# Section 1.2, Solutions of Some Differential Equations (expanded)

## Types of Solutions

A solution to a differential equation may be represented in different forms, often depending on the
method used to obtain it.

**Analytical Solutions:** An **analytical representation** of a solution may take one of two forms:
1. In the **explicit form** $y = f(t)$, the dependent variable is completely isolated and appears only to
the first power on one side of the equation. The other side of the equation is an expression
involving only the independent variable t and constants.

$y(t) = Ce^{2t}$ is an explicit solution, but $y(t) = \dfrac{C}{\sqrt{y-t}}$ is not.

2. The **implicit form** is an equation $h(t,y) = 0$ involving both the dependent and independent
variables *but no derivatives*. In this form the dependent variable $y$ is not expressly given as a
function of the independent variable $t$. We assume that the implicit form is satisfied by at least
one function that also satisfies the differential equation.

The equation $t^2 + y^2 = C$ represents an implicit solution to the differential equation $\dfrac{dy}{dt} = -\dfrac{t}{y}$.

In general the analytic solution of a first-order o.d.e. will have one arbitrary constant, the
solution of a second-order o.d.e. will have two arbitrary constants, while the solution of an nth-
order o.d.e. will have n arbitrary constants. We refer to these as **one-parameter, two-
parameter, or n-parameter solutions**. Because the arbitrary constants (parameters) can take on
infinitely many values, these
one-, two-, or n-parameter solutions represent *families* of solutions.

A **general solution** to an nth-order o.d.e. is an n-parameter analytic solution (expressed explicitly or
implicitly) that contains all possible solutions over an interval I.   All linear nth-order o.d.e.s
have general solutions.

*Example 1.* Solve the o.d.e. $y''(t) = 6t + 2$ to obtain a 2-parameter general solution.

**Graphical Solutions:** A **graphical solution of a first-order o.d.e.** is a curve whose slope at any
point is the value of the derivative there as given by the differential equation.

Graphical solutions may be *quantitative* in nature; i.e., the graph may be sufficiently precise so that the values of the solution function can be read directly from the graph.

Graphical solutions may be *qualitative* in nature where the graph is imprecise as far as numerical values are concerned yet still revealing of the general shape and features of the solution curves.

Graphical solutions can be produced in different ways: from a table of numerical values, by plotting an analytic solution, or by using a *direction* or *tangent field* of the differential equation.

**Numerical Approximations**: A solution to a differential equation may also be approximated numerically. In this case the form of the solution is *a sequence or table of values* of the dependent variable $y$ for a *preselected sequence of values* of the independent variable $t$.

**Initial Value Problems**: An **nth-order Initial Value Problem (I.V.P.)** consists of an nth-order o.d.e.
$$y^{(n)} = f(t, y, y', y'', ..., y^{(n-1)})$$ together with $n$ initial conditions,
$$y'(t_0) = y'_0, \ y''(t_0) = y''_0, \ \cdots, \ y^{(n-1)}(t_0) = y_0^{(n-1)}$$
The **solution of an nth-order I.V.P.** is a **specific solution** where the arbitrary constants have been assigned number values in order to satisfy the initial conditions. This solution must be continuous on an interval containing the initial $t$ value, $t_0$, and must have the value $y_0$ at $t_0$.

*Example 2.* Solve the initial value problem $y''(t) = 6t + 2, \ y(1) = 1, \ y'(1) = 2$.

The graphical solution of a first-order I.V.P. will pass through the point of the initial condition $(t_0, y_0)$. Observe that the solution to Example 2 passes through the point $(1,1)$. What is the slope of the tangent line to the solution at the point $(1,1)$?

*Example 3.* a) Solve the differential equation, $y'(t) = 2t$ to obtain a one-parameter family of curves. Graph several representative curves of this family. Note that the solution of $y'(t) = 2t$ represents an infinite family of solutions. Geometrical representations of the infinite family of curves are called **integral curves**.

b) Solve the initial value problem, $y'(t) = 2t,\ y(2) = 1$ to find a specific solution. Sketch the integral curve for this specific solution. Sketch two other integral curves that do *not* pass through the point (2, 1).

**More Model Examples**

*Example 4. Compound Interest*: When interest is compounded continuously, the rate of change of the principal is proportional to the principal. The constant of proportionality is called the *interest rate*.

a) Set up a differential equation to model the principal $y(t)$ in an account accruing interest at 8% per year, compounded continuously.

b) How does the model change if money is deposited into the account at the constant rate of $1000 per year?

*Example 5. Radioactive Decay*: The atoms of a radioactive substance tend to decompose into atoms of a more stable substance at a rate proportional to the number $y(t)$ of unstable atoms present. Suppose that the initial amount present of the substance is $y(0)=y_0$.

    a)  Set up an initial value problem to model radioactive decay.

    b)  If the solution of the differential equation of part a) is $y(t) = y_0 e^{rt}$, find the value of $r$ if we know that the half-life is 10 years.

    c)  Sometimes you will see the solution to a radioactive decay model written with a negative exponent as $y(t) = y_0 e^{-rt}$. Does the negative exponent change the final solution result?

*Example 6.* A drug is absorbed by the body at a rate proportional to the amount $y(t)$ present in the bloodstream. Suppose that initially there is no drug in the bloodstream but at time $t = 0$ the patient begins to receive the drug intravenously at the constant rate of 15 milligrams per hour. The drug is absorbed at the rate of $0.5y(t)$ per hour.

a)  Set up an I.V.P. (differential equation plus initial condition) to model this situation.

b)  Find and classify the equilibrium solution of the differential equation.

c)  Express the long-term behavior (or limiting value) of the solution below as a limit.
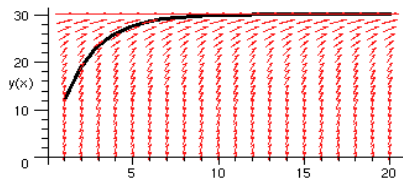


d)  Sketch a phase line for the model
e)  Suppose that 10 mg of the drug is initially in the blood stream. Will this change the long-term behavior of the drug in the blood?

Phase line