for Hilbert modular forms and some kinds of Rankin-Selberg p-adic L-functions, see [30], [31]. C. Skinner has recently been able to use the result of [30] to prove a converse of a theorem of Gross-Zagier-Kolyvagin which states that if the rank of the Selmer group of an elliptic curve is one and the p-part of the Shafarevich-Tate group is finite, then the Heegner point is non-torsion and the central L-value vanishes at order exactly one ([27]). The first step towards the plan outlined above, is to construct the family of Klingen-Eisenstein series and studying the p-adic properties of its Fourier-Jacobi coefficients, which is the main task of the present paper.

In [2] the interpolation formulas are proved at all arithmetic points. However in this paper we are only able to understand the pullback Eisenstein sections in the "generic case" (to be defined in Definition 4.42, basically this puts restrictions on the ramification of the form at primes dividing p). The reason is it seems difficult in general to describe the nearly ordinary Klingen Eisenstein sections. Fortunately, since along a Hida family, the set of forms that are "generic" is Zariski dense, these computations are enough to construct the whole Hida family of Klingen Eisenstein series (similar to the [28] case). Thus we only work with a Hida family of forms instead of a single cusp form due to this "generic" condition. We remark that when $s=0$ by working with forms of general vector-valued weights, we are able to construct a class of the p-adic L-function and Klingen Eisenstein family for a single form unramified at p (not necessarily ordinary, see [3]).

Now we state the main results. Let $\mathcal{K}_\infty$ be the maximal abelian pro-p-extension of $\mathcal{K}$ unramified outside p. We write $\Gamma_\mathcal{K}=\mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K})$. This is a free $\mathbb{Z}_p$-module whose rank should be $d+1$ assuming the Leopoldt conjecture. Take a finite extension L over $\mathbb{Q}_p$. Let $\mathcal{O}_L$ be the integer ring of L. Let $\mathcal{O}_L^{ur}$ be the completion of the integer ring of the maximal unramfied extension of L. We define $\Lambda_\mathcal{K}=\mathcal{O}_L[[\Gamma_\mathcal{K}]]$. Let $\kappa>4$ be an integer and τ_0 a Hecke character of $\mathcal{K}^\times\backslash\mathbb{A}_\mathcal{K}^\times$ whose infinite types are $(-\frac{\kappa}{2},\frac{\kappa}{2})$ at all infinite places. We have a $\Lambda_\mathcal{K}$-valued family of Hecke characters of $\mathcal{K}^\times\backslash\mathbb{A}_\mathcal{K}^\times$ containing τ_0 as a specialization (to be precise later). Let Λ be the weight algebra for $\mathrm{U}(r,s)$ defined later and $\mathbb{I}$ a normal domain containing Λ which is finite over Λ. Let $\mathbb{I}^{ur}$ be the normalization of an irreducible component of $\mathbb{I}\hat{\otimes}_{\mathcal{O}_L}\mathcal{O}_L^{ur}$. (In fact for each such irreducible component we can make the following construction). Let $\Omega_\infty\in\mathbb{C}^{\Sigma_\infty}$ be the CM period of the CM field $\mathcal{K}$ and $\Omega_p\in(\mathbb{Z}_p^{ur})^{\Sigma_\infty}$ be the p-adic period (we refer to [11] for the definition). We write $\Omega_\infty^{\Sigma_\infty}$ for the product of the d elements of Ω_∞ and define $\Omega_p^{\Sigma_\infty}$ similarly. Throughout this paper we write $z_\kappa=\frac{\kappa-r-s-1}{2}$, $z'_\kappa=\frac{\kappa-r-s}{2}$.

Theorem 1.1. *Let* $\mathbf{f}$ *be an* $\mathbb{I}$*-coefficient nearly ordinary cuspidal eigenform on* $\mathrm{GU}(r,s)$ *such that the specialization* $\mathbf{f}_\phi$ *at a Zariski dense set of "generic" arithmetic points* ϕ *is classical and generates an irreducible automorphic representation of* $\mathrm{U}(r,s)$. *Let* Σ *be a finite set of primes containing all primes dividing the any entry of* ζ *or the conductor of* $\mathbf{f}$ *or* $\mathcal{K}$. *In the case when* $s\neq 0$ *we make the assumptions* TEMPERED, $\mathrm{Proj}_{\mathbf{f}^\vee}$ *and* DUAL*; or assumptions* TEMPERED, $\mathrm{Proj}_\mathbf{f},\mathrm{Proj}_{\mathbf{f}^\vee}$ *(to be defined in Subsection* 5.1*). Then*

(i) There is an element $\mathcal{L}_{\mathbf{f},\tau_0}^\Sigma\in\mathbb{I}^{ur}[[\Gamma_\mathcal{K}]]\otimes_{\mathbb{I}^{ur}}F_{\mathbb{I}^{ur}}$ *such that for a Zariski dense subset of arithmetic points* $\phi\in\mathrm{Spec}\mathbb{I}^{ur}[[\Gamma_\mathcal{K}]]$ *(to be specified in the Definition* 4.42*) we have if*

$s=0$, then $\mathcal{L}^{\Sigma}_{\mathbf{f},\tau_0}\in\mathbb{I}^{ur}[[\Gamma_{\mathcal{K}}]]$ and

$$\begin{aligned}\phi(\mathcal{L}^{\Sigma}_{\mathbf{f},\tau_0})= \quad & c'_\kappa(z'_{\kappa_\phi})(\frac{(-2)^{-d(a+2b)}(2\pi i)^{d(a+2b)\kappa_\phi}(2/\pi)^{d(a+2b)(a+2b-1)/2}}{\prod_{j=0}^{a+2b-1}(\kappa_\phi-j-1)^d})^{-1}\cdot C^p_{\mathbf{f}_\phi}\\ &\times\prod_{v|p}(|p^{t_1+\ldots+t_r}|^{-\frac{\kappa_\phi}{2}}\times p^{-\frac{r+1}{2}\sum_{j=1}^r t_j}\prod_{j=1}^r\mathfrak{g}(\chi_j\tau_1^{-1})\chi_j^{-1}\tau_1(p^{t_j}))\\ &\times\frac{L^\Sigma(\tilde{\pi}_{\mathbf{f}_\phi},\bar{\tau}^c_\phi,\kappa_\phi-r)\Omega_p^{r\kappa_\phi\Sigma_\infty}}{\Omega_\infty^{r\kappa_\phi\Sigma_\infty}}.\end{aligned}$$

If $s\neq 0$, then

$$\begin{aligned}\phi(\mathcal{L}^{\Sigma}_{\mathbf{f},\tau_0})= \quad & c'_\kappa(z'_{\kappa_\phi})(\frac{(-2)^{-d(a+2b)}(2\pi i)^{d(a+2b)\kappa_\phi}(2/\pi)^{d(a+2b)(a+2b-1)/2}}{\prod_{j=0}^{a+2b-1}(\kappa_\phi-j-1)^d})^{-1}\cdot C^p_{\mathbf{f}_\phi}\\ &\times\prod_{v|p}(\frac{p^{(r+s)(r+s-1)/2}\cdot(p-1)^{r+s}}{(\prod_{i=1}^r p^{t_i\cdot(r+s-i)})\cdot(\prod_{i=1}^s p^{t_{r+i}(r+s-i)})\cdot\prod_{j=1}^{r+s}(p^j-1)}\\ &\times p^{-ss_2(\frac{1+a+2b}{2})}p^{-\sum_{j=1}^r t_j(\frac{a+1}{2})}p^{\sum_{i=1}^s t_{r+i}(a+b)}\times|p^{t_1+\ldots+t_r+s\cdot s_2}|^{-\frac{\kappa_\phi}{2}}\\ &\times\prod_{i=r+1}^{r+s}\mathfrak{g}(\chi_i^{-1}\tau_2)\chi_i\tau_2^{-1}(p^{s_2})\prod_{j=1}^r\mathfrak{g}(\chi_j\tau_1^{-1})\chi_j^{-1}\tau_1(p^{t_j}))\cdot\frac{L^\Sigma(\tilde{\pi}_{\mathbf{f}_\phi},\bar{\tau}^c_\phi,\kappa_\phi-r-s)}{\langle\tilde{\varphi}^{ord}_\phi,\varphi_\phi\rangle}\end{aligned}$$

where χ_i's are defined in Definition 4.42, $\tau_{\phi,p}=(\tau_1,\tau_2^{-1})$ such that τ_i has conductor p^{s_i} with $s_2>s_1$. The

$$C^p_{\mathbf{f}_\phi}=\prod_{v\nmid p,v\in\Sigma}\tau(y_v\bar{y}_vx_v)|(y_v\bar{y}_v)^2x_v\bar{x}_v|_v^{-z_{\kappa'_\phi}-\frac{a+2b}{2}}\mathrm{Vol}(\mathfrak{Y}_v)$$

(the x_v and y_v are the x and y in Subsubsection 4.3.1 and $\mathfrak{Y}_v$ is defined in Definition 4.11.) The $c_\kappa(z),c'_\kappa(z)$ are defined in Lemma 4.3 and κ_ϕ is the weight associated to the arithmetic point ϕ. The φ_ϕ and $\tilde{\varphi}^{ord}_\phi$ are the specialization of $\mathbf{f}$ and the $\mathbf{f}^\vee$ provided by the assumption $\mathrm{Proj}_{\mathbf{f}^\vee}$ (notice that they are ordinary vectors with respect to different Borel groups, e.g. when $s=0$ the level group for φ_ϕ at p is with respect to the upper triangular Borel subgroup while that for $\tilde{\varphi}^{ord}_\phi$ is with respect to the lower triangular Borel subgroup). The factor

$$\frac{p^{(r+s)(r+s-1)/2}\cdot(p-1)^{r+s}}{(\prod_{i=1}^r p^{t_i\cdot(r+s-i)})\cdot(\prod_{i=1}^s p^{t_{r+i}(r+s-i)})\cdot\prod_{j=1}^{r+s}(p^j-1)}$$

is the volume of a set $\tilde{K}'$ defined in Definition 4.34 (this is smaller than the level group for $\tilde{\varphi}^{ord}_\phi$). The $F_{\mathbb{I}^{ur}}$ is the fraction field of $\mathbb{I}^{ur}$. The τ_ϕ are specializations of the family of CM characters containing τ_0. The p^{t_i}'s are conductors of some characters defined in Definition 4.21. The $\tilde{\lambda}_{\beta,v}$ is defined in (17) whose p-order is $\sum_{i=1}^b t_{a+b+i}(a+b-\kappa)$.

(ii) There is a set of formal q-expansions $\mathbf{E}_{\mathbf{f},\tau_0}:=\{\sum_\beta a^h_{[g]}(\beta)q^\beta\}_{([g],h)}$ for $\sum_\beta a^h_{[g]}(\beta)q^\beta\in(\mathbb{I}^{ur}[[\Gamma_{\mathcal{K}}]]\hat{\otimes}_{\mathbb{Z}_p}\mathcal{R}_{[g],\infty})\otimes_{\mathbb{I}^{ur}}F_{\mathbb{I}^{ur}}$ where $\mathcal{R}_{[g],\infty}$ is some ring to be defined later in equation (5), $([g],h)$ are p-adic cusp labels (Definition 2.6), such that for a Zariski dense set of arithmetic points $\phi\in\mathrm{Spec}\mathbb{I}[[\Gamma_{\mathcal{K}}]]$, $\phi(\mathbf{E}_{\mathbf{f},\tau_0})$ is the Fourier-Jacobi expansion of the holomorphic nearly ordinary Klingen Eisenstein series $E(f_{Kling,\phi},z_{\kappa_\phi},-)$ we construct in Subsection 5.3 (see the interpolation formula in Proposition 5.8). Here f_{Kling} is a certain "Klingen section" to be defined there.

(iii) The terms $a^t_{[g]}(0)$ are divisible by $\mathcal{L}^\Sigma_{\mathbf{f},\tau_0}\cdot\mathcal{L}^\Sigma_{\bar{\tau}_0'}$ where $\mathcal{L}^\Sigma_{\bar{\tau}_0'}$ is the p-adic L-function of a Dirichlet character to be defined in the text.

The assumption "TEMPERED" is included so that we can easily write down the explicit range of absolute convergence for pullback formulas. It is not serious and may be relaxed using ideas of [9]. Besides the theorem, we also make some preliminary computations for the Fourier-Jacobi coefficients for Siegel Eisenstein series. This is crucial for analyzing the p-adic properties of the Klingen Eisenstein series we construct. When doing arithmetic application we need to prove that certain Fourier-Jacobi coefficient of this Eisenstein family is prime to the p-adic L-function.

This paper is organized as follows. In Section 2 we recall various backgrounds. In Section 3 we recall the notion of p-adic automorphic forms on unitary groups and Fourier-Jacobi expansion. In Section 4 we recall the notion of Klingen and Siegel Eisenstein series, the pullback formulas relating them and their Fourier-Jacobi coefficients, and then do the local calculations. (This is the most technical part of this paper). We manage to take the Siegel sections so that when we are moving our Eisentein datum p-adically, these Siegel Eisenstein series also move p-adic analytically. The hard part is to choose the sections at p-adic places. At Non-Archimedean cases prime to p the choice is more flexible (We might change this choice whenever doing arithmetic applications, see[30], [31]). At the Archimedean places we restrict ourselves to the parallel scalar weight case which is enough for doing Hida theory. In Section 5 we make the global calculations and construct the nearly ordinary Klingen Eisenstein series by the pullbacks of a Siegel Eisenstein series from a larger unitary group. Finally we include an appendix by Kai-Wen Lan for detailed proofs of some facts used for the p-adic q-expansion principle. (This is not strictly needed in our construction. But we think it is good to include it for completeness and for convenience of readers).

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2 Background

In this section we recall notations for holomorphic automorphic forms on unitary groups, Eisenstein series and Fourier-Jacobi expansions.

2.1 Notations

Suppose F is a totally real field such that $[F:\mathbb{Q}]=d$ and $\mathcal{K}$ is a totally imaginary quadratic extension of F. For a finite place v of F or $\mathcal{K}$ we usually write ϖ_v for a uniformizer and q_v for the cardinality of its residue field. Let c be the non trivial element of $\mathrm{Gal}(\mathcal{K}/F)$. Let r,s be two integers with $r\geq s\geq 0$. We fix an odd prime p that splits completely in $\mathcal{K}/\mathbb{Q}$. We fix $i_\infty:\bar{\mathbb{Q}}\hookrightarrow\mathbb{C}$ and $\iota:\mathbb{C}\simeq\mathbb{C}_p$ and write i_p for $\iota\circ i_\infty$. Denote Σ_∞ to be the set of Archimedean places of F. We take a CM type of $\mathcal{K}$, still denoted as Σ_∞ (thus $\Sigma_\infty\sqcup\Sigma_\infty^c$ are all embeddings $\mathcal{K}\to\mathbb{C}$ where $\Sigma^c_\infty=\{\tau\circ c,\tau\in\Sigma_\infty\}$).

We use ϵ to denote the cyclotomic character and ω the Techimuller character. We will often adopt the following notation: for an idele class character $\chi=\otimes_v\chi_v$ we write $\chi_p(x)=\prod_{v|p}\chi_v(x_v)$. For a character ψ of $\mathcal{K}_v^\times$ or $\mathbb{A}_\mathcal{K}^\times$ we often write ψ' for the restriction to $F_v^\times$ or $\mathbb{A}_F^\times$. For a character τ of

$\mathcal{K}^\times$ or $\mathbb{A}_\mathcal{K}^\times$ we define τ^c by $\tau^c(x) = \tau(x^c)$. (Note: we will write $\bar{\tau}(x)$ for the complex conjugation of $\tau(x)$ while the "c" means taking complex conjugation for the source).

(Gauss sum) If v is a prime of F with characteristic ℓ and $\mathfrak{d}_v\mathcal{O}_{F,v} = (d_v), d_v \in F_v^\times$ is the different of $F/\mathbb{Q}$ at v and if ψ_v is a character of $F_v^\times$ and $(c_{\psi,v}) \subset \mathcal{O}_{F,v}$ is the conductor, then we define the local Gauss sums:

$$\mathfrak{g}(\psi_v, c_{\psi,v}d_v) := \sum_{a\in(\mathcal{O}_{F,v}/c_{\psi,v})^\times} \psi_v(a)e(\mathrm{Tr}_{F_v/\mathbb{Q}_\ell}(\frac{a}{c_{\psi,v}d_v}))$$

where ℓ is the rational prime above v. If $\otimes\psi_v$ is an idele class character of $\mathbb{A}_F^\times$ then we set the global Gauss sum:

$$\mathfrak{g}(\otimes\psi_v) := \prod_v \psi_v^{-1}(c_{\psi,v}d_v)\mathfrak{g}(\psi, c_{\psi,v}d_v).$$

This is independent of all the choices of d_v ad $C_{\psi,v}$. Also if $F_v \simeq \mathbb{Q}_p$ and (p^t) is the conductor for ψ_v, then we write $\mathfrak{g}(\psi_v) := \mathfrak{g}(\psi_v, p^t)$. We define the Gauss sums for $\mathcal{K}$ similarly.

Let $\mathcal{K}_\infty$ be the maximal abelian $\mathbb{Z}_p$-extension of $\mathcal{K}$ unramified outside p. Write $\Gamma_\mathcal{K} := \mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K})$ and $G_\mathcal{K}$ the absolute Galois group of $\mathcal{K}$. We define: $\Lambda_\mathcal{K} := \mathbb{Z}_p[[\Gamma_\mathcal{K}]]$. For any A a finite extension of $\mathbb{Z}_p$ define $\Lambda_{\mathcal{K},A} := A[[\Gamma_\mathcal{K}]]$. Let $\boldsymbol{\varepsilon}_\mathcal{K} : G_\mathcal{K} \to \Gamma_\mathcal{K} \hookrightarrow \Lambda_\mathcal{K}^\times$ be the canonical character. We define $\Psi_\mathcal{K}$ to be the composition of $\boldsymbol{\varepsilon}_\mathcal{K}$ with the reciprocity map of global class field theory, which we denote as $rec_\mathcal{K} : \mathcal{K}^\times\backslash\mathbb{A}_\mathcal{K}^\times \to G_\mathcal{K}^{ab}$. Here we used the geometric normalization of class field theory. We make the corresponding definitions for F as well.

Let $S_m(R)$ be the set of matrices $S \in M_m(R \otimes_{\mathcal{O}_F} \mathcal{O}_\mathcal{K})$ such that $S = {}^t\bar{S}$ where conjugation is with respect to the second variable of $R \otimes_{\mathcal{O}_F} \mathcal{O}_\mathcal{K}$. We write $B = B_n$ and $N = N_n$ for the upper triangular Borel and unipotent radical of the group GL_n. Let N^{opp} or N^- be the opposite unipotent radical of N. We define the function $e_{\mathbb{A}_\mathbb{Q}} = \prod_v e_v$ and e_v the function on $\mathbb{Q}_v^\times$ such that $e_v(x_v) = e^{2\pi i\cdot\{x_v\}}$ for $\{x_v\}$ the fractional part of x_v and $e_\infty(x) = e^{-2\pi ix}$. We will usually write $\eta = \begin{pmatrix} & 1_m \\ -1_m & \end{pmatrix}$ if m is clear from the context.

2.2 Unitary Groups

We define:

$$\theta_{r,s} = \begin{pmatrix} & & 1_s \\ & \zeta & \\ -1_s & & \end{pmatrix}$$

where ζ is a fixed diagonal matrix such that $i^{-1}\zeta$ is totally positive. Let $V = V(r, s)$ be the skew Hermitian space over $\mathcal{K}$ with respect to this metric, i.e. $\mathcal{K}^{r+s}$ equipped with the metric given by $\langle u, v\rangle := u\theta_{r,s}{}^t\bar{v}$. We define algebraic groups $\mathrm{GU}(r, s)$ and $\mathrm{U}(r, s)$ as follows: for any $\mathcal{O}_F$-algebra R, the R points are:

$$\mathrm{GU}(r, s)(R) := \{g \in \mathrm{GL}_{r+s}(\mathcal{O}_\mathcal{K} \otimes_{\mathcal{O}_F} R)|g\theta_{r,s}g^* = \mu(g)\theta_{r,s}, \mu(g) \in R^\times\}.$$

(The $g^* = {}^t\bar{g}$ and $\mu : \mathrm{GU}(r, s) \to \mathbb{G}_{m,F}$ is called the similitude character) and

$$\mathrm{U}(r, s)(R) := \{g \in \mathrm{GU}(r, s)(R)|\mu(g) = 1\}.$$

So the unitary group $\mathrm{U}(r, s)$ in this paper really means the unitary group with respect to our fixed metric $\theta_{r,s}$. Sometimes we write GU_n and U_n for $\mathrm{GU}(n, n)$ and $\mathrm{U}(n, n)$. For two forms

φ_1, φ_2 on $\mathrm{U}(r,s)(\mathbb{A}_F)$ we define the inner product by

$$\langle \varphi_1, \varphi_2 \rangle := \int_{\mathrm{U}(r,s)(F)\backslash \mathrm{U}(r,s)(\mathbb{A}_F)} \varphi_1(g)\varphi_2(g)dg$$

where the measure is chosen such that $\mathrm{U}(r,s)(\mathcal{O}_{F_v}) = 1$ for all finite v and we take the measure at Archimedean places as [24, (7.14.5)].

We have the following embedding:

$$\mathrm{GU}(r,s) \times \mathrm{Res}_{\mathcal{O}_\mathcal{K}/\mathcal{O}_F}\mathbb{G}_m \to \mathrm{GU}(r+1,s+1)$$

$$g \times x = \begin{pmatrix} a & b & c \\ d & e & f \\ h & l & k \end{pmatrix} \times x \mapsto \begin{pmatrix} a & & b & c & \\ & \mu(g)\bar{x}^{-1} & & & \\ d & & e & f & \\ h & & l & k & \\ & & & & x \end{pmatrix}.$$

We write $m(g,x)$ for the right hand side. The image of the above map is the Levi subgroup of the Klingen parabolic subgroup P of $\mathrm{GU}(r+1,s+1)$, which consists of matrices in $\mathrm{GU}(r+1,s+1)$ such that the off diagonal entries of the $(s+1)$-th column and the last row are 0. We denote this Levi subgroup by M_P . We also write N_P for the unipotent radical of P. We also define $B = B(r,s)$ to be the standard Borel consisting of matrices $g = \begin{pmatrix} A_g & B_g \\ & D_g \end{pmatrix}$ where the blocks are with respect to the partition $r+s$ and we require that A_g is lower triangular and D_g is upper triangular.

We write $-V(r,s) = V(s,r)$ for the hermitian space whose metric is $-\theta_{r,s}$. We define some embeddings of $\mathrm{GU}(r+1,s+1) \times \mathrm{GU}(-V(r,s))$ into some larger unitary groups. This will be used in the doubling method. We define $\mathrm{GU}(r+s+1,r+s+1)'$ to be the unitary similitude group associated to (recall we wrote $a = r-s, b = s$ at the beginning of the introduction):

$$\begin{pmatrix} & & & & 1_b & & & \\ & & & & & 1 & & \\ & & \zeta & & & & & \\ & & & & & & & -1_b \\ -1_b & & & & & & & \\ & -1 & & & & & & \\ & & & & & & -\zeta & \\ & & & 1_b & & & & \end{pmatrix}$$

and $G(r+s,r+s)'$ to be associated to

$$\begin{pmatrix} & & & 1_b & & \\ & \zeta & & & & \\ & & & & & -1_b \\ -1_b & & & & & \\ & & & & -\zeta & \\ & & 1_b & & & \end{pmatrix}.$$

We define an embedding

$$\alpha : \{g_1 \times g_2 \in \mathrm{GU}(r+1,s+1) \times \mathrm{GU}(-V(r,s)), \mu(g_1) = \mu(g_2)\} \to \mathrm{GU}(r+s+1,r+s+1)'$$

as follows: we consider g_1 as a block matrix with respect to the partition $s+1+(r-s)+s+1$ (this means we use this partition to divide both all the rows and all the columns into blocks) and

g_2 as a block matrix with respect to $s+(r-s)+s$, then we define α by requiring the $1,2,3,4,5$-th (block wise) rows and columns of $\mathrm{GU}(r+1,s+1)$ embeds to the $1,2,3,5,6$-th (block wise) rows and columns of $\mathrm{GU}(r+s+1,r+s+1)'$ and the $1,2,3$-th (block wise) rows and columns of $\mathrm{GU}(V(s,r))$ embeds to the $8,7,4$-th rows and columns (block-wise) of $\mathrm{GU}(r+s+1,r+s+1)'$.

We also define an embedding:

$$\alpha' : \{g_1 \times g_2 \in \mathrm{GU}(r,s) \times \mathrm{GU}(-V(r,s)), \mu(g_1) = \mu(g_2)\} \to \mathrm{GU}(r+s,r+s)'$$

in a similar way as above: consider $\mathrm{GU}(r,s)$ and $\mathrm{GU}(-V(r,s))$ as block matrices with respect to the partition $s+(r-s)+s$. Putting the 1,2,3-th (block wise) rows and columns of the first $\mathrm{GU}(r,s)$ into the 1,2,4-th (block wise) rows and columns of $\mathrm{GU}(r+s,r+s)'$ and putting the 1,2,3-th (block wise) rows and columns of the second $\mathrm{GU}(r,s)$ into the 6,5,4-th rows and columns of $\mathrm{GU}(r+s,r+s)'$.

We also define an isomorphism:

$$\beta : \mathrm{GU}(r+s+1,r+s+1)' \xrightarrow{\sim} \mathrm{GU}(r+s+1,r+s+1)$$

and

$$\beta' : \mathrm{GU}(r+s,r+s)' \xrightarrow{\sim} \mathrm{GU}(r+s,r+s)$$

by:

$$g \mapsto S^{-1}gS$$

or

$$g \mapsto S'^{-1}gS',$$

where

$$S = \begin{pmatrix} 1_b & & & & & & & -\frac{1}{2}\cdot 1_b \\ & 1 & & & & & & \\ & & 1_a & & & & -\frac{\zeta}{2} & \\ & & & -1_b & \frac{1}{2}\cdot 1_b & & & \\ & & & 1_b & \frac{1}{2}\cdot 1_b & & & \\ & & & & & 1 & & \\ & & -1_a & & & & -\frac{\zeta}{2} & \\ -1_b & & & & & & & -\frac{1}{2}\cdot 1_b \end{pmatrix} \tag{1}$$

and

$$S' = \begin{pmatrix} 1_b & & & & & -\frac{1}{2}\cdot 1_b \\ & 1_a & & & -\frac{\zeta}{2} & \\ & & -1_b & \frac{1}{2}\cdot 1_b & & \\ & & 1_b & \frac{1}{2}\cdot 1_b & & \\ & -1_a & & & -\frac{\zeta}{2} & \\ -1_b & & & & & -\frac{1}{2}\cdot 1_b \end{pmatrix}. \tag{2}$$

Remark 2.1. (About Unitary Groups) In order to have Shimura varieties for doing p-adic modular forms and Galois representations, we need to use a unitary group defined over $\mathbb{Q}$. More precisely consider V as a skew-Hermitian space over $\mathbb{Q}$ and still denote $\theta_{r,s}$ to be the metric on it. Let T be a $\mathcal{O}_{\mathcal{K}}$ lattice that we use to define $\mathrm{GU}(r,s)$. Then the correct unitary similitude group should be

$$\mathrm{GU}^0(r,s)(A) := \{g \in \mathrm{GL}_{\mathcal{O}_{\mathcal{K}}\otimes_{\mathbb{Z}}A}(T \otimes_{\mathbb{Z}} A) | g\theta_{r,s}g^* = \mu(g)\theta_{r,s}, \mu(g) \in A\}$$

for any commutative ring A. This group is smaller than the one we defined before. However this group is not convenient for local computations since we can not treat the primes of F each independently. So what we do (implicitly) is: for analytic construction, we write down forms on the larger unitary similitude group defined above and then restrict to the smaller one. For the algebraic construction, we only do the pullbacks for unitary (instead of similitude) groups.

We are going to fix some basis of the various Hermitian spaces. We let

$$y^1,...,y^s,w^1,...,w^{r-s},x^1,...,x^s$$

be the standard basis of V such that the Hermitian forms is given above. Let W be the span over $\mathcal{K}$ of $w^1,...,w^{r-s}$. Let $X^\vee=\mathcal{O}_\mathcal{K}x^1\oplus...\oplus\mathcal{O}_\mathcal{K}x^s$ and $Y=\mathcal{O}_\mathcal{K}y^1\oplus...\oplus\mathcal{O}_\mathcal{K}y^s$. Let L be an $\mathcal{O}_\mathcal{K}$-maximal lattice such that $L_p:=L\otimes_\mathbb{Z}\mathbb{Z}_p=\sum_{i=1}^{r-s}(\mathcal{O}_\mathcal{K}\otimes_\mathbb{Z}\mathbb{Z}_p)w^i$. We define a $\mathcal{O}_\mathcal{K}$-lattice M of V by

$$M:=Y\oplus L\oplus X^\vee.$$

Let $M_p=M\otimes_\mathbb{Z}\mathbb{Z}_p$. A pair of sublattice $\mathrm{Pol}_p=\{N^{-1},N^0\}$ of M_p is called an ordered polarization of M_p if N^{-1} and N^0 are maximal isotropic direct summands in M_p and they are dual to each other with respect to the Hermitian pairing. Moreover we require that for each $v=ww^c$, $w\in\Sigma_p$, $\mathrm{rank}N_w^{-1}=\mathrm{rank}N_{w^c}^0=r$ and $\mathrm{rank}N_{w^c}^{-1}=\mathrm{rank}N_w^0=s$. The standard polarization of M_p is given by: $M_v^{-1}=Y_w\oplus L_w\oplus Y_{w^c}$ and $M_v^0=X_{w^c}\oplus L_{w^c}\oplus X_w$. We let $-V$ be the Hermitian space V with the metric given by the negative of V. We let $\tilde{y}^1,\cdots,\tilde{y}^s,\tilde{w}^1,\cdots,\tilde{w}^{r-s},\tilde{x}^1,\cdots,\tilde{x}^s$ be the corresponding basis. Let $\mathcal{K}y^{s+1}\oplus\mathcal{K}x^{s+1}$ be a two dimensional Hermitian space with metric $\begin{pmatrix} & 1\\ -1 & \end{pmatrix}$. We define

$$\mathbf{W}:=V\oplus\mathcal{K}y^{s+1}\oplus\mathcal{K}x^{s+1}\oplus(-V).$$

Let $\Upsilon\in\mathrm{U}(n+1,n+1)(F_p)$ be such that for each $v|p$ such that $v=ww^c$ where w is in our p-adic CM type Σ_p, $\Upsilon_w=S_w^{-1}$. We define another basis of $\mathbf{W}$ given by:

$$\begin{aligned}&{}^t(y^1,\cdots,y^{s+1},w^1,\cdots,w^{r-s},x^1,\cdots,x^{s+1},y^1,\cdots,y^s,w^1,\cdots,w^{r-s},x^1,\cdots,x^s)\Upsilon\\ =\;&{}^t(\mathbf{y}^1,\cdots,\mathbf{y}^{r+s+1},\mathbf{x}^1,\cdots,\mathbf{x}^{r+s+1}).\end{aligned}$$

Then $\mathbf{Y}:=\oplus_{i=1}^{r+s+1}(\mathcal{O}_\mathcal{K}\otimes_\mathbb{Z}\mathbb{Z}_p)\mathbf{y}^i$ and $\mathbf{X}:=\oplus_{i=1}^{r+s+1}(\mathcal{O}_\mathcal{K}\otimes_\mathbb{Z}\mathbb{Z}_p)\mathbf{x}^i$ gives another polarization $(\mathbf{Y},\mathbf{X})$ of $\mathbf{L}_p:=M_p\oplus(-M_p)\oplus\mathcal{O}_\mathcal{K}y^{s+1}\oplus\mathcal{O}_\mathcal{K}x^{s+1}$.

2.3 Automorphic Forms

2.3.1 Hermitian Symmetric Domain

Suppose $r\geq s>0$. Then the Hermitian symmetric domain for $G:=\mathrm{GU}(r,s)$ is

$$X^+=X_{r,s}=\{\tau=\begin{pmatrix}x\\y\end{pmatrix}|x\in M_s(\mathbb{C}^\Sigma),y\in M_{(r-s)\times s}(\mathbb{C}^\Sigma),i(x^*-x)>-iy^*\theta^{-1}y\}.$$

For $\alpha\in\mathrm{GU}(r,s)(F_\infty)$) (here $F_\infty:=F\otimes_\mathbb{Q}\mathbb{R}$) we write

$$\alpha=\begin{pmatrix}a&b&c\\d&e&f\\h&l&d\end{pmatrix}$$

according to the standard basis of V together with the block decomposition with respect to $s+(r-s)+s$. There is an action of $\alpha\in G(F_\infty)^+$ (here the superscript $+$ means the component with positive similitude factor at all Archimedean places) on $X_{r,s}$ defined by:

$$\alpha\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}ax+by+c\\gx+ey+f\end{pmatrix}(hx+ly+d)^{-1}.$$

If $rs=0$, $X_{r,s}$ consists of a single point written $\boldsymbol{x}_0$ with the trivial action of $G(F_\infty^+)$. For an open compact subgroup U of $G(\mathbb{A}_{F,f})$ put

$$M_G(X^+,U):=G(F)^+\backslash X^+\times G(\mathbb{A}_{F,f})/U$$

where U is an open compact subgroup of $G(\mathbb{A}_{F,f})$. We let $\mathbb{C}^{r,s} = \mathbb{C}(\Sigma^c)^s \otimes \mathbb{C}(\Sigma^c)^{r-s} \otimes \mathbb{C}(\Sigma)^s$ and define a map $c_{r,s}$ on it by $(u_1, u_2, u_3)c_{r,s} = (\bar{u}_1, \bar{u}_2, u_3)$. We define $p(\tau) : V \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{C}^{r,s}$ by $p(\tau)v = vB(\tau)c_{r,s}$. Let

$$B(\tau) = \begin{pmatrix} x^* & y^* & x \\ 0 & \zeta & y \\ 1_s & 0 & 1_s \end{pmatrix}.$$

We define the automorphic factors $\kappa(\alpha,\tau), \mu(\alpha,\tau)$ by $\alpha B(\tau) = B(\alpha\tau)(\overline{\kappa(\alpha,\tau)}, \mu(\alpha,\tau))$ for $\alpha \in G(\mathbb{R}), \tau \in X^+$. We sometimes write $\kappa_{r,s}(\alpha,\tau), \mu_{r,s}(\alpha,\tau)$ to emphasis the group $\mathrm{U}(r,s)$. We define $j(g,z) := \det(\mu(g,z))$. For $z \in X_{r+1,s+1}$, we define $\wp(z) \in X_{r,s}$ to be the lower right $(r \times s)$ submatrix. For $z_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $z \in \begin{pmatrix} x \\ y \end{pmatrix}$, we define $\eta(z_1, z) = i(x_1^* - x) - y_1^*(i\zeta^{-1})y$ and $\delta(z_1, z) = 2^{-s}\det(\eta(z_1, z))$.

2.3.2 Automorphic forms

We will mainly follow [15] to define the space of automorphic forms, with slight modifications. We define a cocycle: $J : R_{F/\mathbb{Q}}G(\mathbb{R})^+ \times X^+ \to \mathrm{GL}_r(\mathbb{C}^\Sigma) \times \mathrm{GL}_s(\mathbb{C}^\Sigma) := H(\mathbb{C})$ by $J(\alpha,\tau) = (\kappa(\alpha,\tau), \mu(\alpha,\tau))$ where for $\tau = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\alpha = \begin{pmatrix} a & b & c \\ g & e & f \\ h & l & d \end{pmatrix}$,

$$\kappa(\alpha,\tau) = \begin{pmatrix} \bar{h}^t x + \bar{d} & \bar{h}^t y + l\bar{\theta} \\ -\bar{\theta}^{-1}(\bar{g}^t x + \bar{f}) & -\bar{\theta}^{-1}\bar{g}^t y + \bar{\theta}^{-1}\bar{e}\bar{\theta} \end{pmatrix},\ \mu(\alpha,\tau) = hx + ly + d.$$

Let $\boldsymbol{i}$ be the point $\begin{pmatrix} i1_s \\ 0 \end{pmatrix}$ on the Hermitian symmetric domain for $\mathrm{GU}(r,s)$ (here 0 means the $(r-s) \times s$ matrix 0). Let $\mathrm{GU}(r,s)(\mathbb{R})^+$ be the subgroup of $\mathrm{GU}(r,s)(\mathbb{R})$ whose similitude factor is totally positive. Let K_∞^+ be the compact subgroups of $\mathrm{U}(r,s)(\mathbb{R})$ stabilizing $\boldsymbol{i}$ and let K_∞ be the groups generated by K_∞^+ and $\mathrm{diag}(1_{r+s}, -1_s)$. Then $J : K_\infty^+ \to H(\mathbb{C}), k_\infty \mapsto J(k_\infty, \boldsymbol{i})$ defines an algebraic representation of K_∞^+.

Definition 2.2. *A weight $\underline{k}$ is defined by a set $\{\underline{k}_\sigma\}_{\sigma\in\Sigma_\infty}$ where each*

$$\underline{k}_\sigma = (c_{r+s,\sigma}, ..., c_{s+1,\sigma}; c_{1,\sigma}, ..., c_{s,\sigma})$$

with $c_{1,\sigma} \geq ... \geq c_{s,\sigma} \geq c_{s+1,\sigma} + r + s \geq ... \geq c_{s+r,\sigma} + r + s$ for the $c_{i,\sigma}$'s in $\mathbb{Z}$

Remark 2.3. Our convention is different from others in the literature. For example in [15] the a_{r+1-i} there for $1 \leq i \leq r$ is our $-c_{s+i}$ and b_{s+1-j} there for $1 \leq j \leq s$ is our c_j. We let $\underline{k}' := (a_1, ..., a_r; b_1, ..., b_s)$. We also note that if each $\underline{k}_\sigma = (0, ..., 0; \kappa, ..., \kappa)$ then $L^{\underline{k}}(\mathbb{C})$ is one dimensional with $\rho^{\underline{k}}(h) = \det \mu(h, i)^\kappa$.

For a weight $\underline{k} = (c_{r+s}, ..., c_{s+1}; c_1, ..., c_s)$, we define the representation of $\mathrm{GL}_r \times \mathrm{GL}_s$ with minimal weight $-\underline{k}$ by

$$L_{\underline{k}} = \{f \in \mathcal{O}_{\mathrm{GL}_r\times\mathrm{GL}_s} | f(tn_+g) = \underline{k}'^{-1}(t)f(g), t \in T_r \times T_s, n_+ \in N_r \times {}^tN_s\}$$

(the $\mathcal{O}_{\mathrm{GL}_r\times\mathrm{GL}_s}$ is the structure sheaf of the algebraic group $\mathrm{GL}_r \times \mathrm{GL}_s$. See [15, Section 3]). The group action is denoted by $\rho_{\underline{k}}$. We define the functional $l_{\underline{k}}$ on $L_{\underline{k}}$ by evaluating at the identity. and define a model $L^{\underline{k}}(\mathbb{C})$ of the representation $H(\mathbb{C})$ with the highest weight $\underline{k}$ as follows. The underlying space of $L^{\underline{k}}(\mathbb{C})$ is $L_{\underline{k}}(\mathbb{C})$ and the group action is defined by

$$\rho^{\underline{k}}(h) = \rho_{\underline{k}}({}^th^{-1}), h \in H(\mathbb{C}).$$

For a weight $\underline{k}$, define $\|\underline{k}\| = \{\|\underline{k}\|_\sigma\}_{\sigma\in\Sigma} \in \mathbb{Z}^\Sigma$ by:

$$\|\underline{k}\|_\sigma := -c_{s+1,\sigma} - ... - c_{s+r,\sigma} + c_{1\sigma} + ... + c_{s,\sigma}$$

and $|\underline{k}| \in \mathbb{Z}^{\Sigma\sqcup\Sigma^c}$ by:

$$|\underline{k}| = \sum_{\sigma\in\Sigma}(c_{1,\sigma} + ... + c_{s,\sigma}).\sigma - (c_{s+1,\sigma} + ... + c_{s+r,\sigma}).\sigma^c.$$

Let χ be a Hecke character of $\mathcal{K}$ with infinite type $|\underline{k}|$, i.e. the Archimedean part of χ is given by:

$$\chi(z_\infty) = (\prod_\sigma z_\sigma^{(c_{1,\sigma}+...+c_{s,\sigma})}.z_{\sigma^c}^{-(c_{s+1,\sigma}+...+c_{s+r,\sigma})}).$$

Definition 2.4. *Let U be an open compact subgroup in $G(\mathbb{A}_{F,f})$. We denote by $M_{\underline{k}}(U,\mathbb{C})$ the space of holomorphic $L^{\underline{k}}(\mathbb{C})$-valued functions f on $X^+ \times G(\mathbb{A}_{F,f})$ such that for $\tau \in X^+$, $\alpha \in G(F)^+$ and $u \in U$ we have:*

$$f(\alpha\tau, \alpha gu) = \mu(\alpha)^{-\|\underline{k}\|}\rho^{\underline{k}}(J(\alpha,\tau))f(\tau, g).$$

Now we consider automorphic forms on unitary groups in the adèlic language. Let $\boldsymbol{i} \in X^+$ and $K_\infty^+ \subset \mathrm{U}(r,s)(F_\infty)$ be the stabilizer of $\boldsymbol{i}$. The space of automorphic forms of weight $\underline{k}$ and level U with central character χ consists of smooth and slowly increasing functions $F : G(\mathbb{A}_F) \to L_{\underline{k}}(\mathbb{C})$ such that for every $(\alpha, k_\infty, u, z) \in G(F) \times K_\infty^+ \times U \times Z(\mathbb{A}_F)$,

$$F(z\alpha g k_\infty u) = \rho^{\underline{k}}(J(k_\infty, \boldsymbol{i})^{-1})F(g)\chi^{-1}(z).$$

2.3.3 The Group $\mathrm{GU}(s,r)$

Now we consider the unitary group $\mathrm{GU}(s,r)$ which has the same Hermitian space as $\mathrm{GU}(r,s)$ but with the metric $\langle,\rangle_{s,r} := -\langle,\rangle_{r,s}$. We define the symmetric domain $X_{s,r} = X_{r,s}$ but with the complex structure such that a function is holomorphic on $X_{s,r}$ if and only if it is holomorphic on $X_{r,s}$ after composed with the following map

$$X_{r,s} \to X_{s,r}, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -\bar{x} \\ -\bar{y} \end{pmatrix}.$$

We let $\mathbb{C}^{s,r} = \mathbb{C}(\Sigma)^s \otimes \mathbb{C}(\Sigma)^{r-s} \otimes \mathbb{C}(\Sigma^c)^s$ and define $c_{s,r}$ by $(u_1, u_2, u_3)c_{s,r} = (u_1, u_2, \bar{u}_3)$. For $\mathrm{GU}(s,r)$ we define $p(\tau) : V \otimes_\mathbb{Q} \mathbb{R} \simeq \mathbb{C}^{s,r}$ by $p(\tau)v = \overline{vB(\tau)}c_{s,r}$. We require the automorphic factors $\kappa_{s,r}(\alpha,\tau), \mu_{s,r}(\alpha,\tau)$ by $\alpha B(\tau) = B(\alpha\tau)(\overline{\mu_{s,r}(\alpha,\tau)}, \kappa_{s,r}(\alpha,\tau))$. We define a weight $\underline{k}$ of $\mathrm{U}(r,s)$ to be such that $\underline{k} = (c_{r+1,\sigma}, ..., c_{r+s,\sigma}; c_{1,\sigma}, ..., c_{r,\sigma})_\sigma$ such that $c_{1,\sigma} \geq ... \geq c_{r,\sigma} \geq c_{r+1,\sigma} + r + s \geq ...c_{r+s,\sigma} + r + s$. Using these we can develop the theory of holomorphic automorphic forms on $\mathrm{GU}(s,r)$ similar to the $\mathrm{GU}(r,s)$ case.

2.3.4 Embeddings of Symmetric Domains

We still follow [24]. Pick one Archimedean place. Write $z = \begin{pmatrix} x \\ y \end{pmatrix} \in X_{r+1,s+1}$ or $X_{r,s}$ and $w = \begin{pmatrix} u \\ v \end{pmatrix} \in X_{s,r}$. We define the embeddings ι from $X_{r+1,s+1} \times X_{s,r}$ or $X_{r,s} \times X_{s,r}$ to $X_{r+s+1,r+s+1}$ or $X_{r+s,r+s}$ by

$$\iota(z,w) \to \begin{pmatrix} x & 0 & 0 \\ y & \frac{\zeta}{2} & 0 \\ -\zeta^{-1}v^*y & -v^* & -u^* \end{pmatrix}.$$

(The ζ really means the image of ζ at this Archimedean place). Let $U = RTQ$ for $Q = \begin{pmatrix} 1 & & & & & & & -\frac{1}{2} \\ & 1 & & & & & & \\ & & 1 & & & & \frac{\zeta}{2} & \\ & & & 1 & \frac{1}{2} & & & \\ -1 & & & & & & & -\frac{1}{2} \\ & & & & & 1 & & \\ & & -1 & & & & -\frac{\zeta}{2} & \\ & & & -1 & \frac{1}{2} & & & \end{pmatrix}$ and $RT = \begin{pmatrix} 1_{1+s} & & & & & \\ & 2^{-1}1_{r-s} & & & -2^{-1}1_{r-s} & \\ & & & 1_s & & \\ & & A & & & \\ & -\zeta^{-1} & & & -\zeta^{-1} & \\ & & & & & 1_s \end{pmatrix}$

for $A = \begin{pmatrix} & 1 \\ 1_s & \end{pmatrix}$. (The U here is the U_v defined in [24, Section 22] and other notations are slightly different.) We also define Q' to be Q with the second and sixth rows and columns (block-wise) deleted. Let $R'T' = \begin{pmatrix} 1_s & & & & & \\ & 2^{-1}1_{r-s} & & & -2^{-1}1_t & \\ & & & 1_s & & \\ & & A' & & & \\ & -\zeta^{-1} & & & -\zeta^{-1} & \\ & & & & & 1_s \end{pmatrix}$ with $A' = 1_s$. Define $U' = R'T'Q'$. Let $\wp(z)$ be the lower right $r \times s$ block for $z \in X_{r+1,s+1}$, $\iota_U(z,w) = (U^{-1}\iota(z,w)$ as [24, 22.2.1]. If $z = \begin{pmatrix} x \\ y \end{pmatrix}$, $z_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, let $\delta(z_1, z) = 2^{s-r}\det[i(x_1^* - x) - y_1^*\theta^{-1}y]$. If we write $[h]_S$ for $S^{-1}hS$ then we have $[\mathrm{diag}(g, g_1)]_S\iota_U(z,w) = \iota_U(gz, g_1w)$, $([\mathrm{diag}(g,g_1)]_{S'}\iota_{U'}(z,w) = \iota_{U'}(gz, g_1w))$ and

$$j([\mathrm{diag}(g, g_1)]_S, \iota_U(z,w)) = \delta(w, \wp(z))^{-1}\delta(gw, \wp(g_1z))\det(\gamma)\overline{j_g(w)}j_{g_1}(z). \tag{3}$$

For a function g on $X_{r+s+1,r+s+1}$ or $X_{r+s,r+s}$ we define the pullback g° to be the function on $X_{r+1,s+1} \times X_{s,r}$ or $X_{r,s} \times X_{s,r}$ given by

$$g^\circ(z,w) = \delta(w, \wp(z))^{-k}g(\iota_U(z,w)).$$

Definition 2.5. *We define a scalar weight κ of $\mathrm{U}(s,r)$ to be the weight $(-\kappa, \cdots, -\kappa; 0, \cdots, 0)$ (in total s κ's and r 0's).*

2.4 Shimura varieties and Igusa varieties

Fix a neat open compact subgroup K of $\mathrm{GU}^0(r,s)(\mathbb{A}_f)$ whose p-component is $\mathrm{GU}^0(r,s)(\mathbb{Z}_p)$, we refer to [15] for the definitions and arithmetic models of Shimura varieties over the reflex field E which we denote as $S_G(K)$. It parameterizes isomorphism classes of the quadruples $(A, \lambda, \iota, \bar{\eta}^{(\Box)})_{/S}$ where $\Box$ is a finite set of primes, (A, λ) is a polarized abelian variety over some base ring S, λ is an orbit (see [15, Definition 2.1]) of prime to $\Box$ polarizations of A, ι is an embedding of $\mathcal{O}_{\mathcal{K}}$ into the endomorphism ring of A and $\bar{\eta}^{(\Box)}$ is some prime to $\Box$ level structure of A. To each point $(\tau, g) \in X^+ \times G(\mathbb{A}_{F,f})$ we attach the quadruple as follows:

- The abelian variety $\mathcal{A}_g(\tau) := V \otimes_{\mathbb{Q}} \mathbb{R}/M_{[g]}(M_{[g]} := H_1(\mathcal{A}_g(\tau), \hat{\mathbb{Z}}^p))$.
- The polarization of $\mathcal{A}$ is given by the pullback of $-\langle,\rangle_{r,s}$ on $\mathbb{C}^{r,s}$ to $V \otimes_{\mathbb{Q}} \mathbb{R}$ via $p(\tau)$.
- The complex multiplication ι is the $\mathcal{O}_{\mathcal{K}}$-action induced by the action on V.
- The prime to p level structure: $\eta_g^{(p)} : M \otimes \hat{\mathbb{Z}}^p \simeq M_{[g]}$ is defined by $\eta_g^{(p)}(x) = g * x$ for $x \in M$.

We have a similar theory for Shimura varieties for $\mathrm{GU}(s,r)$ as well.

There is also a theory of compactifications of $S_G(K)$ developed in [18]. We denote $\bar{S}_G(K)$ a fixed choice of a toroidal compactification and $S^*_G(K)$ the minimal compactification.

We define some level groups at p as in [15, 1.10]. Recall that $G(\mathbb{A}_f) \supseteq K = \prod_V K_v$ is an open compact subgroup such that $K_p = G(\mathbb{Z}_p)$ and let Σ be a finite set of primes including all primes above p such that K_v is spherical for all $v \notin \Sigma$. If we write $g_p = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ for the p-component of g, then define

$$K^n = \{g \in K | g_p \equiv \begin{pmatrix} 1_r & * \\ 0 & 1_s \end{pmatrix} \mathrm{mod} p^n\},$$

$$K^n_1 = \{g \in K | A \in N_r(\mathbb{Z}_p) \mathrm{mod} p^n, D \in N^-_s(\mathbb{Z}_p) \mathrm{mod} p^n, C = 0\},$$

$$K^n_0 = \{g \in K | A \in B_r(\mathbb{Z}_p) \mathrm{mod} p^n, D \in B^-_s(\mathbb{Z}_p) \mathrm{mod} p^n, C = 0\}.$$

Now we recall briefly the notion of Igusa schemes over $\mathcal{O}_{v_0}$ (the localization of the integer ring of the reflex field at the p-adic place v_0 determined by $\iota_p : \mathbb{C} \simeq \mathbb{C}_p$) in [15, Subsection 2]. Let V be the Hermitian space for $\mathrm{U}(r, s)$ and M be a standard lattice of V and $M_p = M \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Let $\mathrm{Pol}_p = \{N^{-1}, N^0\}$ be a polarization of M_p. The Igusa variety $I_G(K^n)$ of level p^n is the scheme representing the usual quadruple for Shimura variety together with a

$$j : \mu_{p^n} \otimes_{\mathbb{Z}} N^0 \hookrightarrow A[p^n]$$

where A is the abelian variety in the quadruple. Note that the existence of j implies that if p is nilpotent in the base ring then A must be ordinary. For any integer $m > 0$ let $\mathcal{O}_m := \mathcal{O}_{v_0}/p^m$.

Igusa Schemes over $\bar{S}_G(K)$:
To define p-adic automorphic forms on needs Igusa Schemes over $\bar{S}_G(K)$. We fix such a toroidal compactification and refer to [15, 2.7.6] for the construction. We still denote it as $I_G(K^n)$. Then over $\mathcal{O}_m$ the $I_G(K^n)$ is a Galois covering of the ordinary locus of the Shimura variety with Galois group $\prod_{v|p} \mathrm{GL}_r(\mathcal{O}_{F,v}/p^n) \times \mathrm{GL}_s(\mathcal{O}_{F,v}/p^n)$. We write $I_G(K^n_0) = I_G(K^n)^{K^n_0}$ and $I_G(K^n_1) = I_G(K^n)^{K^n_1}$ over $\mathcal{O}_m$.

Cusps
Let $1 \le t \le s$. We let P_t be the maximal parabolic subgroup of $\mathrm{GU}(r, s)$ consisting of matrices such that in the block form with respect to $t + (r + s - 2t) + t$, it is of the form $\begin{pmatrix} \times & \times & \times \\ & \times & \times \\ & & \times \end{pmatrix}$. Let G_{P_t} be the unitary similitude group with respect to the skew-Hermitian space for ζ. Let Y_t be the $\mathcal{O}_{\mathcal{K}}$ span of $\{y^1, \cdots, y^t\}$. We define the set of cusp labels by:

$$C_t(K) := (\mathrm{GL}(Y_t) \times G_{P_t}(\mathbb{A}_f))N_{P_t}(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K.$$

This is a finite set. We denote $[g]$ for the class represented by $g \in G(\mathbb{A}_f)$. For each such g whose p-component is 1 we define $K^g_{P_t} = G_{P_t}(\mathbb{A}_f) \cap gKg^{-1}$ and denote $S_{[g]} := S_{G_{P_t}}(K^g_{P_t})$ the corresponding Shimura variety for the group G_P with level group $K^g_{P_t}$). By the strong approximation we can choose a set $\underline{C}_t(K)$ of representatives of $C_t(K)$ consisting of elements $g = pk^0$ for $p \in P_t(\mathbb{A}^{\Sigma}_f)$ and $G(\mathbb{A}_\Sigma) \ni k^0 \in K^0$ for K^0 the spherical compact subgroup.

p-adic Cusps

Definition 2.6. *As in [15] each pair $(g, h) \in C_t(K) \times H(\mathbb{Z}_p)$ can be regarded as a p-adic cusp, i.e. cusps of the Igusa tower.*

Igusa Schemes for Unitary Groups

We refer to [15, 2.5] for the notion of Igusa Schemes for the unitary groups $\mathrm{U}(r,s)$ (not the similitude group). It parameterizes quintuples $(A,\lambda,\iota,\bar{\eta}^{(p)},j)_{/S}$ similar to the Igusa Schemes for unitary similitude groups but requiring λ to be a prime to p-polarization of A (instead of an orbit). In order to use the pullback formula algebraically we need a map of Igusa schemes given by:

$$i([(A_!,\lambda_!,\iota_1,\eta_!^pK_!,j_1)],[(A_2,\lambda_2,\iota_2,\eta_2^pK_2,j_2)])=[(A_1\times A_2,\lambda_1\times\lambda_2,\iota_1,\iota_2,(\eta_1^p\times\eta_2^p)K_3,j_1\times j_2)].$$

Similar to [15], we know that taking the change of polarization into consideration

$$i([z,g],[w,h])=[\iota(z,w),(g,h)\Upsilon].$$

(Υ is defined at the end of Subsection 2.2.)

2.4.1 Geometric Modular Forms

Let $H=\prod_{v|p}(\mathrm{GL}_r\times\mathrm{GL}_s)$ and $N\subset H$ being $\prod_{v|p}(N_r\times{}^tN_s)$. To save notation we will also write $H=\prod_{v|p}\mathrm{GL}_r(\mathcal{O}_{F,v})\times\mathrm{GL}_s(\mathcal{O}_{F,v})$ and $N\subset H$ be $\prod_{v|p}N_r(\mathcal{O}_{F,v})\times{}^tN_s(\mathcal{O}_{F,v})$. We define $\underline{\omega}=e^*\Omega_{\mathcal{G}/\bar{S}_G(K)}$ for Ω the sheaf of differentials on the universal semi-abelian scheme $\mathcal{G}$ over the toroidal compactification (see [15, 2.7.2] for a brief discussion). Recall that for $v|p$ we have $v=w\bar{w}$ in $\mathcal{K}$ with $w\in\Sigma_p$. Let e_w and $e_{\bar{w}}$ be the corresponding projections for $\mathcal{K}_v\simeq\mathcal{K}_w\times\mathcal{K}_{\bar{w}}$ then $\underline{\omega}=e_w\underline{\omega}\oplus e_{\bar{w}}\underline{\omega}$. We also define:

$$\mathcal{E}^+:=\underline{\mathrm{Isom}}(\mathcal{O}^r_{\bar{S}_G(K)},e_w\underline{\omega}),$$

$$\mathcal{E}^-:=\underline{\mathrm{Isom}}(\mathcal{O}^s_{\bar{S}_G(K)},e_{\bar{w}}\underline{\omega}),$$

$$\mathcal{E}:=\mathcal{E}^+\oplus\mathcal{E}^-.$$

This is an H-torsor over $\bar{S}_G(K)$. We can define the automorphic sheaf $\omega_{\underline{k}}=\mathcal{E}\times^H L_{\underline{k}}$. A section f of $\underline{\omega}_{\underline{k}}$ is a morphism $f:\mathcal{E}\to L_{\underline{k}}$ such that

$$f(x,h\boldsymbol{w})=\rho_{\underline{k}}(h)f(x,\boldsymbol{\omega}),h\in H,x\in\bar{S}_G(K).$$

2.5 p-adic Automorphic Forms on Unitary Groups

Let R be a p-adic $\mathbb{Z}_p$-algebra and let $R_m:=R/p^m$. Let $T_{n,m}:=I_G(K^n)_{/R_m}$. Define:

$$V_{n,m}=H^0(T_{n,m},\mathcal{O}_{T_{n,m}}),$$

$$V_{\underline{k}}(K_\bullet^n,R_m)=H^0(T_{n,m/R_m},\omega_{\underline{k}})^{K_\bullet^n}.$$

Let $V_{\infty,m}=\varinjlim_n V_{n,m}$ and $V_{\infty,\infty}=\varprojlim_m V_{\infty,m}$. Define $V_p(G,K):=V_{\infty,\infty}^N$ the space of p-adic modular forms. Let $\mathbf{T}=T(\mathbb{Z}_p)\subset H$ and let $\Lambda_{\mathbf{T}}:=\mathbb{Z}_p[[\mathbf{T}]]$. The Galois action of $\mathbf{T}$ on $V_{\infty,m}^N$ makes the space of p-adic modular forms a discrete $\Lambda_{\mathbf{T}}$-module.

Suppose $n\geq m$. To each R_m-quintuple $(\underline{A},j)$ of level K^n we can attach a canonical basis $\boldsymbol{\omega}(j)$ of $H^0(A,\Omega_A)$. Therefore we have a canonical isomorphism

$$H^0(T_{n,m}/_{R_m},\omega_{\underline{k}})\simeq V_{n,m}\otimes L_{\underline{k}}(R_m)$$

given by

$$f\mapsto\hat{f}(\underline{A},j)=f(\underline{A},j,\boldsymbol{\omega}(j)).$$

We call $\hat{f}$ the p-adic avatar of f.

Similarly we can define an embedding of geometric modular forms into p-adic modular forms by

$$f \mapsto \hat{f}(\underline{A}, j) = f(\underline{A}, \boldsymbol{\omega}(j)).$$

We also define the morphism

$$\beta_{\underline{k}} : V_{\underline{k}}(K_1^n, R_m) \to V_{n,m}^N$$

by

$$f \mapsto \beta_{\underline{k}}(f) := l_{\underline{k}}(\hat{f}).$$

We can also pass to the limit for $m \to \infty$ to get the embedding of $V_{\underline{k}}(K_1^n, R)$ into $V_{\infty,\infty}^N$. We refer to [15, 3.8, 3.9] for the definition of an U_p Hecke operator and define the Hida's ordinary projector

$$e := \lim_n U_p^{n!}.$$

2.6 Algebraic Theory for Fourier-Jacobi Expansions

We suppose $s > 0$ in this subsection. Let $X_t^\vee = \mathrm{span}_{\mathcal{O}_\mathcal{K}}\{x^1, \cdots, x^t\}$ and $Y_t = \mathrm{span}_{\mathcal{O}_\mathcal{K}}\{y^1, \cdots, y^t\}$. Let W_t be the skew-Hermitian space $\mathrm{span}_{\mathcal{O}_\mathcal{K}}\{y^{t+1}, \cdots, y^s, w_1, \cdots, x^{t+1}, \cdots, x^s\}$. Let G_t^0 be the unitary similitude group of W_t. Let $[g] \in C_t(K)$ and $K_{G_{P_t}} = G_{P_t}(\mathbb{A}_f) \cap gKg^{-1}$ (we suppress the subscript $[g]$ so as not to make the notation too cumbersome). Let $\mathcal{A}_t$ be the universal abelian scheme over the Shimura variety $S_{G_{P_t}}(K_{G_{P_t}})$. Write $g^\vee = kg_i^\vee \gamma$ for $\gamma \in G(F)^+$ and $k \in K$. Define $X_g^\vee = X_t^\vee g_i^\vee \gamma, Y_g = Y_t g_i^\vee \gamma$. Let $X_g = \{y \in (Y_t \otimes_\mathbb{Q} \mathbb{Z}) \cdot \gamma | \langle y, X_g^\vee \rangle \in \mathbb{Z}\}$. Then we have

$$i : Y_g \hookrightarrow X_g.$$

Let $\mathcal{Z}_{[g]}$ be

$$\underline{\mathrm{Hom}}_{\mathcal{O}_\mathcal{K}}(X_g, \mathcal{A}_t^\vee) \times_{\underline{\mathrm{Hom}}_{\mathcal{O}_\mathcal{K}}(Y_g, \mathcal{A}_t^\vee)} \underline{\mathrm{Hom}}_{\mathcal{O}_\mathcal{K}}(Y_g, \mathcal{A}_t) := \{(c, c^t)|, c(i(y)) = \lambda(c^t(y)), y \in Y_g\}.$$

Here $\underline{\mathrm{Hom}}$'s are the obvious sheaves over the big étale site of $S_{G_{P_t}}$, represented by Abelian schemes. Let $\mathbf{c}$ and $\mathbf{c}^\vee$ be the universal morphisms over $\underline{\mathrm{Hom}}_{\mathcal{O}_\mathcal{K}}(X_g, \mathcal{A}_t^\vee)$ and $\underline{\mathrm{Hom}}_{\mathcal{O}_\mathcal{K}}(Y_g, \mathcal{A}_t)$. Let N_{P_t} be the unipotent radical of P_t and $Z(N_{P_t})$ be its center. Let $H_{[g]} := Z(N_{P_t}(F)) \cap g_i K g_i^{-1}$. Note that if we replace the components of K at $v|p$ by K_1^n then the set $H_{[g]}$ remain unchanged. Let $\Gamma_{[g]} := \mathrm{GL}_\mathcal{K}(Y_t) \cap g_i K g_i^{-1}$. Let $\mathcal{P}_{\mathcal{A}_t}$ be the Poincaré sheaf over $\mathcal{A}_t^\vee \times \mathcal{A}_t/_{\mathcal{Z}_{[g]}}$ and $\mathcal{P}_{\mathcal{A}_t}^\times$ its associated $\mathbb{G}_m$-torsor. Let $S_{[g]} := \mathrm{Hom}(H_{[g]}, \mathbb{Z})$. For any $h \in S_{[g]}$ let $c(h)$ be the tautological map $\mathcal{Z}_{[g]} \to \mathcal{A}_t^\vee \times \mathcal{A}_t$ and $\mathcal{L}(h) := c(h)^* \mathcal{P}_{\mathcal{A}_t}^\times$ its associated $\mathbb{G}_m$ torsor over $\mathcal{Z}_{[g]}$.

It is well-known (see e.g. [18, Chapter 7]) that the minimal compactification $S_G^*(K)$ is the disjoint union of boundary components corresponding to t's for all $1 \leq t \leq s$. Let $\mathcal{O}_{\mathbb{C}_p}$ be the valuation ring for $\mathbb{C}_p$. The following proposition is proved in [18, Proposition 7.2.3.16]. Let $[g] \in C_t(K)$ and $\bar{x}$ is a $\mathcal{O}_{\mathbb{C}_p}$-point of the t-stratum of $S_G^*(K)(1/E)$ corresponding to $[g]$.

Proposition 2.7. *Let $[g]$ and $\bar{x}$ be as above. We write the subscript $\bar{x}$ to mean formal completion along $\bar{x}$. Let π be the map $\bar{S}_G(K) \to S_G^*(K)$. Then $\pi_*(\mathcal{O}_{\bar{S}_G(K)})_{\bar{x}}$ is isomorphic to*

$$\{\sum_{h \in S_{[g]}^+} H^0(\mathcal{Z}_{[g]}, \mathcal{L}(h))_{\bar{x}} q^h\}^{\Gamma_{[g]}}.$$

Here $S_{[g]}^+$ means the totally non-negative elements in $S_{[g]}$. The q^h is just regarded as a formal symbol and $\Gamma_{[g]}$ acts on the set by a certain formula which we omit.

For each $[g] \in C_t(K)$ we fix a $\bar{x}$ corresponding to it as above. Now we consider the diagram

$$\begin{array}{ccc} T_{n,m} & \xrightarrow{\pi_{n,m}} & T^*_{n,m} \\ \downarrow & & \downarrow \\ \bar{S}_G(K)[1/E]_{\mathcal{O}_m} & \xrightarrow{\pi} & S^*_G(K)[1/E]_{\mathcal{O}_m} \end{array}$$

where $T_{n,m} \to T^*_{n,m} \to S^*_G(K)[1/E]_{\mathcal{O}_m}$ is the Stein factorization. By [19, Corollary 6.2.2.8] $T^*_{n,m}$ is finite étale over $S^*_G(K)[1/E]_{\mathcal{O}_m}$. Taking a preimage of $\bar{x}$ in $T^*_{n,m}$ which we still denote as $\bar{x}$. (For doing this we have to extend the field of definition to include the maximal unramified extension of L). Then the formal completion of the structure sheaf of $T^*_{n,m}$ and $S^*_G(K)[1/E]_{\mathcal{O}_m}$ at $\bar{x}$ are isomorphic. So for any p-adic automorphic form $f \in \varprojlim_m \varinjlim_n H^0(T_{n,m}, \mathcal{O}_{n,m})$ (with trivial coefficients) we have a Fourier-Jacobi coefficient

$$FJ(f) \in \{\prod_{h \in S^+_{[g]}} \varprojlim_m \varinjlim_n H^0(\mathcal{Z}_{[g]}, \mathcal{L}(h))_{\bar{x}} \cdot q^h\}_{[g]} \tag{4}$$

by considering f as a global section of $\pi^*_{n,m}(\mathcal{O}_{T_{n,m}}) = \mathcal{O}_{T^*_{n,m}}$ and pullback at $\bar{x}$'s. Note that if $t = s = 1$ then there is no need to choose the $\bar{x}$'s and pullback since the Shimura varieties for G_t is 0-dimensional (see [15, (2.18)]). In application when we construct families of Klingen Eisenstein series in terms off Fourier-Jacobi coefficients, we will take $t = 1$ and define

$$\mathcal{R}_{[g],\infty} := \prod_{h \in S^+_{[g]}} \varprojlim_m \varinjlim_n H^0(\mathcal{Z}_{[g]}, \mathcal{L}(h))_{\bar{x}} \cdot q^h. \tag{5}$$

We remark that the map FJ is injective on the space of forms with prescribed nebentypus at p (this is not needed for our result though). This can be seen using the discussion of [28] right before Section 6.2 of *loc.cit* (which in turn uses result of Hida in [12] about the irreducibility of Igusa towers for the group $\mathrm{SU}(r,s) \subset \mathrm{U}(r,s)$ (kernel of the determinant)). In particular to see this injectivity we need the fact that there is a bijection between the irreducible components of generic and special fiber of $S^*_G(K)$ (see [18, Subsection 6.4.1]) and that there is at least one cusp of any given genus on the ordinary locus of each irreducible component (Note that the signature is (r,s) for $r \geq s > 0$ at all Archimedean places so there is at least one cusp in $C_t(K)$ at each irreducible component. Since p splits completely in $\mathcal{K}$ the cusps of minimal genus must be in the ordinary locus. On the other hand by the construction of minimal compactification the closure of the stratum of any genus r is the union of all stratums of genus less than or equal to r. Note also that since the geometric fibers of the minimal compactification are normal, their irreducible componenents are also connected components. This implies the existence of such cusp on the ordinary locus.) See the appendix of this paper for more details.

3 Eisenstein Series and Fourier-Jacobi Coefficients

The materials of this section are straightforward generalizations of parts of [28, Section 9 and 11] and we use the same notations as *loc.cit*; so everything in this section should eventually be the same as [28] when specializing to the group $\mathrm{GU}(2,2)_{/\mathbb{Q}}$.

3.1 Klingen Eisenstein Series

Let $\mathfrak{gu}(\mathbb{R})$ be the Lie algebra of $\mathrm{GU}(r,s)(\mathbb{R})$. Let δ be a character of the Klingen parabolic subgroup P such that $\delta^{a+2b+1} = \delta_P$ (the modulus character of P).

3.1.1 Archimedean Picture

Let v be an infinite place of F so that $F_v \simeq \mathbb{R}$. Let $\boldsymbol{i}'$ and $\boldsymbol{i}$ be the points on the Hermitian symmetric domain for $\mathrm{GU}(r,s)$ and $\mathrm{GU}(r+1,s+1)$ which are $\begin{pmatrix} i1_s \\ 0 \end{pmatrix}$ and $\begin{pmatrix} i1_{s+1} \\ 0 \end{pmatrix}$ respectively (here 0 means the $(r-s)\times s$ or $(r-s)\times(s+1)$ matrix 0). Let $\mathrm{GU}(r,s)(\mathbb{R})^+$ be the subgroup of $\mathrm{GU}(r,s)(\mathbb{R})$ whose similitude factor is positive. Let K_∞^+ and $K_\infty^{+,\prime}$ be the compact subgroups of $\mathrm{U}(r+1,s+1)(\mathbb{R})$ and $\mathrm{U}(r,s)(\mathbb{R})$ stabilizing $\boldsymbol{i}$ or $\boldsymbol{i}'$ and let K_∞ (K_∞') be the groups generated by K_∞^+ ($K_\infty^{+,\prime}$) and $\mathrm{diag}(1_{r+s+1},-1_{s+1})$ (resp. $\mathrm{diag}(1_{r+s},-1_s)$).

Now let (π,H) be a unitary tempered Hilbert representation of $\mathrm{GU}(r,s)(\mathbb{R})$ with H_∞ the space of smooth vectors. We define a representation of $P(\mathbb{R})$ on H_∞ as follows: for $p=mn, n\in N_P(\mathbb{R}), m=m(g,a)\in M_P(\mathbb{R})$ with $a\in\mathbb{C}^\times, g\in \mathrm{GU}(r+1,s+1)(\mathbb{R})$, put

$$\rho(p)v := \tau(a)\pi(g)v, v\in H_\infty.$$

We define a representation by smooth induction $I(H_\infty) := \mathrm{Ind}_{P(\mathbb{R})}^{\mathrm{GU}(r+1,s+1)(\mathbb{R})}\rho$ and denote $I(\rho)$ as the space of K_∞-finite vectors in $I(H_\infty)$. For $f\in I(\rho)$ we also define for each $z\in\mathbb{C}$ a function

$$f_z(g) := \delta(m)^{(a+2b+1)/2+z}\rho(m)f(k), g=mk\in P(\mathbb{R})K_\infty,$$

and an action of $\mathrm{GU}(r+1,s+1)(\mathbb{R})$ on it by

$$(\sigma(\rho,z)(g)f)(k) := f_z(kg).$$

Let $(\pi^\vee,V)$ be the irreducible $(\mathfrak{gu}(\mathbb{R}),K_\infty')$-module given by $\pi^\vee(x)=\pi(\eta^{-1}x\eta)$ for $\eta=\begin{pmatrix} & & 1_b \\ & 1_a & \\ -1_b & & \end{pmatrix}$ and x in $\mathfrak{gu}(\mathbb{R})$ or K_∞' (this does not mean the contragradient representation!). Denote $\rho^\vee, I(\rho^\vee), I^\vee(H_\infty)$ and $\sigma(\rho^\vee,z), I(\rho^\vee))$ the representations and spaces defined as above but with π,τ replaced by $\pi^\vee\otimes(\tau\circ\det), \bar{\tau}^c$. We are going to define an intertwining operator. Let $w=\begin{pmatrix} & & 1_{b+1} \\ & 1_a & \\ -1_{b+1} & & \end{pmatrix}$.

For any $z\in\mathbb{C}$, $f\in I(H_\infty)$ and $k\in K_\infty$ consider the integral:

$$A(\rho,z,f)(k) := \int_{N_P(\mathbb{R})} f_z(wnk)dn. \tag{6}$$

This is absolutely convergent when $\mathrm{Re}(z) > \frac{a+2b+1}{2}$ and $A(\rho,z,-)\in \mathrm{Hom}_{\mathbf{C}}(I(H_\infty),I^\vee(H_\infty))$ intertwines the actions of $\sigma(\rho,z)$ and $\sigma(\rho^\vee,-z)$.

Suppose π is the holomorphic discrete series representation associated to the (scalar) weight $(0,...,0;\kappa,...,\kappa)$, then it is well known that there is a unique (up to scalar) vector $v\in\pi$ such that $k\cdot v=\det\mu(k,i)^{-\kappa}$ (here μ means the second component of the automorphic factor J instead of the similitude character) for any $k\in K_\infty^{+,\prime}$. Then by Frobenius reciprocity law there is a unique (up to scalar) vector $\tilde{v}\in I(\rho)$ such that $k\cdot\tilde{v}=\det\mu(k,i)^{-\kappa}\tilde{v}$ for any $k\in K_\infty^+$. We fix v and multiply $\tilde{v}$ by a constant so that $\tilde{v}(1)=v$. In $\pi^\vee$, $\pi(w)v$ has the action of K_∞^+ given by multiplying by $\det\mu(k,i)^{-\kappa}$. We define $w'\in \mathrm{U}(r+1,s+1)$ by $w'=\begin{pmatrix} 1_b & & & & \\ & & & & 1 \\ & & 1_a & & \\ & & & 1_b & \\ & -1 & & & \end{pmatrix}$.

There is a unique vector $\tilde{v}^\vee\in I(\rho^\vee)$ such that the action of K_∞^+ is given by $\det\mu(k,i)^{-\kappa}$ and $\tilde{v}^\vee(w')=\pi(w)v$. Then by uniqueness there is a constant $c(\rho,z)$ such that $A(\rho,z,\tilde{v})=c(\rho,z)\tilde{v}^\vee$.

Definition 3.1. *We define $F_\kappa\in I(\rho)$ to be the $\tilde{v}$ as above.*

3.1.2 Prime to p Picture

Our discussion here follows [28, 9.1.2]. Let (π, V) be an irreducible, admissible representation of $\mathrm{GU}(r,s)(F_v)$ which is unitary and tempered. Let ψ and τ be unitary characters of $\mathcal{K}_v^\times$ such that ψ is the central character for π. We define a representation ρ of $P(F_v)$ as follows. For $p = mn, n \in N_P(F_v)$, $m = m(g,a) \in M_P(F_v), a \in K_v^\times, g \in \mathrm{GU}(F_v)$ let

$$\rho(p)v := \tau(a)\pi(g)v, v \in V.$$

Let $I(\rho)$ be the representation defined by admissible induction: $I(\rho) = \mathrm{Ind}_{P(F_v)}^{\mathrm{GU}(r+1,s+1)(F_v)}\rho$. As in the Archimedean case, for each $f \in I(\rho)$ and each $z \in \mathbb{C}$ we define a function f_z on $\mathrm{GU}(r+1,s+1)(F_v)$ by

$$f_z(g) := \delta(m)^{(a+2b+1)/2+z}\rho(m)f(k), g = mk \in P(F_v)K_v$$

and a representation $\sigma(\rho, z)$ of $\mathrm{GU}(r+1,s+1)(F_v)$ on $I(\rho)$ by

$$(\sigma(\rho,z)(g)f)(k) := f_z(kg).$$

Let $(\pi^\vee, V)$ be given by $\pi^\vee(g) = \pi(\eta^{-1}g\eta)$. This representation is also tempered and unitary. We denote by $\rho^\vee, I(\rho^\vee)$, and $(\sigma(\rho^\vee, z), I(\rho^\vee))$ the representations and spaces defined as above but with π and τ replaced by $\pi^\vee \otimes (\tau \circ \det)$, and $\bar{\tau}^c$, respectively.

For $f \in I(\rho), k \in K_v$, and $z \in \mathbb{C}$ consider the integral

$$A(\rho, z, v)(k) := \int_{N_P(F_v)} f_z(wnk)dn. \tag{7}$$

As a consequence of our hypotheses on π this integral converges absolutely and uniformly for z and k in compact subsets of $\{z : \mathrm{Re}(z) > (a+2b+1)/2\} \times K_v$. Moreover, for such z, $A(\rho, z, f) \in I(\rho^\vee)$ and the operator $A(\rho, z, -) \in \mathrm{Hom}_{\mathbb{C}}(I(\rho), I(\rho^\vee))$ intertwines the actions of $\sigma(\rho, z)$ and $\sigma(\rho^\vee, -z)$.

For any open subgroup $U \subseteq K_v$ let $I(\rho)^U \subseteq I(\rho)$ be the finite-dimensional subspace consisting of functions satisfying $f(ku) = f(k)$ for all $u \in U$. Then the function

$$\{z \in \mathbb{C} : \mathrm{Re}(z) > (a+2b+1)/2\} \to Hom_{\mathbb{C}}(I(\rho)^U, I(\rho^\vee)^U), z \mapsto A(\rho, z, -)$$

is holomorphic. This map has a meromorphic continuation to all of $\mathbb{C}$.

We finally remark that when π and τ are unramified, there is a unique up to scalar unramified vector $F_{\rho_v} \in I(\rho)$.

3.1.3 Global Picture

We follow [28, 9.1.4]. Let (π, V) be an irreducible cuspidal tempered automorphic representation of $\mathrm{GU}(r,s)(\mathbb{A}_F)$. It is an admissible $(\mathfrak{gu}(\mathbb{R}), K'_\infty)_{v|\infty} \times \mathrm{GU}(r,s)(\mathbb{A}_f)$-module which is a restricted tensor product of local irreducible admissible representations. Let $\psi, \tau : \mathbb{A}_{\mathcal{K}}^\times \to \mathbb{C}^\times$ be Hecke characters such that ψ is the central character of π. Let $\tau = \otimes\tau_w$ and $\psi = \otimes\psi_w$ be their local decompositions, w running over places of F. Define a representation of $(P(F_\infty) \cap K_\infty) \times P(\mathbb{A}_{F,f})$ by putting:

$$\rho(p)v := \otimes(\rho_w(p_w)v_w),$$

Let $I(\rho)$ be the restricted product $\otimes I(\rho_w)$'s with respect to the F_{ρ_w}'s at those w at which τ_w, ψ_w, π_w are unramified. As before, for each $z \in \mathbb{C}$ and $f \in I(\rho)$ we define a function f_z on $\mathrm{GU}(r+1, s+1)(\mathbb{A}_F)$ as

$$f_z(g) := \otimes f_{w,z}(g_w)$$

where $f_{w,z}$ are defined as before and an action $\sigma(\rho, z)$ of $(\mathfrak{gu}, K_\infty) \otimes \mathrm{GU}(r+1, s+1)(\mathbb{A}_f)$ on $I(\rho)$ by $\sigma(\rho, z) := \otimes \sigma(\rho_w, z)$. Similarly we define $\rho^\vee, I(\rho^\vee)$, and $\sigma(\rho^\vee, z)$ but with the corresponding things replaced by their $\vee$'s and we have global versions of the intertwining operators $A(\rho, f, z)$.

Definition 3.2. *Let Σ be a finite set of primes of F containing all the infinite places, primes dividing p and places where π or τ is ramified. Then we call the triple $\mathcal{D} = (\pi, \tau, \Sigma)$ an Eisenstein Datum.*

3.1.4 Klingen-Type Eisenstein Series on G

We follow [28, 9.1.5] in this subsubsection. Let π, ψ, and τ be as above. For $f \in I(\rho), z \in \mathbb{C}$, there are maps from $I(\rho)$ and $I(\rho^\vee)$ to spaces of automorphic forms on $P(\mathbb{A}_F)$ given by

$$f \mapsto (g \mapsto f_z(g)(1)).$$

In the following we often write f_z for the automorphic form on $P(\mathbb{A}_F)$ given by this recipe.

If $g \in \mathrm{GU}(r+1, s+1)(\mathbb{A}_F)$ it is well known that

$$E(f, z, g) := \sum_{\gamma \in P(F) \backslash G(F)} f_z(\gamma g) \tag{8}$$

converges absolutely and uniformly for (z, g) in compact subsets of $\{z \in \mathbb{C} : \mathrm{Re}(z) > \frac{a+2b+1}{2}\} \times \mathrm{GU}(r+1, s+1)(\mathbb{A}_F)$. Therefore we get some automorphic forms which are called Klingen Eisenstein series.

Definition 3.3. *For any parabolic subgroup R of $\mathrm{GU}(r+1, s+1)$ and an automorphic form φ we define φ_R to be the constant term of φ along R defined by*

$$\varphi_R(g) = \int_{n \in N_R(F) \backslash N_R(\mathbb{A}_F)} \varphi(ng) dn.$$

The following lemma is well-known (see [28, Lemma 9.2]).

Lemma 3.4. *Let R be a standard F-parabolic subgroup of $\mathrm{GU}(r+1, s+1)$ (i.e, $R \supseteq B$ where B is the standard Borel subgroup). Suppose $\mathrm{Re}(z) > \frac{a+2b+1}{2}$.*
(i) If $R \neq P$ then $E(f, z, g)_R = 0$;
(ii) $E(f, z, -)_P = f_z + A(\rho, f, z)_{-z}$.

3.2 Siegel Eisenstein Series on G_n

3.2.1 Local Picture

Our discussion in this subsection follows [28, 11.1-11.3] closely. Let $Q = Q_n$ be the Siegel parabolic subgroup of GU_n consisting of matrices $\begin{pmatrix} A_q & B_q \\ 0 & D_q \end{pmatrix}$. It consists of matrices whose lower-left $(n \times n)$ block is zero.

For a finite place v of F and a character χ of $\mathcal{K}_v^\times$ we let $I_n(\chi)$ be the space of smooth $K_{n,v}$-finite functions (here $K_{n,v}$ means the open compact group $G_n(\mathcal{O}_{F,v})$) $f : K_{n,v} \to \mathbb{C}$ such that $f(qk) = \chi(\det D_q) f(k)$ for all $q \in Q_n(F_v) \cap K_{n,v}$ (we write q as block matrix

$q = \begin{pmatrix} A_q & B_q \\ 0 & D_q \end{pmatrix}$). For $z \in \mathbb{C}$ and $f \in I(\chi)$ we also define a function $f(z,-) : G_n(F_v) \to \mathbb{C}$ by $f(z, qk) := \chi(\det D_q))|\det A_q D_q^{-1}|_v^{z+n/2} f(k), q \in Q_n(F_v)$ and $k \in K_{n,v}$.

For $f \in I_n(\chi), z \in \mathbb{C}$, and $k \in K_{n,v}$, the intertwining integral is defined by:

$$M(z,f)(k) := \bar{\chi}^n(\mu_n(k)) \int_{N_{Q_n}(F_v)} f(z, w_n r k) dr.$$

For z in compact subsets of $\{\mathrm{Re}(z) > n/2\}$ this integral converges absolutely and uniformly, with the convergence being uniform in k. In this case it is easy to see that $M(z,f) \in I_n(\bar{\chi}^c)$. A standard fact from the theory of Eisenstein series says that this has a continuation to a meromorphic section on all of $\mathbb{C}$.

Let $\mathcal{U} \subseteq \mathbb{C}$ be an open set. By a meromorphic section of $I_n(\chi)$ on $\mathcal{U}$ we mean a function $\varphi : \mathcal{U} \mapsto I_n(\chi)$ taking values in a finite dimensional subspace $V \subset I_n(\chi)$ and such that $\varphi : \mathcal{U} \to V$ is meromorphic.

For Archimedean places there is a similar picture (see *loc.cit*).

3.2.2 Global Picture

For an idele class character $\chi = \otimes \chi_v$ of $\mathbb{A}_{\mathcal{K}}^{\times}$ we define a space $I_n(\chi)$ to be the restricted tensor product defined using the spherical vectors $f_v^{sph} \in I_n(\chi_v), f_v^{sph}(K_{n,v}) = 1$, at the finite places v where χ_v is unramified.

For $f \in I_n(\chi)$ we consider the Eisenstein series

$$E(f; z, g) := \sum_{\gamma \in Q_n(F) \backslash G_n(F)} f(z, \gamma g).$$

This series converges absolutely and uniformly for (z, g) in compact subsets of $\{\mathrm{Re}(z) > n/2\} \times G_n(\mathbb{A}_F)$. The automorphic form defined is called Siegel Eisenstein series.

Let $\varphi : \mathcal{U} \to I_n(\chi)$ be a meromorphic section, then we put $E(\varphi; z, g) = E(\varphi(z); z, g)$. This is defined at least on the region of absolute convergence and it is well known that it can be meromorphically continued to all $z \in \mathbb{C}$.

Now for $f \in I_n(\chi), z \in \mathbb{C}$, and $k \in \prod_{v \nmid \infty} K_{n,v} \prod_{v | \infty} K_\infty$ there is a similar intertwining integral $M(z,f)(k)$ as above but with the integration being over $N_{Q_n}(\mathbb{A}_F)$. This again converges absolutely and uniformly for z in compact subsets of $\{\mathrm{Re}(z) > n/2\} \times K_n$. Thus $z \mapsto M(z,f)$ defines a holomorphic section $\{Re(z) > n/2\} \to I_n(\bar{\chi}^c)$. This has a continuation to a meromorphic section on $\mathbb{C}$. For $\mathrm{Re}(z) > n/2$, we have

$$M(z,f) = \otimes_v M(z, f_v), f = \otimes f_v.$$

The functional equation for Siegel Eisenstein series is:

$$E(f, z, g) = \chi^n(\mu(g)) E(M(z,f); -z, g)$$

in the sense that both sides can be meromorphically continued to all $z \in \mathbb{C}$ and the equality is understood as of meromorphic functions of $z \in \mathbb{C}$.

3.2.3 The Pullback Formulas

Let χ be a unitary idele class character of $\mathbb{A}_{\mathcal{K}}^\times$. Given a unitary tempered cuspidal eigenform φ on $\mathrm{GU}(r,s)$ which is a pure tensor we formally define the integral

$$F_\varphi(f;z,g) := \int_{\mathrm{U}(r,s)(\mathbb{A}_F)} f(z, S^{-1}\alpha(g,g_1h)S)\bar{\chi}(\det g_1 g)\varphi(g_1h)dg_1,$$

$$f \in I_{r+s+1}(\chi), g \in \mathrm{GU}(r+1,s+1)(\mathbb{A}_F), h \in \mathrm{GU}(r,s)(\mathbb{A}_F), \mu(g) = \mu(h).$$

This is independent of h. (We suppress the χ in the notation for F_φ since its choice is implicitly given by f). We also formally define

$$F'_\varphi(f;z,g) := \int_{\mathrm{U}(r,s)(\mathbb{A}_F)} f(z, S'^{-1}\alpha(g,g_1h)S')\bar{\chi}(\det g_1 g)\varphi(g_1h)dg_1,$$

$$f \in I_{r+s}(\chi), g \in \mathrm{GU}(r,s)(\mathbb{A}_F), h \in \mathrm{GU}(r,s)(\mathbb{A}_F), \mu(g) = \mu(h)$$

The pullback formulas are the identities in the following proposition.

Proposition 3.5. *Let χ be a unitary idele class character of $\mathbb{A}_{\mathcal{K}}^\times$.*
(i) If $f \in I_{r+s}(\chi)$, then $F_\varphi(f;z,g)$ converges absolutely and uniformly for (z,g) in compact sets of $\{\mathrm{Re}(z) > r+s\} \times \mathrm{GU}(r,s)(\mathbb{A}_F)$, and for any $h \in \mathrm{GU}(r,s)(\mathbb{A}_F)$ such that $\mu(h) = \mu(g)$

$$\int_{\mathrm{U}(r,s)(F)\backslash \mathrm{U}(r,s)(\mathbb{A}_F)} E(f;z,S'^{-1}\alpha(g,g_1h)S')\bar{\chi}(\det g_1h)\varphi(g_1h)dg_1 = F'_\varphi(f;z,g). \tag{9}$$

(ii) If $f \in I_{r+s+1}(\chi)$, then $F_\varphi(f;z,g)$ converges absolutely and uniformly for (z,g) in compact sets of $\{\mathrm{Re}(z) > r+s+1/2\} \times \mathrm{GU}(r+1,s+1)(\mathbb{A}_F)$ such that $\mu(h) = \mu(g)$

$$\begin{aligned}\int_{\mathrm{U}(r,s)(F)\backslash \mathrm{U}(r,s)(\mathbb{A}_F)} & E(f;z,S^{-1}\alpha(g,g_1h)S)\bar{\chi}(\det g_1h)\varphi(g_1h)dg_1 \\ &= \sum_{\gamma\in P(F)\backslash G(r+1,s+1)(F)} F_\varphi(f;z,\gamma g),\end{aligned} \tag{10}$$

with the series converging absolutely and uniformly for (z,g) in compact subsets of $\{\mathrm{Re}(z) > r+s+1/2\} \times \mathrm{GU}(r+1,s+1)(\mathbb{A}_F)$.

Proof. The global integral F_φ and F'_φ can be written as a product of local integrals. The absolute convergence of local integrals for F'_φ is proved in [21, Lemma 2]. The absolute convergence for the global integral F'_φ follows from this and the explicit computations in [21] at all unramified places, together with the temperedness assumption of φ. The absolute convergence for F_φ is proved in the same way. Then part (i) is proved by Piatetski-Shapiro and Rallis [7] and (ii) is a straight-forward generalization by Shimura [24], which is in turn due to Garrett (in earlier works [5], [6]). Both are straightforward consequences of the double coset decompositions in [24, Proposition 2.4, 2.7]. ☐

3.3 Fourier-Jacobi Expansion

3.3.1 Fourier-Jacobi Expansion

We will usually use the notation $e_{\mathbb{A}}(x) = e_{\mathbb{A}_\mathbb{Q}}(\mathrm{Tr}_{\mathbb{A}_F/\mathbb{A}_\mathbb{Q}} x)$ for $x \in \mathbb{A}_F$. For any automorphic form φ on $\mathrm{GU}(r,s)(\mathbb{A}_F)$, $\beta \in S_m(F)$ for $m \leq s$. We define the Fourier-Jacobi coefficient at $g \in \mathrm{GU}(r,s)(\mathbb{A}_F)$ as

$$\varphi_\beta(g) = \int_{S_m(F)\backslash S_m(\mathbb{A}_F)} \varphi(\begin{pmatrix} 1_s & 0 & \begin{matrix} S & 0 \\ 0 & 0 \end{matrix} \\ 0 & 1_{r-s} & 0 \\ 0 & 0 & 1_s \end{pmatrix} g)e_{\mathbb{A}}(-\mathrm{Tr}(\beta S))dS.$$

In fact we are mainly interested in two cases: $m = s$ or $r = s$ and arbitary $m \leq s$. In particular, suppose $G = G_n = \mathrm{GU}(n,n)$, $0 \leq m \leq n$ are integers, $\beta \in S_m(F)$. Let φ be a function on $G(F)\backslash G(\mathbb{A})$. The β-th Fourier-Jacobi coefficient φ_β of φ at g is defined by

$$\varphi_\beta(g) := \int \varphi(\begin{pmatrix} 1_n & \begin{matrix} S & 0 \\ 0 & 0 \end{matrix} \\ & 1_n \end{pmatrix} g) e_{\mathbb{A}}(-\mathrm{Tr}\beta S) dS.$$

Now we prove a useful formula on the Fourier-Jacobi coefficients for Siegel Eisenstein series.

Definition 3.6. *Put:*

$$Z := \{\begin{pmatrix} 1_n & \begin{matrix} z & 0 \\ 0 & 0 \end{matrix} \\ 0_n & 1_n \end{pmatrix} | z \in S_m(\mathcal{K})\}$$

$$V := \{\begin{pmatrix} 1_m & x & z & y \\ & 1_{n-m} & y^* & 0 \\ & & 1_m & \\ 0_n & & -x^* & 1_{n-m} \end{pmatrix} | x, y \in M_{m(n-m)}(\mathcal{K}), z - xy^* \in S_m(\mathcal{K})\}$$

$$X := \{\begin{pmatrix} 1_m & x & & \\ & 1_{n-m} & & 0_n \\ & & 1_m & \\ 0_n & & -x^* & 1_{n-m} \end{pmatrix} | x \in M_{m(n-m)}(\mathcal{K})\}$$

$$Y := \{\begin{pmatrix} 1_n & \begin{matrix} z & y \\ y^* & 0 \end{matrix} \\ 0_n & 1_n \end{pmatrix} | y \in M_{m(n-m)}(\mathcal{K})\}.$$

From now on we will usually write w_n for $\begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}$.

Proposition 3.7. *Let f be in $I_n(\tau)$ and Suppose $\beta \in S_m(F)$ is totally positive. If $E(f; z, g)$ is the Siegel Eisenstein Series on GU_n defined by f for some $\mathrm{Re}(z)$ sufficiently large then the β-th Fourier-Jacobi coefficient $E_\beta(f; z, g)$ satisfies:*

$$E_\beta(f; z, g) = \sum_{\gamma \in Q_{n-m}(F)\backslash \mathrm{GU}_{n-m}(F)} \sum_{y \in Y} \int_{S_m(\mathbb{A})} f(w_n \begin{pmatrix} 1_n & \begin{matrix} S & y \\ {}^t\bar{y} & 0 \end{matrix} \\ & 1_n \end{pmatrix} \alpha_{n-m}(1, \gamma) g) e_{\mathbb{A}}(-\mathrm{Tr}\beta S) dS$$

where α_{n-m} is defined by: if $g_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ then $\alpha_{n-m}(\gamma) = \begin{pmatrix} 1 & & & \\ & D & & C \\ & & 1 & \\ & B & & A \end{pmatrix}$ where A, B, C, D are $(n-m) \times (n-m)$ matrices.

Proof. We follow [16, Section 3]. Let H be the normalizer of V in G. Then

$$G_n(F) = \sqcup_{i=1}^m Q_n(F) \xi_i H(F)$$

for $\xi_i := \begin{pmatrix} 0_{m-i} & 0 & -1_{m-i} & 0 \\ 0 & 1_{n-m+i} & 0 & 0 \\ 1_{m-i} & 0 & 0_{m-i} & 0 \\ 0 & 0_{n-m+i} & 0 & 1_{n-m+i} \end{pmatrix}$. Unfolding the Eisenstein series we get:

$$\begin{aligned} E_\beta(f;z,g) = & \sum_{i>0} \sum_{\gamma \in Q_n(F)\backslash Q_n(F)\xi_i H(F)} \int f(\gamma \begin{pmatrix} 1_n & \begin{matrix} S & 0 \\ 0 & 0 \end{matrix} \\ & 1_n \end{pmatrix} g) e_{\mathbb{A}}(-Tr(\beta S))dS \\ & + \sum_{\gamma \in Q_n(F)\backslash Q_n(F)\xi_0 H(F)} \int f(\gamma \begin{pmatrix} 1_n & \begin{matrix} S & 0 \\ 0 & 0 \end{matrix} \\ & 1_n \end{pmatrix} g) e_{\mathbb{A}}(-Tr(\beta S))dS. \end{aligned}$$

By [16, Lemma(3.1)] (see *loc.cit* P628), the first term vanishes. Also, we have (*loc.cit*)

$$\begin{aligned} & Q_n(F)\backslash Q_n(F)\xi_0 H(F) \\ = & \quad \xi_0 Z(F)X(F)Q_{n-m}(F)\backslash G_{n-m}(F) \\ = & \quad \xi_0 X(F).Q_{n-m}(F)\backslash G_{n-m}(F).Z(F) \\ = & \quad w_n Y(F) S_m(F) w_{n-m} Q_{n-m}(F)\backslash G_{n-m}(F) \end{aligned}$$

(note that S_m commutes with X and G_{n-m}). So

$$E_\beta(f;z,g) = \sum_{\gamma \in Q_{n-m}(F)\backslash G_{n-m}(F)} \sum_{y \in Y(F)} \int_{S_m(\mathbb{A})} f(w_n \begin{pmatrix} 1_n & \begin{matrix} S & y \\ {}^t\bar{y} & 0 \end{matrix} \\ & 1_n \end{pmatrix} \alpha_{n-m}(1,\gamma)g) e_{\mathbb{A}}(-\mathrm{Tr}(\beta S))dS$$

Note that the final integral is essentially a product of local ones. □

Now we record some useful formulas:

Definition 3.8. *If $g_v \in \mathrm{U}_{n-m}(F_v), x \in \mathrm{GL}_m(\mathcal{K}_v)$, then define:*

$$FJ_\beta(f_v;z,y,g,x) = \int_{S_m(F_v)} f(w_n \begin{pmatrix} 1_n & \begin{matrix} S & y \\ {}^t\bar{y} & 0 \end{matrix} \\ & 1_n \end{pmatrix} \alpha(\mathrm{diag}(x, {}^t\bar{x}^{-1}), g)) e_{F_v}(-\mathrm{Tr}\beta S)dS$$

where if $g_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, g_2 = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$ then:

$$\alpha(g_1,g_2) = \begin{pmatrix} A & & B & \\ & D' & & C' \\ C & & D & \\ & B' & & A' \end{pmatrix}.$$

We also define

$$f_{v,\beta,z}(g) := f(z, w_n \begin{pmatrix} 1_n & S \\ & 1_n \end{pmatrix} g) e_v(-\mathrm{Tr}\beta S)dS.$$

Since

$$\begin{pmatrix} 1_n & \begin{matrix} S & X \\ {}^t\bar{X} & \end{matrix} \\ & 1_n \end{pmatrix} \begin{pmatrix} 1_m & & & \\ & \bar{A}^{-1} & & \\ & & 1_m & \\ & B\bar{A}^{-1} & & A \end{pmatrix} = \begin{pmatrix} 1_m & XB\bar{A}^{-1} & & \\ & \bar{A}^{-1} & & \\ & & 1_m & \\ & B\bar{A}^{-1} & & A \end{pmatrix} \begin{pmatrix} 1_n & \begin{matrix} S - XB{}^t\bar{X} & XA \\ \bar{A}{}^t\bar{X} & \end{matrix} \\ & 1_n \end{pmatrix},$$

it follows that:

$$FJ_\beta(f;z,X,\begin{pmatrix}A & B\bar{A}^{-1}\\ & \bar{A}^{-1}\end{pmatrix}g,Y)=$$

$$\tau_v^c(\det A)^{-1}|\det A\bar{A}|_v^{z+n/2}e_v(-tr({}^t\bar{X}\beta XB))FJ_\beta(f;z,XA,g,Y).$$

Also we have:

$$FJ_\beta(f;z,y,g,x)=\tau_v(\det x)|\det x\bar{x}|_{\mathbb{A}}^{-(z+\frac{n}{2}-m)}FJ_{{}^t\bar{x}\beta x}(f;z,x^{-1}y,g,1).$$

3.3.2 Weil Representations

We define the Weil representations which will be used in calculating local Fourier-Jacobi coefficients in the next section.

The local set-up.
Let v be a place of F. Let $h\in S_m(F_v),\det h\neq 0$. Let U_h be the unitary group of this metric and denote V_v to be the corresponding Hermitian space. Let $V_{n-m}:=\mathcal{K}_v^{(n-m)}\oplus\mathcal{K}_v^{(n-m)}:=X_v\oplus Y_v$ be the skew-Hermitian space associated to $\mathrm{U}(n-m,n-m)$. Let $W:=V_v\otimes_{\mathcal{K}_v}V_{n-m,v}$. Then $(-,-):=\mathrm{Tr}_{\mathcal{K}_v/F_v}(\langle -,-\rangle_h\otimes_{\mathcal{K}_v}\langle -,-\rangle_{n-m})$ is a F_v linear pairing on W that makes W into an $4m(n-m)$-dimensional symplectic space over F_v. The canonical embeding of $\mathrm{U}_h\times\mathrm{U}_{n-m}$ into $\mathrm{Sp}(W)$ realizes the pair $(\mathrm{U}_h,\mathrm{U}_{n-m})$ as a dual pair in $\mathrm{Sp}(W)$. Let λ_v be a character of $\mathcal{K}_v^\times$ such that $\lambda_v|_{F_v^\times}=\chi^m_{\mathcal{K}/F,v}$. It is well known (see [17]) that there is a splitting $\mathrm{U}_h(F_v)\times\mathrm{U}_{n-m}(F_v)\hookrightarrow\mathrm{Mp}(W,F_v)$ of the metaplectic cover $\mathrm{Mp}(W,F_v)\to\mathrm{Sp}(W,F_v)$ determined by the character λ_v. This gives the Weil representation $\omega_{h,v}(u,g)$ of $\mathrm{U}_h(F_v)\times\mathrm{U}_{n-m}(F_v)$ where $u\in\mathrm{U}_h(F_v)$ and $g\in\mathrm{U}_{n-m}(F_v)$, via the Weil representation of $\mathrm{Mp}(W,F_v)$ on the space of Schwartz functions $\mathcal{S}(V_v\otimes_{\mathcal{K}_v}X_v)$. Moreover we write $\omega_{h,v}(g)$ to mean $\omega_{h,v}(1,g)$. For $X\in M_{m\times(n-m)}(\mathcal{K}_v)$, we define $\langle X,X\rangle_h:={}^t\bar{X}\beta X$ (note that this is a $(n-m)\times(n-m)$ matrix). We record here some useful formulas for $\omega_{h,v}$ which are generalizations of the formulas in [28, section 10].

- $\omega_{h,v}(u,g)\Phi(X)=\omega_{h,v}(1,g)\Phi(u^{-1}X)$,
- $\omega_{h,v}(\mathrm{diag}(A,{}^t\bar{A}^{-1}))\Phi(X)=\lambda(\det A)|\det A|_{\mathcal{K}}\Phi(XA)$,
- $\omega_{h,v}(r(S))\Phi(x)=\Phi(x)e_v(\mathrm{tr}\langle X,X\rangle_hS)$,
- $\omega_{h,v}(\eta)\Phi(x)=|\det h|_v\int\Phi(Y)e_v(\mathrm{tr}_{\mathcal{K}_v/F_v}(\mathrm{tr}\langle Y,X\rangle_h))dY$.

Global setup:
Let $h\in S_m(F)$ be positive definite. We can define global versions of $\mathrm{U}_h,\mathrm{GU}_h,X,Y,W$, and $(-,-)$, analogously to the local case. Fixing an idele class character $\lambda=\otimes\lambda_v$ of $\mathbb{A}_{\mathcal{K}}^\times/\mathcal{K}^\times$ such that $\lambda|_{F^\times}=\chi^m_{\mathcal{K}/F}$, the associated local splitting described above then determines a global splitting $\mathrm{U}_h(\mathbb{A}_F)\times\mathrm{U}_1(\mathbb{A}_F)\hookrightarrow\mathrm{Mp}(W,\mathbb{A}_F)$ and hence an action $\omega_h:=\otimes\omega_{h,v}$ of $\mathrm{U}_h(\mathbb{A}_F)\times\mathrm{U}_1(\mathbb{A}_F)$ on the Schwartz space $\mathcal{S}(V_{\mathbb{A}_{\mathcal{K}}}\otimes_{\mathcal{K}}X)$.

4 Local Computations

In this section we do the local computations for Klingen Eisenstein sections realized as the pullbacks of Siegel Eisenstein sections. We will compute the Fourier and Fourier-Jacobi coefficients for the Siegel sections and the pullback Klingen Eisenstein sections.

4.1 Archimedean Computations

Let v be an Archimedean place of F.

4.1.1 Fourier Coefficients

Definition 4.1.

$$f_{\kappa,n}(z,g)=J_n(g,i1_n)^{-\kappa}|J_n(g,i1_n)|^{\kappa-2z-n}.$$

Now we recall [28, Lemma 11.4]. Let $J_n(g,i1_n):=\det(C_g i1_n+D_g)$ for $g=\begin{pmatrix}A_g & B_g\\ C_g & D_g\end{pmatrix}$.

Lemma 4.2. *Suppose $\beta\in S_n(\mathbb{R})$. Then the function $z\to f_{\kappa,\beta}(z,g)$ has a meromorphic continuation to all of $\mathbb{C}$. Furthermore, if $\kappa\geq n$ then $f_{\kappa,n,\beta}(z,g)$ is holomorphic at $z_\kappa:=(\kappa-n)/2$ and for $y\in\mathrm{GL}_n(\mathbb{C})$, $f_{\kappa,n,\beta}(z_\kappa,\mathrm{diag}(y,{}^t\bar{y}^{-1}))=0$ if $\det\beta\leq 0$ and if $\det\beta>0$ then*

$$f_{\kappa,n,\beta}(z_\kappa,\mathrm{diag}(y,{}^t\bar{y}^{-1}))=\frac{(-2)^{-n}(2\pi i)^{n\kappa}(2/\pi)^{n(n-1)/2}}{\prod_{j=0}^{n-1}(\kappa-j-1)!}e_v(i\mathrm{Tr}(\beta y{}^t\bar{y}))\det(\beta)^{\kappa-n}\det\bar{y}^\kappa.$$

4.1.2 Pullback Sections

Now we assume that our π is the holomorphic discrete series representation associated to the (scalar) weight $(0,...,0;\kappa,...,\kappa)$ and let φ be the unique (up to scalar) vector such that the action of $K_\infty^{+,'}$ (see subsection 3.1) is given by $\det\mu(k,i)^{-\kappa}$. Recall also that in subsection 3.1 we have defined the Klingen section $F_\kappa(z,g)$ there (denoted as F_κ). Recall we have defined S and S' in equations (1) and (2). Let $\mathbf{i}:=\begin{pmatrix}\frac{i}{2}1_b &&&\\ & i &&\\ && \frac{\zeta}{2} &\\ &&& \frac{i}{2}1_b\end{pmatrix}$ or $\begin{pmatrix}\frac{i}{2}1_b &&\\ & \frac{\zeta}{2} &\\ && \frac{i}{2}1_b\end{pmatrix}$ be the distinguished point in the symmetric domain for $\mathrm{GU}(n,n)$ or $\mathrm{GU}(n+1,n+1)$ for $n=a+2b$. We define archimedean sections to be:

$$f_\kappa(g)=J_{n+1}(g,\mathbf{i})^{-\kappa}|J_{n+1}(g,\mathbf{i})|^{\kappa-2z-n-1}$$

and

$$f'_\kappa(g)=J_n(g,\mathbf{i})^{-\kappa}|J_n(g,\mathbf{i})|^{\kappa-2z-n}$$

and the pullback sections on $\mathrm{GU}(a+b+1,b+1)$ and $\mathrm{GU}(a+b,a)$ to be

$$F_\kappa(z,g):=\int_{\mathrm{U}(a+b,b)(\mathbb{R})}f_\kappa(z,S^{-1}\alpha(g,g_1)S)\bar{\tau}(\det g_1)\pi(g_1)\varphi dg_1$$

and

$$F'_\kappa(z,g):=\int_{\mathrm{U}(a+b,b)(\mathbb{R})}f'_\kappa(z,S'^{-1}\alpha(g,g_1)S')\bar{\tau}(\det g_1)\pi(g_1)\varphi dg_1.$$

Lemma 4.3. *The integrals F_κ and F'_κ are absolutely convergent for $\mathrm{Re}(z)$ sufficiently large and for such z, we have:*
(i)

$$F_\kappa(z,g)=c_\kappa(z)F_{\kappa,z}(g);$$

(ii)

$$F'_\kappa(z,g)=c'_\kappa(z)\pi(g)\varphi;$$

where

$$c'_\kappa(z,g)=2^\nu|\det\zeta|_v^b\begin{cases}\pi^{(a_v+b_v)b_v}\Gamma_{b_v}(z+\frac{n+\kappa}{2}-a_v-b_v)\Gamma_{b_v}(z+\frac{n+k}{2})^{-1}, & b>0\\ 1, & \textit{otherwise.}\end{cases}$$

and $c_\kappa(z,g)=c'_\kappa(z+\frac{1}{2},g)$. Here $\Gamma_m(s):=\pi^{\frac{m(m+1)}{2}}\prod_{k=0}^{m-1}\Gamma(s-k)$ and $\nu:=(a+2b)db$ (recall that $d=[F:\mathbb{Q}]$).

Proof. See [24, 22.2, A2.9]. Note that the action of $(\beta,\gamma)\in \mathrm{U}(r,s)\times\mathrm{U}(s,r)$ is given by (β',γ') defined there. Taking this into consideration, our conjugation matrix S are Shimura's S times Σ^{-1} (notation as in *loc.cit*), which is defined in (22.1.2) in [24]. Also our result differs from [28, Lemma 11.6] by some powers of 2 since we are using a different S here. $\square$

4.1.3 Fourier-Jacobi Coefficients

We write $FJ_{\beta,\kappa}$ for the Fourier-Jacobi coefficient defined in Definition 3.8 with f_v chosen as $f_{\kappa,n}$.

Lemma 4.4. *Let* $z_\kappa=\frac{\kappa-n}{2}$*,* $\beta\in S_m(\mathbb{R})$*,* $m<n$*,* $\det\beta>0$*. then:*

(i) $FJ_{\beta,\kappa}(z_\kappa,x,\eta,1)=f_{\kappa,m,\beta}(z_\kappa+\frac{n-m}{2},1)e(i\mathrm{Tr}({}^t\bar{X}\beta X))$*;*

(ii) If $g\in U_{n-m}(\mathbb{R})$*, then*

$$FJ_{\beta,\kappa}(z_\kappa,X,g,1)=e(iTr\beta)c_m(\beta,\kappa)f_{\kappa-m,n-m}(z_\kappa,g')w_\beta(g')\Phi_{\beta,\infty}(x).$$

where $g'=\begin{pmatrix}1_n & \\ & -1_n\end{pmatrix}g\begin{pmatrix} & 1_n\\ 1_n & \end{pmatrix}$, $c_t(\beta,\kappa)=\frac{(-2)^{-t}(2\pi i)^{t\kappa}(2/\pi)^{t(t-1)/2}}{\prod_{j=0}^{t-1}(\kappa-j-1)}\det\beta^{\kappa-t}$ *and* $\Phi_{\beta,\infty}(x)=e^{-2\pi\mathrm{Tr}(\langle x,x\rangle_\beta)}$.

Proof. Our proof is similar to [28, Lemma 11.5]. For (i) we first assume that $m\le n/2$, then there is a matrix $U\in\mathrm{U}_{n-m}$ such that $XU=(0,A)$ for A a $(m\times m)$ positive semi-definite Hermitian matrix . It then follows that $FJ_{\beta,\kappa}(z,X,\eta,1)=FJ_{\beta,\kappa}(z,(0,A),\eta,1)$ and $e(iTr({}^t\bar{X}\beta X))=e(i\mathrm{Tr}(U^{-1}{}^t\bar{X}\beta XU))$, so we are reduced to the case when $X=(0,A)$.

Let C be a $(m\times m)$ positive definite Hermitian matrix defined by $C=\sqrt{A^2+1}$. (Since A is positive semi-definite Hermitian, this C exists by linear algebra.) We have

$$\begin{pmatrix} & & & A\\ 1_n & & & \\ & A & & \\ & & 1_n & \end{pmatrix}=\begin{pmatrix} C & & & & & \\ & 1 & & & & \\ & & C & & & \\ & & AC^{-1} & C^{-1} & & \\ & & & & 1 & \\ AC^{-1} & & & & & C^{-1}\end{pmatrix}\begin{pmatrix} C^{-1} & & & & & C^{-1}A\\ & 1 & & & & \\ & & C^{-1} & C^{-1}A & & \\ & & -C^{-1}A & C^{-1} & & \\ & & & & 1 & \\ -C^{-1}A & & & & & C^{-1}\end{pmatrix}.$$

Write $k(A)$ for the second matrix in the right of above which belongs to $K^+_{n,\infty}$, then as in [28, Lemma 11.5],

$$w_n\begin{pmatrix} & S & X\\ 1_n & {}^t\bar{X} & \\ & 1_n & \end{pmatrix}=\begin{pmatrix} C^{-1} & & & \times & \times & \times\\ & 1 & & \times & \times & \times\\ \times & \times & C^{-1} & \times & \times & \times\\ & & & C & & \\ & & & & 1 & \\ & & & \times & \times & C\end{pmatrix}w_n\begin{pmatrix} & U^{-1}SU^{-1} & \\ 1_n & & \\ & & 1_n\end{pmatrix}k(A).$$

Thus

$$\begin{aligned} FJ_{\beta,\kappa}(z_\kappa,(0,A),\eta,1) &= (\det C)^{2m-2\kappa}FJ_{\beta',\kappa}(z_\kappa,0,\eta,1),\beta'=C\beta C\\ &= (\det C)^{2m-2\kappa}f_{\kappa,m,\beta'}(z_\kappa+\frac{n-m}{2},1)\\ &= f_{\kappa,m,\beta}(z+\frac{n-m}{2},1)e(i\mathrm{Tr}(C\beta C-\beta)).\end{aligned}$$

But

$$e(i\mathrm{Tr}(C\beta C-\beta))=e(i\mathrm{Tr}(C^2\beta-\beta))=e(i\mathrm{Tr}((C^2-1)\beta))=e(i\mathrm{Tr}(A^2\beta))=e(i\mathrm{Tr}(A\beta A)).$$

This proves part (i).

Part (ii) is proved completely the same as in [28, Lemma 11.5].

In the case when $m>\frac{n}{2}$ we proceed similarly as in [28, Lemma 11.5], replacing a and u there by corresponding block matrices just as above. We omit the details.

□

4.2 Finite Primes, Unramified Case

4.2.1 Pullback Integrals

Lemma 4.5. *Suppose π,ψ and τ are unramified and $\varphi\in\pi$ is a new vector. If $\mathrm{Re}(z)>(a+b)/2$ then the pullback integral converges and*

$$F_\varphi(f_v^{sph};z,g)=\frac{L(\tilde{\pi},\bar{\tau}^c,z+1)}{\prod_{i=0}^{a+2b-1}L(2z+a+2b+1-i,\bar{\tau}'\chi_{\mathcal{K}}^i)}F_{\rho,z}(g)$$

where $F_{\rho,z}$ is the spherical section taking value φ at the identity and

$$F_\varphi(f_v^{sph};z,g)=\frac{L(\tilde{\pi},\bar{\tau}^c,z+\frac{1}{2})}{\prod_{i=0}^{a+2b-1}L(2z+a+2b-i,\bar{\tau}'\chi_{\mathcal{K}}^i)}\pi(g)\varphi.$$

This is computed in [21, Proposition 3.3].

4.2.2 Fourier-Jacobi Coefficients

Let v be a prime of F not dividing p and τ a character of $\mathcal{K}_v^\times$. For $f\in I_n(\tau)$ and $\beta\in S_m(F_v), 0\leq m\leq n$, we define the local Fourier-Jacobi coefficient to be

$$f_\beta(z;g):=\int_{S_m(F_v)}f(z,w_n\begin{pmatrix}1_n & \begin{matrix}S & 0\\ 0 & 0\end{matrix}\\ & 1_n\end{pmatrix}g)e_v(-\mathrm{Tr}\beta S)dS.$$

We first record straightforward generalizations of [28, Lemma 11.7, Lemma 11.8] to any fields (Propositions 18.14 and 19.2 of [24]).

Lemma 4.6. *Let $\beta\in S_n(F_v)$ and let $r:=\mathrm{rank}(\beta)$. Then for $y\in\mathrm{GL}_n(\mathcal{K}_v)$,*

$$\begin{aligned}f_{v,\beta}^{sph}(z,diag(y,{}^t\bar{y}^{-1}))= & \quad\tau(\mathrm{det}y)|\mathrm{det}y\bar{y}|_v^{-z+n/2}D_v^{-n(n-1)/4}\\ & \times\frac{\prod_{i=r}^{n-1}L(2z+i-n+1,\bar{\tau}'\chi_{\mathcal{K}}^i)}{\prod_{i=0}^{n-1}L(2z+n-i,\bar{\tau}'\chi_{\mathcal{K}}^i)}h_{v,{}^t\bar{y}\beta y}(\bar{\tau}'(\varpi)q_v^{-2z-n}).\end{aligned}$$

where $h_{v,{}^t\bar{y}\beta y}\in\mathbb{Z}[X]$ is a monic polynomial depending on v and ${}^t\bar{y}\beta y$ but not on τ. If $\beta\in S_n(\mathcal{O}_{F,v})$ and $\det\beta\in\mathcal{O}_{F,v}^\times$, then we say that β is v-primitive and in this case $h_{v,\beta}=1$.

Lemma 4.7. *Suppose v is unramified in $\mathcal{K}$. Let $\beta\in S_m(F_v)$ such that $\det\beta\neq 0$. Let $\beta\in S_m(\mathcal{O}_{F_v})$, let λ be an unramified character of $\mathcal{K}_v^\times$ such that $\lambda|_{F_v^\times}=1$. If $\beta\in\mathrm{GL}_m(\mathcal{O}_v)$, then for $u\in\mathrm{U}_\beta(F_v)$:*

$$FJ_\beta(f_n^{sph};z,x,g,u)=\tau(\det u)|\det u\bar{u}|_v^{-z+1/2}\frac{f_{n-m}^{sph}(z,g)\omega_\beta(u,g)\Phi_0(x)}{\prod_{i=0}^{m-1}L(2z+n-i,\bar{\tau}'\chi_{\mathcal{K}}^i)}.$$

4.3 Prime to p Ramified Case

4.3.1 Pullback integrals

Again let v be a prime of F not dividing p. We fix some x and y in $\mathcal{K}$ which are divisible by some high power of ϖ_v (can be made precise from the proof of the following two lemmas). (When we are moving things p-adically the x and y are not going to change). We define $f^\dagger \in I_{n+1}(\tau)$ to be the Siegel section supported on the cell $Q(F_v)w_{a+2b+1}N_Q(\mathcal{O}_{F,v})$ where $w_{a+2b+1} = \begin{pmatrix} & 1_{a+2b+1} \\ -1_{a+2b+1} & \end{pmatrix}$ and the value at $N_Q(\mathcal{O}_{F,v})$ equals 1. Similarly we define $f^{\dagger,\prime} \in I_n(\tau)$ to be the section supported in $Q(F_v)w_{a+2b}N_Q(\mathcal{O}_{F,v})$ and takes value 1 on $N_Q(\mathcal{O}_{F,v})$.

Definition 4.8.

$$f_{v,sieg}(g) := f^\dagger(g\tilde{S}_v^{-1}\tilde{\gamma}_v) \in I_{n+1}(\tau)$$

where $\tilde{\gamma}_v$ is defined to be:

$$\begin{pmatrix} 1_b & & & & & & & \frac{1}{x}1_b \\ & 1 & & & & & & \\ & & 1_a & & & & \frac{1}{y\bar{y}}1_a & \\ & & & 1_b & \frac{1}{\bar{x}}1_b & & & \\ & & & & 1_b & & & \\ & & & & & 1 & & \\ & & & & & & 1_a & \\ & & & & & & & 1_b \end{pmatrix}$$

and

$$\tilde{S}_v = \begin{pmatrix} 1_b & & & & & & & -\frac{1}{2}1_b \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & -1_b & \frac{1}{2}1_b & & & \\ & & & 1_b & \frac{1}{2}1_b & & & \\ & & & & & 1 & & \\ & & & & & & 1_a & \\ -1_b & & & & & & & -\frac{1}{2}1_b \end{pmatrix}.$$

Similarly we define $f'_{v,sieg}(g) := f^{\dagger,\prime}(g\tilde{S}_v^{-1}\tilde{\gamma}'_v)$ for

$$\tilde{S}'_v := \begin{pmatrix} 1_b & & & & & -\frac{1}{2}1_b \\ & 1_a & & & & \\ & & -1_b & \frac{1}{2}1_b & & \\ & & 1_b & \frac{1}{2}1_b & & \\ & & & & 1_a & \\ -1_b & & & & & -\frac{1}{2}1_b \end{pmatrix}$$

and

$$\tilde{\gamma}_v = \begin{pmatrix} 1_b & & & & & \frac{1}{x}1_b \\ & 1_a & & & \frac{1}{y\bar{y}}1_a & \\ & & 1_b & \frac{1}{\bar{x}}1_b & & \\ & & & 1_b & & \\ & & & & 1_a & \\ & & & & & 1_b \end{pmatrix}.$$

Lemma 4.9. *Let $K_v^{(2)}$ be the subgroup of $G(F_v)$ of the form* $\begin{pmatrix} 1_b & & & & d \\ a & 1 & f & b & c \\ & & 1_a & & g \\ & & & 1_b & e \\ & & & & 1 \end{pmatrix}$ *where* $e = -{}^t\bar{a}$, $b = {}^t\bar{d}$, $g = -\zeta {}^t\bar{f}$, $b \in M(\mathcal{O}_v)$, $c - f\zeta {}^t\bar{f} \in \mathcal{O}_{F,v}$, $a \in (x)$, $e \in (\bar{x})$, $f \in (y\bar{y})$, $g \in (2\zeta y\bar{y})$. *Then $F_\varphi(z; g, f)$ is supported in $PwK_v^{(2)}$ and is invariant under the action of $K_v^{(2)}$.*

Proof. Let $S_{x,y}$ consist of matrices $S := \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{pmatrix}$ in the space of Hermitian $(a+2b+1) \times (a+2b+1)$ matrices (the blocks are with respect to the partition $b+1+a+b$) such that the entries of S_{13}, S_{23} are divisible by y, the entries of S_{14}, S_{24} are divisible by x, the entries of S_{31}, S_{32} are divisible by $\bar{y}$, the entries of S_{41}, S_{42} are divisible by $\bar{x}$, the entries of S_{33} are divisible by $y\bar{y}$, the entries of S_{34} are divisible by $x\bar{y}$, the entries of S_{43} are divisible by $\bar{x}y$, and the entries of S_{44} are divisible by $x\bar{x}$. Let $Q_{x,y} := Q(F_v) \cdot \begin{pmatrix} 1 & \\ S_{x,y} & 1 \end{pmatrix}$.

Write $\eta = \begin{pmatrix} & & 1_b \\ & 1_a & \\ -1_b & & \end{pmatrix}$. As in [28, Proposition 11.16] for $g = \begin{pmatrix} a_1 & a_2 & a_3 & b_1 & b_2 \\ a_4 & a_5 & a_6 & b_3 & b_4 \\ a_7 & a_8 & a_9 & b_5 & b_6 \\ c_1 & c_2 & c_3 & d_1 & d_2 \\ c_4 & c_5 & c_6 & d_3 & d_4 \end{pmatrix}$, we have:

$$\begin{aligned} & \gamma(g,1) \in \mathrm{supp} f_{v,sieg} \\ \Leftrightarrow \quad & S_v^{-1}\alpha(g,1)Aw_{a+2b+1}d_{x,y}\tilde{\gamma}^{-1} \in Q_{x,y} \\ \Leftrightarrow \quad & S_v^{-1}\alpha(gw, \eta\mathrm{diag}(\bar{x}^{-1}, 1, x))Aw'd_y\tilde{\gamma}^{-1} \in Q_{x,y}. \end{aligned}$$

Here $A = \begin{pmatrix} 1_b & & & & & & & \\ & 1 & & & & & & \\ & & 1_a & & & & -\frac{\zeta}{2} & \\ & & & 1_b & & & & \\ & & & & 1_b & & & \\ & & & & & 1 & & \\ & & -1_a & & & & -\frac{\zeta}{2} & \\ & & & & & & & 1_b \end{pmatrix}$, $d_{x,y} = \mathrm{diag}(1, 1, y, x, 1, 1, \bar{y}^{-1}, \bar{x}^{-1})$, $d_y =$

$$\operatorname{diag}(1,1,y,1,1,1,\bar{y}^{-1},1),\ w' = \begin{pmatrix} 1_b &&&&&&& \\ & 1 &&&&&& \\ &&&&&& 1_a & \\ &&& 1_b &&&& \\ &&&& 1_b &&& \\ &&&&& 1 && \\ && -1_a &&&&& \\ &&&&&&& 1_b \end{pmatrix} \text{ and}$$

$$\tilde{\gamma} = \begin{pmatrix} 1_b &&&&&&& \\ & 1 &&&&&& \\ && 1_a &&&&& \\ &&& 1_b &&&& \\ &&& 1_b & 1_b &&& \\ &&&&& 1 && \\ && 1_a &&&& 1_a & \\ 1_b &&&&&&& 1_b \end{pmatrix}.$$

Here x and y stand for the corresponding block matrices of the corresponding size. Recall that $\gamma((m(g_1,1),g_1) \in Q$, by multiplying this to the left for $g_1 = \operatorname{diag}(\bar{x},1,x^{-1})\eta^{-1}$, we are reduced to proving that if $\gamma(g,1)w'd_y\tilde{\gamma}^{-1} \in Q_{x,y}$, then $g \in PwK_v^{(2)}w^{-1}$. A computation tells us that: $\gamma(g,1)w'd_y\tilde{\gamma}^{-1}$ equals:

$$\begin{pmatrix} 1_b &&&&&&& -\frac{1}{2}1_b \\ & 1 &&&&&& \\ && 1_a &&&&& \\ &&& -1_b & \frac{1}{2}1_b &&& \\ &&&& 1_b &&& \\ &&&&& 1 && \\ &&&&&& 1_a & \\ &&&&&&& -1_b \end{pmatrix} \times$$

$$\begin{pmatrix} a_1 & a_2 & \frac{\zeta}{2}a_3y - a_3\bar{y}^{-1} & -b_1 & b_1 & b_2 & a_3\bar{y}^{-1} & \\ a_4 & a_5 & \frac{\zeta}{2}a_6y - a_6\bar{y}^{-1} & -b_3 & b_3 & b_4 & a_6\bar{y}^{-1} & \\ a_7/2 & a_8/2 & \frac{\zeta y(a_9-1)}{4} - \frac{(a_9+1)\bar{y}^{-1}}{2} & -\frac{b_5}{2} & \frac{b_5}{2} & \frac{b_6}{2} & \frac{(a_9+1)\bar{y}^{-1}}{2} & \\ &&& 1 &&&& \\ c_1 & c_2 & \frac{\zeta}{2}c_3y - c_3\bar{y}^{-1} & 1-d_1 & d_1 & d_2 & c_3\bar{y}^{-1} & \\ c_4 & c_5 & \frac{\zeta}{2}c_6y - c_6\bar{y}^{-1} & -d_3 & d_3 & d_4 & c_6\bar{y}^{-1} & \\ -\zeta^{-1}a_7 & -\zeta^{-1}a_8 & -\frac{(a_9+1)}{2}y + \zeta^{-1}(a_9-1)\bar{y}^{-1} & \zeta^{-1}b_5 & -\zeta^{-1}b_5 & -\zeta^{-1}b_6 & \zeta^{-1}(1-a_9)\bar{y}^{-1} & \\ a_1-1 & a_2 & \frac{\zeta}{2}a_3y - a_3\bar{y}^{-1} & -b_1 & b_1 & b_2 & a_3\bar{y}^{-1} & 1 \end{pmatrix}.$$

One first proves that $d_4 \neq 0$ by looking at the second row of the lower left of the above matrix, so by left multiplying g by some matrix in N_P, we may assume that $d_2 = b_2 = b_4 = b_6 = 0$, then the result follows by an argument similarly to the proof of Lemma 4.36 later on. $\square$

Now recall that $g = \begin{pmatrix} a_5 & a_6 & a_4 \\ a_8 & a_9 & a_7 \\ a_2 & a_3 & a_1 \end{pmatrix}$. Let $\mathfrak{Y}$ be the set of g's so that the entries of a_2 are integers, the entries of a_3 are divisible by $y\bar{y}$, the entries of $a_1 - 1$ are divisible by $\bar{x}$, the entries of $1 - a_5$ are divisible by x, the entries of a_6 are divisible by $\bar{x}y$, the entries of a_4 are divisible by $x\bar{x}$, $1 - a_9 = y\bar{y}\zeta(1 + y\bar{y}N)$ for some N with integral entries, the entries of a_8 are divisible by $\frac{\bar{y}y\zeta}{2}$, and the entries of a_7 are divisible by $\bar{y}yx\zeta$.

Lemma 4.10. *Let $\varphi_x = \pi(\mathrm{diag}(\bar{x}, 1, x^{-1})\eta^{-1})\varphi$ where φ is invariant under the action of $\mathfrak{Y}$ defined above, then*

(i) $F_{\varphi_x}(f_{v,sieg}; z, w) = \tau(y\bar{y}x)|(y\bar{y})^2 x\bar{x}|_v^{-z-\frac{a+2b+1}{2}}\mathrm{Vol}(\mathfrak{Y})\cdot\varphi.$

(ii) $F'_{\varphi_x}(f'_{v,sieg}; z, w) = \tau(y\bar{y}x)|(y\bar{y})^2 x\bar{x}|_v^{-z-\frac{a+2b}{2}}\mathrm{Vol}(\mathfrak{Y})\cdot\varphi.$

Proof. First one computes:

$$
\begin{pmatrix}
1 &&&&&&& \\
& 1 &&&&&& \\
&& \frac{1}{2} &&&& -\frac{1}{2} & \\
&&& 1 &&&& \\
&&& 1 & 1 &&& \\
&&&&& 1 && \\
&& -\zeta^{-1} &&&& -\zeta^{-1} & \\
1 &&&&&&& 1
\end{pmatrix}
\begin{pmatrix}
1 &&&&&&& \\
& 1 &&&&&& \\
&& 1 &&&&& \\
&&& a_1 &&& a_3 & a_2 \\
&&&& 1 &&& \\
&&&&& 1 && \\
&&& a_7 &&& a_9 & a_8 \\
&&& a_4 &&& a_6 & a_5
\end{pmatrix}
\begin{pmatrix}
1 &&&&&&& \\
& 1 &&&&&& \\
&& 1 &&&& -\frac{\zeta}{2} & \\
&&& 1 &&&& \\
&&&& 1 &&& \\
&&&&& 1 && \\
&& -1 &&&& -\frac{\zeta}{2} & \\
&&&&&&& 1
\end{pmatrix}
$$

$$
\times
\begin{pmatrix}
1 &&&&&&& \\
& 1 &&&&&& \\
&&&&&& 1 & \\
&&& 1 &&&& \\
&&&& 1 &&& \\
&&&&& 1 && \\
&& -1 &&&&& \\
&&&&&&& 1
\end{pmatrix}
\begin{pmatrix}
1 &&&&&&& \\
& 1 &&&&&& \\
&& y &&&&& \\
&&& 1 &&&& \\
&&&& 1 &&& \\
&&&&& 1 && \\
&&&&&& \bar{y}^{-1} & \\
&&&&&&& 1
\end{pmatrix}
\begin{pmatrix}
1 &&&&&&& \\
& 1 &&&&&& \\
&& 1 &&&&& \\
&&& 1 &&&& \\
&&& -1 & 1 &&& \\
&&&&& 1 && \\
&& -1 &&&& 1 & \\
-1 &&&&&&& 1
\end{pmatrix}
$$

$$
=
\begin{pmatrix}
1 &&&&&&& \\
& 1 &&&&&& \\
\frac{a_8}{2} && \frac{\zeta y}{4}(1-a_9) - \frac{\bar{y}^{-1}}{2}(1+a_9) & -\frac{a_7}{2} &&& \frac{\bar{y}^{-1}}{2}(1+a_9) & -\frac{a_8}{2} \\
-a_2 && \frac{a_3\zeta y}{2} + a_3\bar{y}^{-1} & a_1 &&& -a_3\bar{y}^{-1} & a_2 \\
-a_2 && \frac{a_3\zeta y}{2} + a_3\bar{y}^{-1} & a_1 - 1 & 1 && -a_3\bar{y}^{-1} & a_2 \\
&&&&& 1 && \\
\zeta^{-1}a_8 && -\frac{y}{2}(1+\zeta^{-1}a_9\zeta) + \zeta^{-1}\bar{y}^{-1}(1-a_9) & -\zeta^{-1}a_7 &&& -\zeta^{-1}\bar{y}^{-1}(1-a_9) & -\zeta^{-1}a_8 \\
1-a_5 && \frac{a_6\zeta y}{2} + a_6\bar{y}^{-1} & a_4 &&& -a_6\bar{y}^{-1} & a_5
\end{pmatrix}.
$$

One checks the above matrix belongs to $Q_{x,y}$ if and only if the a_i's satisfy the conditions required by the definition of $\mathfrak{Y}$. The lemma follows by a similar argument as in Lemma 4.38 below. ☐

Definition 4.11. *We will sometimes write $\mathfrak{Y}_v$ for the $\mathfrak{Y}$ above to emphasis the dependence on v.*

4.3.2 Fourier-Jacobi Coefficient

We first give a formula for the Fourier coefficients for $\tilde{f}_{v,sieg} := \rho(\tilde{\gamma}_v)f^{\dagger}_{v,sieg}$ and $\tilde{f}'_{v,sieg} := \rho(\tilde{\gamma}'_v)f^{\dagger,\prime}_{v,sieg}$.

Lemma 4.12. *(i) Let $\beta = (\beta_{ij}) \in S_{n+1}(F_v)$ then for all $z \in \mathbb{C}$ we have:*

$$\tilde{f}_{v,sieg,\beta}(z,1) = \mathrm{Vol}(S_{n+1}(\mathcal{O}_{F,v}))e_v(\mathrm{Tr}_{\mathcal{K}_v/F_v}(\frac{\beta_{a+b+2,1}+...+\beta_{a+2b+1,b}}{x}) + \frac{\beta_{b+2,b+2}+...+\beta_{b+1+a,b+1+a}}{y\bar{y}}).$$

(ii) Let $\beta = (\beta_{ij}) \in S_v(F_v)$. Then

$$\tilde{f}'_{v,sieg,\beta}(z,1) = \mathrm{Vol}(S_n(\mathcal{O}_{F,v}))e_v(\mathrm{Tr}_{\mathcal{K}_v/F_v}(\frac{\beta_{a+b+1,1}+...+\beta_{a+2b,b}}{x}) + \frac{\beta_{b+1,b+1}+...+\beta_{b+a,b+a}}{y\bar{y}}).$$

The proof is straightforward.

Here we record a lemma on the Fourier-Jacobi coefficient for $f_v^\dagger \in I_n(\tau_v)$ and $\beta \in S_m(F_v)$.

Lemma 4.13. *If $\beta \notin S_m(\mathcal{O}_{F_v})^*$ then $FJ_\beta(f^\dagger; z, u, g, hy) = 0$. If $\beta \in S_n(\mathcal{O}_{F_v})^*$ then*

$$FJ_\beta(f^\dagger; z, u, g, h) = f^\dagger(z, g'\eta)\omega_\beta(h, g'\eta)\Phi_{0,y}(u).\mathrm{Vol}(S_m(\mathcal{O}_{F_v})),$$

where $g' = \begin{pmatrix} 1_{n-m} & \\ & -1_{n-m} \end{pmatrix} g \begin{pmatrix} 1_{n-m} & \\ & -1_{n-m} \end{pmatrix}$.

The proof is similar to [28, Lemma 11.15].

4.4 p-adic Computations

In this subsection we first prove that under some 'generic conditions' the unique up to scalar nearly ordinary vector in $I(\rho)$ is just the unique up to scalar vector with certain prescribed action of level subgroup. Then we construct a section $F^\dagger$ in $I(\rho^\vee)$ which is the pullback of a Siegel section $f^\dagger$ supported in the big cell. We can understand the action of the level group of this section. Then we define F^0 to be the image of $F^\dagger$ under the intertwining operator. By checking the action of the level subgroup on F^0 we can prove that it is just the nearly ordinary vector.

In our calculations we will usually use the projection to the first component of $\mathcal{K}_v \simeq \mathcal{K}_w \times \mathcal{K}_{\bar{w}} \simeq \mathbb{Q}_p \times \mathbb{Q}_p$.

4.4.1 Nearly Ordinary Sections

Let $\lambda_1, ..., \lambda_n$ be n characters of $F_v^\times$ which we identify with $\mathbb{Q}_p^\times$, $\pi = \mathrm{Ind}_B^{GL_n}(\lambda_1, ..., \lambda_n)$.

Definition 4.14. *Let $n = r+s$ and $\underline{k} = (c_{r+s}, ..., c_{s+1}; c_1, ..., c_s)$ be a weight. We say $(\lambda_1, ...\lambda_n)$ is nearly ordinary with respect to $\underline{k}$ if the set:*

$$\begin{aligned} & \{\mathrm{val}_p\lambda_1(p), ..., \mathrm{val}_p\lambda_n(p)\} \\ = \ & \{c_1 + s - 1 - \tfrac{n}{2} + \tfrac{1}{2}, c_2 + s - 2 - \tfrac{n}{2} + \tfrac{1}{2}, ..., c_s - \tfrac{n}{2} + \tfrac{1}{2}, c_{s+1} + r + s - 1 - \tfrac{n}{2} + \tfrac{1}{2}, ..., c_{r+s} + s - \tfrac{n}{2} + \tfrac{1}{2}\} \end{aligned}$$

We denote the set as $\{\kappa_1, ..., \kappa_{r+s}\}$. Thus $\kappa_1 > ... > \kappa_{r+s}$.

Let $\mathcal{A}_p := \mathbb{Z}_p[t_1, t_2, ..., t_n, t_n^{-1}]$ be the Atkin-Lehner ring of $G(\mathbb{Q}_p)$, where t_i is defined by $t_i = N(\mathbb{Z}_p)\alpha_i N(\mathbb{Z}_p)$, $\alpha_i = \begin{pmatrix} 1_{n-i} & \\ & p1_i \end{pmatrix}$. Then t_i acts on $\pi^{N(\mathbb{Z}_p)}$ by

$$v|t_i = \sum_{x \in N|\alpha_i^{-1}N\alpha_i} x_i\alpha_i^{-1}v.$$

We also define a normalized action with respect to the weight $\underline{k}$ following ([10]):

$$v\|t_i := \delta(\alpha_i)^{-1/2}p^{\kappa_1+...+\kappa_i}v|t_i$$

Definition 4.15. *A vector $v \in \pi$ is called nearly ordinary if it is an eigenvector for all $||t_i$'s with eigenvalues that are p-adic units.*

We identify π as a set of smooth functions on $\mathrm{GL}_n(\mathbb{Q}_p)$:

$$\pi = \{f : \mathrm{GL}_n(\mathbb{Q}_p) \to \mathbb{C}, f(bx) = \lambda(b)\delta_B(b)^{1/2}f(x)\}.$$

Here $\lambda(b) := \prod_{i=1}^n \lambda_i(b_i)$ for $b = \begin{pmatrix} b_1 & \times & \times \\ & ... & \times \\ & & b_n \end{pmatrix}$ and δ_B is the modulus function for the upper triangular Borel subgroup. Let w_ℓ be the longest Weyl element $\begin{pmatrix} & & & 1 \\ & & 1 & \\ & ... & & \\ 1 & & & \end{pmatrix}$, and let f^ℓ be the element in π such that f^ℓ is supported in $Bw_\ell N(\mathbb{Z}_p)$ and invariant under $N(\mathbb{Z}_p)$. The f^ℓ is unique up to scalar. We have:

Lemma 4.16. *f^ℓ is an eigenvector for all t_i's.*

Proof. Note that for any i, $f^\ell|t_i$ is invariant under $N(\mathbb{Z}_p)$. By looking at the definition of $v|t_i$ for the above model of π, it is not hard to see that $f^\ell|t_i$ is supported in $B(\mathbb{Q}_p)w_\ell B(\mathbb{Z}_p)$. So $f^\ell|t_i$ must be a multiple of f^ℓ. $\square$

Lemma 4.17. *Suppose that $(\lambda_1, ..., \lambda_n)$ is nearly ordinary with respect to $\underline{k}$ and suppose*

$$\nu_p(\lambda_1(p)) > \nu_p(\lambda_2(p)) > ... > \nu_p(\lambda_n(p))$$

then the eigenvalues of $\|t_i$ acting on f^ℓ are p-adic units. In other words f^ℓ is an ordinary vector.

Proof. A straightforward computation gives that

$$f^\ell||t_i = \lambda_1...\lambda_i(p^{-1})p^{\kappa_1+...+\kappa_i} f^\ell$$

which is clearly a p-adic unit by the definition of $(\lambda_1, ..., \lambda_n)$ to be nearly ordinary with respect to $\underline{k}$. $\square$

Remark 4.18. Hida proved in [10, Theorem 5.3] that the nearly ordinary vector is unique up to scalar.

Lemma 4.19. *Let $\lambda_1, ..., \lambda_{a+2b}$ be a set characters of $\mathbb{Q}_p^\times$ such that $\mathrm{cond}(\lambda_{a+2b}) > ...,> \mathrm{cond}(\lambda_{b+1}) > \mathrm{cond}(\lambda_1) > ... > \mathrm{cond}(\lambda_b)$. We define a subgroup: K_λ of $\mathrm{GL}_{a+2b}(\mathbb{Z}_p)$ to be those matrices whose below diagonal entries of the i-th column are divisible by cond($\lambda_{a+2b+1-i}$) for $1 \leq i \leq a+b$, and the left to diagonal entries of the j-th row are divisible by cond($\lambda_{a+2b+1-j}$) for $a+b+2 \leq j \leq a+2b$. Let λ^{op} be the character of K_λ defined by:*

$$\lambda_{a+2b}(g_{11})\lambda_{a+2b-1}(g_{22})...\lambda_1(g_{a+2b\ a+2b}).$$

Then f^ℓ is the unique (up to scalar) vector in π such that the action of K_λ is given by multiplying λ^{op}.

Proof. We only need to prove the uniqueness. We use the model of induced representation as above. Let $n = a+2b$ and $e_1, \cdots, e_n$ be the standard basis of the standard representation of GL_n. Let p^{t_i} be the conductor of λ_i. So $t_{a+2b} = \max\{t_i\}_i$. Write $K_0(p) \subset \mathrm{GL}_n(\mathbb{Z}_p)$ for the subgroup consisting of elements in $B(\mathbb{Z}_p)$ modulo p. Suppose f is any vector satisfying the requirement of the lemma. Let w be a Weyl element of GL_n such that f is not identically 0 on $wK_0(p)$. Then we see that $w \cdot e_1 = e_{a+2b}$ by considering right multiplication by $\mathrm{diag}(1 + p^{t_{a+2b}-1}, 1, \cdots, 1)$. Continue this argument, we see that $w \cdot e_2 = e_{a+2b-1}, \cdots$. Finally we have $w = w^\ell$ and the lemma is clear by Bruhat decomposition. $\square$

We let $w_1 := \begin{pmatrix} 1 &&&&& \\ & \ddots &&&& \\ && 1 &&& \\ &&&&& 1 \\ &&&& \iddots & \\ &&& 1 && \end{pmatrix}$. Now let $\tilde{B} = B^{w_1}$ and $\tilde{K}_\lambda = K_\lambda^{w_1}$.

Corollary 4.20. *Denote $a_i := \nu_p(\lambda_i(p))$. Suppose $\lambda_1, ..., \lambda_{a+2b}$ are such that $\mathrm{cond}(\lambda_1) > ... > \mathrm{cond}(\lambda_{a+2b})$ and $a_1 < ... < a_{a+b} < a_{a+2b} < ... < a_{a+b+1}$. Then the unique (up to scalar) ordinary section with respect to $\tilde{B}$ is*

$$f^{ord}(x) = \begin{cases} \lambda_1(g_{11})...\lambda_{a+2b}(g_{a+2b,a+2b}), & g \in \tilde{K}_\lambda, \\ 0 & \textit{otherwise} . \end{cases}$$

Proof. We only need to prove that $\pi(w_1)f^{ord}(x)$ is ordinary with respect to $\tilde{B}^{w_1} = B$. Let $\lambda'_1 = \lambda_{a+b+1}, ..., \lambda'_b = \lambda_{a+2b}, \lambda'_{b+1} = \lambda_{a+b}, ..., \lambda'_{a+2b} = \lambda_1$. Then λ' satisfies Lemma 4.17 and thus the ordinary section for B (up to scalar) is $f^\ell_{\lambda'}$. Since λ' also satisfies the assumptions of Lemma 4.19, $f^\ell_{\lambda'}$ is the unique section such that the action of K_λ is given by $\lambda'_{a+2b}(g_{11})...\lambda'_1(g_{a+2b,a+2b})$. But λ is clearly regular, so $\mathrm{Ind}_B^{\mathrm{GL}_{a+2b}}(\lambda) \simeq \mathrm{Ind}_B^{\mathrm{GL}_{a+2b}}(\lambda')$. So the ordinary section of $\mathrm{Ind}_B^{\mathrm{GL}_{a+2b}}(\lambda)$ for B also has the action of K_λ given by this character. It is easy to check that $\pi(w_1)f^{ord}$ has this property and the uniqueness (up to scalar) gives the result. □

4.4.2 Pullback Sections

In this subsubsection we construct a Siegel section on $\mathrm{U}(a+2b+1, a+2b+1)$ which pulls back to the nearly ordinary Klingen sections on $\mathrm{U}(a+b+1, b+1)$. We need to re-arrange the basis since we are going to study large block matrices and the new basis will simplify the explanation. One can check that the Klingen Eisenstein series we construct in this subsection, when going back to our previous basis, is nearly ordinary with respect to the Borel subgroup $B_1 := \begin{pmatrix} * & * & & * & * \\ & * & & * & * \\ & * & * & * & * \\ & & & * & \\ & & & * & * \end{pmatrix}$, where the first to fifth diagonal blocks are upper, upper, upper, lower, lower triangular, respectively. But the one we need is nearly ordinary with respect to the Borel subgroup $B_2 := \begin{pmatrix} * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & \\ & & & * & * \end{pmatrix}$ (it is for this one we can use the Λ-adic Fourier-Jacobi expansions). (Here the blocks are with respect to the partition: $b+1+a+b+1$.) There is a Weyl element w_{Borel} of GL_{a+2b+2} such that $w_{Borel}^{-1} B_2 w_{Borel} = B_1$. This w_{Borel} is in fact in the Weyl group of GL_{b+1+a} embedded as the upper left minor. In the case of doubling method ($\mathrm{U}(r,s) \times \mathrm{U}(s,r) \hookrightarrow \mathrm{U}(r+s, r+s)$) we have a corresponding change of index and we write w'_{Borel} for the corresponding Weyl element. In Subsubsection 4.4.4 we will come back to the original basis.

Now we explain the new basis. Let $V_{a,b}$ be the hermitian space with matrix $\begin{pmatrix} \zeta 1_a & & \\ & & 1_b \\ & -1_b & \end{pmatrix}$ and $V_{a,b+1}$ be the hermitian space with metric $\begin{pmatrix} \zeta & & \\ & & 1_{b+1} \\ & -1_{b+1} & \end{pmatrix}$. (These are our skew-

Hermitian spaces for $\mathrm{U}(r,s)$ and $\mathrm{U}(r+1,s+1)$ under the new basis). The matrix S for the embedding: $\mathrm{U}(V_{a,b}) \times \mathrm{U}(V_{a,b+1}) \hookrightarrow \mathrm{U}(V_{a+2b+1})$ becomes:

$$\begin{pmatrix} 1 & & & & -\frac{\zeta}{2} & & & \\ & 1 & & & & & & \frac{1}{2} \\ & & 1 & & & & & \\ & & & -1 & & \frac{1}{2} & & \\ -1 & & & & -\frac{\zeta}{2} & & & \\ & & & 1 & & \frac{1}{2} & & \\ & & & & & & 1 & \\ & -1 & & & & & & -\frac{1}{2} \end{pmatrix}.$$

Godement Sections at p

Let $v|p$ be a prime of F and $\mathcal{K}_v \simeq \mathbb{Q}_p \times \mathbb{Q}_p$. Let τ be character of $\mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$. Suppose $\tau = (\tau_1, \tau_2^{-1})$ and let p^{s_i} be the conductor of $\tau_i, i = 1,2$. Let $\chi_1, ...\chi_a, \chi_{a+1}, ...\chi_{a+2b}$ be characters of $\mathbb{Q}_p^\times$ whose conductors are $p^{t_1}, ..., p^{t_{a+2b}}$. Suppose we are in the:

Definition 4.21. *(Generic case):*

$$t_1 > t_2 > ... > t_{a+b} > s_1 > t_{a+b+1} > ... > t_{a+2b} > s_2.$$

Also, let $\xi_i = \chi_i \tau_1^{-1}$ *for* $1 \le i \le a+b$, $\xi_j = \chi_j^{-1}\tau_2$ *for* $a+b+2 \le j \le a+2b+1$. *Let* $\xi_{a+b+1} = 1$.

Let Φ_1 be the following Schwartz function on $M_{a+2b+1}(\mathbb{Q}_p)$: let Γ be the subgroup of $\mathrm{GL}_{a+2b+1}(\mathbb{Z}_p)$ consisting of matrices $\gamma = (\gamma_{ij})$ such that p^{t_k} divides the below diagonal entries (i.e. $i > j$) of the k^{th} column for $1 \le k \le a+b$ and p^{s_1} divides γ_{ij} when $a+b+2 \le i \le a+2b+1$, $j = a+b+1$; $p^{t_{j-1}}$ divides γ_{ij} when $a+b+2 \le j \le a+2b+1$, $i \le a+b+1$ or $i > j$.

Let $\xi'_i = \chi_i \tau_2^{-1}$, $1 \le i \le a+b$, $\xi'_j = \chi_j^{-1}\tau_1$, $a+b+2 \le j \le a+2b+1$, and $\xi'_{a+b+1} = \tau_1\tau_2^{-1}$. (Thus $\xi'_k = \xi_k \tau_1 \tau_2^{-1}$ for any k).

Definition 4.22.

$$\Phi_1(x) = \begin{cases} 0 & x \notin \Gamma \\ \prod_{k=1}^{a+b+1} \xi'_k(x_{kk}) & x \in \Gamma. \end{cases}$$

Now we define another Schwartz function Φ_2 on $M_{a+2b+1}(\mathbb{Q}_p)$.

Let $\mathfrak{X}$ be the following set: if $\mathfrak{X} \ni x = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$ is in the block matrix form with respect to the partition: $a+2b+1 = a+b+1+b$, then:

- x has entries in $\mathbb{Z}_p$;
- $\begin{pmatrix} A_{11} & A_{14} \\ A_{21} & A_{24} \end{pmatrix}$ has the i-th-upper-left minors A_i such that $(\det A_i) \in \mathbb{Z}_p^\times$ for $i = 1,2,...,a+b$;
- and A_{42} has i-upper-left minors B_i so that $(\det B_i) \in \mathbb{Z}_p^\times$ for $i = 1,2,...,b$.

We define:

$$\Phi_\xi(x) = \begin{cases} 0 & x \notin \mathfrak{X}, \\ \xi_1/\xi_2(\det A_1)...\xi_{a+b-1}/\xi_{a+b}(\det A_{a+b-1})\xi_{a+b}(A_{a+b}) & \\ \times\xi_{a+b+2}/\xi_{a+b+3}(\det B_1)...\xi_{a+2b}/\xi_{a+2b+1}(\det B_{b-1})\xi_{a+2b+1}(\det B_b). & x \in \mathfrak{X}. \end{cases} \tag{11}$$

This is a locally constant function with compact support. Let

$$\Phi_2(x) := \tilde{\Phi}_\xi(x) = \int_{M_{a+2b+1}(\mathbb{Q}_p)} \Phi_\xi(y) e_p(\mathrm{tr} y^t x) dy$$

(tilde stands for Fourier transform). Let Φ be the Schwartz function on $M_{a+2b+1,2(a+2b+1)}(\mathbb{Q}_p)$ defined by:

$$\Phi(X,Y) := \Phi_1(X)\Phi_2(Y),$$

and define a Godement section (terminology of Jacquet) by:

$$f^{\Phi}(g) = \tau_2(\det g)|\det g|_p^{-s+\frac{a+2b+1}{2}} \times \int_{\mathrm{GL}_{a+2b+1}(\mathbb{Q}_p)} \Phi((0,X)g)\tau_1^{-1}\tau_2(\det X)|\det X|_p^{-2s+a+2b+1}d^{\times}X.$$

Lemma 4.23. *If $\gamma \in \Gamma$, then:*

$$\Phi_\xi({}^t\gamma X) = \prod_{k=1}^{a+2b+1}(\xi_k(\gamma_{kk}))\Phi_\xi(X).$$

Proof. Straightforward. For example to see that the A_{42}-block of ${}^t\gamma X$ has invertible upper-left minors (i.e. having determinants in $\mathbb{Z}_p^\times$) for $\gamma \in \Gamma, x \in \mathfrak{X}$, one notes that all entries of the upper-right block of γ is zero modulo p, and that multiplying by invertible matrices which are lower-triangular modulo p does not change the property that all upper-left minors are invertible. $\square$

Fourier Coefficients:
Let z be in the absolutely convergent range. For $\beta \in S_{a+2b+1}(\mathbb{Q}_p)$ (isomorphic to $M_{a+2b+1}(\mathbb{Q}_p)$ through the first projection) the Fourier coefficient is defined by:

$$\begin{aligned}
f_\beta^{\Phi}(1,z) &= \int_{M_{a+2b+1}(\mathbb{Q}_p)} f^{\Phi}(\begin{pmatrix} & 1_{a+2b+1} \\ -1_{a+2b+1} & \end{pmatrix}\begin{pmatrix} 1 & N \\ & 1\end{pmatrix})e_p(-\mathrm{tr}\beta N)dN \\
&= \int_{M_{a+2b+1}(\mathbb{Q}_p)}\int_{\mathrm{GL}_{a+2b+1}\mathbb{Q}_p)} \Phi((0,X)\begin{pmatrix} & 1_{a+2b+1} \\ -1_{a+2b+1} & -N\end{pmatrix})\tau_1^{-1}\tau_2(\det X) \\
&\quad \times|\det X|_p^{-2z+a+2b+1}e_p(-\mathrm{tr}\beta N)dNd^{\times}X \\
&= \int_{\mathrm{GL}_{a+2b+1}(\mathbb{Q}_p)} \Phi_1(-X)\Phi_\xi(-{}^tX^{-1}{}^t\beta)\tau_1^{-1}\tau_2(\det X)|\det X|_p^{-2z}d^{\times}X \\
&= \tau_1^{-1}\tau_2(-1)\mathrm{vol}(\Gamma)\Phi_\xi({}^t\beta). \qquad (12)
\end{aligned}$$

Definition 4.24. *Let $\tilde{f}^\dagger = \tilde{f}^\dagger_{a+2b+1}$ be the Siegel section supported on $Q(\mathbb{Q}_p)w_{a+2b+1}\begin{pmatrix} 1 & M_{a+2b+1}(\mathbb{Z}_p) \\ & 1\end{pmatrix}$ and $\tilde{f}^\dagger(w_{a+2b+1}\begin{pmatrix} 1 & X \\ & 1\end{pmatrix}) = 1$ for $X \in M_{a+2b+1}(\mathbb{Z}_p)$.*

Lemma 4.25.

$$\tilde{f}^\dagger_\beta(1) = \begin{cases} 1 & \beta \in M_{a+2b+1}(\mathbb{Z}_p) \\ 0 & \beta \notin M_{a+2b+1}(\mathbb{Z}_p)\end{cases}$$

(here we used the projection of β onto its first component in $\mathcal{K}_v = F_v \times F_v$) where the first component correspond to the element inside our CM-type Σ_∞ under $\iota := \mathbb{C} \simeq \mathbb{C}_p$ (see subsection 2.1).

Definition 4.26.

$$f^\dagger := \frac{f^\Phi}{\tau_1^{-1}\tau_2(-1)\mathrm{Vol}(\Gamma)}.$$

Thus $f^\dagger_\beta = \Phi_\xi({}^t\beta)$.

We define

$$c_n(\tau', z) := \begin{cases} \tau'(p^{nt})p^{2ntz-tn(n+1)/2} & t > 0 \\ p^{2nz-n(n+1)/2} & t = 0. \end{cases} \tag{13}$$

Now we recall a lemma from [28, Lemma 11.12] which will be useful later.

Lemma 4.27. *Suppose $v|p$ and $\beta \in S_n(\mathbb{Q}_v), \det \beta \neq 0$.*
(i) If $\beta \notin S_n(\mathbb{Z}_v)$ then $M(z, \tilde{f}_n^\dagger)_\beta(-z, 1) = 0$.
(ii) Suppose $\beta \in S_n(\mathbb{Z}_v)$. Let $t := \mathrm{ord}_v(\mathrm{cond}(\tau'))$. Then:

$$M(z, \tilde{f}_n^\dagger)_\beta(-z, 1) = \tau'(\det \beta)|\det \beta|_v^{-2z}\mathfrak{g}(\bar{\tau}')^n c_n(\tau', z).$$

Note that our $\tilde{f}^\dagger$ is the $f^\dagger$ in [28] and our τ is their χ.

Now we want to write down our Godement section f^Φ in terms of $\tilde{f}^\dagger$. First we prove the following:

Lemma 4.28. *Suppose $\Phi_{\xi,n}$ is the function on $M_n(\mathbb{Q}_p)$ defined as follows: if $\mathrm{cond}(\xi_i) = (p^{t_i})$ for $t_1 > ... > t_n$, and ξ_i are characters of $\mathbb{Q}_p^\times$ with conductor p^{t_i}. Let $\mathfrak{X}_n$ be the subset of $M_n(\mathbb{Z}_p)$ such that the ith upper-left minor M_i has determinant in $\mathbb{Z}_p^\times$. Define $\Phi_{\xi,n}$ to be*

$$\frac{\xi_1}{\xi_2}(\det M_1)...\frac{\xi_{n-1}}{\xi_n}(\det M_{n-1})\xi_n(\det M_n)$$

on $\mathfrak{X}_n$ and 0 otherwise. Let

$$\tilde{\mathfrak{X}}_{\xi,n} = \tilde{\mathfrak{X}}_\xi := N(\mathbb{Z}_p)\begin{pmatrix} p^{-t_1}\mathbb{Z}_p^\times & & \\ & ... & \\ & & p^{-t_n}\mathbb{Z}_p^\times \end{pmatrix} N^{opp}(\mathbb{Z}_p).$$

Then the Fourier transform $\hat{\Phi}_\xi$ of Φ_ξ is the following function:

$$\tilde{\Phi}_\xi(x) = \begin{cases} 0 & x \notin \tilde{\mathfrak{X}}_\xi \\ \prod_{i=1}^n \mathfrak{g}(\xi_i)\prod_{i=1}^n \bar{\xi}_i(x_i p^{t_i}) & \tilde{\mathfrak{X}}_\xi \ni x = \begin{pmatrix} 1 & ... & ... \\ & ... & ... \\ & & 1 \end{pmatrix}\begin{pmatrix} x_1 & & \\ & ... & \\ & & x_n \end{pmatrix}\begin{pmatrix} 1 & & \\ ... & ... & \\ ... & ... & 1 \end{pmatrix}. \end{cases}$$

Proof. First suppose x is in the "big cell": $N(\mathbb{Q}_p)T(\mathbb{Q}_p)N^{opp}(\mathbb{Q}_p)$. It is easily seen that we can write x in terms of block matrices:

$$x = \begin{pmatrix} 1_{n-1} & u \\ & 1 \end{pmatrix}\begin{pmatrix} z & \\ & w \end{pmatrix}\begin{pmatrix} 1_{n-1} & \\ v & 1 \end{pmatrix}$$

where $z \in \mathrm{GL}_{n-1}(\mathbb{Q}_p)$ and $w \in \mathbb{Q}_p^\times$, $u \in M_{n-1,1}(\mathbb{Q}_p), v \in M_{1,n-1}(\mathbb{Q}_p)$. A first observation is that $\tilde{\Phi}_\xi$ is invariant under right multiplication by $N^{opp}(\mathbb{Z}_p)$ and left multiplication by $N(\mathbb{Z}_p)$.

We show that $v \in M_{1\times(n-1)}(\mathbb{Z}_p)$ if $\tilde{\Phi}_\xi(x) \neq 0$. By definition:

$$\begin{aligned}
\tilde{\Phi}_\xi(x) &= \int_{M_n(Q_p)} \Phi_\xi(y) e_p(\mathrm{tr} y {}^t x) dy \\
&\quad (\text{ writing } y = \begin{pmatrix} 1_{n-1} & \\ \ell & 1 \end{pmatrix} \begin{pmatrix} a & \\ & b \end{pmatrix} \begin{pmatrix} 1 & m \\ & 1 \end{pmatrix} = \begin{pmatrix} a & am \\ \ell a & \ell am + b \end{pmatrix}) \\
&= \int_{a\in\mathfrak{X}_{\xi,n-1}, m\in M(\mathbb{Z}_p), \ell\in M(\mathbb{Z}_p), b\in\mathbb{Z}_p^\times} \Phi_\xi(\begin{pmatrix} 1 & \\ \ell & 1 \end{pmatrix} \begin{pmatrix} a & \\ & b \end{pmatrix} \begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}) \\
&\times \quad e_p(\mathrm{tr}(\begin{pmatrix} 1 & \\ {}^t m & 1 \end{pmatrix} \begin{pmatrix} {}^t a & \\ & b \end{pmatrix} \begin{pmatrix} 1 & {}^t\ell \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} z & \\ & w \end{pmatrix} \begin{pmatrix} 1 & \\ v & 1 \end{pmatrix}))dy \\
&= \int \Phi_\xi(\begin{pmatrix} a & \\ & b \end{pmatrix}) e_p(\mathrm{tr} \begin{pmatrix} 1 & \\ {}^t m + v & 1 \end{pmatrix} \begin{pmatrix} {}^t a & \\ & b \end{pmatrix} \begin{pmatrix} 1 & {}^t\ell + u \\ & 1 \end{pmatrix} \begin{pmatrix} z & \\ & w \end{pmatrix})dy \\
&= \int \Phi_\xi(\begin{pmatrix} a & \\ & b \end{pmatrix}) e_p(\mathrm{tr}(\begin{pmatrix} {}^t a & {}^t a({}^t\ell + u) \\ ({}^t m + v){}^t a & ({}^t m + v){}^t a({}^t\ell + u) + b \end{pmatrix} \begin{pmatrix} z & \\ & w \end{pmatrix}))dy \\
&= \int \Phi_\xi(\begin{pmatrix} a & \\ & b \end{pmatrix}) e_p(\mathrm{tr}({}^t az + (({}^t m + v){}^t a({}^t\ell + u) + b)w))dy.
\end{aligned}$$

(Note that Φ_ξ is invariant under transpose.)

If $\tilde{\Phi}_\xi(x) \neq 0$, then it follows from the last expression that $w \in p^{-t_n}\mathbb{Z}_p^\times$. Suppose $v \notin M_{1\times(n-1)}(\mathbb{Z}_p)$, then ${}^t m + v \notin M_{1\times(n-1)}(\mathbb{Z}_p)$. We let a, m, b to be fixed and let ℓ to vary in $M_{1\times(n-1)}(\mathbb{Z}_p)$, we find that this integral must be 0. (Notice that $a \in \mathfrak{X}_{\xi,n-1}$ and $w \in p^{-t_n}\mathbb{Z}_p^\times$, thus $({}^t m + v){}^t aw \notin M_{1\times n-1}(\mathbb{Z}_p)$). Thus a contradiction. Therefore, $v \in M_{1\times n-1}(\mathbb{Z}_p)$, similarly $u \in M_{n-1,1}(\mathbb{Z}_p)$. Thus by the observation at the beginning of the proof we may assume $u = 0$ and $v = 0$ without lose of generality.

Thus if we write $\Phi_{\xi,n-1}$ as the restriction of Φ_ξ to the upper-left $(n-1)\times(n-1)$ minor,

$$\begin{aligned}
\tilde{\Phi}_\xi(x) &= \int \Phi_\xi(\begin{pmatrix} a & \\ & b \end{pmatrix}) e_p(\mathrm{tr}({}^t az + ({}^t m {}^t a {}^t \ell + b)w))dy \\
&= p^{-nt_n}\mathfrak{g}(\xi_n)\bar{\xi}_n(wp^{t_n}) \int_{a\in\mathfrak{X}_{\xi,n-1}} \Phi_{\xi,n-1}(a) e_p({}^t az)dy.
\end{aligned}$$

By an induction procedure one gets:

$$\tilde{\Phi}_\xi(x) = \begin{cases} 0 & x \notin \tilde{\mathfrak{X}}_{\xi,n} \\ p^{-\sum_{i=1}^n it_i} \prod_{i=1}^n \mathfrak{g}(\xi_i) \prod_{i=1}^n \bar{\xi}_i(x_i p^{t_i}) & x \in \tilde{\mathfrak{X}}_{\xi,n}. \end{cases}$$

We have thus proved that $\tilde{\Phi}_{\xi,n}$ when restricting to the "big cell" has support in $\tilde{\mathfrak{X}}_{\xi,n}$. Since $\tilde{\mathfrak{X}}_{\xi,n}$ is compact, therefore $\tilde{\Phi}_{\xi,n}$ itself must be supported in $\tilde{\mathfrak{X}}_{\xi,n}$. $\square$

Lemma 4.29. *Let $\tilde{\mathfrak{X}}_\xi$ be the support of $\Phi_2 = \hat{\Phi}_\xi$, then a complete representative of $\tilde{\mathfrak{X}}_\xi$* mod $M_{a+2b+1}(\mathbb{Z}_p)$ *is given by:*

$$\begin{pmatrix} A & & B \\ C & & D \\ & & \\ & E & \end{pmatrix}$$

where the blocks are with respect to the partition $a+b+1+b$ where $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ runs over the *following set:*

$$\begin{pmatrix} 1 & m_{12} & \dots & m_{1,a+b} \\ & \dots & \dots & \dots \\ & & \dots & m_{a+b-1,a+b} \\ & & & 1 \end{pmatrix} \begin{pmatrix} x_1 & & & \\ & \dots & & \\ & & \dots & \\ & & & x_{a+b} \end{pmatrix} \begin{pmatrix} 1 & & & \\ n_{21} & \dots & & \\ \dots & \dots & \dots & \\ n_{a+b,1} & \dots & n_{a+b,a+b-1} & 1 \end{pmatrix}$$

where x_i runs over $p^{-t_i}\mathbb{Z}_p^\times$ mod $\mathbb{Z}_p$, m_{ij} runs over $\mathbb{Z}_p$ mod p^{t_j} and n_{ij} runs over $\mathbb{Z}_p$ mod p^{t_i}, and E runs over the following set:

$$\begin{pmatrix} 1 & k_{12} & \dots & k_{1,b} \\ & \dots & \dots & \dots \\ & & \dots & k_{b-1,b} \\ & & & 1 \end{pmatrix} \begin{pmatrix} y_1 & & & \\ & \dots & & \\ & & \dots & \\ & & & y_b \end{pmatrix} \begin{pmatrix} 1 & & & \\ \ell_{21} & \dots & & \\ \dots & \dots & \dots & \\ \ell_{b,1} & \dots & \ell_{b,b-1} & 1 \end{pmatrix}$$

where y_i runs over $p^{-t_{i+a+b}}\mathbb{Z}_p^\times$ mod $\mathbb{Z}_p$; k_{ij} runs over $\mathbb{Z}_p$ mod $p^{t_{a+b+j}}$; ℓ_{ij} runs over $\mathbb{Z}_p$ mod $p^{t_{a+b+i}}$.

Proof. This is elementary and we omit it here. □

Also we define for $g \in \mathrm{GL}_{a+2b}(\mathbb{Q}_p)$, $g^\iota = \begin{pmatrix} & 1_{a\times a} & \\ & & 1_{b\times b} \\ 1_{b\times b} & & \end{pmatrix} g \begin{pmatrix} & & 1_{b\times b} \\ 1_{a\times a} & & \\ & 1_{b\times b} & \end{pmatrix}$ and

$$g_\iota = \begin{pmatrix} & 1_{a\times a} & \\ & & 1_{b\times b} \\ 1_{b\times b} & & \end{pmatrix}^{-1} g \begin{pmatrix} & & 1_{b\times b} \\ 1_{a\times a} & & \\ & 1_{b\times b} & \end{pmatrix}^{-1}.$$

Corollary 4.30.

$$\begin{aligned} f^\dagger(z,g) = \quad & p^{-\sum_{i=1}^{a+b} it_i - \sum_{i=1}^{b} it_{a+b+i}} \prod_{i=1}^{a+b} \mathfrak{g}(\xi_i)\xi_i(-1) \prod_{i=1}^{b} \mathfrak{g}(\xi_{a+b+1+i})\xi_{a+b+1+i}(-1) \\ & \times \sum_{A,B,C,D,E} \prod_{i=1}^{a} \bar{\xi}_i(\frac{\det A_i}{\det A_{i-1}} p^{t_i}) \prod_{i=1}^{b} \bar{\xi}_{a+i,a+i}(\frac{\det D_i}{\det D_{i-1}} p^{t_{a+i}}) \times \prod_{i=1}^{b} \bar{\xi}_{a+b+1+i}(\frac{\det E_i}{\det E_{i-1}} p^{t_{a+b+i}}) \\ & \tilde{f}^\dagger(z, g \begin{pmatrix} & & A & B \\ & & C & D \\ 1_{a+2b+1} & & & \\ & & E & \\ & & 1_{a+2b+1} & \end{pmatrix}). \end{aligned}$$

Here A_i is the i-th upper-left minor of A, D_i is the $(a+i)$-th upper left minor of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ (not D), E_i is the i-th upper-left minor of E, and the sum is running over the set of representative of Lemma 4.29.

Proof. We only need to check the Siegel Eisenstein sections on both hand sides coincide on $wN_{a+2b+1}(\mathbb{Q}_p)$ since the big cell $Q_{a+2b+1}(\mathbb{Q}_p)wN_{a+2b+1}(\mathbb{Q}_p)$ is dense in $\mathrm{GL}_{2a+4b+2}$. To see this we just need to know that they have the same β-th Fourier coefficients for all $\beta \in S_{a+2b+1}(\mathbb{Q}_p)$. But this is seen by (12), Lemma 4.28 and 4.29. □

Now we define several sets: Let $\mathfrak{B}'$ be the set of $(a+b)\times(a+b)$ upper triangular matrices of the form

$$\begin{pmatrix} 1 & m_{12} & \dots & m_{1,a+b} \\ & \dots & \dots & \dots \\ & & \dots & m_{a+b-1,a+b} \\ & & & 1 \end{pmatrix} \begin{pmatrix} x_1 & & & \\ & \dots & & \\ & & \dots & \\ & & & x_{a+b} \end{pmatrix}$$

where x_i runs over $\mathbb{Z}_p^\times$ mod p^{t_i}, m_{ij} runs over $\mathbb{Z}_p$ mod p^{t_j}.

Let $\mathfrak{D}'$ be the set of $b\times b$ lower triangular matrices of the form

$$\begin{pmatrix} 1 & & & \\ n_{21} & ... & & \\ ... & ... & ... & \\ n_{a+b,1} & ... & n_{a+b,a+b-1} & 1 \end{pmatrix}$$

where n_{ij} runs over $\mathbb{Z}_p$ mod $p^{t_{i+a+b}}$.

Let $\mathfrak{E}'$ be the set of $b\times b$ upper triangular matrices of the form

$$\begin{pmatrix} 1 & k_{12} & ... & k_{1,b} \\ & ... & ... & ... \\ & & ... & k_{b-1,b} \\ & & & 1 \end{pmatrix}$$

where k_{ij} runs over $\mathbb{Z}_p$ mod $p^{t_{a+b+j}}$.

Let $\mathfrak{C}'$ be the set of $(a+b)\times(a+b)$ lower triangular matrices of the form

$$\begin{pmatrix} y_1 & & & \\ & ... & & \\ & & ... & \\ & & & y_b \end{pmatrix}\begin{pmatrix} 1 & & & \\ \ell_{21} & ... & & \\ ... & ... & ... & \\ \ell_{b,1} & ... & \ell_{b,b-1} & 1 \end{pmatrix}$$

where y_i runs over $\mathbb{Z}_p^\times$ mod $p^{t_{i+a+b}}\mathbb{Z}_p$; ℓ_{ij} runs over $\mathbb{Z}_p$ mod $p^{t_{a+b+i}}$.

Thus if B', C', D', E' run over the set $\mathfrak{B}', \mathfrak{C}', \mathfrak{D}', \mathfrak{E}'$, then

$$\begin{aligned} f^\dagger(z,g) &= p^{-\sum_{i=1}^{a+b} it_i - \sum_{i=1}^{b} it_{a+b+i}} \sum_{B',C',D',E'} \prod_{i=1}^{a+b} \mathfrak{g}(\xi_i)\xi_i(-1)\prod_{i=1}^{b}\mathfrak{g}(\xi_{a+b+1+i})\xi_{a+b+1+i}(-1) \\ &\quad \times \sum_{B',C',D',E'} \prod_{i=1}^{a+b}\bar{\xi}_i(B'_{ii})\prod_{i=1}^{b}\xi_{a+b+i}(C'_{ii}) \\ &\quad \times \tilde{f}^\dagger(z, g\alpha(\begin{pmatrix} B' & & & \\ & 1 & & \\ & & C' & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} E' & \\ & D' \end{pmatrix}^\iota)\begin{pmatrix} 1 & A' \\ & 1 \end{pmatrix}\alpha(\begin{pmatrix} B' & & & \\ & 1 & & \\ & & C' & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} E' & \\ & D' \end{pmatrix}^\iota)^{-1}) \\ &= p^{-\sum_{i=1}^{a+b} it_i - \sum_{i=1}^{b} it_{a+b+1+i}} \prod_{B',C',D',E'} \prod_{i=1}^{a+b} \mathfrak{g}(\xi_i)\xi_i(-1)\prod_{i=1}^{b}\mathfrak{g}(\xi_{a+b+1+i})\xi_{a+b+1+i}(-1) \\ &\quad \times \sum_{B',C',D',E'} \prod_{i=1}^{a+b}\bar{\xi}_i(B'_{ii})\prod_{i=1}^{b}\xi_{a+b+i}(C'_{ii})\prod_{i=1}^{a+b}\bar{\tau}_1(B'_{ii})\prod_{i=1}^{b}\bar{\tau}_2(C'_{ii}) \\ &\quad \times \tilde{f}^\dagger(z, g\alpha(\begin{pmatrix} B' & & & \\ & 1 & & \\ & & C' & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} E' & \\ & D' \end{pmatrix}^\iota)\begin{pmatrix} 1 & A' \\ & 1 \end{pmatrix}), \end{aligned}$$

where $A' = \begin{pmatrix} p^{-t_1} &&&&&&&&& \\ & \cdots &&&&&&&& \\ && p^{-t_a} &&&&&&& \\ &&&&&&& p^{-t_{a+1}} && \\ &&&&&&&& \cdots & \\ &&&&&&&&& p^{-t_{a+b}} \\ &&&&&& 0 &&& \\ &&& p^{-t_{a+b+1}} &&&&&& \\ &&&& \cdots &&&&& \\ &&&&& p^{-t_{a+2b}} &&&& \end{pmatrix}.$

We let

$$\gamma = \begin{pmatrix} \zeta^{-1} &&&& -\zeta^{-1} &&& \\ & 1 &&&&&& \\ && 1 &&&&& \\ &&& 1 &&&& \\ \frac{1}{2} &&&& \frac{1}{2} &&& \\ &&& 1 && 1 && \\ &&&&&& 1 & \\ & 1 &&&&&& 1 \end{pmatrix},$$

and $w' = \begin{pmatrix} &&&& 1 &&& \\ & 1 &&&&&& \\ && 1 &&&&& \\ &&& 1 &&&& \\ -1 &&&&&&& \\ &&&&& 1 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{pmatrix}.$

Definition 4.31. *(pullback section) If f is a Siegel section and $\varphi \in \pi_p$, then*

$$F_\varphi(z, f, g) := \int_{\mathrm{GL}_{a+2b}(\mathbb{Q}_p)} f(z, \gamma\alpha(g, g_1)\gamma^{-1})\bar{\tau}(\det g_1)\rho(g_1)\varphi dg_1.$$

Now we define a subset K of $\mathrm{GL}_{a+2b+2}(\mathbb{Z}_p)$ so that $k \in K$ if and only if p^{t_i} divides the below diagonal entries of the i-th column for $1 \le i \le a+b$, p^{s_1} divides the below diagonal entries of the $(a+b+1)$-th column, and $p^{t_{a+b+j}}$ divides the right to diagonal entries of the $(a+b+1+j)$-th row for $1 \le j \le b-1$. We also define ν, a character of K by:

$$\nu(k) = \tau_1(k_{a+b+1,a+b+1})\tau_2(k_{a+2b+2,a+2b+2}) \prod_{i=1}^{a+b} \chi_i(k_{ii}) \prod_{i=1}^{b} \chi_{a+b+i}(k_{a+b+i+1,a+b+i+1})$$

for any $k \in K$.

Definition 4.32. *We define Υ to be the element in $\mathrm{U}(n,n)(F_v)(= \mathrm{U}(n,n)(\mathbb{Q}_p))$ such that the projection to the first component of $\mathcal{K}_v = F_v \times F_v$ equals that of γ (note that $\gamma \notin \mathrm{U}(n,n)(F_v)$).*

Lemma 4.33. *Let $K' \subset K$ be the compact subgroup defined by:*
$K' \ni k = \begin{pmatrix} a_1 & a_2 & a_3 & b_1 & b_2 \\ a_4 & a_5 & a_6 & b_3 & b_4 \\ a_7 & a_8 & a_9 & b_5 & b_6 \\ c_1 & c_2 & c_3 & d_1 & d_2 \\ c_4 & c_5 & c_6 & d_3 & d_4 \end{pmatrix} \in K$ *(here the blocks are with respect to the partition $(a+b+1+b+1)$) if and only if: $p^{t_{a+b+i}+t_j}$ divides the (i,j)th entry of c_1 for $1 \le i \le b$, $1 \le j \le a$*

and $p^{t_{a+b+i}+t_{a+j}}$ divides the (i,j)th entry of c_2 for $1 \le i \le b$, $1 \le j \le b$. (it is not hard to check that this is a group).
Then: $F_\varphi(z, \rho(\Upsilon)f^\dagger, gk) = \nu(k)F_\varphi(z, \rho(\Upsilon)f^\dagger, g)$ for any $\varphi \in \pi$ and $k \in K'$.

Proof. This follows directly from the action of K' on the Godement section $f^\dagger$. $\square$

We define K'' to be the subgroup of K consists of matrices

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ c_1 & c_2 & & 1 & \\ & & & & 1 \end{pmatrix}$$

such that p^{t_j} divides the (i,j)th entry of c_1 for $1 \le i \le b$, $1 \le j \le a$ and $p^{t_{a+j}}$ divides the (i,j)th entry of c_2 for $1 \le i \le b$, $1 \le j \le b$.

Definition 4.34. *Let $\tilde{K} \subset \mathrm{GL}_{a+2b}(\mathbb{Z}_p)$ be the set of matrices $\begin{pmatrix} a_1 & a_3 & a_2 \\ a_7 & a_9 & a_8 \\ a_4 & a_6 & a_5 \end{pmatrix}$ (blocks are with respect to the partition $(b+a+b)$) such that the columns of a_3, a_6 are divisible by $p^{t_1}, ..., p^{t_a}$; the column's of a_4 are divisible by $p^{t_{a+1}}, ..., p^{t_{a+b}}$; $p^{t_{a+j}}$ divides the below diagonal entries of the i-th column of a_1 $(1 \le i \le b)$; p^{t_j} divides the below diagonal entries of the j-th column of a_9 $(1 \le j \le a)$; $p^{t_{a+b+k}}$ divides the above diagonal entries of the k-th row of a_5.*
Let $\tilde{K}' \subset \tilde{K}$ be the set such that $p^{t_{a+b+i}+t_{a+j}}$ divides the (i,j)-th entry of a_4 for $1 \le i \le b$, $1 \le j \le b$ and $p^{t_{a+b+i}+t_j}$ divides the (i,j)-th entry of a_6 for $1 \le i \le b$, $1 \le j \le a$. We also define $\tilde{K}''$ to be the subset of $\tilde{K}$ consisting of matrices:

$$\begin{pmatrix} 1 & & \\ & 1 & \\ a_4 & a_6 & 1 \end{pmatrix}$$

such that $p^{t_{a+j}}$ divides the (i,j)th entry of a_4 for $1 \le i \le b$, $1 \le j \le b$ and p^{t_j} divides the (i,j)th entry of a_6 for $1 \le i \le b$, $1 \le j \le a$. We also define $\tilde{\nu}$ a character of $\tilde{K}$ by:

$$\tilde{\nu}(k) = \prod_{i=1}^{b} \chi_{a+i}(k_{i,i}) \prod_{i=1}^{a} \chi_i(k_{b+i,b+i}) \prod_{i=1}^{b} \chi_{a+b+i}(k_{a+b+i,a+b+i}).$$

The following lemma would be useful in identifying our pullback section:

Lemma 4.35. *Suppose $F_\varphi(z, \rho(\Upsilon)f^\dagger, g)$ as a function of g is supported in PwK and*

$$F_\varphi(z, \rho(\Upsilon)f^\dagger, gk) = \nu(k)F_\varphi(z, \rho(\Upsilon)f^\dagger, g)$$

for $k \in K'$, and $F_\varphi(z, \rho(\Upsilon)f^\dagger, w)$ is invariant under the action of $(\tilde{K}'')^\iota$. then $F_\varphi(a, \rho(\Upsilon)f^\dagger, g)$ is the unique section (up to scalar) whose action by $k \in K$ is given by multiplying by $\nu(k)$.

Proof. This is easy from the fact that $K = K'K'' = K''K'$. The uniqueness follows from Lemma 4.19. $\square$

From now on in this subsection we use w to denote $\begin{pmatrix} 1_a & & \\ & & 1_{b+1} \\ & -1_{b+1} & \end{pmatrix}$ or $\begin{pmatrix} 1_a & & \\ & & 1_b \\ & -1_b & \end{pmatrix}$.

Lemma 4.36. *If $\gamma\alpha(g,1)\gamma^{-1} \in supp(\rho(\Upsilon)f^\dagger)$ then $g \in PwK$. (Here ρ denotes the action of $\mathrm{GU}_{a+2b+1}(F_v)$ on the Siegel sections given by right translation.)*

Proof. Since $f^\dagger$ is of the form $f^\dagger(g) = \sum_{A\in\mathfrak{X}} \tilde{f}^\dagger(g\begin{pmatrix}1 & A\\ & 1\end{pmatrix})$, where $\mathfrak{X}$ is some set, we only have to check the lemma for each term in the summation.

Recall we defined: $A' = \begin{pmatrix} \begin{matrix}-p^{-t_1} & & \\ & \cdots & \\ & & p^{-t_a}\end{matrix} & & & \\ & & & \begin{matrix}-p^{t_{a+1}} & & \\ & \cdots & \\ & & -p^{t_{a+b}}\end{matrix} \\ & & 0 & \\ & \begin{matrix}-p^{-t_{a+b+1}} & & \\ & \cdots & \\ & & -p^{-t_{a+2b}}\end{matrix} & & \end{pmatrix}$

where the blocks are with respect to the partition $(a+b+1+b)$. Let ζ_v and γ_v be the projection of ζ and γ_v to the first component of $\mathcal{K}_v \simeq F_v \times F_v$, then:

$$\begin{aligned}
\gamma_v &= \begin{pmatrix}
\zeta_v^{-1} & & & & -\zeta_v^{-1} & & & \\
 & 1_b & & & & & & \\
 & & 1 & & & & & \\
 & & & 1_b & & & & \\
\frac{1}{2}1_a & & & & \frac{1}{2}1_a & & & \\
 & & & 1_b & & 1_b & & \\
 & & & & & & 1 & \\
 & 1_b & & & & & & 1_b
\end{pmatrix}\\
&= \begin{pmatrix}
2\zeta_v^{-1} & & & & -\zeta_v^{-1} & & & \\
 & 1_b & & & & & & \\
 & & 1 & & & & & \\
 & & & 1_b & & & & \\
 & & & & \frac{1}{2}1_a & & & \\
 & & & & & 1_b & & \\
 & & & & & & 1 & \\
 & & & & & & & 1_b
\end{pmatrix}
\begin{pmatrix}
1_a & & & & & & & \\
 & 1_b & & & & & & \\
 & & 1 & & & & & \\
 & & & 1_b & & & & \\
1_a & & & & 1_a & & & \\
 & & & 1_b & & 1_b & & \\
 & & & & & & 1 & \\
 & 1_b & & & & & & 1_b
\end{pmatrix}.
\end{aligned}$$

We denote the last term $\tilde{\gamma}_v$ (different from the definition in the prime to p case).

Using the expression for $f^\dagger$ involving the B', C', D', E''s as above and the fact that $\gamma(m(g,1), g) \in Q$ and that K is invariant under the right multiplication of B's and C's, we only need to check that if $\tilde{\gamma}_v\alpha(g,1)\tilde{\gamma}_v^{-1} \in \mathrm{supp}(\rho(\Upsilon)\rho(\begin{pmatrix}1 & A'\\ & 1\end{pmatrix}))\tilde{f}^\dagger$, then $g \in PwK$. Our calculations below are generalizations of the proof of [28, Proposition 11.16]. If $gw = \begin{pmatrix} a_1 & a_2 & a_3 & b_1 & b_2\\ a_4 & a_5 & a_6 & b_3 & b_4\\ a_7 & a_8 & a_9 & b_5 & b_6\\ c_1 & c_2 & c_3 & d_1 & d_2\\ c_4 & c_5 & c_6 & d_3 & d_4\end{pmatrix}$ then this is equivalent to

$$\begin{pmatrix} 1_a &&&&&&& \\ & 1_b &&&&&& \\ && 1 &&&&& \\ &&& 1_b &&&& \\ 1_a &&&& 1_a &&& \\ &&& 1_b && 1_b && \\ &&&&&& 1 & \\ & 1_b &&&&&& 1_b \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 &&& b_1 & b_2 & \\ a_4 & a_5 & a_6 &&& b_3 & b_4 & \\ a_7 & a_8 & a_9 &&& b_5 & b_6 & \\ &&& 1_b &&&& \\ &&&& 1_a &&& \\ c_1 & c_2 & c_3 &&& d_1 & d_2 & \\ c_4 & c_5 & c_6 &&& d_3 & d_4 & \\ &&&&&&& 1_b \end{pmatrix} \alpha(1, w^{-1})w' \times$$

$$\mathrm{diag}(p^{t_1}, ..., p^{t_a}, 1_b, 1, p^{t_{a+1}}, ..., p^{t_{a+b}}, 1_a, 1_b, 1, p^{-t_{a+b+1}}, ..., p^{-t_{a+2b}}) \times \begin{pmatrix} 1_a &&&&&&& \\ & 1_b &&&&&& \\ && 1 &&&&& \\ &&& 1_b &&&& \\ -1_a &&&& 1_a &&& \\ &&& -1_b && 1_b && \\ &&&&&& 1 & \\ & -1_b &&&&&& 1_b \end{pmatrix}$$

$$\times\, \mathrm{diag}(p^{-t_1}, ..., p^{-t_a}, 1_b, 1, p^{-t_{a+1}}, ..., p^{-t_{a+b}}, 1_a, 1_b, 1, p^{t_{a+b+1}}, ..., p^{t_{a+2b}}) w_{a+2b+1}^{-1}$$

being in $\mathrm{supp}\tilde{f}^{\dagger}$, which is equivalent to

$$\tilde{\gamma}_v \alpha(gw, \begin{pmatrix} && -1_b \\ & 1_a & \\ 1_b && \end{pmatrix} \begin{pmatrix} p^{-t_{a+b+1}} &&&&& \\ & ... &&&& \\ && p^{t_1} &&& \\ &&& ... && \\ &&&& p^{t_{a+1}} & \\ &&&&& ... \end{pmatrix}) w' \tilde{\gamma}_v^{-1},$$

belonging to:

$$\mathrm{supp}(\rho(\mathrm{diag}(p^{-t_1}, ..., p^{-t_a}, 1_b, 1, p^{-t_{a+1}}, ..., p^{-t_{a+b}}, 1_a, 1_b, 1, p^{t_{a+b+1}}, p^{t_{a+2b}}) w_{a+2b+1}) \tilde{f}^{\dagger}).$$

The right hand side is contained in: $Q_t := Q \cdot \{ \begin{pmatrix} 1 & \\ S & 1 \end{pmatrix} : S \in S_t = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{pmatrix} \}$ where the blocks for S_t are with respect to the partition $a+b+1+b$ and consist of matrices such that $S_{ij} \in M(\mathbb{Z}_p)$, and such that p^{t_i} divides the i-th column for $1 \leq i \leq a$ of the matrix S, $p^{t_{a+i}}$ divides the $(a+b+1+i)$-th column for $1 \leq i \leq b$, $p^{t_{a+b+i}}$ divides the $(a+b+1+i)$-th row for $1 \leq i \leq b$, and the (i,j)-th entry of S_{41} and S_{44} are divisible by $p^{t_{a+b+i}+t_j}$ and $p^{t_{a+b+i}+t_{a+j}}$ respectively. Observe that we have only to show that if $\tilde{\gamma}\alpha(gw,1)w'\tilde{\gamma}^{-1} \in Q_t$ then $g \in PwK$, i.e. $gw \in PK^w$ for $K^w := wKw$ (note that $\gamma(m(g_1,1),g_1) \in Q$).

Let

$$\tilde{\gamma}_v \alpha(gw, 1) w' \tilde{\gamma}_v^{-1} = \begin{pmatrix} -a_1 & a_2 & a_3 & -b_1 & a_1 & b_1 & b_2 & \\ -a_4 & a_5 & a_6 & -b_3 & a_4 & b_3 & b_4 & \\ -a_7 & a_8 & a_9 & -b_5 & a_7 & b_5 & b_6 & \\ &&& 1 &&&& \\ -1-a_1 & a_2 & a_3 & -b_1 & a_1 & b_1 & b_2 & \\ -c_1 & c_2 & c_3 & 1-d_1 & c_1 & d_1 & d_2 & \\ -c_4 & c_5 & c_6 & -d_3 & c_4 & d_3 & d_4 & \\ -a_4 & a_5 - 1 & a_6 & -b_3 & a_4 & b_3 & b_4 & 1 \end{pmatrix} := H.$$

Thus if $H \in Q_t$, then $\begin{pmatrix} a_1 & b_1 & b_2 & \\ c_1 & d_1 & d_2 & \\ c_4 & d_3 & d_4 & \\ a_4 & b_3 & b_4 & 1 \end{pmatrix}$ is invertible and there exists $S \in S_t$ such that:

$$\begin{pmatrix} -1-a_1 & a_2 & a_3 & -b_1 \\ -c_1 & c_2 & c_3 & 1-d_1 \\ -c_4 & c_5 & c_6 & -d_3 \\ -a_4 & a_5-1 & a_6 & -b_3 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & b_2 & \\ c_1 & d_1 & d_2 & \\ c_4 & d_3 & d_4 & \\ a_4 & b_3 & b_4 & 1 \end{pmatrix} S.$$

By looking at the 3rd row (block-wise), one finds $d_4 \neq 0$, so by left multiplying g by a matrix $\begin{pmatrix} 1_a & & & & \times \\ & 1_b & & & \times \\ & & 1 & & \times \\ & & & 1_b & \times \\ & & & & d_4^{-1} \end{pmatrix}$ (which does not change the assumption and conclusion) we may assume that $d_4 = 1$ and $d_2 = 0, b_2 = 0, b_4 = 0, b_6 = 0$. So we assume that gw is of the form:

$$\begin{pmatrix} a_1 & a_2 & a_3 & b_1 & \\ a_4 & a_5 & a_6 & b_3 & \\ a_7 & a_8 & a_9 & & \\ c_1 & c_2 & c_3 & d_1 & \\ c_4 & c_5 & c_6 & d_3 & 1 \end{pmatrix}.$$

Next by looking at the 2nd row (block-wise) and noting that $d_2 = 0$ we find that d_1 is of the form

$$\begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p & ... & ... & \mathbb{Z}_p \\ p^{t_{a+1}}\mathbb{Z}_p & \mathbb{Z}_p^\times & ... & ... & \mathbb{Z}_p \\ ... & p^{t_{a+2}}\mathbb{Z}_p & \mathbb{Z}_p^\times & ... & ... \\ ... & ... & ... & ... & ... \\ p^{t_{a+1}}\mathbb{Z}_p & ... & ... & ... & \mathbb{Z}_p^\times \end{pmatrix}.$$

So by multiplying a matrix of the form $\begin{pmatrix} 1_a & & & & \\ & 1_b & & & \\ & & 1 & \times & \\ & & & 1_b & \\ & & & & 1 \end{pmatrix}$ from the left we may assume that $b_5 = 0$. And by looking at the 3rd row again we see $c_4 = (p^{t_1}\mathbb{Z}_p, ..., p^{t_a}\mathbb{Z}_p), c_5, c_6 \in M(\mathbb{Z}_p)$, $d_3 \in (p^{t_{a+1}}, ..., p^{t_{a+b}})$. From the 2nd row: $c_1 \in (M_{b\times 1}(p^{t_1}\mathbb{Z}_p), M_{b\times 1}(p^{t_2}\mathbb{Z}_p), ..., M_{b\times 1}(p^{t_a}\mathbb{Z}_p))$, $c_2 \in M_{b\times b}(\mathbb{Z}_p)$, $c_3 \in M_{b\times 1}(\mathbb{Z}_p)$.

By looking at the 1st row and note that $b_2 = 0$ we know $a_1 \in \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p & ... & ... & \mathbb{Z}_p \\ p^{t_1}\mathbb{Z}_p & \mathbb{Z}_p^\times & ... & ... & \mathbb{Z}_p \\ ... & p^{t_2}\mathbb{Z}_p & \mathbb{Z}_p^\times & ... & ... \\ ... & ... & ... & ... & ... \\ p^{t_1}\mathbb{Z}_p & ... & ... & ... & \mathbb{Z}_p^\times \end{pmatrix}$, $a_2, a_3 \in M(\mathbb{Z}_p)$, $b_1 \in (M_{a\times 1}(p^{t_{a+1}}\mathbb{Z}_p), M_{a\times 1}(p^{t_{a+2}}\mathbb{Z}_p), ..., M_{a\times 1}(p^{t_{a+b}}\mathbb{Z}_p))$. Finally looking at the 4-th row (block-wise), we note that $b_4 = 0$. Similarly, $a_4 \in (M_{b\times 1}(p^{t_1}\mathbb{Z}_p), M_{b\times 1}(p^{t_2}\mathbb{Z}_p), ..., M_{b\times 1}(p^{t_a}\mathbb{Z}_p))$, $b_3 \in (M_{b\times 1}(p^{t_{a+1}}\mathbb{Z}_p), M_{b\times 1}(p^{t_{a+2}}\mathbb{Z}_p), ..., M_{b\times 1}(p^{t_{a+b}}\mathbb{Z}_p))$,
and $a_5 - 1 \in \begin{pmatrix} M_{1\times b}(p^{t_{a+b+1}}\mathbb{Z}_p) \\ M_{1\times b}(p^{t_{a+b+2}}\mathbb{Z}_p) \\ ... \\ M_{1\times b}(p^{t_{a+2b}}\mathbb{Z}_p) \end{pmatrix}$, $a_6 \in \begin{pmatrix} p^{t_{a+b+1}}\mathbb{Z}_p \\ p^{t_{a+b+2}}\mathbb{Z}_p \\ ... \\ p^{t_{a+2b}}\mathbb{Z}_p \end{pmatrix}$.
Now we prove that $gw \in PK^w$ using the properties proven above. First we right multiply gw

by $\begin{pmatrix} 1_a &&&& \\ & 1_b &&& \\ && 1 && \\ -d_1^{-1}c_1 & -d_1^{-1}c_2 & -d_1^{-1}c_3 & d_1^{-1} & \\ -c_4 & -c_5 & -c_6 & -d_3 & 1 \end{pmatrix} \in K^w$, which does not change the above properties or what needs to be proven, so without loss of generality we assume that $c_4 = 0, c_5 = 0, c_6 = 0, d_3 = 0, c_1 = 0, c_2 = 0, c_3 = 0, d_1 = 1$. Moreover we set $\begin{pmatrix} a_1 & a_2 \\ a_4 & a_5 \end{pmatrix}^{-1} \begin{pmatrix} a_3 \\ a_6 \end{pmatrix} := T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$. Then $\begin{pmatrix} 1_a && T_1 && \\ & 1_b & T_2 && \\ && 1 && \\ &&& 1_b & \\ &&&& 1 \end{pmatrix} \in K^w$. By multiplying $\begin{pmatrix} 1_a && -T_1 && \\ & 1_b && -T_2 & \\ && 1 && \\ &&& 1_b & \\ &&&& 1 \end{pmatrix}$ to the right we get an element in P. So it is clear that $gw \in PK^w$. $\square$

Now suppose that π is nearly ordinary with respect to $\underline{k}$. We denote φ to be the unique (up to scalar) nearly ordinary vector in π with respect to the Borel $\tilde{B}$. Let $\varphi_w = \pi(w)\varphi$.

Now write

$$\varphi' = \pi(\begin{pmatrix} p^{-t_{a+b+1}} &&&&& \\ & \dots &&&& \\ && p^{t_1} &&& \\ &&& \dots && \\ &&&& p^{t_{a+1}} & \\ &&&&& \dots \end{pmatrix}^{\iota} \begin{pmatrix} && -1_b \\ & 1_a & \\ 1_b && \end{pmatrix}^{\iota})\varphi_w.$$

<u>Compute the value $F_{\varphi'}(z, \rho(\Upsilon)f^\dagger, w)$</u>

In fact $F_{\varphi'}(z, \rho(\Upsilon)f^\dagger, w)$ it is equal to:

$$\sum_{B,C,D,E} \int_{\mathrm{GL}_{a+2b}(\mathbb{Q}_p)} \tilde{f}^\dagger(\tilde{\gamma}\alpha(w \begin{pmatrix} B &&& \\ & 1 && \\ && C & \\ &&& 1 \end{pmatrix} w, (g_1 \begin{pmatrix} E & \\ & D \end{pmatrix} \begin{pmatrix} p^{-t_{a+b+1}} &&&&& \\ & \dots &&&& \\ && p^{t_1} &&& \\ &&& \dots && \\ &&&& p^{t_{a+1}} & \\ &&&&& \dots \end{pmatrix}$$

$$\times \begin{pmatrix} && 1_b \\ & 1_a & \\ -1_b && \end{pmatrix})^{\iota})w'\tilde{\gamma}^{-1}\mathrm{diag}(p^{-t_1}, ..., p^{-t_a}, 1_b, 1, p^{-t_{a+1}}, ..., p^{-t_{a+b}}, 1_a, 1_b, 1, p^{t_{a+b+1}}, p^{t_{a+2b}}$$

$$\times w_{a+2b+1}^{-1})\bar{\tau}(\det g_1)\rho(g_1^\iota)\varphi' dg_1$$

where sum is over $B \in \mathfrak{B}', C \in \mathfrak{C}', D \in \mathfrak{D}', E \in \mathfrak{E}'$. A direct computation gives: $\tilde{\gamma}\alpha(1, \begin{pmatrix} a_1 & a_3 & a_2 \\ a_7 & a_9 & a_8 \\ a_4 & a_6 & a_5 \end{pmatrix}^{\iota})w'\tilde{\gamma}^{-1}$ equals

$$\begin{pmatrix} -1_a &&&& 1_a &&& \\ & 1_b &&&&&& \\ && 1 &&&&& \\ -a_3 & -a_2 && a_1 &&&& a_2 \\ -a_9 - 1_a & -a_8 && a_7 & 1_a &&& a_8 \\ -a_3 & -a_2 && a_1 - 1_b && 1_b && a_2 \\ &&&&&& 1 & \\ -a_6 & 1_b - a_5 && a_4 &&&& a_5 \end{pmatrix}.$$

Now we define $\mathfrak{Y}$ to be the subset of $\mathrm{GL}_{a+2b}(\mathbb{Z}_p)$ consisting of block matrices $\begin{pmatrix} a_1 & a_3 & a_2 \\ a_7 & a_9 & a_8 \\ a_4 & a_6 & a_5 \end{pmatrix}$ such that $\tilde{\gamma}\alpha(1, \begin{pmatrix} a_1 & a_3 & a_2 \\ a_7 & a_9 & a_8 \\ a_4 & a_6 & a_5 \end{pmatrix}^{\iota})w'\tilde{\gamma}^{-1}$ is in the Q_t defined in the proof of Lemma 4.36. It is not hard to prove that it can be described as follows: the i-th column of $-a_9 - 1$ and a_3 are divisible by p^{t_i} for $1 \le i \le a$, the i-th column of $a_7, a_1 - 1$ are divisible by $p^{t_{a+i}}$, the (i,j)-th entry of a_6 is divisible by $p^{t_{a+b+i}+t_j}$, the (i,j)-th entry of a_4 is divisible by $p^{t_{a+b+i}+t_{a+j}}$, the i-th row of $1 - a_5$ is divisible by $p^{t_{a+b+i}}$. The entries in a_2 and a_8 are in $\mathbb{Z}_p$. Then the pullback section is equal to

$$\begin{aligned}\sum_{B,C,D,E} \int \tilde{f}^{\dagger}(\tilde{\gamma}\alpha(1, g_1^{\iota})w'\tilde{\gamma}^{-1}\mathrm{diag}(p^{-t_1}, ..., p^{-t_a}, 1, 1, p^{-t_{a+1}}, ..., p^{-t_{a+b}}, 1, 1, 1, p^{t_{a+b+1}}, ..., p^{t_{a+2b}}) \\ \times w_{a+2b+1}^{-1})\bar{\tau}(\det g_1)\pi(g_1^{\iota})\varphi dg_1\end{aligned}$$

(superscript w means conjugation by w) where the integration is over the set:

$$g_1 \in \begin{pmatrix} B & \\ & C \end{pmatrix}_{\iota}^{w} \mathfrak{Y} \begin{pmatrix} E & \\ & D \end{pmatrix}_{conj} \begin{pmatrix} & & 1_b \\ & 1_a & \\ -1_b & & \end{pmatrix} \begin{pmatrix} p^{t_{a+b+1}} & & & & & \\ & \cdots & & & & \\ & & p^{-t_1} & & & \\ & & & \cdots & & \\ & & & & p^{-t_{a+b}} & \\ & & & & & \cdots \end{pmatrix}$$

for:

$$\begin{aligned}\begin{pmatrix} E & \\ & D \end{pmatrix}_{conj} := & \begin{pmatrix} & & 1_b \\ & 1_a & \\ -1_b & & \end{pmatrix} \begin{pmatrix} p^{t_{a+b+1}} & & & & & \\ & \cdots & & & & \\ & & p^{-t_1} & & & \\ & & & \cdots & & \\ & & & & p^{-t_{a+1}} & \\ & & & & & \cdots \end{pmatrix} \begin{pmatrix} E & \\ & D \end{pmatrix} \\ & \times \begin{pmatrix} p^{-t_{a+b+1}} & & & & & \\ & \cdots & & & & \\ & & p^{t_1} & & & \\ & & & \cdots & & \\ & & & & p^{t_{a+1}} & \\ & & & & & \cdots \end{pmatrix} \begin{pmatrix} & & -1_b \\ & 1_a & \\ 1_b & & \end{pmatrix}.\end{aligned}$$

Lemma 4.37. *If φ_w is invariant under the action of $(\tilde{K}'')^{\iota}$. Then*

$$F_{\varphi'}(z, \rho(\Upsilon)f^{\dagger}, w)$$

is such that the action of $\tilde{K}^{\iota}$ on it is given by $\tilde{\nu}$.

Proof. By the above two lemmas we only need to check that $F_{\varphi'}(z, \rho(\Upsilon)f^{\dagger}, w)$ is invariant under the action of $\tilde{K}''$. We first claim that $\sum_{D,E} \pi(\begin{pmatrix} E & \\ & D \end{pmatrix}_{conj}^{\iota})\varphi'$ is invariant under $(\tilde{K}'')^{\iota}$. The claim follows from direct checking. Also for any $k_1 \in \tilde{K}''$, we can find a $k_2 \in \tilde{K}''$ such that $k_1 \begin{pmatrix} B & \\ & C \end{pmatrix}_{\iota}^{w} k_2^{-1}$ runs over the same set of representatives as $\begin{pmatrix} B & \\ & C \end{pmatrix}_{\iota}^{w}$. For any $k_1 \in \tilde{K}''$, we can find a $k_2 \in \tilde{K}''$ such that $k_1\mathfrak{Y}k_2^{-1} = \mathfrak{Y}$. The lemma follows from these observations. $\square$

The value of $\tilde{f}^\dagger$ at $g_1 = \begin{pmatrix} & & 1_b \\ & 1_a & \\ -1_b & & \end{pmatrix} \begin{pmatrix} p^{t_{a+b+1}} & & & & & \\ & \cdots & & & & \\ & & p^{-t_1} & & & \\ & & & \cdots & & \\ & & & & p^{-t_{a+b}} & \\ & & & & & \cdots \end{pmatrix}$ is

$$\tau((p^{t_{a+b+1}+\dots+t_{a+2b}}, p^{t_1+\dots+t_{a+b}}))|p^{t_1+\dots+t_{a+2b}}|^{-z-\frac{a+2b+1}{2}}.$$

So a straightforward computation making use of the model for $\pi = \pi(\chi_1, \dots, \chi_{a+2b})$ tells us the following:

Lemma 4.38. *If φ and φ' are defined as after the proof of Lemma* 4.36 *then:*

$$\begin{aligned} F_{\varphi'}(z, \rho(\Upsilon)f^\dagger, w) = \quad & \tau((p^{t_1+\dots+t_{a+b}}, p^{t_{a+b+1}+\dots+t_{a+2b}}))|p^{t_1+\dots+t_{a+2b}}|^{-z-\frac{a+2b+1}{2}}\mathrm{Vol}(\tilde{K}') \\ & \times p^{-\sum_{i=1}^{a+b} it_i - \sum_{i=1}^{b} it_{a+b+i}} \prod_{i=1}^{a+b} \mathfrak{g}(\xi_i)\xi_i(-1) \prod_{i=1}^{b} \mathfrak{g}(\xi_{a+b+1+i})\xi_{a+b+1+i}(-1)\varphi_w. \end{aligned}$$

Combining the 3 lemmas above, we get the following:

Proposition 4.39. *Assumptions are as in the above lemma. $F_{\varphi'}(z, \rho(\Upsilon)f^\dagger, g)$ is the unique section supported in PwK such that the right action of K is given by multiplying the character ν and its value at w is:*

$$\begin{aligned} F_{\varphi'}(z, \rho(\Upsilon)f^\dagger, w) = \quad & \tau((p^{t_1+\dots+t_{a+b}}, p^{t_{a+b+1}+\dots+t_{a+2b}}))|p^{t_1+\dots+t_{a+2b}}|^{-z-\frac{a+2b+1}{2}}\mathrm{Vol}(\tilde{K}') \\ & \times p^{-\sum_{i=1}^{a+b} it_i - \sum_{i=1}^{b} it_{a+b+1+i}} \prod_{i=1}^{a+b} \mathfrak{g}(\xi_i)\xi_i(-1) \prod_{i=1}^{b} \mathfrak{g}(\xi_{a+b+1+i})\xi_{a+b+1+i}(-1)\varphi_w. \end{aligned}$$

Proof. ϕ_w is clearly invariant under $(\tilde{K}'')^\iota$. □

This $F_{\varphi'}(z, \rho(\Upsilon)f^\dagger, g)$ we constructed is not going to be the nearly ordinary vector unless we apply the intertwining operator to it. So now we start with a $\rho = (\pi, \tau)$. We define our Siegel section $f^0 \in I_{a+2b+1}(\tau)$ to be:

$$f^0(z; g) := M(-z, f^\dagger)_z(g)$$

where $f^\dagger \in I_{a+2b+1}(\bar{\tau}^c)$. We recall the following generalization of a proposition from [28] (in a generalized form).

Proposition 4.40. *Suppose our data (π, τ) comes from the local component at v of a global data. Then there is a meromorphic function $\gamma^{(2)}(\rho, z)$ such that*

$$F_{\varphi^\vee}(-z, M(z, f), g) = \gamma^{(2)}(\rho, z)A(\rho, z, F_\varphi(f; z, -))_{-z}(g).$$

Moreover if $\pi_v \simeq \pi(\chi_1, \dots, \chi_{a+2b})$ then if we write $\gamma^{(1)}(\rho, z) = \gamma^{(2)}(\rho, z + \frac{1}{2})$ then

$$\gamma^{(1)}(\rho, z) = \psi(-1)c_n(\tau', z)\mathfrak{g}(\tau'_p)^n \epsilon(\pi, \tau^c, z + \frac{1}{2})\frac{L(\tilde{\pi}, \bar{\tau}^c, 1/2 - z)}{L(\pi, \tau^c, z + 1/2)}$$

where $c_n(\tau', z)$ is the constant appearing in lemma 4.27.

Proof. The same as [28, Proposition 11.13]. □

Remark 4.41. Note that here we are using the L-factors for the base change from the unitary groups while [28] uses the GL_2 L-factor for π so our formula appears slightly differently.

Now we are going to show that:

$$F_v^0(z;g) := F_{\varphi'}(z,\rho(\Upsilon)f^0,g)$$

is a constant multiple of the nearly ordinary vector if our ρ comes from the local component of the global Eisenstein data (see subsection 3.1). Return to the situation of our Eisenstein Data. Suppose that at the Archimedean places our representation is a holomorphic discrete series associated to the (scalar) weight: $\underline{k}=(0,...0;\kappa,...\kappa)$ with r 0's and s κ's. Here $r=a+b, s=b$. Suppose $\pi\simeq \mathrm{Ind}(\chi_1,...,\chi_{a+2b})$ is nearly ordinary with respect to the weight $\underline{k}$. We may reorder the χ_i's such that $\nu_p(\chi_1(p))=s-\frac{n}{2}+\frac{1}{2},...,\nu_p(\chi_r(p))=r+s-1-\frac{n}{2}+\frac{1}{2},\nu_p(\chi_{r+s}(p))=\kappa-\frac{n}{2}+\frac{1}{2},...,\nu_p(\chi_{r+1}(p))=\kappa+s-1-\frac{n}{2}+\frac{1}{2}$, and $\tau=(\tau_1,\tau_2^{-1})$ a character of $\mathbb{Q}_p^\times\times\mathbb{Q}_p^\times$ with $\nu_p(\tau_1(p))=\frac{\kappa}{2},\nu_p(\tau_2(p))=\frac{\kappa}{2}$, so

$$\nu_p(\chi_1(p))<...<\nu_p(\chi_{a+b}(p))<\nu_p(\tau_2(p)p^{-z_\kappa})<\nu_p(\tau_1(p)p^{z_\kappa})<\nu_p(\chi_{a+2b}(p))<...<\nu_p(\chi_{a+b+1}(p))$$

where $z_\kappa=\frac{\kappa-r-s-1}{2}$. It is easy to see that $I(\rho_v,z_\kappa)\simeq \mathrm{Ind}(\chi_1,...\chi_{r+s},\tau_2|.|^{z_\kappa},\tau_1|.|^{-z_\kappa})$. By definition $I(\rho_v,z_\kappa)$ is nearly ordinary with respect to the weight $(0,...,0;\kappa,...,\kappa)$ with $(r+1)$ 0's and $(s+1)$ κ's.

Definition 4.42. *Assumptions and conventions are as above. We say (π,τ) is generic if*

$$\mathrm{cond}(\chi_1)>\cdots>\mathrm{cond}(\chi_{a+b})>\mathrm{cond}(\tau_2)>\mathrm{cond}(\chi_{a+b+1})>\cdots>\mathrm{cond}(\chi_{a+2b})>\mathrm{cond}(\tau_1).$$

We suppose also the conductor of τ_i is p^{s_i}. Notice that we have $s_2>s_1$ by our assumption which is different from Definition 4.21 *(since we have applied the intertwining operator here).*

Let us record the following formula for the ϵ-factor in Proposition 4.40:

$$\begin{aligned}\epsilon(\pi,\tau^c,z+\frac{1}{2})= &\prod_{i=1}^r\mathfrak{g}(\chi_i^{-1}\tau_2)\chi_i\tau_2^{-1}(p^{t_i})\cdot\prod_{i=1}^s\mathfrak{g}(\chi_{r+i}^{-1}\tau_2)\chi_{r+i}\tau_2^{-1}(p^{s_2})\cdot|p^{\sum_{i=1}^r t_i+s\cdot s_2}|^{z+\frac{1}{2}}\\ &\times\prod_{i=1}^{r+s}\mathfrak{g}(\chi_i\tau_1^{-1})\chi_i^{-1}\tau_1(p^{t_i})\cdot|p^{\sum_{i=1}^{r+s}t_i}|^{z+\frac{1}{2}}. \qquad (14)\end{aligned}$$

From the form of $F_{\varphi'}(z,\rho(\Upsilon)f^\dagger;g)$ and the above proposition we have a description in the "generic" case for $F_v^0(z,g)$ as in [28, Lemma 9.6]: it is supported in $P(\mathbb{Q}_p)K_v$,

$$\begin{aligned}F_v^0(z,1)= &\ \gamma^{(2)}(\rho,-z)\bar{\tau}^c((p^{t_1+...+t_{a+b}},p^{t_{a+b+1}+...+t_{a+2b}}))|p^{t_1+...+t_{a+2b}}|^{z-\frac{a+2b+1}{2}}\mathrm{Vol}(\tilde{K}')\\ &\times p^{-\sum_{i=1}^{a+b}(i+1)t_i-\sum_{i=1}^b(i+1)t_{a+b+1+i}}\prod_{i=1}^{a+b}\mathfrak{g}(\xi_i^\dagger)\xi_i(-1)\prod_{i=1}^b\mathfrak{g}(\xi_{a+b+1+i}^\dagger)\xi_{a+b+1+i}^\dagger(-1)\varphi\\ = &\ c_n(\tau_p',-z-\frac{1}{2})\mathfrak{g}(\tau_p')^n\bar{\tau}^c((p^{t_1+...+t_{a+b}},p^{t_{a+b+1}+...+t_{a+2b}}))|p^{t_1+...+t_{a+2b}}|^{z-\frac{a+2b+1}{2}}\mathrm{Vol}(\tilde{K}')\\ &\times p^{-\sum_{i=1}^{a+b}it_i-\sum_{i=1}^b it_{a+b+1+i}}\prod_{i=r+1}^{r+s}\mathfrak{g}(\chi_i^{-1}\tau_2)\chi_i\tau_2^{-1}(p^{s_2})\prod_{j=1}^r\mathfrak{g}(\chi_j\tau_1^{-1})\chi_j^{-1}\tau_1(p^{t_j})\epsilon(\pi,\tau^c,z)\varphi\\ &\times|p^{\sum_{i=1}^r t_i+s\cdot s_2}|^{-z}\cdot|p^{\sum_{i=1}^{r+s}t_i}|^{-z}\end{aligned}$$

where the $\xi_i^\dagger$ are the ξ_i defined in Definition 4.21 but using $(\pi,\bar{\tau}^c)$ instead of (π,τ). Here we also used Proposition 4.40 and the formula for the epsilon factor there. Notice that we have absorbed a factor $p^{-\sum_{i=1}^{a+b}t_i-\sum_{i=1}^b t_{a+b+1+i}}$ which comes from computing the image under intertwining operator of $F_{\varphi'}(z,\rho(\Upsilon)f^\dagger;g)$ to get the factor $p^{-\sum_{i=1}^{a+b}(i+1)t_i-\sum_{i=1}^{b}(i+1)t_{a+b+1+i}}$ in the above expression. The right action of K_v is given by the character

$$\chi_1(g_{11})...\chi_{a+b}(g_{a+b\ a+b})\tau_2(g_{a+b+1\ a+b+1})\chi_{a+b+1}(g_{a+b+2\ a+b+2})...\\ \times\chi_{a+2b}(g_{a+2b+1\ a+2b+1})\tau_1(g_{a+2b+2\ a+2b+2}).$$

(It is easy to compute $A(\rho,z,F_{\varphi'}(\rho(\Upsilon)f^\dagger;z,-))_{-z}(1)$ and we use the uniqueness of the vector with the required K_v action. Here on the second row of the above formula for $F_v^0(z,1)$ the power for p is slightly different from that for the section $F(z,f^\dagger,w)$. This comes from the computations for the intertwining operators for Klingen Eisenstein sections.)

Thus Corollary 4.20 tells us that $F_v^0(z,g)$ is a nearly ordinary vector in $I(\rho)$.

Now we describe f^0:

Definition 4.43. *Suppose $(p^t)=\mathrm{cond}(\tau')$ for $t\geq 1$ then define f_t to be the section supported in $Q(\mathbb{Q}_p)K_Q(p^t)$ and $f_t(k)=\tau(\det d_k)$ on $K_Q(p^t)$.*

Lemma 4.44.

$$\tilde{f}^0:=M(-z,\tilde{f}^\dagger)_z=f_{t,z}.$$

Proof. This is just [28, Lemma 11.10]. □

Corollary 4.45.

$$\begin{aligned}f^0(z,g)=\ & p^{-\sum_{i=1}^{a+b}it_i-\sum_{i=1}^{b}it_{a+b+i}}\prod_{i=1}^{a+b}\mathfrak{g}(\xi_i)\xi_i(-1)\prod_{i=1}^{b}\mathfrak{g}(\xi_{a+b+1+i})\xi_{a+b+1+i}(-1)\\ &\times\sum_{A,B,C,D,E}\prod_{i=1}^{a}\bar{\xi}_i(\frac{\det A_i}{\det A_{i-1}}p^{t_i})\prod_{i=1}^{b}\bar{\xi}_{a+i,a+i}(\frac{\det D_i}{\det D_{i-1}}p^{t_{a+i}})\times\prod_{i=1}^{b}\bar{\xi}_{a+b+1+i}(\frac{\det E_i}{\det E_{i-1}}p^{t_{a+b+i}})\\ &\times\tilde{f}_t(z,g\begin{pmatrix}1_{a+2b+1} & \begin{matrix}A & B\\ C & D\end{matrix}\\ & \begin{matrix}E\\ 1_{a+2b+1}\end{matrix}\end{pmatrix}).\end{aligned}$$

Here A_i is the i-th upper-left minor of A, D_i is the $(a+i)$-th upper left minor of $\begin{pmatrix}A & B\\ C & D\end{pmatrix}$, E_i is the i-th upper-left minor of E.

We define the Siegel section $f^{0\prime}\in I_{a+2b}(\tau)$ by

$$\begin{aligned}f^{0\prime}(z,g)=\ & p^{-\sum_{i=1}^{a+b}it_i-\sum_{i=1}^{b}it_{a+b+i}}\prod_{i=1}^{a+b}\mathfrak{g}(\xi_i)\xi_i(-1)\prod_{i=1}^{b}\mathfrak{g}(\xi_{a+b+1+i})\xi_{a+b+1+i}(-1)\\ &\times\sum_{A,B,C,D,E}\prod_{i=1}^{a}\bar{\xi}_i(\frac{\det A_i}{\det A_{i-1}}p^{t_i})\prod_{i=1}^{b}\bar{\xi}_{a+i,a+i}(\frac{\det D_i}{\det D_{i-1}}p^{t_{a+i}})\times\prod_{i=1}^{b}\bar{\xi}_{a+b+1+i}(\frac{\det E_i}{\det E_{i-1}}p^{t_{a+b+i}})\\ &\times\tilde{f}_t(z,gw'^{-1}_{Borel}\begin{pmatrix}1_{a+2b+1} & \begin{matrix}A & B\\ C & D\end{matrix}\\ & \begin{matrix}E\\ 1_{a+2b+1}\end{matrix}\end{pmatrix}w'_{Borel}).\end{aligned}$$

Then similar as before the corresponding pullback section $F'_{\varphi'}(z,\rho(\Upsilon')f^{0\prime},1)$ is

$$\begin{aligned}
= \quad & c_n(\tau'_p,-z)\mathfrak{g}(\tau'_p)^n\bar{\tau}^c((p^{t_1+\ldots+t_{a+b}},p^{t_{a+b+1}+\ldots+t_{a+2b}}))|p^{t_1+\ldots+t_{a+2b}}|^{z-\frac{a+2b}{2}}\mathrm{Vol}(\tilde{K}')\\
& \times p^{-\sum(i-1)t_i-\sum(i-1)t_{a+b+i}}\prod_{i=r+1}^{r+s}\mathfrak{g}(\chi_i^{-1}\tau_2)\chi_i\tau_2^{-1}(p^{s_2})\prod_{j=1}^{r}\mathfrak{g}(\chi_j\tau_1^{-1})\chi_j^{-1}\tau_1(p^{t_j})\epsilon(\pi,\tau^c,z+\frac{1}{2})\varphi\\
& \times|p^{\sum_{i=1}^{r}t_i+s\cdot s_2}|^{-z+\frac{1}{2}}\cdot|p^{\sum_{i=1}^{r+s}t_i}|^{-z+\frac{1}{2}}.
\end{aligned}$$

Fourier Coefficients for f^0

We record a formula here for the Fourier Coefficients for f^0 which will be used in p-adic interpolation.

Lemma 4.46. *Suppose* $|\det\beta|\neq 0$ *then:*
(i) If $\beta\notin S_{a+2b+1}(\mathbb{Z}_p)$ *then* $f^0_\beta(z,1)=0$*;*
(ii) Let $t:=\mathrm{ord}_p(\mathrm{cond}(\tau')$*. If* $\beta\in S_{a+2b+1}(\mathbb{Z}_p)$*, then:*

$$f^0_\beta(z,1)=\bar{\tau}'(\det\beta)|\det\beta|_p^{2z}\mathfrak{g}(\tau')^{a+2b+1}c_{a+2b+1}(\bar{\tau}',-z)\Phi_\xi({}^t\beta).$$

where $c_{a+2b+1}(-,-)$ *is defined in equation (*13*) and* Φ_ξ *is defined in (*11*).*

Proof. This follows from [28, 11.4.12] and the argument of corollary 4.30 where we deduce the form of $f^\dagger$ from the section $\tilde{f}^\dagger$. □

4.4.3 Fourier-Jacobi Coefficients

Now let $m=b+1$. For $\beta\in S_m(F_v)\cap\mathrm{GL}_m(\mathcal{O}_v)$ we are going to compute the Fourier-Jacobi coefficient for f_t at β.

Lemma 4.47. *Let* $x:=\begin{pmatrix}1&\\D&1\end{pmatrix}$ *(this is a block matrix with respect to* $(a+b)+(a+b)$*).*
(a) $FJ_\beta(f_t;z,v,x\eta^{-1},1)=0$ *if* $D\notin p^tM_{a+b}(\mathbb{Z}_p)$*;*
(b) if $D\in p^tM_n(\mathbb{Z}_p)$ *then* $FJ_\beta(f_t;z,v,x\eta^{-1},1)=c(\beta,\tau,z)\Phi_0(v)$*, where*

$$c(\beta,\tau,z):=\bar{\tau}(-\det\beta)|\det\beta|_v^{2z+n-m}\mathfrak{g}(\tau')^mc_m(\tau',-z-\frac{n-m}{2})$$

and c_m *is defined in Lemma* 4.27*.*

Proof. Similar to the proof of [28, Lemma 11.20]. We only give the detailed proof for the case when $a=0$. The case when $a>0$ is even easier to treat.

Assuming $a=0$, we temporarily write n for b and save the letter b for other use. We have:

$$w_{2n+1}\begin{pmatrix}1_{2n+1}&\begin{matrix}S&v\\{}^t\bar{v}&D\end{matrix}\\&1_{2n+1}\end{pmatrix}\alpha(1,\eta^{-1})=\begin{pmatrix}&&1_{n+1}&\\&&&-1_n\\-1_{n+1}&v&-S&\\&D&-{}^t\bar{v}&-1_n\end{pmatrix}.$$

This belongs to $Q_{2n+1}(\mathbb{Q}_p)K_{Q_{2n+1}}(p^t)$ ($K_{Q_{2n+1}}(p^t)$ consists of matrices in $Q_{2n+1}(\mathbb{Z}_p)$ modulo p^t) if and only if S is invertible, $S^{-1}\in p^tM_{n+1}(\mathcal{O}_v)$, $S^{-1}v\in p^tM_{(n+1)\times n}(\mathcal{O}_v)$ and ${}^t\bar{v}S^{-1}v-D\in p^tM_n(\mathbb{Z}_p)$. Since $v=\gamma{}^t(b,0)$ for some $\gamma\in\mathrm{SL}_{n+1}(\mathcal{O}_v)$ and $b\in M_n(\mathcal{K}_v)$ we are reduced to the case $v={}^t(b,0)$. Writing $b=(b_1,b_2)$ with $b_i\in M_n(\mathbb{Q}_p)$ and $S=(T,{}^tT)$ with $T\in M_{n+1}(\mathbb{Q}_p)$ and $T^{-1}=\begin{pmatrix}a_1&a_2\\a_3&a_4\end{pmatrix}$ where $a_1\in M_n(\mathbb{Q}_p)$, $a_2\in M_{n\times 1}(\mathbb{Q}_p)$,$a_3\in M_{1\times n}(\mathbb{Q}_p)$, $a_4\in M_1(\mathbb{Q}_p)$, the conditions on S and v can be rewritten as:

(*) $\det T \neq 0$, $a_i \in p^t M_n(\mathbb{Z}_p)$, $a_1 b_1 \in p^t M_n(\mathbb{Z}_p)$, $a_3 b_1 \in p^t M_{1\times n}(\mathbb{Z}_p)$, ${}^t a_1 b_2 \in p^t M_n(\mathbb{Z}_p)$, ${}^t a_2 b_2 \in p^t \mathbb{Z}_p$, ${}^t b_2 a_1 b_1 - D \in p^t M_n(\mathbb{Z}_p)$.

Now we prove that: if the integral for $FJ_\beta(f_t; z, v, x\eta^{-1}, 1)$ is non-zero then $b_1, b_2 \in M_n(\mathbb{Z}_p)$. Suppose otherwise, then without loss of generality we assume b_1 has an entry which has the maximal p-adic absolute value among all entries of b_1 and b_2, Suppose it is p^w for $w > 0$ (throughout the paper w means this only inside this lemma). Also, for any matrix A of given size we say $A \in {}^t b_2^\vee$ if and only ${}^t b_2 A$ has all entries in $\mathbb{Z}_p$ (of course we assume the sizes of the matrices are correct so that the product makes sense).

Now let

$$\Gamma := \left\{ \begin{array}{ll} \gamma = \begin{pmatrix} h & j \\ k & l \end{pmatrix} \in \mathrm{GL}_n(\mathbb{Z}_p): & h \in \mathrm{GL}_{n+1}(\mathbb{Z}_p), l \in \mathbb{Z}_p^\times, \\ h - 1 \in {}^t b_2^\vee \cap p^t M_n(\mathbb{Z}_p), & j \in \mathbb{Z}_p^n \cap {}^t b_2^\vee, k \in p^t M_{1\times n}(\mathbb{Z}_p) \end{array} \right\}.$$

Suppose that our b_1, b_2, D are such that there exist a_i's satisfying (*), then one can check that Γ is a subgroup, and if T satisfies (*), so does $T\gamma$ for any $\gamma \in \Gamma$. Let $\mathcal{T}$ denote the set of $T \in M_{n+1}(\mathbb{Q}_p)$ satisfying (*). Then $FJ_\beta(f_t; z, v, \begin{pmatrix} 1 & \\ D & 1 \end{pmatrix} \eta^{-1}, 1)$ equals

$$\sum_{T \in \mathcal{T}/\Gamma} |\det T|_p^{3n+2-2z} \int_\Gamma \tau'(-\det T\gamma) e_p(-\mathrm{tr}\beta T\gamma) d\gamma.$$

Let $T' := \beta T = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$ (blocks with respect to the partition $(n+1)$), then the above integral is zero unless $c_1 \in p^{-t} M_n(\mathbb{Z}_p) \oplus [{}^t b_2]_{n\times n}, c_4 \in p^{-t}\mathbb{Z}, c_2 \in p^{-t} M_{n+1}(\mathbb{Z}_p), c_3 \in [{}^t b_2]_{1\times n} \oplus M_{1\times n}(\mathbb{Z}_p)$. Here $[{}^t b_2]_{i\times n}$ means the set of $i \times n$ matrices such that each row is a $\mathbb{Z}_p$-linear combination of the rows of ${}^t b_2$.

But then

$$\beta \begin{pmatrix} b_1 \\ 0 \end{pmatrix} = T' T^{-1} \begin{pmatrix} b_1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 a_1 b_1 + c_2 a_3 b_1 \\ c_3 a_1 b_1 + c_4 a_3 b_1 \end{pmatrix}.$$

Since $\beta \in \mathrm{GL}_{n+1}(\mathbb{Z}_p)$, the left must contain some entry with p-adic absolute value p^w. But it is not hard to see that all entries on the right hand side have p-adic values strictly less than p^w, a contradiction. Thus we conclude that $b_1 \in M_n(\mathbb{Z}_p)$ and $b_2 \in M_n(\mathbb{Z}_p)$. By (*): $b_2{}^t a_1 b_1 - D \in p^t M_n(\mathbb{Z}_p), a_1 \in p^t M_n(\mathbb{Z}_p)$. So $D \in p^t M_n(\mathbb{Z}_p)$.

The value claimed in part (ii) can be deduced similarly as in [28, Lemma 11.20] □

4.4.4 Original Basis

Recall that we have changed the basis at the beginning of this subsection. Now we go back. We define the corresponding sections (we use the same notations)

$$\begin{aligned}
f^\dagger(z,g) = \quad & p^{-\sum_{i=1}^{a+b} it_i - \sum_{i=1}^{b} it_{a+b+i}} \prod_{i=1}^{a+b} \mathfrak{g}(\xi_i)\xi_i(-1) \prod_{i=1}^{b} \mathfrak{g}(\xi_{a+b+1+i})\xi_{a+b+1+i}(-1) \\
& \times \sum_{A,B,C,D,E} \prod_{i=1}^{a} \bar{\xi}_i(\frac{\det A_i}{\det A_{i-1}} p^{t_i}) \prod_{i=1}^{b} \bar{\xi}_{a+i,a+i}(\frac{\det D_i}{\det D_{i-1}} p^{t_{a+i}}) \times \prod_{i=1}^{b} \bar{\xi}_{a+b+1+i}(\frac{\det E_i}{\det E_{i-1}} p^{t_{a+b+i}}) \\
& \times \tilde{f}^\dagger(z, g w_{Borel}^{-1} \begin{pmatrix} 1_b & & & & & & C & D \\ & 1 & & & & & & \\ & & 1_a & & & & A & B \\ & & & 1_b & E & & & \\ & & & & 1_b & & & \\ & & & & & 1 & & \\ & & & & & & 1_a & \\ & & & & & & & 1_b \end{pmatrix} w_{Borel}).
\end{aligned}$$

$$\begin{aligned}
f^0(z,g) = \quad & p^{-\sum_{i=1}^{a+b} it_i - \sum_{i=1}^{b} it_{a+b+i}} \prod_{i=1}^{a+b} \mathfrak{g}(\xi_i)\xi_i(-1) \prod_{i=1}^{b} \mathfrak{g}(\xi_{a+b+1+i})\xi_{a+b+1+i}(-1) \\
& \times \sum_{A,B,C,D,E} \prod_{i=1}^{a} \bar{\xi}_i(\frac{\det A_i}{\det A_{i-1}} p^{t_i}) \prod_{i=1}^{b} \bar{\xi}_{a+i,a+i}(\frac{\det D_i}{\det D_{i-1}} p^{t_{a+i}}) \times \prod_{i=1}^{b} \bar{\xi}_{a+b+1+i}(\frac{\det E_i}{\det E_{i-1}} p^{t_{a+b+i}}) \\
& \times \tilde{f}_t(z, g w_{Borel}^{-1} \begin{pmatrix} 1_b & & & & & & C & D \\ & 1 & & & & & & \\ & & 1_a & & & & A & B \\ & & & 1_b & E & & & \\ & & & & 1_b & & & \\ & & & & & 1 & & \\ & & & & & & 1_a & \\ & & & & & & & 1_b \end{pmatrix} w_{Borel}).
\end{aligned}$$

Here A_i is the i-th upper-left minor of A, D_i is the $(a+i)$-th upper left minor of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, E_i is the i-th upper-left minor of E. The w_{Borel} is the element in $G(F_p)$ such that for any $v = w\bar{w}$ dividing p, $w \in \Sigma_p$, its projection to the first factor of $\mathcal{K}_v \simeq \mathcal{K}_w \times \mathcal{K}_{\bar{w}}$ is the Weyl element defined at the beginning of Subsection 4.4.2. We also define

$$\begin{aligned}
f^{\dagger\prime}(z,g) = \quad & p^{-\sum_{i=1}^{a+b} it_i - \sum_{i=1}^{b} it_{a+b+i}} \prod_{i=1}^{a+b} \mathfrak{g}(\xi_i)\xi_i(-1) \prod_{i=1}^{b} \mathfrak{g}(\xi_{a+b+1+i})\xi_{a+b+1+i}(-1) \\
& \times \sum_{A,B,C,D,E} \prod_{i=1}^{a} \bar{\xi}_i(\frac{\det A_i}{\det A_{i-1}}) \prod_{i=1}^{b} \bar{\xi}_{a+i,a+i}(\frac{\det D_i}{\det D_{i-1}}) \times \prod_{i=1}^{b} \bar{\xi}_{a+b+1+i}(\frac{\det E_i}{\det E_{i-1}}) \\
& \times \tilde{f}^\dagger(z, g w_{Borel}^{\prime -1} \begin{pmatrix} 1_b & & & & C & D \\ & 1_a & & & A & B \\ & & 1_b & E & & \\ & & & 1_b & & \\ & & & & 1_a & \\ & & & & & 1_b \end{pmatrix} w'_{Borel}).
\end{aligned}$$

$$f^{0\prime}(z,g)= \quad p^{-\sum_{i=1}^{a+b} it_i-\sum_{i=1}^{b} it_{a+b+i}}\prod_{i=1}^{a+b}\mathfrak{g}(\xi_i)\xi_i(-1)\prod_{i=1}^{b}\mathfrak{g}(\xi_{a+b+1+i})\xi_{a+b+1+i}(-1)$$

$$\times\sum_{A,B,C,D,E}\prod_{i=1}^{a}\bar{\xi}_i(\frac{\det A_i}{\det A_{i-1}})\prod_{i=1}^{b}\bar{\xi}_{a+i,a+i}(\frac{\det D_i}{\det D_{i-1}})\times\prod_{i=1}^{b}\bar{\xi}_{a+b+1+i}(\frac{\det E_i}{\det E_{i-1}})$$

$$\times\tilde{f}_t(z,gw'^{-1}_{Borel}\begin{pmatrix}1_b & & & & C & D\\ & 1_a & & & A & B\\ & & 1_b & E & & \\ & & & 1_b & & \\ & & & & 1_a & \\ & & & & & 1_b\end{pmatrix}w'_{Borel}).$$

The corresponding pullback section $F_{\varphi'}(f^0,z,-)$ is the nearly ordinary section with respect to the Borel B_2 defined in subsubsection 4.4.2 such that $F_{\varphi'}(f^0,z,w_{Borel})$ is given by

$$c_{n+1}(\tau'_p,-z-\tfrac{1}{2})\mathfrak{g}(\tau'_p)^{n+1}\bar{\tau}^c((p^{t_1+\ldots+t_{a+b}},p^{t_{a+b+1}+\ldots+t_{a+2b}}))|p^{t_1+\ldots+t_{a+2b}}|^{-z-\frac{a+2b+1}{2}}\mathrm{Vol}(\tilde{K}')$$
$$\times p^{-\sum it_i-\sum it_{a+b+i}}\prod_{i=r+1}^{r+s}\mathfrak{g}(\chi_i^{-1}\tau_2)\chi_i\tau_2^{-1}(p^{t_i})\prod_{j=1}^{r}\mathfrak{g}(\chi_j\tau_1^{-1})\chi_j^{-1}\tau_1(p^{t_j})\epsilon(\pi,\tau^c,z)\varphi.$$

Also we have $F'_{\varphi'}(z,\rho(\Upsilon')f^{0'},w'_{Borel})$ is given by

$$c_n(\tau'_p,-z)\mathfrak{g}(\tau'_p)^n\bar{\tau}^c((p^{t_1+\ldots+t_{a+b}},p^{t_{a+b+1}+\ldots+t_{a+2b}}))|p^{t_1+\ldots+t_{a+2b}}|^{-z-\frac{a+2b}{2}}\mathrm{Vol}(\tilde{K}')$$
$$\times p^{-\sum it_i-\sum it_{a+b+i}}\prod_{i=r+1}^{r+s}\mathfrak{g}(\chi_i^{-1}\tau_2)\chi_i\tau_2^{-1}(p^{t_i})\prod_{j=1}^{r}\mathfrak{g}(\chi_j\tau_1^{-1})\chi_j^{-1}\tau_1(p^{t_j})\epsilon(\pi,\tau^c,z+\tfrac{1}{2})\varphi.$$

5 Global Computations

5.1 p-adic Interpolation

5.1.1 Weight Space and Eisenstein Datum

Recall that we have the algebraic group $H=\prod_{v|p}\mathrm{GL}_r\times\mathrm{GL}_s$ such that $H(/\mathbb{Z}_p)$ is the Galois group of the Igusa tower over the ordinary locus of the toroidal compactified Shimura variety. Let $T_{/\mathbb{Z}_p}$ be the diagonal torus. Let $\mathbf{T}:=T(1+\mathbb{Z}_p)$. We define the weight ring $\Lambda=\Lambda_{r,s}$ as $\mathcal{O}_L[[\mathbf{T}]]$. Fix throughout a finite order character χ_0 of $T(\mathbb{F}_p)$ (the torsion part of $T(\mathbb{Z}_p)$), a $\bar{\mathbb{Q}}_p$-point $\phi\in\mathrm{Spec}\Lambda$ is called arithmetic if there is a weight $\underline{k}=(c_{s+1},...,c_{s+r};c_1,...,c_s)=(0,...,0;\kappa,...,\kappa)$ such that ϕ is given by a character $\chi_0\chi_\phi t_1^{c_{s+1}}...t_r^{c_{r+s}}t_{r+1}^{-c_1}...t_{r+s}^{-c_s}$ of $\mathbf{T}$ for χ_ϕ a character of order and conductor powers of p, $\kappa\geq 2(a+b+1)$. We write this κ as κ_ϕ. Let $\Lambda_\mathcal{K}=\mathcal{O}_L[[\Gamma_\mathcal{K}]]$.

Definition 5.1. *For $\mathbb{I}$ a normal domain over Λ which is also a finite module over Λ, a $\bar{\mathbb{Q}}_p$-point $\phi\in\mathrm{Spec}\mathbb{I}$ is called arithmetic if its image in* $\mathrm{Spec}\Lambda$ *is arithmetic.*

(i) If $s>0$, let $V^N_{\infty,\infty}(K,\mathbb{I},\chi_0)$ be the set of $\mathbb{I}$-adic formal Fourier-Jacobi expansions

$$\{\mathbf{f}_x=\sum_\beta a_\beta(x,\mathbf{f})q^\beta\}_x$$

such that for a Zariski dense set of generic arithmetic points $\phi\in\mathrm{Spec}\mathbb{I}$, the specialization $\mathbf{f}_\phi$ is the formal Fourier-Jacobi expansion of a form on $\mathrm{U}(r,s)$ whose p-part nebentype at is given by

$$\chi_0\chi_\phi\omega(t_1^{c_{s+1}}...t_r^{c_{s+r}}t_{r+1}^{-c_1}...t_{r+s}^{-c_s})$$

for the weight $(c_{s+1},...,c_{s+r};c_1,...,c_s)=(0,...,0;\kappa_\phi,...,\kappa_\phi)$. Here by χ_ϕ we also mean the character of $T(\mathbb{Z}_p)$ restricting to χ_ϕ on $\mathbf{T}$ and trivially on the torsion part of $T(\mathbb{Z}_p)$. We say $\mathbf{f}\in V^N_{\infty,\infty}(K,\mathbb{I})$ is a family of eigenforms if the specializations $\mathbf{f}_\phi$'s above are eigenforms. We define $V^{N,ord}_{\infty,\infty}(K,\mathbb{I},\chi_0)$ for the subspace such that the specializations above are nearly ordinary.

(ii) If $s=0$, then let $K=\prod_v K_v$ and $K_0(p)=\prod_{v\nmid p}K_v\prod_{v|p}K_0(p)_v$ ($K_0(p)_v\subset G(\mathcal{O}_{F_v})$ being the set of matrices which are in $B(\mathcal{O}_{F,v})$ modulo p). Then $G(F)\backslash G(\mathbb{A}_F)/K_0(p)$ is a finite set with $\{g_i\}_i$ a set of representatives. We identify the set

$$S^N_G(K):=G(F)\backslash G(\mathbb{A}_F)/K^pN(\mathcal{O}_{F,p})$$

with the disjoint union of $g_i\cdot N^-(p\mathcal{O}_{F,p})T(\mathcal{O}_{F,p})$ and endow the latter with the p-adic topology on $N^-(p\mathcal{O}_{F,p})T(\mathcal{O}_{F,p})$. We define $V^N_{\infty,\infty}(K,\mathbb{I},\chi_0)$ to be the set of continuous $\mathbb{I}$-valued functions on $S^N_G(K)$ such that for a Zariski dense set of arithmetic points $\phi\in\mathrm{Spec}\mathbb{I}$, the specialization $\mathbf{f}_\phi$ is a form on $\mathrm{U}(r,0)$ whose p-part nebentype at is given by

$$\chi_0\chi_\phi\omega(t_1^{c_1}...t_r^{c_r})$$

for the weight $(0,...,0)$. Note that by the description of nebentypus at p such family is determined for its values on $g_i\cdot N^-(p\mathcal{O}_{F,p})$'s. Similarly we define $V^{N,ord}_{\infty,\infty}(K,\mathbb{I},\chi_0)$ for the nearly ordinary part.

Remark 5.2. To see this is a good definition, we have to compare it with the notion of Hida families in the literature. We refer to [10, Chapter 8] and [15, Sections 3, 4] for the definition of Hida families. We have to see that a Hida family in *loc.cit* does give a Hida family here. We need to show that if $\kappa_\phi >> 0$ (depending on the p-part of the conductor at ϕ) when $s>0$, then any nearly ordinary p-adic cusp form is classical. If $s>0$ this is proved by the argument of [15, Theorem 4.19]. (Although it is assumed that $s=1$ in *loc.cit*, however the proof for this particular theorem works in the general case.) If $s=0$ the situation is even easier: the contraction property of the U_p operator [15, Proposition 4.4] (which again works in our case as well), shows that the specialization at ϕ is right invariant under an open subgroup of $\mathrm{U}(r)(\mathbb{Z}_p)$ depending only on the conductor of the nebentypus (note also that we have trivial weight if $s=0$), and is thus classical.

Definition 5.3. *We define an Eisenstein data as a quadruple $\mathbf{D}:=(\mathbb{I},\mathbf{f},\tau_0,\chi_0)$ where χ_0 is a finite order character of $T(\mathbb{Z}_p)$; τ_0 is a finite order character of $\mathcal{K}^\times\backslash\mathbb{A}^\times_\mathcal{K}$ whose conductors at primes above p divides (p); $\mathbf{f}\in V^{N,ord}_{\infty,\infty}(K,\mathbb{I})$ is a Hida family of eigenforms defined as above; We define $\Lambda_\mathbf{D}:=\Lambda\otimes_{\mathcal{O}_L}\Lambda_\mathcal{K}$. We call a $\bar{\mathbb{Q}}_p$-point $\phi\in\mathrm{Spec}\Lambda_\mathbf{D}$ is arithmetic if $\phi|_\mathbb{I}$ is arithmetic with some weight κ_ϕ and $\phi(\gamma^+)=(1+p)^{\frac{\kappa_\phi}{2}}\zeta_+$, $\phi(\gamma^-)=(1+p)^{\frac{\kappa_\phi}{2}}\zeta_-$ for p-power roots of unity $\zeta_\pm$. We define $\tau_\phi=\phi\circ\Psi_\mathcal{K}$.*

Let $\mathcal{X}$ be the set of arithmetic points. If $\mathbf{f}_\phi$ is classical and generates an irreducible automorphic representation $\pi_{\mathbf{f}_\phi}$ of $\mathrm{U}(r,s)$, we say that ϕ is generic if $(\pi_{\mathbf{f}_\phi},\tau)$ is generic (see Definition 4.42). Let $\mathcal{X}^{gen}$ be the set of generic arithmetic points.

5.2 Some Assumptions

5.2.1 Including Types

Consider the group $\mathrm{U}(s,r)$. Suppose $K^p=K_\Sigma K^\Sigma\subset G(\mathbb{A}^p_f)$ for a finite set of primes Σ and let W_Σ be a finite $\mathcal{O}_L$ module on which K_Σ acts through a finite quotient. Let $K'_\Sigma\subset K_\Sigma$ be a normal subgroup containing $\prod_{v\in\Sigma\backslash\{v|p\}}\mathfrak{Y}_v$ defined in Definition 4.11 and acting trivially on

W_Σ and let $K' = G(\mathbb{Z}_p)K'_\Sigma K^\Sigma$. The modules of modular forms on weight κ and type W_Σ and character ψ are

$$M_\kappa(K, W_\Sigma; \mathcal{O}_L) = (M_\kappa(K'; \mathcal{O}_L) \otimes_{\mathcal{O}_L} W_\Sigma)^{K_\Sigma}.$$

Suppose for $v \in \Sigma\backslash\{v|p\}$, we have open compact subgroups $\tilde{K}'_v \subset \tilde{K}_v \subset G(F_v)$ such that $\tilde{K}'_v$ is a normal subgroup of $\tilde{K}_v$ and an irreducible finite dimensional representation W_v of $\tilde{K}_v/\tilde{K}'_v$. Suppose $\varphi_v \in \pi_v$ is a vector in W_v. We fix a $\tilde{K}_v$-invariant measure and let $v_1, v_2, \ldots$ be a basis such that φ_v is v_1. We also assume that $\tilde{K}'_v$ includes the $\mathfrak{Y}^t_v$ defined in section 4. We let $W_v^\vee$ be the dual representation and we write $v_1^\vee, v_2^\vee, \ldots$ for the dual basis. We first prove the following lemma.

Lemma 5.4. *Let G be a finite group and $\rho : G \to \mathrm{Aut}(V)$ an irreducible representation on an n-dimensional vector space V. We fix a G-invariant norm and a unitary basis $v_1, \ldots, v_n$. Let $\rho^\vee$ be the dual representation on $V^\vee$ with dual basis $v_1^\vee, \ldots, v_n^\vee$. Then as elements in $V \otimes V^\vee$*

$$\sum_{g\in G}(gv_i \otimes gv_j^\vee) = 0, i \neq j,$$

$$\sum_g (gv_i \otimes gv_i^\vee) = |G| \sum_{i=1}^n v_i \otimes v_i^\vee.$$

Proof. This is a straightforward application of the Schur orthogonal relation. □

Definition 5.5. *We define $W_{\Sigma\backslash\{p\}} = \prod_{v\in\Sigma\backslash\{p\}} W_v$ and $v_1 = \prod_{v\in\Sigma\backslash\{p\}} v_{v,1} \in W_{\Sigma\backslash\{p\}}$.*

We can also make a notion of $W_{\Sigma\backslash\{p\}}$-valued Hida families in a similar manner as Definition 5.1.

5.2.2 Assumption TEMPERED

Let $\mathbf{f}$ be a Hida family of eigenforms as defined in Definition 5.1. We say it satisfies the assumption "TEMPERED" if the specializations $\mathbf{f}_\phi$ in the definition are tempered eigenforms.

5.2.3 Assumption DUAL

We first define an $\mathcal{O}_L$-involution $\circ : \Lambda \to \Lambda$ sending any $\mathrm{diag}(t_1, \cdots, t_n) \in T(1 + \mathbb{Z}_p)$ to $\mathrm{diag}(t_n^{-1}, \cdots, t_1^{-1})$. We define $\mathbb{I}^\circ$ to be the ring $\mathbb{I}$ but with the Λ-algebra structure given by composing the involution $\circ$ with the original Λ structure map of $\mathbb{I}$.

Let $\mathbf{f}$ be an $\mathbb{I}$-adic cuspidal eigenform on $\mathrm{U}(r, s)$ such that for a Zariski dense set of generic arithmetic points ϕ the specialization $\mathbf{f}_\phi$ is classical and generates an irreducible automorphic representation $\pi_{\mathbf{f}_\phi}$ of $\mathrm{U}(r, s)$, we say it satisfies assumption DUAL if there is an $\mathbb{I}^\circ$-adic nearly ordinary cusp form $\mathbf{f}^\vee$ on $\mathrm{U}(s, r)$ such that for all the arithmetic points $\phi \in \mathrm{Spec}\mathbb{I}$ such that ϕ is in the image of some point in $\mathcal{X}^{gen}$, $\mathbf{f}_\phi^\vee \in \pi_{\mathbf{f}_\phi}^\vee$. (Here we identified $\mathrm{U}(r, s)$ and $\mathrm{U}(s, r)$ in the obvious way. At an arithmetic point both f_ϕ and $f_\phi^\vee$ have scalar weight κ. Note also that we only require the specialization $\mathbf{f}_\phi$ to be "generic" (not required for $\mathbf{f}_\phi^\vee$)).

5.2.4 Assumption $\mathrm{Proj}_{f^\vee}$ and $\mathrm{Proj}_{\mathbf{f}^\vee}$

For a nearly ordinary cuspidal eigenform $f^\vee$ on $\mathrm{U}(s, r)$. We say it satisfies assumption $\mathrm{Proj}_{f^\vee}$ if $(\pi_{f^\vee} \otimes W_{\Sigma\backslash\{p\}})^K$ is 1-dimensional and there is a Hecke operator $1_{f^\vee}$ on $\mathrm{U}(s, r)$ that is an L-coefficient polynomial of Hecke operators outside Σ such that for any $g \in M_\kappa(K, W_{\Sigma\backslash\{p\}})$, we have $e^{ord} \cdot g - 1_{f^\vee} e^{ord} \cdot g$ is a sum of forms in irreducible automorphic representations which are orthogonal to $\pi_{f^\vee}$.

For a nonzero nearly ordinary cuspidal $\mathbb{I}^\circ$-adic family of eigenforms $\mathbf{f}^\vee \in (V^N_{\infty,\infty}(K,\mathbb{I}^\circ,\chi_0^{-1})\otimes W_{\Sigma\backslash\{p\}})^{K_\Sigma}$, we say it satisfies assumption $\mathrm{Proj}_{\mathbf{f}^\vee}$ if there is an action e^{ord} acting on $(V^N_{\infty,\infty}(K,\mathbb{I}^\circ,\chi_0^{-1})\otimes W_{\Sigma\backslash\{p\}})^{K_\Sigma}$ interpolating the e^{ord}'s of specializations and there is a Hecke operator $1_{\mathbf{f}^\vee}$ which is an $F_{\mathbb{I}}$ polynomial of Hecke operators outside Σ such that for Zariski dense set of arithmetic points $\phi\in\mathrm{Spec}\mathbb{I}^\circ$ in the image of $\mathcal{X}^{gen}$, $(\pi_{\mathbf{f}^\vee_\phi}\otimes W_{\Sigma\backslash\{p\}})^K$ is 1-dimensional and, for any $\mathbf{g}\in(V^N_{\infty,\infty}(K,\mathbb{I}^\circ,\chi_0^{-1})\otimes W_\Sigma)^{K_\Sigma}$, $(e^{ord}\cdot\mathbf{g}-1_{\mathbf{f}^\vee}e^{ord}\mathbf{g})_\phi$ is a sum of forms in irreducible automorphic representations which are orthogonal to $\pi_{\mathbf{f}^\vee_\phi}$.

Remark 5.6. If $r+s=2$ then these assumptions often hold since the unitary group is closely related to GL_2 or quaternion algebras. It is easy to see DUAL by simply taking $\mathbf{f}^\vee=\mathbf{f}\otimes(\chi)^{-1}$ for $\boldsymbol{\chi}$ being the central character of $\mathbf{f}$. To see $\mathrm{Proj}_{\mathbf{f}}$ and $\mathrm{Proj}_{\mathbf{f}^\vee}$, we first suppose $r=s=1$ and $\mathbf{f}$ is a Hida family of GL_2 newforms with tame level M such that $(M,p\delta_{\mathcal{K}})=1$ and trivial character. The existence of e^{ord} is the as in [28, Lemma 12.2] Since we have isomorphism of algebraic groups over F

$$\mathrm{GU}(1,1)\sim\mathrm{GL}_2\times_{\mathbb{G}_m}\mathrm{Res}_{\mathcal{K}/F}\mathbb{G}_m,$$

we can obtain a family on $\mathrm{U}(1,1)$ from $\mathbf{f}$ and the trivial character of $\mathbb{A}^\times_{\mathcal{K}}/\mathcal{K}^\times$, which we still denote as $\mathbf{f}$. Take an arithmetic point ϕ and a GL_2 Hecke operator t involving only Hecke operators T_v at primes v outside Σ which are split in $\mathcal{K}/F$ such that the t-eigenvalue $t(\mathbf{f}_\phi)$ is different from its eigenvalues on other forms on $S^{ord}_{\kappa_\phi}(\Gamma_0(M)\cap\Gamma_1(p^{t_\phi}),\mathbb{C})$ (the space of ordinary cusp forms on $\mathrm{U}(1,1)$ of weight $(0,\kappa_\phi)$ and level $\Gamma_0(M)\cap\Gamma_1(p^{t_\phi})$ with p^{t_ϕ} being the p-part level at ϕ. Also here we use the $\mathrm{U}(1,1)$ Hecke operators at split primes $v=w\bar{w}$ which are associated to the elements $(\mathrm{diag}(\varpi_w,1),\mathrm{diag}(1,\varpi_{\bar{w}}^{-1})))$. This is possible since any form in $S^{ord}_{\kappa_\phi}(\Gamma_0(M)\cap\Gamma_1(p^{t_\phi}),\mathbb{C})$ is the restriction of a form on $\mathrm{GU}(1,1)$ obtained from a GL_2 form of conductor dividing $\mathrm{Nm}_{\mathcal{K}/F}\delta_{\mathcal{K}/F}Mp^{t_\phi}$ and a character of $\mathbb{A}^\times_{\mathcal{K}}/\mathcal{K}^\times$ unramified outside p. Note that any cuspidal automorphic representation on GL_2/F with the same Hecke eigenvalue with $\mathbf{f}_\phi$ on split primes are $\pi_{\mathbf{f}_\phi}$ and $\pi_{\mathbf{f}_\phi}\otimes\chi_{\mathcal{K}/F}$, and that any element $g\in\mathrm{GL}_2(F_v)$ such that $\det(g)\in\mathrm{Nm}_{\mathcal{K}/F}(\mathcal{K}_v^\times)$ can be written as ag' with $a\in\mathcal{K}_v^\times, g'\in\mathrm{U}(1,1)(F_v)$. A simple representation theoretic argument shows that the only forms in $S^{ord}_{\kappa_\phi}(\Gamma_0(M)\cap\Gamma_1(p^{t_\phi}),\mathbb{C})$ with the same Hecke eigenvalues with $\mathbf{f}_\phi$ at split primes are in the one dimensional space spanned by $\mathbf{f}_\phi$. Let Λ be the weight space for $\mathrm{U}(1,1)$ and define

$$S^{ord}(\Gamma_0(M),\mathbb{I}):=S^{ord}(\Gamma_0(M),\Lambda)\otimes_\Lambda\mathbb{I}.$$

It follows from Hida's control theorem for unitary groups (see e.g. [15, Theorem 4.21]) that this is a free module over $\mathbb{I}$ of finite rank, and the specialization of this free module to ϕ gives the space $S^{ord}_{\kappa_\phi}(\Gamma_0(M)\cap\Gamma_1(p^{t_\phi}),\mathcal{O}_L)$ for some L finite over $\mathbb{Q}_p$ provided $\kappa_\phi>>0$ with respect to the p-part of the conductor of ϕ. We consider $\det(T-t)$ where T is a variable and we regard t as an operator on this free $\mathbb{I}$-module. We thus obtain an $\mathbb{I}$-coefficient polynomial of T. Moreover we can write $\det(T-t)=(T-t(\mathbf{f}))\cdot g(T)$ for some polynomial $g(T)$. Then we define

$$1_{\mathbf{f}}=\frac{g(t)}{g(\mathbf{f})}$$

(note that $g(t(\mathbf{f}))$ is not identically zero.) This proves $\mathrm{Proj}_{\mathbf{f}}$ and $\mathrm{Proj}_{\mathbf{f}^\vee}$ is seen in a similar way. If $(r,s)=(2,0)$ we observe that if we set

$$D=\{g\in M_2(\mathcal{K})|g^t\zeta\bar{g}=\det(g)\zeta\},$$

then D is a definite quaternion algebra over $\mathbb{Q}$ with local invariants $\mathrm{inv}_v(D)=(-\mathfrak{s},-D_{\mathcal{K}/\mathbb{Q}})_v$ (the Hilbert symbol). The relation between $\mathrm{GU}(2)$ and D is explained by

$$\mathrm{GU}(2)=D^\times\times_{\mathbb{G}_m}\mathrm{Res}_{\mathcal{K}/\mathbb{Q}}\mathbb{G}_m.$$

We can similarly show that if $\mathbf{f}$ is a Hida family of newforms on $D^\times$ with trivial character, tame level prime to p and all primes of $\delta_{\mathcal{K}}$ such that D is unramified, and is the trivial representation at primes where D is ramified, then we can produce a family $\mathbf{f}$ on $\mathrm{U}(2,0)$ from $\mathbf{f}$ and the trivial character of $\mathbb{A}_{\mathcal{K}}^\times/\mathcal{K}^\times$. A similar argument proves that $\mathrm{Proj}_{\mathbf{f}}$ and $\mathrm{Proj}_{\mathbf{f}^\vee}$ is true.

5.3 Klingen Eisenstein Series and p-adic L-functions

5.3.1 Construction

Now we are going to construct the nearly ordinary Klingen Eisenstein series (and will p-adically interpolate them in families). First of all, recall that for a Hecke character τ which is of infinite type $(-\frac{\kappa}{2}, \frac{\kappa}{2})$ at all infinite places (here the convention is that the first infinite place of $\mathcal{K}$ is inside our CM type). Recall that we write $\mathcal{D} := \{\pi, \tau, \Sigma\}$ for the Eisenstein data (see definition 3.2). We define the normalization factor:

$$B_{\mathcal{D}}: \quad = \frac{\Omega_p^{r\kappa\Sigma_\infty}}{\Omega_\infty^{r\kappa\Sigma_\infty}}\left(\frac{(-2)^{-d(a+2b+1)}(2\pi i)^{d(a+2b+1)\kappa}(2/\pi)^{d(a+2b+1)(a+2b)/2}}{\prod_{j=0}^{a+2b}(\kappa-j-1)^d}\right)^{-1}\prod_{i=0}^{(a+2b)} L^\Sigma(2z_\kappa+a+2b+1-i, \bar{\tau}'\chi_{\mathcal{K}}^i)$$
$$\times\prod_{v|p}(\mathfrak{g}(\bar{\tau}_v')^{a+2b+1}c_{a+2b+1}(\tau_v', -z_\kappa))^{-1},$$

$$B'_{\mathcal{D}}: \quad = \frac{\Omega_p^{r\kappa\Sigma_\infty}}{\Omega_\infty^{r\kappa\Sigma_\infty}}\left(\frac{(-2)^{-d(a+2b)}(2\pi i)^{d(a+2b)\kappa}(2/\pi)^{d(a+2b)(a+2b-1)/2}}{\prod_{j=0}^{a+2b-1}(\kappa-j-1)^d}\right)^{-1}\prod_{i=0}^{(a+2b-1)} L^\Sigma(2z_\kappa+a+2b-i, \bar{\tau}'\chi_{\mathcal{K}}^i)$$
$$\times\prod_{v|p}(\mathfrak{g}(\bar{\tau}_v')^{a+2b}c_{a+2b}(\tau_v', -z'_\kappa))^{-1}.$$

The $z_\kappa = \frac{\kappa-a-2b-1}{2}$ and $z'_\kappa = \frac{\kappa-a-2b}{2}$. The c_m is defined in equation (13). The Ω_∞ is the CM period in subsection 2.1.

We construct a Siegel Eisenstein series E_{sieg} associated to the Siegel section:

$$f_{\mathcal{D},sieg} = B_{\mathcal{D}}\prod_{v|\infty} f_\kappa \prod_{v|p}\rho(\Upsilon_v)f_v^0 \prod_{v\in\Sigma, v\nmid p}\tilde{f}_{v,sieg}\prod_v f_v^{sph} \in I_{a+2b+1}(\tau, z)$$

and E'_{sieg} associated to the section

$$f'_{\mathcal{D},sieg} = B'_{\mathcal{D}}\prod_{v|\infty} f'_\kappa \prod_{v|p}\rho(\Upsilon'_v)f_v^{0,\prime} \prod_{v\in\Sigma, v\nmid p}\tilde{f}'_{v,sieg}\prod_v f_v^{sph,\prime} \in I_{a+2b}(\tau, z).$$

Here Υ_v and Υ'_v are defined in Definition 4.32. First note that since π is nearly ordinary with respect to the scalar weight κ and $\varphi = \varphi^{ord}$ is a holomorphic nearly ordinary vector. Then its contragradient is also nearly ordinary on $\mathrm{U}(s,r)$ with respect to the scalar weight κ. We denote this representation as $\tilde{\pi}$. We consider $E(\gamma(g,-))$ as an automorphic form on $\mathrm{U}(s,r)$. For each $v \nmid p$ we choose an open compact group $K_{v,s} \subset \mathrm{U}(s,r)_v$ such that

$$\prod_{v\in\Sigma, v\nmid p}\rho(\gamma(1, \eta\mathrm{diag}(\bar{x}_v^{-1}, 1, x_v).\tilde{S}_v^{-1}))(E(\gamma(g,-))\otimes\bar{\tau}(\det -))$$

is invariant under its action. We have the following lemma

Lemma 5.7. *There is a bounded measure $\mathcal{E}_{\mathcal{D},sieg}$ on $\Gamma_{\mathcal{K}}\times T(1+\mathbb{Z}_p)$ with values in the space of p-adic automorphic forms on $\mathrm{U}(r+s+1, r+s+1)$, such that for all arithmetic points $\phi \in \mathcal{X}^{gen}$ with the associated character $\hat{\phi}$ on $\Gamma_{\mathcal{K}}\times T(1+\mathbb{Z}_p)$, we have*

$$\int_{\Gamma_{\mathcal{K}}\times T(1+\mathbb{Z}_p)}\hat{\phi}\, d\mathcal{E}_{\mathcal{D},sieg}$$

is the Siegel Eisenstein series $\rho(\prod_{v\in\Sigma,v\nmid p}\gamma(1, \eta\mathrm{diag}(\bar{x}_v^{-1}, 1, x_v)\tilde{S}_v^{-1}))E_{sieg,\mathcal{D}_\phi}$ where $E_{sieg,\mathcal{D}_\phi}$ is the Siegel Eisenstein series we construct using the characters $(\chi_{1,\phi}, ..., \chi_{n,\phi}, \tau_\phi)$. Similarly we can define a measure $\mathcal{E}'_{\mathbf{D},sieg}$ interpolating the $E'_{sieg,\mathcal{D}_\phi}$'s.

Proof. It follows from our computations for Fourier coefficients Lemmas 4.2, 4.6, 4.12 and 4.46 and [28, Lemma 11.2] that, all the Fourier coefficients of E_{sieg} and E'_{sieg} are interpolated by elements in $\Lambda_{r,s}[[\Gamma_{\mathcal{K}}]]$. Then the lemma follows from the abstract Kummer congruence. We refer to [14, Lemma 3.15, Theorem 3.16] for a detailed proof. $\square$

Now we define our Klingen Eisenstein series using the pullback formula. Note that by (3) the pullback of the Siegel Eisenstein series are still holomorphic automorphic forms. Let β be the embedding given in subsection 2.2. Let $\tilde{K}_s$ be the open compact subgroup of $G(\mathcal{O}_{F,\Sigma})$ which is $\tilde{K}_{v,s}$ as above for $v \in \Sigma\backslash\{v|p\}$, $\tilde{K}_v$ for $v|p$ and spherical otherwise. We define $\mathbf{E}_{\mathbf{D},Kling}$ by: for any x, x_1 points on the Igusa schemes of $\mathrm{U}(r+1, s+1)$ and $\mathrm{U}(s, r)$,

$$e^{ord,low} \cdot 1^{low}_{\mathbf{f}^\vee}\mathrm{tr}_{\tilde{K}/\tilde{K}_s}(e^{low}(\beta^{-1}(\mathcal{E}_{\mathbf{D},sieg}) \cdot \bar{\tau}(\det(g_1))) \otimes v_1)(x, x_1) = \mathbf{E}_{\mathbf{D},Kling}(x) \boxtimes \mathbf{f}^\vee(x_1)$$

(as a $W_{\Sigma\backslash\{p\}}$-valued form. Recall $v_1 \in W_{\Sigma\backslash\{p\}}$ (see Subsubsection 5.2.1).). Here we let $\tilde{K}_{\Sigma\backslash\{p\}}$ acts on both $\mathcal{E}_{\mathbf{D},sieg}$ and $W_{\Sigma\backslash\{p\}}$. We get a $\Lambda_{\mathcal{D}}$-adic formal Fourier-Jacobi expansion from the measure $e^{low}\beta^{-1}(\mathcal{E}_{\mathbf{D},sieg})$ and then apply the Hecke operators to the expansion. We also define the Σ-primitive p-adic L-function $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K},\tau_0} \in \mathbb{I}^{ur}[[\Gamma_{\mathcal{K}}]]$ by: for x, x_1 elements in the Igusa schemes of $\mathrm{U}(r, s)$ and $\mathrm{U}(s, r)$,

$$e^{ord,low} \cdot 1^{low}_{\mathbf{f}^\vee}\mathrm{tr}_{\tilde{K}/\tilde{K}_s}(e^{low}\beta^{-1}(\mathcal{E}'_{\mathbf{D},sieg}) \cdot \bar{\tau}(\det g_1) \otimes v_1)(x, x_1) = \mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K},\tau_0}\mathbf{f}_1(x) \boxtimes \mathbf{f}^\vee(x_1).$$

The $\mathbf{f}_1$ is the $v_1^\vee$-component of $\mathbf{f}$ (see Subsubsection 5.2.1). This is possible by Lemma 5.4. Here note that the necessity of enlarging the coefficient ring to include $\mathcal{O}_L^{ur}$ is caused when specifying points on Igusa schemes (recall Subsection 2.6).

Here we used the superscript "*low*" to mean that under $\mathrm{U}(a+b+1, b+1) \times \mathrm{U}(b, a+b) \hookrightarrow \mathrm{U}(a+2b+1, a+2b+1)$ the action is for the group $\mathrm{U}(b, a+b)$.

5.3.2 Identify with Klingen Eisenstein Series Constructed Before

We define a Klingen Eisenstein section by

$$f^{\square}_{\mathcal{D}_\phi,Kling}(z, g) = B_{\mathcal{D}} \prod_v F_{\varphi_v}(z; f_{v,sieg}, g)$$

where $F_{\varphi_v}(z; f_{v,sieg}, g)$'s are the pullback sections we computed in section 5 and φ_v's for $v \in \Sigma\backslash\{v|p\}$ are the $v_1^\vee$-components as in Subsubsection 5.2.1 and 5.2.4. We first look at places dividing p. The pairing $\langle,\rangle$ induces natural pairing between π and $\tilde{\pi}$. Write

$$\varphi_w = \prod_{v|\infty} \varphi_v \prod_{v \notin \Sigma} \varphi^{sph} \prod_{v \in \Sigma, v \nmid p} \varphi_v \prod_{v|p} \varphi_{w,v}.$$

Then

$$\langle\prod_{v\nmid p}tr_{\tilde{K}_v/\tilde{K}_{v,s}}\rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v).\tilde{S}_v^{-1}))(E_{sieg}(\gamma(g,-))\bar{\tau}(\det -)),$$

$$\rho(\prod_{v|p}(\begin{pmatrix}p^{-t_{a+b+1}}&&&&&\\&\cdots&&&&\\&&p^{t_1}&&&\\&&&\cdots&&\\&&&&p^{t_{a+1}}&\\&&&&&\cdots\end{pmatrix}^{\iota}\begin{pmatrix}&&-1_b\\&1_a&\\1_b&&\end{pmatrix}^{\iota}))\varphi_w\rangle$$

$$= \langle\rho^{low}(\prod_{v|p}\begin{pmatrix}p^{t_{a+b+1}}&&&&\\&\cdots&&&\\&&p^{t_{a+2b}}&&\\&&&1_a&\\&&&&1_b\end{pmatrix}^{\iota})\prod_{v\nmid p}\mathrm{tr}_{\tilde{K}_v/\tilde{K}_{v,s}}\rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v)\tilde{S}_v^{-1}))(E(\gamma(g,-))\bar{\tau}(\det -),$$

$$\rho(\prod_{v|p}(\begin{pmatrix}1_b&&&&\\&p^{t_1}&&&\\&&\cdots&&\\&&&p^{t_{a+1}}&\\&&&&\cdots\end{pmatrix}^{\iota}\begin{pmatrix}&&-1_b\\&1_a&\\1_b&&\end{pmatrix}^{\iota}))\varphi_w\rangle.$$

Since $E_{sieg}(\gamma(g,-))\bar{\tau}(\det -)$ satisfies the property that if $\tilde{K}'''$ is the subgroup of $\mathrm{GL}_{a+2b}(\mathbb{Z}_p)$ (defined in the last section) consisting of matrices $\begin{pmatrix}a_1&a_3&a_2\\a_7&a_9&a_8\\a_4&a_6&a_5\end{pmatrix}$ such that the (i,j)-th entry of a_7 is divisible by $p^{t_i+t_{a+b+j}}$ and the (i,j)-th entry of a_4 is divisible by $p^{t_{a+i}+t_{a+b+j}}$, the i-th row of a_8 and the right to diagonal entries of a_9 are divisible by p^{t_i} for $i=1,\cdots,a$, the i-th column of the below diagonal entries of a_1 are divisible by $p^{t_{a+b+i}}$, the i-th row of the up to diagonal entries of a_5 are divisible by $p^{t_{a+i}}$, $a_2,a_3,a_6\in M(\mathbb{Z}_p)$, then the right action of h^ι for $h\in\tilde{K}'''$ on $E(\gamma(g,-))\bar{\tau}(\det -)$ is given by the character

$$\lambda(h^\iota)=\bar{\chi}_{a+b+1}(h_{11})\cdots\bar{\chi}_{a+2b}(h_{bb})\bar{\chi}_1(h_{b+1,b+1})\cdots\bar{\chi}(h_{a+b,a+b})\bar{\chi}_{a+1}(h_{a+b+1,a+b+1})\cdots\bar{\chi}_{a+b}(h_{a+2b,a+2b}).$$

(This is easily checked from the definition of the Godement section). It is elementary to check that the above expression equals:

$$\begin{aligned}(\prod_{v|p}\frac{1}{\prod_{i=1}^{b}p^{t_{a+b+i}(a+b)}})\quad&\langle\prod_{v|p}(\sum_y\rho^{low}(y)\rho^{low}(\begin{pmatrix}p^{t_{a+b+1}}&&&\\&\cdots&&\\&&1_a&\\&&&1_b\end{pmatrix}^{\iota})\\&\prod_{v\nmid p}\mathrm{tr}_{\tilde{K}_v/\tilde{K}_{v,s}}\rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v)\tilde{S}_v^{-1}))(E_{sieg}(\gamma(g,-))\bar{\tau}(\det -)),\\&\rho(\prod_{v|p}(\begin{pmatrix}1_b&&&&\\&p^{t_1}&&&\\&&\cdots&&\\&&&p^{t_{a+1}}&\\&&&&\cdots\end{pmatrix}^{\iota}\begin{pmatrix}&&-1_b\\&1_a&\\1_b&&\end{pmatrix}^{\iota}))\varphi_w\rangle\end{aligned}\tag{15}$$

where y runs over $N(\mathbb{Z}_p)/\beta N(\mathbb{Z}_p)\beta^{-1}$ for N consisting of matrices of the form $\begin{pmatrix}1_s & 0\\ * & 1_r\end{pmatrix}$ with $*$ having $\mathbb{Z}_p$-entries and $\beta = \begin{pmatrix}p^{t_{a+b+1}} & & & \\ & \cdots & & \\ & & 1_a & \\ & & & 1_b\end{pmatrix}$. Write the following expression

$$
\begin{aligned}
& e^{ord,low}\prod_{v|p}(\sum_y \rho^{low}(y)\rho^{low}(\begin{pmatrix}p^{t_{a+b+1}} & & & \\ & \cdots & & \\ & & 1_a & \\ & & & 1_b\end{pmatrix}^{\iota})\\
\times \quad & \prod_{v\nmid p}\mathrm{tr}_{\tilde{K}_v/\tilde{K}_v,s}\rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v)\tilde{S}_v^{-1}))(E_{sieg}(\gamma(g,-))\bar{\tau}(\det -)).
\end{aligned}
\tag{16}
$$

Now let $\tilde{K}^\flat$ consists of matrices in $\mathrm{GL}_b(\mathbb{Z}_p)$ whose below diagonal entries of the i-th row are divisible by p^t_{a+b+i} for $1\le i\le s$. Let $\tilde{K}^\sharp$ be the set of elements in $\mathrm{GL}_{a+2b}(\mathbb{Z}_p)$ whose right to diagonal entries of the i-th row are divisible by p^{t_i} for $1\le i\le a+b$; whose lower right $(b\times b)$ block is in

$$\mathrm{diag}(p^{t_{a+b+1}},\cdots,p^{t_{a+2b}})\tilde{K}^\flat\mathrm{diag}(p^{t_{a+b+1}},\cdots,p^{t_{a+2b}})^{-1}.$$

Then a similar argument as in Subsubsection 4.4.1 shows that there is a unique up to scalar vector $\tilde{\varphi}_v^\sharp\in\pi(\chi_1^{-1},\cdots,\chi_{a+2b}^{-1})$ such that the action of $(k_{ij})\in K^\sharp$ is given by the character $\mathrm{diag}(\chi_1^{-1}(k_{11}),\cdots,\chi_{a+2b}^{-1}(k_{a+2ba+2b})$. We use the model of the induced representation from $\chi_1^{-1}\otimes\cdots\otimes\chi_{a+2b}^{-1}$ on the space of smooth functions on $\mathrm{GL}_{a+2b}(\mathbb{Z}_p)$. We take the $\tilde{\varphi}_v^\sharp$ such that if $\tilde{\varphi}_v^{ord}$ takes value 1 on identity in this model, then $\tilde{\varphi}_v^\sharp$ also takes value 1 on identity (and has support $K^\sharp\subset\mathrm{GL}_{a+2b}(\mathbb{Z}_p)$). From the action of the level group we know that the action of $\rho^{low}(K^\sharp)$ on the left part of the inner product in (15) is given by the character $\mathrm{diag}(\chi_1^{-1}(k_{11}),\cdots,\chi_{a+2b}^{-1}(k_{a+2ba+2b})$. For $v|p$ define $T_{\beta,v}^{low}$ to be the Hecke operator corresponding to β just in terms of double cosets acting on $\pi_\varphi^\vee$ (no normalization factors involved). By checking the actions of the level groups at primes dividing p (certain open compact subgroups of $G(\mathcal{O}_{F,p})$) we can see that the $\tilde{\pi}$ component of the left part when viewed as an automorphic form on $\mathrm{U}(a+b,b)$ is a multiple of $\tilde{\varphi}^{ord}$. Suppose the eigenvalue for the Hecke operator $T_{\beta,v}^{low}$ on $\tilde{\varphi}^{ord}$ is $\tilde{\lambda}_{\beta,v}$. It is easy to compute that

$$\tilde{\lambda}_{\beta,v}=p^{\sum_{i=1}^b t_{a+b+i}(\frac{a+2b+1}{2}-i)}\cdot\prod_{j=1}^b\chi_{a+2b+1-j}^{-1}(p^{t_{a+b+j}}) \tag{17}$$

with the convention of χ_i's after Remark 4.41.

Let

$$\varphi'=\prod_{v|\infty}\varphi_v\prod_{v\notin\Sigma}\varphi^{sph}\prod_{v\in\Sigma,v\nmid p}\varphi_v\prod_{v|p}\rho(\begin{pmatrix}p^{-t_{a+b+1}} & & & & & \\ & \cdots & & & & \\ & & p^{t_1} & & & \\ & & & \cdots & & \\ & & & & p^{t_{a+1}} & \\ & & & & & \cdots\end{pmatrix}^{\iota}\begin{pmatrix} & & -1_b\\ & 1_a & \\ 1_b & & \end{pmatrix}^{\iota})\varphi_{w,v}$$

and

$$\varphi''=\prod_{v|\infty}\varphi_v\prod_{v\notin\Sigma}\varphi^{sph}\prod_{v\in\Sigma,v\nmid p}\varphi_v\prod_{v|p}\rho(\begin{pmatrix}1 & & & & & \\ & \cdots & & & & \\ & & p^{t_1} & & & \\ & & & \cdots & & \\ & & & & p^{t_{a+1}} & \\ & & & & & \cdots\end{pmatrix}^{\iota}\begin{pmatrix} & & -1_b\\ & 1_a & \\ 1_b & & \end{pmatrix}^{\iota})\varphi_{w,v}.$$

Here for $v|\infty$ the φ_v is the unique vector mentioned before definition 3.1. Define the Klingen Eisenstein section promised in the introduction as

$$f_{\mathcal{D}_\phi,Kling} = B_{\mathcal{D}_\phi}(\prod_{v\in\Sigma, v\nmid p}|\tilde{K}_v/\tilde{K}_{v,s}|)f^{\square}_{\mathcal{D}_\phi,Kling}.$$

Then we have

Proposition 5.8. *For a classical generic arithmetic point ϕ we have:*

$$\begin{aligned}\phi(\mathbf{E}_{\mathbf{D},Kling}) = & \prod_{v\in\Sigma,v\nmid p}|\tilde{K}_v/\tilde{K}_{v,s}|\frac{E_{Kling}(f_{\mathcal{D}_\phi,Kling}, z_{\kappa_\phi}, g)}{\langle\tilde{\varphi}^{ord}_\phi, \varphi_\phi\rangle}\\ & \times\prod_{v|p}(\prod_{j=1}^{s}\chi_{r+j}(p^{t_{r+j}})\prod_{j=1}^{r}\chi_j^{-1}(p^{t_j})p^{\sum_{i=1}^{s}t_{a+b+i}(\frac{a-1}{2}+i)}\cdot p^{-\sum_{j=1}^{r}t_j(\frac{a+1}{2}-j)}).\end{aligned}$$

Proof. We have

$$\frac{(15)}{\langle\tilde{\varphi}^\sharp, \varphi''\rangle} = \frac{\langle(16), \varphi''\rangle\tilde{\lambda}_{\beta,v}}{\prod_{v|p}(\prod_{1\le i\le j\le s}p^{t_{a+b+i}-t_{a+b+j}})\langle\tilde{\varphi}^{ord}, \varphi''\rangle}$$

and

$$\langle\tilde{\varphi}^\sharp, \varphi''\rangle = \langle\tilde{\varphi}^{ord}, \varphi''\rangle\cdot\prod_{v|p}(\prod_{1\le i\le j\le s}p^{t_{a+b+i}-t_{a+b+j}})$$

(e.g. using the model of the induced representation). So

$$\frac{\langle(16), \varphi''\rangle}{\langle\tilde{\varphi}^{ord}, \varphi''\rangle} = \frac{(15)\prod_{v|p}(\prod_{1\le i\le j\le s}p^{t_{a+b+i}-t_{a+b+j}})}{\tilde{\lambda}_{\beta,v}\langle\tilde{\varphi}^\sharp, \varphi''\rangle} = \frac{(15)\prod_{v|p}(\prod_{1\le i\le j\le s}p^{t_{a+b+i}-t_{a+b+j}})}{\prod_{v|p}(\prod_{1\le i\le j\le s}p^{t_{a+b+i}-t_{a+b+j}})\tilde{\lambda}_{\beta,v}\langle\tilde{\varphi}^{ord}, \varphi''\rangle}.$$

We also have

$$\langle\tilde{\varphi}^{ord}, \varphi''\rangle = \prod_{j=1}^{r}\chi_j(p^{t_j})\cdot p^{\sum_{j=1}^{r}t_j(\frac{a+1}{2}-j)}.$$

The proposition follows. □

Then part one and two of the Theorem 1.1 is just a corollary of the above proposition (except the statement in the $s=0$ case which we are going to consider next).

Similarly we obtain an interpolation formula for the p-adic L-function as in Theorem 1.1, using also the formula (14).

5.3.3 Interpolating Petersson Inner Products for Definite Unitary Groups

To simplify the exposition we only discuss the case when $F=\mathbb{Q}$ in this subsubsection. In the case when $s=0$ we hope that the periods showing up are CM periods. Thus by our assumption the Archimedean components of π are trivial representations. For this purpose we prove that under certain assumptions the Petersson inner products of two families can be interpolated by elements in the Iwasawa algebra. Let $K=\prod_v K_v$ be an open compact subgroup of $\mathrm{U}(r,s)(\mathbb{A}_f)$ which is $G(\mathbb{Z}_p)$ at all primes dividing p and $K_0(p)$ obtained from K by replacing the v-component by the K_0^1 at all primes v dividing p. Now we take $\{g_i\}_i$ a set of representatives for $\mathrm{U}(r,s)(F)\backslash\mathrm{U}(r,s)(\mathbb{A}_F)/K_0(p)$. We take K sufficiently small so that for all i we have $\mathrm{U}(r,s)(F)\cap g_iKg_i^{-1}=1$. For the nearly ordinary Hida family $\mathbf{f}^\vee$ of eigenforms (recall that this Hida family is nearly ordinary with respect to the lower triangular Borel subgroup) we construct a set of bounded $\mathbb{I}$-valued measure μ_i on $N^-(p\mathbb{Z}_p)$ as follows. Let T^- be the set of elements $\mathrm{diag}(p^{a_1},\cdots,p^{a_r})$ with $a_1\le\cdots\le a_r$. We only need to specify the measure for sets of the form $nt^-N^-(\mathbb{Z}_p)(t^-)^{-1}$ where $n \in N^-(\mathbb{Z}_p)$ and $t^- \in T^-$. We assign its measure $\mu_i(nt^-N^-(\mathbb{Z}_p)(t^-)^{-1})$ by $\mathbf{f}^\vee(g_i n \cdot t^-)\lambda(t^-)^{-1}$ where $\lambda(t^-)$ is the Hecke eigenvalue of $\mathbf{f}^\vee$ for U_{t^-}. This does define a measure. We briefly explain the point when $r = 2$ (the general case is only notationally more complicated). Write $\pi_{\mathbf{f}^\vee_\phi,p} = \pi(\chi_{1,p}, \chi_{2,p})$ such that $\nu_p(\chi_{1,p}(p)) = \frac{1}{2}, \nu_p(\chi_{2,p}(p)) = -\frac{1}{2}$. Then $\lambda(\mathrm{diag}(1, p^n)) = (\chi_{2,p}(p) \cdot p^{\frac{1}{2}})^n$. One checks that

$$\sum_{m \in p^{n-1}\mathbb{Z}_p/p^n\mathbb{Z}_p} \pi(\begin{pmatrix} 1 & \\ m & 1 \end{pmatrix}_p)\pi(\mathrm{diag}(1, p^n)_p)\mathbf{f}^\vee_{\phi,p} = (\chi_{2,p}(p) \cdot p^{\frac{1}{2}})\pi(\mathrm{diag}(1, p^{n-1})_p)\mathbf{f}^\vee_{\phi,p}.$$

This implies that for any $m_1 \in p\mathbb{Z}_p/p^{n-1}\mathbb{Z}_p$,

$$\sum_{m_2 \in p^{n-1}\mathbb{Z}_p/p^n\mathbb{Z}_p} \mu_i(m_1 m_2 \mathrm{diag}(1, p^n)N^-(\mathbb{Z}_p)\mathrm{diag}(1, p^{-n})) = \mu_i(m_1 \mathrm{diag}(1, p^{n-1})N^-(\mathbb{Z}_p)\mathrm{diag}(1, p^{1-n})),$$

i.e. this μ_i does define a measure.

Proposition 5.9. *If we define*

$$\langle \mathbf{f}, \mathbf{f}^\vee \rangle := \sum_i \int_{n \in N^-(p\mathbb{Z}_p)} \mathbf{f}(g_i n) d\mu_i \in \mathbb{I}.$$

Then for all $\phi \in \mathcal{X}^{gen}$ the specialization of $\langle \mathbf{f}, \mathbf{f}^\vee \rangle$ to ϕ is $\langle \mathbf{f}_\phi, \mathbf{f}^\vee_\phi \rangle \cdot \mathrm{Vol}(\tilde{K}_\phi)^{-1}$.

Proof. For each $\phi \in \mathcal{X}^{gen}$, we choose t^- such that $t^-N^-(p\mathbb{Z}_p)(t^-)^{-1} \subseteq \tilde{K}_\phi$. We consider

$$\langle \mathbf{f}_\phi, \pi^\vee_{\mathbf{f}_\phi}(t^-)\mathbf{f}^\vee_\phi \rangle.$$

Unfolding the definitions, note that $\chi_\phi^{-1}(t^-)\delta_B(t^-)$ gives the Hecke eigenvalue $\lambda(t^-)$, this gives $\delta_B(t^-)\chi_\phi^{-1}(t^-)\sum_i \int_{n \in N^-(p\mathbb{Z}_p)} \mathbf{f}(g_i n) d\mu_i \cdot \mathrm{Vol}(\tilde{K}_\phi)$. On the other hand, using the model of $\pi_{\mathbf{f}_\phi,p}$ and $\pi_{\mathbf{f}^\vee_\phi,p}$ as the induced representation $\pi(\chi_{1,\phi}, ..., \chi_{r,\phi})$ and $\pi(\chi^{-1}_{1,\phi}, ..., \chi^{-1}_{r,\phi})$ of $\mathrm{GL}_r(\mathbb{Q}_p)$, we get that

$$\langle \mathbf{f}_\phi, \pi^\vee_{\mathbf{f}_\phi}(t^-)\mathbf{f}^\vee_\phi \rangle = \delta_B(t^-)\chi_\phi^{-1}(t^-)\langle \mathbf{f}_\phi, \mathbf{f}^\vee_\phi \rangle.$$

This proves that the specialization of $\langle \mathbf{f}, \mathbf{f}^\vee \rangle$ to ϕ is $\langle \mathbf{f}_\phi, \mathbf{f}^\vee_\phi \rangle \cdot \mathrm{Vol}(\tilde{K}_\phi)^{-1}$. ☐

So to see the main theorem in the case when $s = 0$, instead of applying the Hecke operator $e^{ord} \cdot 1_{\mathbf{f}^\vee}$ we pair the pullback of Siegel Eisenstein series ($\mathbb{I}^{ur}[[\Gamma_\mathcal{K}]]$-valued) with the measure determined by the Hida family $\mathbf{f}$ using the above lemma (i.e. considering

$$E_{Kling}(g, z_\kappa) = \sum_i \int_{n \in N^-(p\mathcal{O}_{F,p})} E_{sieg}(S^{-1}\alpha(g, g_i n)S, z_\kappa) d\mu_i$$

where $\{d\mu_i\}_i$'s are the measures constructed from $\mathbf{f}$ as above. In our situation when restricting to $\mathrm{U}(s, r)$ the level group at p for Eisenstein series is lower triangular modulo certain power of p while that for $\mathbf{f}$ is upper triangular modulo certain power of p. The above construction works in the same way). The powers of CM and p-adic periods enter when applying the comparison between the standard basis and the Néron basis for differentials of CM abelian varieties while doing pullback (see [15, (3.14)]).

5.4 Constant Terms

We explain part (iii) of the main theorem.

5.4.1 p-adic L-functions for Dirichlet Characters

There is an element $\mathcal{L}_{\bar{\tau}'}$ in $\Lambda_{\mathcal{K},\mathcal{O}_L}$ such that at each arithmetic point $\phi \in \mathcal{X}^{pb}$, $\phi(\mathcal{L}_{\bar{\tau}'}) = L(\bar{\tau}'_\phi, \kappa_\phi - r).\tau'_\phi(p^{-1})p^{\kappa_\phi - r}\mathfrak{g}(\bar{\tau}'_\phi)^{-1}$. For more details see [28, 3.4.3].

5.4.2 Archimedean Computation

As in [28], we calculate the Archimedean part of the intertwining operator for Klingen Eisenstein sections and prove the "intertwining operator"-part (see Lemma 3.4) of the constant term vanishes. Suppose π is associated to the weight $(0,...,0;\kappa,...,\kappa)$, then it is well known that there is a unique (up to scalar) vector $v \in \pi$ such that $k.v = \det \mu(k,i)^{-\kappa}$ for any $k \in K_\infty^{+,\prime}v$ (notations as in Subsubsection 3.1.1). Recall we defined $c(\rho, z)$ in Subsubsection 3.1.1.

Lemma 5.10. *Assumptions are as above, then:*

$$\begin{aligned} c(\rho,z) &= \pi^{a+2b+1}\prod_{i=0}^{b-1}\left(\frac{1}{z+\frac{\kappa}{2}-\frac{1}{2}-i-a}\right)\left(\frac{1}{z-\frac{\kappa}{2}+\frac{1}{2}-i}\right)\prod_{i=0}^{a-1}\left(\frac{1}{-1+i-2z+2b}\right) \\ &\quad \times \frac{\Gamma(2z+a)2^{-1-2z+2b}}{\Gamma(\frac{a+1}{2}+z+\frac{\kappa}{2})\Gamma(\frac{a+1}{2}+z-\frac{\kappa}{2})}\det(i\zeta/2)^{-2}. \end{aligned}$$

Proof. It follows the same way as [28, Lemma 9.3]. □

Corollary 5.11. *In case when $\kappa > \frac{3}{2}a + 2b$ or $\kappa \geq 2b$ and $a = 0$, we have $c(\rho, z) = 0$ at the point $z = \frac{\kappa - a - 2b - 1}{2}$.*

In the case when κ is sufficiently large the intertwining operator:

$$A(\rho, z_\kappa, F) = A(\rho_\infty, z_\kappa, F_\kappa) \otimes A(\rho_f, z_\kappa, F_f)$$

and all terms are absolutely convergent. Thus as a consequence of the above corollary we have $A(\rho, z_\kappa, F) = 0$. Therefore the constant term of E_{Kling} is essentially

$$\frac{L^\Sigma(\tilde{\pi}, \bar{\tau}^c, z_\kappa + 1)}{\Omega_\infty^{2\kappa\Sigma}\langle\tilde{\varphi}^{ord}, \varphi''\rangle}.L^\Sigma(2z_\kappa + 1, \bar{\tau}'\chi_{\mathcal{K}}^{a+2b})\varphi.$$

up to a product of normalization factors at local places. Interpolating the calculations in p-adic families, the part (iii) of the Theorem 1.1 follows from the above discussion, Lemma 3.4 and our local descriptions for $F_{\varphi_v}(z; f_{v,sieg}, g)$'s in Section 4. (See also the proof of [28, Theorem 12.11]).

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Appendix:

Boundary strata of connected components in positive characteristics

Kai-Wen Lan

Abstract

Under the assumption that the PEL datum involves no factor of type D, and that the integral model has good reduction, we show that all boundary strata of the toroidal or minimal compactifications of the integral model (constructed in earlier works of the author) have nonempty pullbacks to connected components of geometric fibers, even in positive characteristics.

A.1 Introduction

Toroidal and minimal compactifications of Shimura varieties and their integral models have played important roles in the study of arithmetic properties of cohomological automorphic representations. While all known models of them are equipped with natural stratifications, they often suffer from some imprecisions or redundancies due to their constructions. The situation is especially subtle in positive or mixed characteristics, or when we need purely algebraic constructions even in characteristic zero (for example, when we study the degeneration of abelian varieties), where the constructions are much less direct than algebraizing complex manifolds created by unions of explicit double coset spaces.

For example, integral models of Shimura varieties defined by moduli problems of PEL structures suffer from the so-called *failure of Hasse's principle*, because there is no known way to tell the difference between two moduli problems associated with algebraic groups which are everywhere locally isomorphic to each other. Similarly, when their toroidal and minimal compactifications are constructed using the theory of degeneration, the data for describing them are also local in nature. Unlike in the case over complex numbers, one cannot just express all the boundary points as the disjoint unions of some double coset spaces labeled by certain standard maximal (rational) parabolic subgroups. (Even the nonemptiness of the *whole boundaries* in positive characteristics was not straightforward—see the introduction to [9].) As we shall see (in Example A.7.2), when factors of type D are allowed, it is unrealistic to expect that the boundary stratifications in the algebraic and complex analytic constructions match with each other.

Our goal here is a simple-minded one—to show that the strata of good reduction integral models of toroidal and minimal compactifications constructed as in [11] have nonempty pullbacks to each connected component of each geometric fiber, under the assumption that the data defining them involve no factors of type D (in a sense we will make precise). We will also answer the analogous question for the integral models constructed by normalization as in [12], allowing arbitrarily deep levels and ramifications (that is, *bad reductions* in general).

Such a goal is motivated by the study of p-adic families of Eisenstein series, for which it is crucial to know that the strata on connected components of the characteristic p fibers are all nonempty. For example, this is useful for the consideration of algebraic Fourier–Jacobi expansions. We expect it to play foundational roles in other applications of a similar nature.

A.2 Main result

We shall formulate our results in the notation system of [11], which we shall briefly review. (We shall follow [11, Notation and Conventions] unless otherwise specified. While for practical reasons we cannot explain everything we need from [11], we recommend the reader to make use of the reasonably detailed index and table of contents there, when looking for the numerous definitions.)

Let $(\mathcal{O}, \star, L, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$ be an *integral PEL datum*, where $\mathcal{O}$, $\star$, and $(L, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$ are as in [11, Def. 1.2.1.3], satisfying [11, Cond. 1.4.3.10], which defines a group functor G over $\mathbb{Z}$ as in [11, Def. 1.2.1.6], and the *reflex field* F_0 (as a subfield of $\mathbb{C}$) as in [11, Def. 1.2.5.4], with rings of integers $\mathcal{O}_{F_0}$. Let p be any *good prime* as in [11, Def. 1.4.1.1]. Let $\mathcal{H}^p$ be any open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^p)$ that is *neat* as in [11, Def. 1.4.1.8]. Then we have a moduli problem $\mathsf{M}_{\mathcal{H}^p}$ over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$ as in [11, Def. 1.4.1.4], which is representable by a scheme quasi-projective and smooth over S_0 by [11, Thm. 1.4.1.11 and Cor. 7.2.3.10]. By [11, Thm. 7.2.4.1 and Prop. 7.2.4.3], we have the *minimal compactification* $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}^p}$ of $\mathsf{M}_{\mathcal{H}^p}$, which is a scheme projective and flat over S_0, with geometrically normal fibers. Moreover, for each compatible collection Σ^p of cone decompositions for $\mathsf{M}_{\mathcal{H}^p}$ as in [11, Def. 6.3.3.4], we also have the *toroidal compactification* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p}$ of $\mathsf{M}_{\mathcal{H}^p}$, which is an algebraic space proper and smooth over S_0, by [11, Thm. 6.4.1.1], which is representable by a scheme projective over M_0 when Σ^p is *projective* as in [11, Def. 7.3.1.3], by [11, Thm. 7.3.3.4]. Any such $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p}$ admits a canonical surjection $\oint_{\mathcal{H}^p} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p} \to \mathsf{M}^{\mathrm{min}}_{\mathcal{H}^p}$, which is constructed by Stein factorization as in [11, Sec. 7.2.3], whose fibers are all geometrically connected. (The superscript "p" indicates that the objects are defined using level structures "away from p". We will also encounter their variants without the superscript "p", which also involve level structures "at p".)

By [11, Thm. 7.2.4.1(4)], there is a stratification of $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}^p}$ by locally closed subschemes $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$, where $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]$ runs through the (finite) set of *cusp labels* for $\mathsf{M}_{\mathcal{H}^p}$ (see [11, Def. 5.4.2.4]). The open dense subscheme $\mathsf{M}_{\mathcal{H}^p}$ is the stratum labeled by $[(0,0)]$; we call all the other strata the *cusps* of $\mathsf{M}_{\mathcal{H}^p}$. Similarly, by [11, Thm. 6.4.1.1(2)], there is a stratification of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p}$ by locally closed subschemes $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]}$, where $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]$ runs through equivalence classes as in [11, Def. 6.2.6.1] with $\sigma^p \subset \mathbf{P}^+_{\Phi_{\mathcal{H}^p}}$ and $\sigma^p \in \Sigma_{\Phi_{\mathcal{H}^p}} \in \Sigma^p$. By [11, Thm. 7.2.4.1(5)], the surjection $\oint_{\mathcal{H}^p}$ induces a surjection from the $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]}$ of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p}$ to the $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$ of $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}^p}$.

Let $s \to \mathsf{S}_0$ be any geometric point with residue field $k(s)$, and let U be any connected component of the fiber $\mathsf{M}_{\mathcal{H}^p} \underset{\mathsf{S}_0}{\times} s$. Since $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}^p} \to \mathsf{S}_0$ is proper and has geometrically normal fibers, the closure U^{min} of U in $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}^p} \underset{\mathsf{S}_0}{\times} s$ is a connected component of $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}^p} \underset{\mathsf{S}_0}{\times} s$. Similarly, since $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p} \to \mathsf{S}_0$ is proper and smooth, the closure U^{tor} of U in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p} \underset{\mathsf{S}_0}{\times} s$ is a connected component of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p} \underset{\mathsf{S}_0}{\times} s$. (In these cases the connected components are also the irreducible components of the ambient spaces.)

The stratifications of $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}^p}$ and $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p}$ induce stratifications of U^{min} and U^{tor}, respectively, by pullback. We shall denote the pullback of $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$ to U^{min} by $U_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$, and call it the $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]$-stratum of U^{min}. Similarly, we shall denote the pullback of $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]}$ to U^{tor} by $U_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]}$, and call it the $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]$-stratum of U^{tor}. By construction, the surjection $\oint_{\mathcal{H}^p}$ induces a surjection $U^{\mathrm{tor}} \to U^{\mathrm{min}}$, which maps the $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]$-stratum $U_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]}$ of U^{tor} surjectively onto the $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]$-stratum $U_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$ of U^{min}. It is natural to ask whether a particular stratum of $U^{\min}$ or U^{tor} is nonempty.

From now on, we shall make the following:

Assumption A.2.1. *The semisimple algebra* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ *over* $\mathbb{Q}$ *involves no factor of type* D (*in the sense of [11, Def. 1.2.1.15]*).

Our main result is the following:

Theorem A.2.2. *With the setting as above, all strata of* $U^{\min}$ *are nonempty.*

An immediate consequence is the following:

Corollary A.2.3. *With the setting as above, all strata of* U^{tor} *are nonempty.*

Proof. Since the canonical morphism $U_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]} \to U_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$ is surjective for each equivalence class $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]$ with underlying cusp label $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]$ as above, the nonemptiness of $U_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$ implies that of $U_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]}$. ☐

Remark A.2.4. Each stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\mathsf{z}_{\mathcal{H}^p})]}$ (resp. $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\mathsf{z}_{\mathcal{H}^p},\sigma^p)]}$) is nonempty by [11, Thm. 7.2.4.1 (4) and (5), Cor. 6.4.1.2, and the explanation of the existence of complex points as in Rem. 1.4.3.14]. The question is whether its pullback to $U^{\min}$ (resp. U^{tor}) is still nonempty for every U as above.

Remark A.2.5. It easily follows from Theorem A.2.2 and Corollary A.2.3 that their analogues are also true when the geometric point $s \to \mathsf{S}_0$ is replaced with morphisms from general schemes, although we shall omit their statements. In particular, we can talk about connected components of fibers rather than geometric fibers.

The proof of Theorem A.2.2 will be carried out in Sections A.3, A.4, and A.5. In Sections A.5 and A.6, we will also state and prove analogues of Theorem A.2.2 in zero and arbitrarily ramified characteristics, respectively (see Theorems A.5.1 and A.6.1). We will give some examples in Section A.7, including one (see Example A.7.2) showing that we cannot expect Theorem A.2.2 to be true without the requirement (in Assumption A.2.1) that $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ involves no factor of type D.

A.3 Reduction to case of characteristic zero

The goal of this section is to prove the following:

Proposition A.3.1. *Suppose Theorem* A.2.2 *is true when* $\mathrm{char}(k(s)) = 0$. *Then it is also true when* $\mathrm{char}(k(s)) = p > 0$.

Remark A.3.2. Proposition A.3.1 holds regardless of Assumption A.2.1.

Remark A.3.3. It might seem that everything in characteristic zero is well known and straightforward. But Proposition A.3.1, which is insensitive to the crucial Assumption A.2.1, shows that the key difficulty is in fact in characteristic zero.

By [11, Thm. 7.2.4.1(4)], each $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$ is isomorphic to a boundary moduli problem $\mathsf{M}_{\mathcal{H}^p}^{\mathsf{Z}_{\mathcal{H}^p}}$ defined in the same way as $\mathsf{M}_{\mathcal{H}^p}$ (but with certain integral PEL datum associated with $\mathsf{Z}_{\mathcal{H}^p}$). Then it makes sense to consider the minimal compactification $\mathsf{Z}^{\min}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$ of $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$, which is proper flat and has geometrically normal fibers over $\mathsf{M}_{\mathcal{H}}$, as in [11, Thm. 7.2.4.1 and Prop. 7.2.4.3]. (So the connected components of the geometric fibers of $\mathsf{Z}^{\min}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]} \to \mathsf{S}_0$ are closures of those of $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]} \to \mathsf{S}_0$.) By considering the Stein factorizations of the structural morphisms $\mathsf{Z}^{\min}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]} \to \mathsf{S}_0$ (see [7, III-1, 4.3.3 and 4.3.4]), we obtain the following:

Lemma A.3.4 (cf. [11, Cor. 6.4.1.2] and [5, Thm. 4.17])**.** *Suppose* $\mathrm{char}(k(s)) = p > 0$*. Then there exists some discrete valuation ring* R *flat over* $\mathcal{O}_{F_0,(p)}$*, with fraction field* K *and residue field* $k(s)$*, the latter lifting the structural homomorphism* $\mathcal{O}_{F_0,(p)} \to k(s)$*, such that, for each cusp label* $[(\Phi_{\mathcal{H}^p}, \delta_{\mathcal{H}^p})]$*, and for each connected component* V *of* $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} R$*, the induced flat morphism* $V \to \mathrm{Spec}(R)$ *has connected special fiber over* $\mathrm{Spec}(k(s))$*.*

Proof of Proposition A.3.1*.* Let R be as in Lemma A.3.4. Let $\tilde{U}$ denote the connected component of $\mathsf{M}_{\mathcal{H}^p} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} R = \mathsf{Z}_{[(0,0)]} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} R$ such that $\tilde{U} \underset{R}{\otimes} k(s) = U$ as subsets of $\mathsf{M}_{\mathcal{H}^p} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} k(s) = \mathsf{M}_{\mathcal{H}^p} \underset{\mathsf{S}_0}{\times} s$, and let $\tilde{U}^{\min}$ denote its closure in $\mathsf{M}^{\min}_{\mathcal{H}^p} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} R$, which is a connected component of $\mathsf{M}^{\min}_{\mathcal{H}^p} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} R$ because $\mathsf{M}^{\min}_{\mathcal{H}^p} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} R$ is normal by [11, Prop. 7.2.4.3(4)]. For each cusp label $[(\Phi_{\mathcal{H}^p}, \delta_{\mathcal{H}^p})]$, let $\tilde{U}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$ denote the pullback of $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$ to $\tilde{U}^{\min}$. Then $\tilde{U}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$ is an open and closed subscheme of $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} R$ such that $\tilde{U}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]} \underset{R}{\otimes} k(s) = U_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$ as subsets of $\mathsf{M}^{\min}_{\mathcal{H}^p} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} k(s)$. By Lemma A.3.4, it suffices to show that $\tilde{U}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]} \underset{R}{\otimes} \bar{K} \neq \emptyset$ for some algebraic closure $\bar{K}$ of K. Also by Lemma A.3.4, $\tilde{U} \underset{R}{\otimes} \bar{K} \neq \emptyset$, and so $\tilde{U}^{\min} \underset{R}{\otimes} \bar{K}$ contains at least one connected component of $\mathsf{M}^{\min}_{\mathcal{H}^p} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \bar{K}$. Thus, $\tilde{U}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]} \underset{R}{\otimes} \bar{K} \neq \emptyset$ under the assumption of the proposition, as desired. □

A.4 Comparison of cusp labels

Let $\mathcal{H}_p := \mathrm{G}(\mathbb{Z}_p)$ and $\mathcal{H} := \mathcal{H}^p\mathcal{H}_p$, the latter being a neat open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}})$. By the same references to [11] as in Section A.2, we have the moduli problem $\mathsf{M}_{\mathcal{H}}$ and its minimal compactification $\mathsf{M}^{\min}_{\mathcal{H}}$ over $\mathsf{S}_{0,\mathbb{Q}} := \mathsf{S}_0 \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \cong \mathrm{Spec}(F_0)$. For each compatible collection Σ' of cone decompositions for $\mathsf{M}_{\mathcal{H}}$, we also have a toroidal compactification $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$, together with a canonical morphism $\oint_{\mathcal{H}} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'} \to \mathsf{M}^{\min}_{\mathcal{H}}$, over $\mathsf{S}_{0,\mathbb{Q}}$. (Here Σ' does not have to be related to Σ^p above.)

Each cusp label $[(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$ for $\mathsf{M}_{\mathcal{H}}$ (where $\mathsf{Z}_{\mathcal{H}}$ has been suppressed in the notation for simplicity) can be described as an equivalence class of the $\mathcal{H}$-orbit $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ of some triple $(\mathsf{Z}, \Phi, \delta)$, where:

1. $\mathsf{Z} = \{\mathsf{Z}_{-i}\}_{i\in\mathbb{Z}}$ is an admissible filtration on $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$ that is fully symplectic as in [11, Def. 5.2.7.1]. In particular, $\mathsf{Z}_{-i} = (\mathsf{Z} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}) \cap (L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}})$; the symplectic filtration $\mathsf{Z} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ on $L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^\infty$ extends to a symplectic filtration $\mathsf{Z}_{\mathbb{A}}$ on $\mathsf{Z} \underset{\mathbb{Z}}{\otimes} \mathbb{A}$; and each graded piece of Z or $\mathsf{Z} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is integrable as in [11, Def. 1.2.1.23], that is, it is the base extension of some $\mathcal{O}$-lattice.
2. $\Phi = (X, Y, \phi, \varphi_{-2}, \varphi_0)$ is a torus argument as in [11, Def. 5.4.1.3], where $\phi : Y \hookrightarrow X$ is an embedding of $\mathcal{O}$-lattices with finite cokernel, and where $\varphi_{-2} : \mathrm{Gr}^{\mathsf{Z}}_{-2} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}, \hat{\mathbb{Z}}(1))$ and $\varphi_0 : \mathrm{Gr}^{\mathsf{Z}}_0 \xrightarrow{\sim} Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$ are isomorphisms matching the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{20} : \mathrm{Gr}^{\mathsf{Z}}_{-2} \times \mathrm{Gr}^{\mathsf{Z}}_0 \to \hat{\mathbb{Z}}(1)$ induced by $\langle\,\cdot\,,\,\cdot\,\rangle$ with the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_\phi : \mathrm{Hom}_{\hat{\mathbb{Z}}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}, \hat{\mathbb{Z}}(1)) \times (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}) \to \hat{\mathbb{Z}}(1)$ induced by ϕ.
3. $\delta : \mathrm{Gr}^{\mathsf{Z}} \xrightarrow{\sim} L$ is an $\mathcal{O}$-equivariant splitting of the filtration Z.
4. Two triples $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ are equivalent (as in [11, Def. 5.4.2.2]) if $\mathsf{Z}_{\mathcal{H}} = \mathsf{Z}'_{\mathcal{H}}$ and if there exists a pair of isomorphisms $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ matching $\Phi_{\mathcal{H}}$ with $\Phi'_{\mathcal{H}}$.

Since $\mathcal{H} = \mathcal{H}^p\mathcal{H}_p$, it makes sense to consider the p-part of $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$, which is the $\mathcal{H}_p$-orbit of some triple $(\mathsf{Z}_{\mathbb{Z}_p}, (\varphi_{-2,\mathbb{Z}_p}, \varphi_{0,\mathbb{Z}_p}), \delta_{\mathbb{Z}_p})$, where:

1. $\mathsf{Z}_{\mathbb{Z}_p} = \{\mathsf{Z}_{\mathbb{Z}_p,-i}\}_{i\in\mathbb{Z}}$ is a symplectic admissible filtration on $L \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$, which determines and is determined by a symplectic admissible filtration $\mathsf{Z}_{\mathbb{Q}_p} = \{\mathsf{Z}_{\mathbb{Q}_p,-i}\}_{i\in\mathbb{Z}}$ of $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$ by $\mathsf{Z}_{\mathbb{Q}_p,-i} = \mathsf{Z}_{\mathbb{Z}_p,-i} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ and $\mathsf{Z}_{\mathbb{Z}_p,-i} = \mathsf{Z}_{\mathbb{Q}_p,-i} \cap (L \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p)$, for all $i \in \mathbb{Z}$.
2. $\varphi_{-2,\mathbb{Z}_p} : \mathrm{Gr}_{-2}^{\mathsf{Z}_{\mathbb{Z}_p}} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(X \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p, \mathbb{Z}_p(1))$ and $\varphi_0 : \mathrm{Gr}_0^{\mathsf{Z}_{\mathbb{Z}_p}} \xrightarrow{\sim} Y \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$ are isomorphisms matching the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{20,\mathbb{Z}_p} : \mathrm{Gr}_{-2}^{\mathsf{Z}_{\mathbb{Z}_p}} \times \mathrm{Gr}_0^{\mathsf{Z}_{\mathbb{Z}_p}} \to \mathbb{Z}_p(1)$ induced by $\langle\,\cdot\,,\,\cdot\,\rangle$ with the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{\phi,\mathbb{Z}_p} : \mathrm{Hom}_{\mathbb{Z}_p}(X \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p, \mathbb{Z}_p(1)) \times (Y \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p) \to \mathbb{Z}_p(1)$ induced by ϕ.
3. $\delta_{\mathbb{Z}_p} : \mathrm{Gr}^{\mathsf{Z}_{\mathbb{Z}_p}} \xrightarrow{\sim} L \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$ is a splitting of the filtration $\mathsf{Z}_{\mathbb{Z}_p}$.

By forgetting its p-part, each representative $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ for $\mathsf{M}_{\mathcal{H}}$ induces a representative $(\mathsf{Z}_{\mathcal{H}^p}, \Phi_{\mathcal{H}^p}, \delta_{\mathcal{H}^p})$ for $\mathsf{M}_{\mathcal{H}^p}$, and this assignment is compatible with the formation of equivalence classes. Therefore, we have well-defined assignments

$$(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \mapsto (\mathsf{Z}_{\mathcal{H}^p}, \Phi_{\mathcal{H}^p}, \delta_{\mathcal{H}^p}) \tag{A.4.1}$$

and

$$[(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})] \mapsto [(\mathsf{Z}_{\mathcal{H}^p}, \Phi_{\mathcal{H}^p}, \delta_{\mathcal{H}^p})]. \tag{A.4.2}$$

By construction, these assignments are compatible with surjections on their both sides (see [11, Def. 5.4.2.12]). We would like to show that they are both bijective.

Lemma A.4.3. *Let k be any field over $\mathbb{Z}_{(p)}$. Consider the assignment to each flag W of totally isotropic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} k$-submodules of $L \underset{\mathbb{Z}}{\otimes} k$ (with respect to $\langle\,\cdot\,,\,\cdot\,\rangle \underset{\mathbb{Z}}{\otimes} k$) its stabilizer subgroup P_W in $\mathrm{G} \underset{\mathbb{Z}}{\otimes} k$. Then each such P_W is a parabolic subgroup of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} k$, and the assignment is bijective. Moreover, given any minimal parabolic subgroup P_{W_0} of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} k$, which is the stabilizer of some maximal flag W_0 of totally isotropic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} k$-submodules of $L \underset{\mathbb{Z}}{\otimes} k$, every parabolic subgroup of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} k$ is conjugate under the action of $\mathrm{G}(k)$ to some parabolic subgroup of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} k$ containing P_{W_0}, which is the stabilizer of some subflag of W_0.*

Although the assertions in this lemma are well know, we provide a proof because we cannot find a convenient reference in the literature in the generality we need.

Proof of Lemma A.4.3. Let k^{sep} be a separable closure of k. Since the characteristic of k is either 0 or p, the latter being a good prime by assumption, it follows from [11, Prop. 1.2.3.11] that each of the simple factors of the adjoint quotient of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} k^{\mathrm{sep}}$ is isomorphic to one of the groups of standard type listed in the proof of [11, Prop. 1.2.3.11]. Then we can make an explicit choice of a Borel subgroup B of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} k^{\mathrm{sep}}$ stabilizing a flag of totally isotropic submodules, with a maximal torus T of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} k^{\mathrm{sep}}$ contained in B which is isomorphic to the group of automorphisms of the graded pieces of this flag. By [16, Thm. 6.2.7 and Thm. 8.4.3(iv)], since all parabolic subgroups of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} k^{\mathrm{sep}}$ are conjugate to one containing B, the parabolic subgroups of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} k^{\mathrm{sep}}$ are exactly the stabilizers of flags of totally isotropic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} k^{\mathrm{sep}}$-submodules of $L \underset{\mathbb{Z}}{\otimes} k^{\mathrm{sep}}$. Then the analogous assertion over k follows, because the assignment of maximal parabolic subgroups of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} k^{\mathrm{sep}}$ is compatible with the actions of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ on the set of flags of totally isotropic submodules of $L \underset{\mathbb{Z}}{\otimes} k^{\mathrm{sep}}$ and on the set of parabolic subgroups of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} k^{\mathrm{sep}}$. The last assertion of the lemma follows from [16, Thm. 15.1.2(ii) and Thm. 15.4.6(i)]. $\square$

Lemma A.4.4. *The assignment*

$$\mathsf{Z}_{\mathcal{H}} \mapsto \mathsf{Z}_{\mathcal{H}_p} \tag{A.4.5}$$

is bijective.

Proof. Let $\mathsf{Z}_{\mathbb{Z}_p} = \{\mathsf{Z}_{\mathbb{Z}_p,-i}\}_{i\in\mathbb{Z}}$ be a symplectic admissible filtration on $L \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$ as above, which determines and is determined by a symplectic filtration $\mathsf{Z}_{\mathbb{Q}_p} = \{\mathsf{Z}_{\mathbb{Q}_p,-i}\}_{i\in\mathbb{Z}}$ on $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$. By Lemma A.4.3, the action of $\mathrm{G}(\mathbb{Q}_p)$ on the set of such filtrations $\mathsf{Z}_{\mathbb{Q}_p}$ is transitive, because the $\mathcal{O}$-multirank (see [11, Def. 1.2.1.25]) of the bottom piece $\mathsf{Z}_{\mathbb{Q}_p,-2}$ of any such $\mathsf{Z}_{\mathbb{Q}_p}$ is determined by the existence of some isomorphism $\varphi_{-2,\mathbb{Z}_p} : \mathrm{Gr}_{-2}^{\mathsf{Z}_{\mathbb{Z}_p}} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(X \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p, \mathbb{Z}_p(1))$. Let P denote the parabolic subgroup of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$ stabilizing any such $\mathsf{Z}_{\mathbb{Q}_p}$ (see Lemma A.4.3). Since p is a good prime by assumption, the pairing $\langle\,\cdot\,,\,\cdot\,\rangle \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$ is self-dual, and hence $\mathrm{G}(\mathbb{Z}_p)$ is a maximal open compact subgroup of $\mathrm{G}(\mathbb{Q}_p)$ by [3, Cor. 3.3.2]. Since $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$ is connected under Assumption A.2.1 (because the kernel of the similitude character of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$ factorizes over an algebraic closure of $\mathbb{Q}_p$ as a product of connected groups, by the proof of [11, Prop. 1.2.3.11]), we have the Iwasawa decomposition $\mathrm{G}(\mathbb{Q}_p) = \mathrm{G}(\mathbb{Z}_p)\mathrm{P}(\mathbb{Q}_p)$, by [3, Prop. 4.4.3] (see also [4, (18) on p. 392] for a more explicit statement). Consequently, $\mathcal{H}_p = \mathrm{G}(\mathbb{Z}_p)$ acts transitively on the set of possible filtrations $\mathsf{Z}_{\mathbb{Z}_p}$ as above, and hence the assignment (A.4.5) is injective.

As for the surjectivity of (A.4.5), it suffices to show that there exists some symplectic admissible filtration $\mathsf{Z}_{\mathbb{Z}_p}$ such that some isomorphism $\varphi_{-2,\mathbb{Z}_p} : \mathrm{Gr}_{-2}^{\mathsf{Z}_{\mathbb{Z}_p}} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(X \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p, \mathbb{Z}_p(1))$ exists. By [14, Thm. 18.10] and [11, Cor. 1.1.2.6], it suffices to show that there exists some symplectic filtration $\mathsf{Z}_{\mathbb{Q}_p}$ such that $\mathsf{Z}_{\mathbb{Q}_p,-2}$ and $\mathrm{Hom}_{\mathbb{Q}_p}(X \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p, \mathbb{Q}_p(1))$ have the same $\mathcal{O}$-multirank. Or rather, we just need to notice that the $\mathcal{O}$-multirank of a totally isotropic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$-submodule can be any $\mathcal{O}$-multirank below a maximal one (with respect to the natural partial order), by Assumption A.2.1 and by the classification in [11, Prop. 1.2.3.7 and Cor. 1.2.3.10]. □

Lemma A.4.6. *The assignment* (A.4.1) *is bijective.*

Proof. It is already explained in the proof of Lemma A.4.4 that an isomorphism $\varphi_{-2,\mathbb{Z}_p} : \mathrm{Gr}_{-2}^{\mathsf{Z}_{\mathbb{Z}_p}} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(X \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p, \mathbb{Z}_p(1))$ exists for any $\mathsf{Z}_{\mathbb{Z}_p}$ considered there. Since p is a good prime, which forces both $[L^\# : L]$ and $[X : \phi(Y)]$ to be prime to p, any choice of $\varphi_{-2,\mathbb{Z}_p}$ above uniquely determines an isomorphism $\varphi_0 : \mathrm{Gr}_0^{\mathsf{Z}_{\mathbb{Z}_p}} \xrightarrow{\sim} Y \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$. Also by the explicit classification in [11, Prop. 1.2.3.7 and Cor. 1.2.3.10] as in the proof of Lemma A.4.4, there exists a splitting $\delta_{\mathbb{Z}_p} : \mathrm{Gr}^{\mathsf{Z}_{\mathbb{Z}_p}} \xrightarrow{\sim} L \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$, and the action of $\mathrm{G}(\mathbb{Z}_p) \cap \mathrm{P}(\mathbb{Q}_p)$ acts transitively on the set of possible triples $(\varphi_{-2,\mathbb{Z}_p}, \varphi_{0,\mathbb{Z}_p}, \delta_{\mathbb{Z}_p})$. Hence the assignment (A.4.1) is bijective, as desired. □

Lemma A.4.7. *The assignment* (A.4.2) *is bijective.*

Proof. By Lemma A.4.6, it suffices to show that (A.4.2) is injective. Suppose we have two representatives $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}}), \delta_{\mathcal{H}})$ and $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}} = (X', Y', \phi', \varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}}), \delta'_{\mathcal{H}})$ such that the induced $(\mathsf{Z}_{\mathcal{H}^p}, \Phi_{\mathcal{H}^p}, \delta_{\mathcal{H}^p})$ and $(\mathsf{Z}'_{\mathcal{H}^p}, \Phi'_{\mathcal{H}^p}, \delta'_{\mathcal{H}^p})$ are equivalent to each other. By definition, $\mathsf{Z}_{\mathcal{H}^p} = \mathsf{Z}'_{\mathcal{H}^p}$, so that $\mathsf{Z}_{\mathcal{H}} = \mathsf{Z}'_{\mathcal{H}}$ by Lemma A.4.4; and there exists a pair $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ matching $\Phi_{\mathcal{H}^p}$ with $\Phi'_{\mathcal{H}^p}$. Hence we may assume that $(X, Y, \phi) = (X', Y', \phi')$, take any Z in $\mathsf{Z}_{\mathcal{H}^p} = \mathsf{Z}'_{\mathcal{H}^p}$, and take any $(\varphi_{-2} : \mathrm{Gr}_{-2}^{\mathsf{Z}} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}, \hat{\mathbb{Z}}(1)), \varphi_0 : \mathrm{Gr}_0^{\mathsf{Z}} \xrightarrow{\sim} Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}})$ and $(\varphi'_{-2} : \mathrm{Gr}_{-2}^{\mathsf{Z}} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}, \hat{\mathbb{Z}}(1)), \varphi'_0 : \mathrm{Gr}_0^{\mathsf{Z}} \xrightarrow{\sim} Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}})$ inducing $(\varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ and

$(\varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$, respectively, and inducing the same $(\varphi_{-2,\mathcal{H}^p}, \varphi_{0,\mathcal{H}^p})$ and $(\varphi'_{-2,\mathcal{H}^p}, \varphi'_{0,\mathcal{H}^p})$. Then the injectivity of (A.4.2) follows from that of (A.4.1). $\square$

Lemma A.4.8. *If* $(\mathsf{Z}_{\mathcal{H}^p}, \Phi_{\mathcal{H}^p}, \delta_{\mathcal{H}^p})$ *is assigned to* $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *under* (A.4.1)*, then we have a canonical isomorphism*

$$\Gamma_{\Phi_{\mathcal{H}}} \xrightarrow{\sim} \Gamma_{\Phi_{\mathcal{H}^p}} \tag{A.4.9}$$

(*see [11, Def. 6.2.4.1]*)*. Moreover, we have a canonical isomorphism*

$$\mathbf{S}_{\Phi_{\mathcal{H}^p}} \xrightarrow{\sim} \mathbf{S}_{\Phi_{\mathcal{H}}}, \tag{A.4.10}$$

which induces a canonical isomorphism

$$(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}} \xrightarrow{\sim} (\mathbf{S}_{\Phi_{\mathcal{H}^p}})^\vee_{\mathbb{R}} \tag{A.4.11}$$

matching $\mathbf{P}_{\Phi_{\mathcal{H}}}$ *(resp.* $\mathbf{P}^+_{\Phi_{\mathcal{H}}}$*) with* $\mathbf{P}_{\Phi_{\mathcal{H}^p}}$ *(resp.* $\mathbf{P}^+_{\Phi_{\mathcal{H}^p}}$*), both isomorphisms being equivariant with the actions of the two sides of* (A.4.9) *above.*

Proof. Since p is a good prime, with $\mathcal{H}_p = \mathrm{G}(\mathbb{Z}_p)$, the levels at p are not needed in the constructions of $\Gamma_{\Phi_{\mathcal{H}}}$ and $\mathbf{S}_{\Phi_{\mathcal{H}}}$ in [11, Sec. 6.2.3–6.2.4], and hence we have the desired the isomorphisms (A.4.9) and (A.4.10). The induced morphism (A.4.11) matches $\mathbf{P}_{\Phi_{\mathcal{H}}}$ (resp. $\mathbf{P}^+_{\Phi_{\mathcal{H}}}$) with $\mathbf{P}_{\Phi_{\mathcal{H}^p}}$ (resp. $\mathbf{P}^+_{\Phi_{\mathcal{H}^p}}$) because both sides of (A.4.11) can be canonically identified with the space of Hermitian forms over $Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, as explained in the beginning of [11, Sec. 6.2.5], regardless of the levels $\mathcal{H}$ and $\mathcal{H}^p$. $\square$

Therefore, we also have assignments

$$(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma) \mapsto (\Phi_{\mathcal{H}^p}, \delta_{\mathcal{H}^p}, \sigma^p) \tag{A.4.12}$$

and

$$[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)] \mapsto [(\Phi_{\mathcal{H}^p}, \delta_{\mathcal{H}^p}, \sigma^p)] \tag{A.4.13}$$

(see [11, Def. 6.2.6.2]), where we have suppressed $\mathsf{Z}_{\mathcal{H}}$ and $\mathsf{Z}_{\mathcal{H}^p}$ from the notation, where $\sigma \subset (\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$, and where $\sigma^p \subset (\mathbf{S}_{\Phi_{\mathcal{H}^p}})^\vee_{\mathbb{R}}$ is the image of σ under isomorphism (A.4.11), which are compatible with (A.4.1) and (A.4.2).

Lemma A.4.14. *The assignment* (A.4.12) *is bijective.*

Proof. This follows from Lemma A.4.6 and the definition of (A.4.12) based on Lemma A.4.8. $\square$

Lemma A.4.15. *The assignment* (A.4.13) *is bijective.*

Proof. By [11, Def. 6.2.6.2], given any representative $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ of cusp label, the collection of the cones $\sigma \subset (\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$ defining the same equivalence class $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ form an $\Gamma_{\Phi_{\mathcal{H}}}$-orbit. Similarly, the collection of the cones $\sigma^p \subset (\mathbf{S}_{\Phi_{\mathcal{H}^p}})^\vee_{\mathbb{R}}$ defining the same equivalence class $[(\Phi_{\mathcal{H}^p}, \delta_{\mathcal{H}^p}, \sigma^p)]$ form an $\Gamma_{\Phi_{\mathcal{H}^p}}$-orbit. Hence, given (A.4.9), the lemma follows from Lemma A.4.7. $\square$

Definition A.4.16. *We say that* Σ *is* **induced** *by* Σ^p *if, for each cusp label* $[(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$ *of* $\mathsf{M}_{\mathcal{H}}$ *represented by some* $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$*, with assigned* $(\mathsf{Z}_{\mathcal{H}^p}, \Phi_{\mathcal{H}^p}, \delta_{\mathcal{H}^p})$ *as in* (A.4.1)*, the cone decomposition* $\Sigma_{\Phi_{\mathcal{H}}}$ *of* $\mathbf{P}_{\Phi_{\mathcal{H}}}$ *is the pullback of the cone decomposition of* $\mathbf{P}_{\Phi_{\mathcal{H}^p}}$ *under* (A.4.11)*.*

By forgetting the p-parts of level structures, we obtain a canonical isomorphism

$$\mathsf{M}_{\mathcal{H}} \xrightarrow{\sim} \mathsf{M}_{\mathcal{H}^p} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \tag{A.4.17}$$

over $\mathsf{S}_{0,\mathbb{Q}}$ (as in [11, (1.4.4.1)]), by [11, Prop. 1.4.4.3 and Rem. 1.4.4.4] and by Assumption A.2.1. Given any Σ^p for $\mathsf{M}_{\mathcal{H}^p}$, with induced Σ for $\mathsf{M}_{\mathcal{H}}$ as in Definition A.4.16, by comparing the universal properties of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ and $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p}$ as in [11, Thm. 6.4.1.1 (5) and (6)], using also the inverse of the isomorphism (A.4.17) above, we obtain a canonical isomorphism

$$\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \xrightarrow{\sim} \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \tag{A.4.18}$$

over $\mathsf{S}_{0,\mathbb{Q}}$, extending (A.4.17), and mapping $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ isomorphically to $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]}$ when $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p},\sigma^p)]$ is assigned to $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$ under (A.4.13), such that the pullback of the tautological semi-abelian scheme over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is canonically isomorphic to the pullback of the tautological semi-abelian scheme over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$. Consequently, by [11, Thm. 7.2.4.1 (3) and (4)], and by the fact that the pullback of the Hodge invertible sheaf over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is canonically isomorphic to the pullback of the Hodge invertible sheaf over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ (because their definitions only use the tautological semi-abelian schemes), the canonical isomorphism (A.4.18) induces a canonical isomorphism

$$\mathsf{M}^{\mathrm{min}}_{\mathcal{H}} \xrightarrow{\sim} \mathsf{M}^{\mathrm{min}}_{\mathcal{H}^p} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \tag{A.4.19}$$

over $\mathsf{S}_{0,\mathbb{Q}}$, extending (A.4.17), compatible with (A.4.18) (under the canonical morphisms $\oint_{\mathcal{H}} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$ and $\oint_{\mathcal{H}^p} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}^p,\Sigma^p} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \to \mathsf{M}^{\mathrm{min}}_{\mathcal{H}^p} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$), and mapping $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ isomorphically to $\mathsf{Z}_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$ when $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]$ is assigned to $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ under (A.4.2) (where we have suppressed $\mathsf{Z}_{\mathcal{H}}$ and $\mathsf{Z}_{\mathcal{H}^p}$ from the notation).

A.5 Complex analytic construction

By Proposition A.3.1, in order to prove Theorem A.2.2, we may and we shall assume that $\mathrm{char}(k(s)) = 0$. Thanks to the isomorphisms (A.4.17) and (A.4.19), we shall identify U with a connected component of $\mathsf{M}_{\mathcal{H}} \underset{F_0}{\otimes} k(s)$, identify U^{min} with the connected component of $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}} \underset{F_0}{\otimes} k(s)$ that is the closure of U, and identify $U_{[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]}$ with $U_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$, the pullback of the stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ of $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$ under the canonical morphism $U^{\mathrm{min}} \to \mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$, when $[(\Phi_{\mathcal{H}^p},\delta_{\mathcal{H}^p})]$ is assigned to $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ under (A.4.2).

Now in characteristic zero we no longer need $\mathcal{H}$ to be of the form $\mathcal{H} = \mathcal{H}^p\mathcal{H}_p$ as in Section A.4. We shall allow $\mathcal{H}$ to be any neat open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}})$. Then $\mathsf{M}_{\mathcal{H}}$ and $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$ are still defined over $\mathsf{M}_{0,\mathbb{Q}} = \mathrm{Spec}(F_0)$, with the stratification on the latter by locally closed subschemes $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ labeled by cusp labels $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ for $\mathsf{M}_{\mathcal{H}}$ (see the same references we made in Section A.2). For any geometric point $s \to \mathsf{S}_{0,\mathbb{Q}}$ with residue field $k(s)$ and for any connected component U of the fiber $\mathsf{M}_{\mathcal{H}^p} \underset{\mathsf{S}_0}{\times} s$, we define U^{min} to be the closure of U in $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} s$, and define $U_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ to be the pullback of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ to of U^{min}, for each cusp label $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$. (These are consistent with what we have done before, when the settings overlap.)

Then we have the following analogue of Theorem A.2.2:

Theorem A.5.1. *With the setting as above, every stratum $U_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ is nonempty.*

Since $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$ is projective over $\mathsf{S}_{0,\mathbb{Q}}$, we may and we shall assume that $k(s) \cong \mathbb{C}$. We shall denote base changes to $\mathbb{C}$ with a subscript, such as $\mathsf{M}_{\mathcal{H},\mathbb{C}} = \mathsf{M}_{\mathcal{H}} \underset{F_0}{\otimes} \mathbb{C}$, etc.

Let X denote the $\mathrm{G}(\mathbb{R})$-orbit of h_0, which is a finite disjoint union of Hermitian symmetric domains, and let X_0 denote the connected component of X containing h_0. Let $\mathrm{G}(\mathbb{Q})_0$ denote the finite index subgroup of $\mathrm{G}(\mathbb{Q})$ stabilizing X_0. Let $\mathsf{Sh}_{\mathcal{H}} := \mathrm{G}(\mathbb{Q})\backslash \mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)/\mathcal{H}$. By [10, Lem. 2.5.1], we have a canonical bijection $\mathrm{G}(\mathbb{Q})_0\backslash \mathsf{X}_0 \times \mathrm{G}(\mathbb{A}^\infty)/\mathcal{H} \to \mathrm{G}(\mathbb{Q})\backslash \mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)/\mathcal{H}$. Let $\{g_i\}_{i\in I}$ be any finite set of elements of $\mathrm{G}(\mathbb{A}^\infty)$ such that $\mathrm{G}(\mathbb{A}^\infty) = \coprod_{i\in I} \mathrm{G}(\mathbb{Q})_0 h_i \mathcal{H}$, which exists because of [2, Thm. 5.1] and because $\mathrm{G}(\mathbb{Q})_0$ is of finite index in $\mathrm{G}(\mathbb{Q})$. Then we have

$$\mathsf{Sh}_{\mathcal{H}} = \mathrm{G}(\mathbb{Q})_0\backslash \mathsf{X}_0 \times \mathrm{G}(\mathbb{A}^\infty)/\mathcal{H} = \coprod_{i\in I} \Gamma^{(g_i)}\backslash \mathsf{X}_0, \tag{A.5.2}$$

where $\Gamma^{(g_i)} := (g_i \mathcal{H} g_i^{-1}) \cap \mathrm{G}(\mathbb{Q})_0$ for each $i \in I$. By applying [1, 10.11] to each $\Gamma^{(g_i)}\backslash \mathsf{X}_0$, we obtain the minimal compactification $\mathsf{Sh}_{\mathcal{H}}^{\min}$ of $\mathsf{Sh}_{\mathcal{H}}$, which is the complex analytification of a normal projective variety $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}^{\min}$ over $\mathbb{C}$. Thus $\mathsf{Sh}_{\mathcal{H}}$ is the analytification of a quasi-projective variety $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}$ (embedded in $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}^{\min}$).

By [10, Lem. 3.1.1], the rational boundary components $\mathsf{X}_{\mathtt{V}}$ of X_0 (see [1, 3.5]) correspond to parabolic subgroups of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ stabilizing symplectic filtrations $\mathtt{V}$ on $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ with $\mathtt{V}_{-3} = 0 \subset \mathtt{V}_{-2} \subset \mathtt{V}_{-1} = \mathtt{V}_{-2}^{\perp} \subset \mathtt{V}_0 = L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. Consider the rational boundary components of $\mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)$ as in [10, Def. 3.1.2], which are $\mathrm{G}(\mathbb{Q})$-orbits of pairs $(\mathtt{V}, g)$, where $\mathtt{V}$ are as above and $g \in \mathrm{G}(\mathbb{A}^\infty)$. Consider the boundary components $\mathrm{G}(\mathbb{Q})\backslash(\mathrm{G}(\mathbb{Q})\mathsf{X}_{\mathtt{V}}) \times \mathrm{G}(\mathbb{A}^\infty)/\mathcal{H} = \mathrm{G}(\mathbb{Q})_0\backslash(\mathrm{G}(\mathbb{Q})_0\mathsf{X}_{\mathtt{V}}) \times \mathrm{G}(\mathbb{A}^\infty)/\mathcal{H}$ of $\mathsf{Sh}_{\mathcal{H}} = \mathrm{G}(\mathbb{Q})_0\backslash \mathsf{X}_0 \times \mathrm{G}(\mathbb{A}^\infty)/\mathcal{H}$. By the construction in [1], each such component defines a nonempty locally closed subset and meets all connected components of $\mathsf{Sh}_{\mathcal{H}}^{\min}$, corresponding to a nonempty locally closed subscheme of $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}^{\min}$ which we call its $\mathrm{G}(\mathbb{Q})(\mathtt{V}, g)\mathcal{H}$-stratum. Thus, we obtain the following:

Proposition A.5.3 (Satake, Baily–Borel)**.** *Each $\mathrm{G}(\mathbb{Q})(\mathtt{V}, g)\mathcal{H}$-stratum as above meets every connected component of $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}^{\min}$.*

For each $g \in \mathrm{G}(\mathbb{A}^\infty)$, let $L^{(g)}$ denote the $\mathcal{O}$-lattice in $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ such that $L^{(g)} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}} = g(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}})$ in $L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^\infty$. Let $r \in \mathbb{Q}_{>0}^\times$ be the unique element such that $\nu(g) = ru$ for some $u \in \hat{\mathbb{Z}}$, and let $\langle\,\cdot\,,\,\cdot\,\rangle^{(g)} : L^{(g)} \times L^{(g)} \to \mathbb{Z}(1)$ denote the pairing induced by $r\langle\,\cdot\,,\,\cdot\,\rangle \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ (see [10, Sec. 2.4]; the key point being that $\langle\,\cdot\,,\,\cdot\,\rangle^{(g)}$ is valued in $\mathbb{Z}(1)$).

Construction A.5.4. As explained in [10, Sec. 3.1], we have an assignment of a fully symplectic admissible filtration $\mathtt{Z}^{(g)}$ on $\mathtt{Z} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$ and a torus argument $\Phi^{(g)} = (X^{(g)}, Y^{(g)}, \phi^{(g)}, \varphi_{-2}^{(g)}, \varphi_0^{(g)})$ to $\mathrm{G}(\mathbb{Q})(\mathtt{V}, g)$, by setting:

1. $\mathtt{F}^{(g)} := \{\mathtt{F}_{-i}^{(g)} := \mathtt{V}_{-i} \cap L^{(g)}\}_{i\in\mathbb{Z}}$.
2. $\mathtt{Z}^{(g)} := \{\mathtt{Z}_{-i}^{(g)} := g^{-1}(\mathtt{F}^{(g)} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}})\}_{i\in\mathbb{Z}} = \{g^{-1}(\mathtt{V} \underset{\mathbb{Q}}{\otimes} \mathbb{A}^\infty) \cap (L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}})\}_{i\in\mathbb{Z}}$.
3. $X^{(g)} := \mathrm{Hom}_{\mathbb{Z}}(\mathtt{F}_{-2}^{(g)}, \mathbb{Z}(1)) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Gr}_{-2}^{\mathtt{F}^{(g)}}, \mathbb{Z}(1))$.
4. $Y^{(g)} := \mathrm{Gr}_0^{\mathtt{F}^{(g)}} = \mathtt{F}_0^{(g)}/\mathtt{F}_{-1}^{(g)}$.
5. $\phi^{(g)} : Y^{(g)} \hookrightarrow X^{(g)}$ is equivalent to the nondegenerate pairing
$$\langle\,\cdot\,,\,\cdot\,\rangle_{20}^{(g)} : \mathrm{Gr}_{-2}^{\mathtt{F}^{(g)}} \times \mathrm{Gr}_0^{\mathtt{F}^{(g)}} \to \mathbb{Z}(1)$$
induced by $\langle\,\cdot\,,\,\cdot\,\rangle^{(g)} : L^{(g)} \times L^{(g)} \to \mathbb{Z}(1)$.
6. $\varphi_{-2}^{(g)} : \mathrm{Gr}_{-2}^{\mathtt{Z}^{(g)}} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}}(X^{(g)} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}, \hat{\mathbb{Z}}(1))$ is the composition
$$\mathrm{Gr}_{-2}^{\mathtt{Z}^{(g)}} \overset{\mathrm{Gr}_{-2}(g)}{\xrightarrow{\sim}} \mathrm{Gr}_{-2}^{\mathtt{F}^{(g)}} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}}(X^{(g)} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}, \hat{\mathbb{Z}}(1)).$$

7. $\varphi_0^{(g)} : \mathrm{Gr}_0^{\mathtt{Z}^{(g)}} \xrightarrow{\sim} Y^{(g)} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$ is the composition

$$\mathrm{Gr}_0^{\mathtt{Z}^{(g)}} \overset{\mathrm{Gr}_0(g)}{\xrightarrow{\sim}} \mathrm{Gr}_0^{\mathtt{F}^{(g)}} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}} \xrightarrow{\sim} Y^{(g)} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}.$$

By the assumption that our integral PEL datum satisfies [11, Cond. 1.4.3.10], and by the fact that maximal orders over Dedekind domains are *hereditary* (see [14, Thm. 21.4 and Cor. 21.5]), there exists some splitting $\varepsilon^{(g)} : \mathrm{Gr}^{\mathtt{F}^{(g)}} \xrightarrow{\sim} L^{(g)}$, whose base extension from $\mathbb{Z}$ to $\hat{\mathbb{Z}}$ defines by pre- and post- compositions with $\mathrm{Gr}(g)$ and g^{-1} a splitting $\delta^{(g)} : \mathrm{Gr}^{\mathtt{Z}^{(g)}} \xrightarrow{\sim} L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$. These define an assignment

$$\mathrm{G}(\mathbb{Q})(\mathtt{V}, g) \mapsto [(\mathtt{Z}^{(g)}, \Phi^{(g)}, \delta^{(g)})], \tag{A.5.5}$$

which is compatible with the formation of $\mathcal{H}$-orbits and induces an assignment

$$\mathrm{G}(\mathbb{Q})(\mathtt{V}, g)\mathcal{H} \mapsto [(\mathtt{Z}_{\mathcal{H}}^{(g)}, \Phi_{\mathcal{H}}^{(g)}, \delta_{\mathcal{H}}^{(g)})]. \tag{A.5.6}$$

Definition A.5.7. *For each cusp label $[(\mathtt{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$, the $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$-stratum of $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}^{\min}$ is the union of all the $\mathrm{G}(\mathbb{Q})(\mathtt{V}, g)\mathcal{H}$-strata such that $[(\mathtt{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$ is assigned to $\mathrm{G}(\mathbb{Q})(\mathtt{V}, g)\mathcal{H}$ under* (A.5.6).

Proposition A.5.8. *Given the $\mathcal{H}$-orbit $\mathtt{Z}_{\mathcal{H}}$ of any $\mathtt{Z} = \{\mathtt{Z}_{-i}\}_{i\in\mathbb{Z}}$ as above, there exists some totally isotropic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$-submodule $\mathtt{V}_{-2}$ of $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ such that $\mathtt{V}_{-2} \underset{\mathbb{Q}}{\otimes} \mathbb{A}^\infty$ lies in the $\mathcal{H}$-orbit of $\mathtt{Z}_{-2} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.*

Proof. Up to replacing $\mathcal{H}$ with an open compact subgroup, which is harmless for proving this proposition, we may and we shall assume that $\mathcal{H} = \mathcal{H}^S\mathcal{H}_S$, where S is a finite set of primes containing all bad ones for the integral PEL datum (see [11, Def. 1.4.1.1]), such that $\mathcal{H}^S = \mathrm{G}(\hat{\mathbb{Z}}^S) = \prod_{\ell\notin S} \mathrm{G}(\mathbb{Z}_\ell)$ and $\mathcal{H}_S \subset \mathrm{G}(\hat{\mathbb{Z}}_S) = \prod_{\ell\in S} \mathrm{G}(\mathbb{Z}_\ell)$, where $\ell \notin S$ means that ℓ runs through all prime numbers not in S.

By Assumption A.2.1, by reduction to the case where $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is a product of division algebras by Morita equivalence (see [11, Prop. 1.2.1.14]), and by the local-global principle for isotropy in [15, table on p. 347, and its references], it follows that, if $\mathtt{Z}_{-2} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is nonzero and extends to some isotropic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{A}$-submodule of $L \underset{\mathbb{Z}}{\otimes} \mathbb{A}$ isomorphic to the base extension of some $\mathcal{O}$-lattice, then there exists some nonzero isotropic element in $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. By induction on the $\mathcal{O}$-multirank of $\mathtt{Z}_{-2} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$—by replacing $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ (resp. $L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^\infty$) with the orthogonal complements modulo the span of a nonzero isotropic element in $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ (resp. $L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^\infty$)—there exists some totally isotropic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$-submodule $\mathtt{V}_{-2}^0$ of $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ such that $\mathtt{V}_{-2}^0 \underset{\mathbb{Q}}{\otimes} \mathbb{A}^\infty$ and $\mathtt{Z}_{-2} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ have the same $\mathcal{O}$-multirank.

Let G' denote the derived subgroup of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ (see [6, $\mathrm{VI_B}$, 7.2(vii) and 7.10]). Then the pullback to G' induces a bijection between the parabolic subgroups of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ and those of G' (see [6, XXII, 6.2.4 and 6.2.8] and [16, Thm. 15.1.2(ii) and Thm. 15.4.6(i)]), and they both are in bijection with the stabilizers of flags of totally isotropic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$-submodules as in Lemma A.4.3. Therefore, there exists some element $h = (h_\ell) \in \mathrm{G}'(\mathbb{A}^\infty)$, where the index ℓ runs through all prime numbers, such that $\mathtt{V}_{-2}^0 \underset{\mathbb{Q}}{\otimes} \mathbb{A}^\infty = h(\mathtt{Z}_{-2} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$.

Since G' is simply connected by Assumption A.2.1 (because the kernel of the similitude character of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ factorizes over an algebraic closure of $\mathbb{Q}$ as a product of groups with simply connected derived groups, by the proof of [11, Prop. 1.2.3.11]), by weak approximation (see [13, Thm. 7.8]), there exists $\gamma \in \mathrm{G}'(\mathbb{Q})$ such that $\gamma(h_\ell)_{\ell\in S} \in \mathcal{H}_S$. On the other hand, by using the

Iwasawa decomposition at the places $\ell \in S$ as in the proof of Lemma A.4.4, up to replacing h_ℓ with a right multiple by an element of $\mathrm{G}'(\mathbb{Q}_\ell)$ stabilizing $\mathtt{Z}_{-2} \underset{\hat{\mathbb{Z}}}{\otimes} \mathbb{Q}_\ell$, we may assume that $\gamma h_\ell \in \mathrm{G}(\mathbb{Z}_\ell)$ for all $\ell \notin S$. Thus, we can conclude by taking $\mathtt{V}_{-2} := \gamma(\mathtt{V}^0_{-2})$. □

Proposition A.5.9. *For each cusp label $[(\mathtt{Z}_\mathcal{H}, \Phi_\mathcal{H}, \delta_\mathcal{H})]$, there exists some rational boundary component $\mathrm{G}(\mathbb{Q})(\mathtt{V}, g)$ of $\mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)$ such that $[(\mathtt{Z}_\mathcal{H}, \Phi_\mathcal{H}, \delta_\mathcal{H})]$ is assigned to $\mathrm{G}(\mathbb{Q})(\mathtt{V}, g)\mathcal{H}$ under* (A.5.6).

Proof. Let $(\mathtt{Z}, \Phi = (X, Y, \phi, \varphi_{-2}, \varphi_0), \delta)$ be any triple whose $\mathcal{H}$-orbit induces $[(\mathtt{Z}_\mathcal{H}, \Phi_\mathcal{H}, \delta_\mathcal{H})]$, and let $\mathtt{V}_{-2}$ be as in Proposition A.5.8. Up to replacing $(\mathtt{Z}, \Phi, \delta)$ with another such triple, we may and we shall assume that

$$\mathtt{Z}_{-2} = (\mathtt{V}_{-2} \underset{\mathbb{Q}}{\otimes} \mathbb{A}^\infty) \cap (L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}) = \mathtt{Z}^{(1)}_{-2}, \tag{A.5.10}$$

where $\mathtt{F}^{(1)} = \{\mathtt{F}^{(1)}_{-i}\}_{i \in \mathbb{Z}}$, $\mathtt{Z}^{(1)} = \{\mathtt{Z}^{(1)}_{-i}\}_{i \in \mathbb{Z}}$, and $\Phi^{(1)} = (X^{(1)}, Y^{(1)}, \phi^{(1)}, \varphi^{(1)}_{-2}, \varphi^{(1)}_0)$ are assigned to $(\mathtt{V}, 1)$ as in Construction A.5.4, together with some noncanonical choices of $\varepsilon^{(1)}$ and $\delta^{(1)}$.

Let P denote the parabolic subgroup of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ stabilizing $\mathtt{V}_{-2}$ (see Lemma A.4.3). By (A.5.10), the elements of $\mathrm{P}(\mathbb{A}^\infty)$ also stabilizes $\mathtt{Z}_{-2} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. Therefore, for each $g \in \mathrm{P}(\mathbb{A}^\infty)$, the filtration $\mathtt{Z}^{(g)}$ defined as in Construction A.5.4 coincides with $\mathtt{Z}$.

Using (A.5.10) and the compatibility among the objects, both $\phi \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$ and $\phi^{(1)} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$ can be identified (under $(\varphi_{-2}, \varphi_0)$ and $(\varphi^{(1)}_{-2}, \varphi^{(1)}_0)$) with the canonical morphism

$$\langle \cdot\,, \cdot \rangle^*_{20} : \mathrm{Hom}_{\hat{\mathbb{Z}}}(\mathrm{Gr}^{\mathtt{Z}}_{-2}, \hat{\mathbb{Z}}(1)) \to \mathrm{Gr}^{\mathtt{Z}}_0 \tag{A.5.11}$$

induced by the pairing $\langle \cdot\,, \cdot \rangle$, which induce compatible isomorphisms

$${}^t(\varphi^{(1)}_{-2} \circ \varphi^{-1}_{-2}) : X^{(1)} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}} \xrightarrow{\sim} X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}} \tag{A.5.12}$$

and

$$\varphi^{(1)}_0 \circ \varphi^{-1}_0 : Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}} \xrightarrow{\sim} Y^{(1)} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}. \tag{A.5.13}$$

By [11, Cond. 1.4.3.10], there exists some maximal order $\mathcal{O}'$ in $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, containing $\mathcal{O}$, such that the $\mathcal{O}$-action on L extends to an $\mathcal{O}'$-action; hence the $\mathcal{O}$-actions on Y and $Y^{(1)}$ also extend to $\mathcal{O}'$-actions. Using the local isomorphisms given by (A.5.13), by [14, Thm. 18.10] (which is applicable because we are now considering modules of the maximal order $\mathcal{O}'$) and [11, Cor. 1.1.2.6], there exists an element $g_0 \in \mathrm{GL}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{A}^\infty}(\mathrm{Gr}^{\mathtt{Z}}_0 \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$ and an $\mathcal{O}$-equivariant embedding $h_0 : Y^{(1)} \hookrightarrow Y \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ such that $(h_0(Y^{(1)})) \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}} = (\varphi_0 \underset{\mathbb{Z}}{\otimes} \mathbb{Q})(g_0(\mathrm{Gr}^{\mathtt{Z}}_0))$ in $Y \underset{\mathbb{Z}}{\otimes} \mathbb{A}^\infty$. Let $g_{-2} := {}^t g_0^{-1} \in \mathrm{GL}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{A}^\infty}(\mathrm{Gr}^{\mathtt{Z}}_{-2} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$, where the transposition is induced by (A.5.11). Then there is a corresponding $\mathcal{O}$-equivariant embedding $h_{-2} : \mathrm{Hom}_\mathbb{Z}(X^{(1)}, \mathbb{Z}(1)) \hookrightarrow \mathrm{Hom}_\mathbb{Z}(X, \mathbb{Z}(1)) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ such that $(h_{-2}(\mathrm{Hom}_\mathbb{Z}(X^{(1)}, \mathbb{Z}(1)))) \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}} = (\varphi_{-2} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})(g_{-2}(\mathrm{Gr}^{\mathtt{Z}}_{-2}))$ in $\mathrm{Hom}_\mathbb{Z}(X, \mathbb{Z}(1)) \underset{\mathbb{Z}}{\otimes} \mathbb{A}^\infty$.

Take $g \in \mathrm{P}(\mathbb{A}^\infty)$ such that $\mathrm{Gr}_{-2}(g) = g_{-2}$, $\mathrm{Gr}_0(g) = g_0$, and $\nu(g) = 1$, which exists thanks to the splitting δ. Then $X^{(g)}$ and $Y^{(g)}$ are realized as the preimages of X and Y under ${}^t h_{-2} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ and $h_0^{-1} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, respectively; and the induced pair $(\gamma_X : X^{(g)} \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y^{(g)})$ matches $\Phi^{(g)}$ with Φ. Such a $(\mathtt{V}, g)$ is what we want. □

As explained in [11, Sec. 2.5], there is a canonical open and closed immersion

$$\mathsf{Sh}_{\mathcal{H},\mathrm{alg}} \hookrightarrow \mathsf{M}_{\mathcal{H},\mathbb{C}}. \tag{A.5.14}$$

As explained in [8, §8, p. 399] (see also [11, Rem. 1.4.3.12]), $\mathsf{M}_{\mathcal{H},\mathbb{C}}$ is the disjoint union of the images of morphisms like (A.5.14), from certain $\mathsf{Sh}^{(j)}_{\mathcal{H},\mathrm{alg}}$ defined by some $(\mathcal{O},\star,L^{(j)},\langle\,\cdot\,,\,\cdot\,\rangle^{(j)},h_0)$ such that $(L^{(j)},\langle\,\cdot\,,\,\cdot\,\rangle^{(j)})\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}\cong(L,\langle\,\cdot\,,\,\cdot\,\rangle)\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}$ and $(L^{(j)},\langle\,\cdot\,,\,\cdot\,\rangle^{(j)})\underset{\mathbb{Z}}{\otimes}\mathbb{R}\cong(L,\langle\,\cdot\,,\,\cdot\,\rangle)\underset{\mathbb{Z}}{\otimes}\mathbb{R}$, but not necessarily satisfying $(L^{(j)},\langle\,\cdot\,,\,\cdot\,\rangle^{(j)})\underset{\mathbb{Z}}{\otimes}\mathbb{Q}\cong(L,\langle\,\cdot\,,\,\cdot\,\rangle)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$, for all j in some index set J (whose precise description is not important for our purpose). (Each $(L^{(j)},\langle\,\cdot\,,\,\cdot\,\rangle^{(j)})$ is determined by its rational version $(L^{(j)},\langle\,\cdot\,,\,\cdot\,\rangle^{(j)})\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ by taking intersection of the latter with $(L^{(j)},\langle\,\cdot\,,\,\cdot\,\rangle^{(j)})\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}\cong(L,\langle\,\cdot\,,\,\cdot\,\rangle)\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}$ in $(L^{(j)},\langle\,\cdot\,,\,\cdot\,\rangle^{(j)})\underset{\mathbb{Z}}{\otimes}\mathbb{A}^\infty\cong(L,\langle\,\cdot\,,\,\cdot\,\rangle)\underset{\mathbb{Z}}{\otimes}\mathbb{A}^\infty$. Due to the *failure of Hasse's principle*, J might have more than one element.)

By [10, Thm. 5.1.1], (A.5.14) extends to a canonical open and closed immersion

$$\mathsf{Sh}^{\min}_{\mathcal{H},\mathrm{alg}}\hookrightarrow\mathsf{M}^{\min}_{\mathcal{H},\mathbb{C}},\tag{A.5.15}$$

respecting the stratifications on both sides labeled by cusp labels (see Definition A.5.7). Again, $\mathsf{M}^{\min}_{\mathcal{H},\mathbb{C}}$ is the disjoint union of the images of morphisms like (A.5.15), from the minimal compactifications $\mathsf{Sh}^{(j),\min}_{\mathcal{H},\mathrm{alg}}$ of $\mathsf{Sh}^{(j)}_{\mathcal{H},\mathrm{alg}}$, for all $j\in J$.

Everything we have proved remain true after replacing the objects defined by $(L,\langle\,\cdot\,,\,\cdot\,\rangle)$ with those defined by $(L^{(j)},\langle\,\cdot\,,\,\cdot\,\rangle^{(j)})$, for each $j\in J$. Thus, in order to show that $U_{[(\Phi_\mathcal{H},\delta_\mathcal{H})]}$ is nonempty, it suffices to note that, by Propositions A.5.3 and A.5.9, the $[(\Phi_\mathcal{H},\delta_\mathcal{H})]$-stratum of $\mathsf{Sh}^{(j),\min}_{\mathcal{H},\mathrm{alg}}$ meets every connected component of $\mathsf{Sh}^{(j),\min}_{\mathcal{H},\mathrm{alg}}$, for all $j\in J$. The proof of Theorem A.5.1 is now complete. $\square$

By Proposition A.3.1, and by the explanations in Section A.4 and in the beginning of this section, the proof of Theorem A.2.2 is also complete. $\square$

A.6 Extension to cases of ramified characteristics

In this section, we shall no longer assume that p is a good prime for the integral PEL datum $(\mathcal{O},\star,L,\langle\,\cdot\,,\,\cdot\,\rangle,h_0)$, but we shall assume that the image $\mathcal{H}^p$ of $\mathcal{H}$ under the canonical homomorphism $\mathrm{G}(\hat{\mathbb{Z}})\to\mathrm{G}(\hat{\mathbb{Z}}^p)$ is *neat*.

Even for such general $\mathcal{H}$ and p, for any collections of lattices stabilized by $\mathcal{H}$ as in [12, Sec. 2], we still have an integral model $\vec{\mathsf{M}}_\mathcal{H}$ of $\mathsf{M}_\mathcal{H}$ flat over S_0 constructed by "taking normalization" (see [12, Prop. 6.1; see also the introduction]). Moreover, we have an integral model $\vec{\mathsf{M}}^{\min}_\mathcal{H}$ of $\mathsf{M}^{\min}_\mathcal{H}$ projective and flat over S_0 (see [12, Prop. 6.4]), with a stratification by locally closed subschemes $\vec{\mathsf{Z}}_{[(\Phi_\mathcal{H},\delta_\mathcal{H})]}$ labeled by cusp labels $[(\Phi_\mathcal{H},\delta_\mathcal{H})]$ for $\mathsf{M}_\mathcal{H}$, which extends the stratification of $\mathsf{M}_\mathcal{H}$ by the locally closed subschemes $\mathsf{Z}_{[(\Phi_\mathcal{H},\delta_\mathcal{H})]}$ (see [12, Thm. 12.1]). For certain (possibly nonsmooth) compatible collections Σ (not the same ones for which we can construct $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ over $\mathsf{M}_{0,\mathbb{Q}}$), we also have the toroidal compactifications $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ of $\vec{\mathsf{M}}_\mathcal{H}$ projective and flat over S_0 (see [12, Sec. 7]), with a stratification by locally closed subschemes $\vec{\mathsf{Z}}_{[(\Phi_\mathcal{H},\delta_\mathcal{H},\sigma)]}$ (see [12, Thm. 9.13]), and with a canonical surjection $\vec{\oint}_\mathcal{H}:\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}\to\vec{\mathsf{M}}^{\min}_\mathcal{H}$ with geometrically connected fibers (see [12, Lem. 12.9 and its proof]), inducing surjections $\vec{\mathsf{Z}}_{[(\Phi_\mathcal{H},\delta_\mathcal{H},\sigma)]}\to\vec{\mathsf{Z}}_{[(\Phi_\mathcal{H},\delta_\mathcal{H})]}$ (see [12, Thm. 12.16]).

As in Section A.2, consider a geometric point $s\to\mathsf{S}_0=\mathrm{Spec}(\mathcal{O}_{F_0,(p)})$ with algebraically closed residue field $k(s)$, and consider a connected component $U^{\min}$ of the fiber $\vec{\mathsf{M}}^{\min}_\mathcal{H}\underset{\mathsf{S}_0}{\times}s$. For each cusp label $[(\Phi_\mathcal{H},\delta_\mathcal{H})]$ for $\mathsf{M}_\mathcal{H}$, we define $U_{[(\Phi_\mathcal{H},\delta_\mathcal{H})]}$ to be the pullback of $\vec{\mathsf{Z}}_{[(\Phi_\mathcal{H},\delta_\mathcal{H})]}$ to $U^{\min}$. Since the fibers of $\vec{\oint}_\mathcal{H}$ are geometrically connected, the preimage of $U^{\min}$ under $\vec{\oint}_\mathcal{H}\underset{\mathsf{S}_0}{\times}s$ is a connected component U^{tor} of $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}\underset{\mathsf{S}_0}{\times}s$. (In general neither $\vec{\mathsf{M}}^{\min}_\mathcal{H}\underset{\mathsf{S}_0}{\times}s$ nor $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}\underset{\mathsf{S}_0}{\times}s$ is normal.) For each equivalence class $[(\Phi_\mathcal{H},\delta_\mathcal{H},\sigma)]$ defining a stratum $\vec{\mathsf{Z}}_{[(\Phi_\mathcal{H},\delta_\mathcal{H},\sigma)]}$ of $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$, we define $U_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ to be the pullback of $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$. Then we also have a canonical surjection $U_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \to U_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ induced by $\vec{\oint}_{\mathcal{H}}$.

Theorem A.6.1. *With the setting as above, all strata of* $U^{\min}$ *are nonempty.*

By using the canonical surjection $U_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \to U_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ (as in the proof of Corollary A.2.3), Theorem A.6.1 implies the following:

Corollary A.6.2. *With the setting as above, all strata of* U^{tor} *are nonempty.*

As in Section A.3, it suffices to prove the following:

Proposition A.6.3. *Suppose Theorem* A.6.1 *is true when* $\mathrm{char}(k(s)) = 0$. *Then it is also true when* $\mathrm{char}(k(s)) = p > 0$.

Remark A.6.4. Since $\vec{\mathsf{M}}_{\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \cong \mathsf{M}_{\mathcal{H}}$ and $\vec{\mathsf{M}}_{\mathcal{H}}^{\min} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \cong \mathsf{M}_{\mathcal{H}}^{\min}$ by construction, by Theorem A.5.1, the assumption in Proposition A.6.3 always holds. Nevertheless, the proof of Proposition A.6.3 will clarify that the deduction of Theorem A.6.1 from Theorem A.5.1 does not require Assumption A.2.1 (cf. Remark A.3.2).

The remainder of this section will be devoted to the proof of Proposition A.6.3. We shall assume that $\mathrm{char}(k(s)) = p > 0$.

While each $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ is isomorphic to some boundary moduli problem $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$, each stratum $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ of $\vec{\mathsf{M}}_{\mathcal{H}}^{\min}$ is similarly isomorphic to some integral model $\vec{\mathsf{M}}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ defined by taking normalization (see [12, Prop. 7.4, and Thm. 12.1 and 12.16]). Hence it also makes sense to consider the minimal compactification $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}^{\min}$ of $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$, which is proper flat (with possibly non-normal geometric fibers) over S_0, and we obtain the following:

Lemma A.6.5 (cf. Lemma A.3.4 and [5, Thm. 4.17(ii)])**.** *There exists some discrete valuation ring R flat over $\mathcal{O}_{F_0,(p)}$, with fraction field K and residue field $k(s)$, the latter lifting the structural homomorphism $\mathcal{O}_{F_0,(p)} \to k(s)$, such that, for each cusp label $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$, and for each connected component V of $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}^{\min} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} R$, the induced flat morphism $V \to$* $\mathrm{Spec}(R)$ *has connected special fiber over* $\mathrm{Spec}(k(s))$.

Proof of Proposition A.6.3. By [12, Cor. 12.4], it suffices to show that $U_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} \neq \emptyset$ when $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ is *maximal* with respect to the surjection relations as in [11, Def. 5.4.2.13]. In this case, by [12, Thm. 12.1], $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ is a closed stratum of $\vec{\mathsf{M}}_{\mathcal{H}}^{\min}$, and so $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} = \vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}^{\min}$. Hence the lemma follows from Theorem A.5.1 and the same argument as in the proof of Proposition A.3.1, with the reference to Lemma A.3.4 replaced with an analogous reference to Lemma A.6.5. ☐

As explained in Remark A.6.4, the proof of Theorem A.6.1 is now complete. ☐

A.7 Examples

Example A.7.1. Suppose $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is a CM field F with maximal totally subfield F^+, with positive involution given by the complex conjugation of F over F^+. Suppose $L = \mathcal{O}_F^{\oplus\, a+b}$, where $a \geq b \geq 0$ are integers. Suppose $(2\pi\sqrt{-1})^{-1}\langle\,\cdot\,,\,\cdot\,\rangle$ is the skew-Hermitian pairing defined in block matrix form $\begin{pmatrix} & & 1_b \\ & S & \\ -1_b & & \end{pmatrix}$ where S is some $(a-b)\times(a-b)$ matrix over F such that $\sqrt{-1}S$ is Hermitian and either positive or negative definite. Then, for each $0 \leq r \leq b$, the $\mathcal{O}$-submodule $\mathtt{Z}_{-2}^{(r)}$ of $L = \mathcal{O}_F^{\oplus(a+b)}$ with the last $(a+b-r)$-entries zero is totally isotropic, and $\mathtt{V}_{-2}^{(r)} := \mathtt{F}_{-2}^{(r)} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is a totally isotropic F-submodule of $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q} = F^{\oplus(a+b)}$, which is maximal when $r = b$. The stabilizer of $\mathtt{V}_{-2}^{(r)}$ either is the whole group (when $r=0$) or defines a maximal (proper) parabolic subgroup $\mathrm{P}^{(r)}$ of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ (when $r>0$), and all maximal parabolic subgroups of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ are conjugate to one of these *standard* ones, by Lemma A.4.3. Similarly, $\mathtt{Z}_{-2}^{(r)} := \mathtt{F}_{-2}^{(r)} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$ is a totally isotropic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$-submodule of $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$, and the left $\mathrm{G}(\mathbb{Q})$- and right $\mathcal{H}$- double orbits of $\mathtt{Z}_{-2}^{(r)}$, for $0 \leq r \leq b$, exhaust all possible $\mathtt{Z}_{\mathcal{H}}$'s appearing in cusp labels $[(\mathtt{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$ for $\mathsf{M}_{\mathcal{H}}$, by Proposition A.5.8. By Lemma A.4.7, by forgetting their p-parts, their left $\mathrm{G}(\mathbb{Q})$- and right $\mathcal{H}^p$- double orbits also exhaust all possible $\mathtt{Z}_{\mathcal{H}^p}$'s appearing in cusp labels $[(\mathtt{Z}_{\mathcal{H}^p}, \Phi_{\mathcal{H}^p}, \delta_{\mathcal{H}^p})]$ for $\mathsf{M}_{\mathcal{H}^p}$. Let us say that a cusp label $[(\mathtt{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$ for $\mathsf{M}_{\mathcal{H}}$ is *of rank* r if $\mathtt{Z}_{\mathcal{H}}$ is in the double orbit of $\mathtt{Z}_{-2}^{(r)}$, and that a cusp $[(\mathtt{Z}_{\mathcal{H}^p}, \Phi_{\mathcal{H}^p}, \delta_{\mathcal{H}^p})]$ for $\mathsf{M}_{\mathcal{H}^p}$ is *of rank* r if it is assigned to one of rank r under (A.4.1). (This is consistent with [11, Def. 5.4.1.2 and 5.4.2.7].) On the other hand, as a byproduct of the proof of Proposition A.5.9, any $\mathtt{Z}_{\mathcal{H}}$ in the double orbit of $\mathtt{Z}_{-2}^{(r)}$ does extend to some cusp label $[(\mathtt{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$ for $\mathsf{M}_{\mathcal{H}}$, inducing some cusp label $[(\mathtt{Z}_{\mathcal{H}^p}, \Phi_{\mathcal{H}^p}, \delta_{\mathcal{H}^p})]$ for $\mathsf{M}_{\mathcal{H}^p}$ under (A.4.1). Then Theorem A.2.2 shows that, in the boundary stratification of every connected component of every geometric fiber of $\mathsf{M}_{\mathcal{H}^p}^{\min} \to \mathsf{S}_0 = \operatorname{Spec}(\mathcal{O}_{F_0,(p)})$, there exist nonempty strata labeled by cusp labels for $\mathsf{M}_{\mathcal{H}^p}$ of all possible ranks $0 \leq r \leq b$. (The theorem shows the more refined nonemptiness for strata labeled by cusp labels, not just by ranks.)

The next example shows that we cannot expect Theorem A.2.2 to be true without the requirement (in Assumption A.2.1) that $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ involves no factor of type D.

Example A.7.2. Suppose $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is a central division algebra D over a totally real field F as in [11, Prop. 1.2.1.13] such that $D \underset{F,\tau}{\otimes} \mathbb{R} \cong \mathbb{H}$, the real Hamiltonian quaternion algebra, for every embedding $\tau : F \to \mathbb{R}$, with $\star = \diamond$ given by $x \mapsto x^{\diamond} := \operatorname{Tr}_{D/F}(x) - x$. Suppose that D is nonsplit at strictly more than two places. Suppose L is chosen such that $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \cong D^{\oplus 2}$. By the Gram–Schmidt process as in [11, Sec. 1.2.4], and by [11, Cor. 1.1.2.6], there is up to isomorphism only one isotropic skew-Hermitian pairing over $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. But we do know the failure of Hasse's principle (see [8, §7, p. 393]) in this case (see [15, Rem. 10.4.6]), which means there exists a choice of $(L, \langle \cdot , \cdot \rangle)$ as above that is globally anisotropic but locally isotropic everywhere. Thus, even when $k(s) \cong \mathbb{C}$, there exists some connected component U of $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}$ and some nonzero cusp label $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$ for $\mathsf{M}_{\mathcal{H}}$ such that $U_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]} = \emptyset$.

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