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TRANSVERSALITY OF GENERIC PROJECTIONS AND SEMINORMALITY OF THE IMAGE HYPERSURFACES

Joel Roberts * and Rahim Zaare-Nahandi **

1. Introduction

Let X be a variety defined over an algebraically closed field, k. We will say that X is *seminormal* if $\mathcal{O}_{X,x}$ is seminormal in its integral closure for every $x \in X$. For the definition and basic properties of seminormal rings we will refer to [14] and [5] but it seems worthwhile to mention one characterization of seminormal varieties which follows from [14, 1.1]. Saying that X is seminormal is equivalent to saying that if $\nu = g \circ f$ is a factorization of the normalization map ν: $\tilde{X} \to X$ such that f and g are finite and birational but g: $Y \to X$ is not an isomorphism, then either g is not bijective (as a map of topological spaces) or there exists $y \in Y$ such that the extension of residue fields $k(x) \to k(y)$, where $x = g(y)$, is nontrivial. The plane curve $y^2 = x^3$ is the simplest example of a variety which is *not* seminormal.

Bombieri [2], and Andreotti and Holm [1, p. 91], have asked whether a projective variety which is the image of a nonsingular variety, under a generic projection, is seminormal. Greco and Traverso [5, Theorem 3.7] proved that if $X \subset \mathbb{P}^n_{\mathbb{C}}$ is a nonsingular r-dimensional projective variety and π: $X \to \mathbb{P}^{r+1}$ is a generic projection, then $X' = \pi(X)$ is seminormal. They also proved [5, Theorem 3.5] that if X is a projective variety over an arbitrary algebraically closed field k, then X is birationally equivalent to a seminormal hypersurface. In the proof of this result, they studied generic projections of a suitable projective embedding of the normalization of X. But they observed that it was not known whether a generic projection of an *arbitrary* embedding of X would yield a seminormal hypersurface. (Of course, this was an open question only when char$(k) > 0$.) Our first main result answers that question.

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THEOREM 1.1: *Let $X \subset \mathbb{P}^n$ be a normal projective variety, defined over an algebraically closed field k, and let $r = \dim(X)$. If $\pi: X \to \mathbb{P}^{r+1}$ is a generic projection and $X' = \pi(X)$, then $\pi: X \to X'$ is finite and birational, and X' is a seminormal hypersurface.*

The proof of this result uses some geometric properties of the singular locus of X'. Thus, if Y is a hypersurface in $\mathbb{P}^{r+1}$ and x is a closed point of Y, we say that Y is *analytically bihyperplanar* at x if $\hat{\mathcal{O}}_{Y,x} \cong k[[x_1,\ldots,x_{r+1}]]/(x_1x_2)$.

THEOREM 1.2: *Let $X \subset \mathbb{P}^n$ be as in Theorem 1.1, and let π: $X \to X' = \pi(X) \subset \mathbb{P}^{r+1}$ be a generic projection. Then $\mathrm{Sing}(X') = \pi(\mathrm{Sing}\ X) \cup V$, where V is purely of dimension $r-1$, and X' is analytically bihyperplanar at every closed point of some dense open subset of V.*

We will prove Theorem 1.2 in Section 3, as a fairly direct consequence of Proposition 3.1. Here, we will use Theorem 1.2 to prove Theorem 1.1. The main ideas of our proof are exactly the same as in the proofs of Theorems 3.5 and 3.7 of [5].

PROOF OF THEOREM 1.1: By [5, Corollary 2.7 (vi)], a hypersurface $X' \subset \mathbb{P}^{r+1}$, or more generally a variety X' whose local rings satisfy Serre's criterion S_2 [9, §17.I], is seminormal if and only if $\mathcal{O}_{X',\xi}$ is seminormal for every point ξ of codimension 1. Since $\mathrm{Sing}(X)$ has dimension $\leqslant r-2$, it will suffice to check seminormality in the case where ξ is the generic point of a component of V. Since a localization of a seminormal ring is seminormal [5, Corollary 2.2], it is enough to show that $\mathcal{O}_{X',x'}$ is seminormal for some closed point x' of each irreducible component of V. Now $\hat{\mathcal{O}}_{X',x'} \cong k[[x_1,\ldots,x_{r+1}]]/(x_1x_2)$ for every x' in some dense open subset of V, and it is well known that this latter ring is seminormal. But seminormality of $\hat{\mathcal{O}}_{X',x'}$ implies seminormality of $\mathcal{O}_{X',x'}$ [5, Corollary 1.8], so this completes the proof.

In Section 2 we will prove some results about embedded projective varieties, which are needed in the proofs of Theorem 1.2 and Proposition 3.1. Those two results are proved in Section 3.

We will say that a ring A is *seminormal* if it is a Mori ring (i.e. A is reduced and the integral closure is finitely generated as an A-module), and coincides with $+A$, the seminormalization of A in its integral closure. We will say that a scheme is *seminormal* if all of its local rings are seminormal. In Section 4, we consider a hypersurface $X' \subset \mathbb{P}^{r+1}$ whose normalization is a nonsingular variety. Let π: $X \to X'$ be the normalization map and let $D \subset X'$ be the double locus, defined as in [13]. Proposition 4.1 says, among other things, that if $\Delta = \pi^{-1}(D)$ is a seminormal scheme, then so is D. When X' is a surface, this implies that the singular points of D must be either nodes or triple points with independent tangent lines. (See Corollary 4.2.) We conclude Section 4 by giving an example which shows that both possibilities can actually occur.

A proof of Theorem 3.1 appeared in [11], in a slightly different form, but the connection with seminormality was not even suspected at the time. Theorem 1.1 was proved in [15] by the same methods used here, but it was assumed there that X is nonsingular. A version of the results of Section 4 also appeared there. Other main results of [15] pertain to seminormality for strongly generic projections $\pi: X \to X' = \pi(X) \subset \mathbb{P}^m$, where $m > (3\dim(X) - 4)/2$ and X is nonsingular. Those results will be published as a revised version of [15].

2. Strange varieties

If $t \geqslant 0$, then an embedded variety $X \subset \mathbb{P}_k^n$ is said to be *t-strange* if there is a t-dimensional linear subspace $L \subset \mathbb{P}_k^n$ with $L \subset t_{X,x}$ for every closed point $x \in X$. (Here, and in the rest of this paper, $t_{X,x}$ denotes the *embedded* tangent space.) If $\mathrm{char}(k) = 0$, the only t-strange varieties are cones. But if $\mathrm{char}(k) > 0$, it is easy to construct examples of t-strange varieties which are not cones, for any $t < \dim(X)$. A variety which is 0-strange is often said to be *strange.*

LEMMA 2.1: *Let X be a closed subvariety of $\mathbb{P}^n$, and set $r = \dim(X)$.*

(a) *If X is nonsingular in codimension* 1, *then X is not $(r-1)$-strange.*

(b) *If X is not $(r-1)$-strange and X is not a hypersurface in some $\mathbb{P}^{r+1} \subset \mathbb{P}^n$, then $\dim(t_{X,x} \cap t_{X,y}) < r - 1$ for all (x, y) in some dense open subset of $X \times X - \Delta$. (Here, Δ is the diagonal.)*

(c) *If X is not $(r-1)$-strange, then the line $\overline{xy}$ meets X only at x and y (and is not tangent to X) for all (x, y) in some dense open subset of $X \times X - \Delta$.*

PROOF: For a proof of (c), we refer to [7, Theorem 2.5] or [8, Lemma 15]. To prove (a) we observe that if $Y = X \cap L$, where $L \subset \mathbb{P}_k^n$ is an $(n - r + 1)$-subspace in general position, then Y is a nonsingular curve and $Y \not\subseteq \mathbb{P}^2$. If X were $(r-1)$-strange, then Y would be a strange curve. But the only nonsingular strange curves are straight lines $\mathbb{P}_k^1 \subset \mathbb{P}_k^n$ and also (nonsingular) plane conics if $\mathrm{char}(k) = 2$. (See [6, Chapter IV, Theorem 3.9].) Both are impossible since $Y \not\subseteq \mathbb{P}^2$.

To prove (b) we fix 2 nonsingular closed points $x_1, x_2 \in X$ such that $L = t_{X,x_1} \cap t_{X,x_2}$ is $(r-1)$-dimensional. Then $M = \mathrm{Span}(t_{X,x_1}, t_{X,x_2})$ is $(r+1)$-dimensional. If the conclusion of (b) were false, then for every $x \in \mathrm{Reg}(X) = \{\text{nonsingular points of } X\}$ the inequalities $\dim(t_{X,x} \cap t_{X,x_i}) \geqslant r - 1$, $i = 1, 2$, imply that either (1) $t_{X,x} \supset L$ or (2) $t_{X,x} \subset M$. But the set of points satisfying each condition is closed. Irreducibility of $\mathrm{Reg}(X)$ implies that X must be $(r-1)$-strange or else that $\mathrm{Reg}(X) \subset M$, which would given $X \subset M = \mathbb{P}^{r+1}$. This would contradict the hypothesis.

Q.E.D.

3. Incidence conditions and generic projections

Consider a variety $Y \subset \mathbb{P}_k^m$ and a closed point $x \in Y$, and let $r = \dim(Y)$. We will say that Y is *transversally biplanar at* x (*in the weak sense*) if the tangent cone $C = \operatorname{Spec}(\operatorname{gr}(\mathcal{O}_{Y,x}))$ is the union of two r-dimensional linear spaces, $C = L_1 \cup L_2$, with $\dim(L_1 \cap L_2) = 2r - m$. We will say that Y is *transversally biplanar at* x *in the strong* (*or analytic*) *sense* if, in addition, $\hat{\mathcal{O}}_{Y,x} \cong \hat{\mathcal{O}}_{C,x}$. If M_1 and M_2 are linear subspaces of $\mathbb{P}^n$, then $\operatorname{Span}(M_1, M_2)$ is defined to be the smallest linear subspace containing both of them.

PROPOSITION 3.1: *Let X be an irreducible closed subvariety of $\mathbb{P}^n$; assume that X is not $(r-1)$-strange, where $r = \dim(X)$. Suppose that $\operatorname{Span}(t_{X,x}, t_{X,y})$ is d-dimensional for (x, y) in a dense open subset of $X \times X - \Delta$. If $r + 1 \leqslant m \leqslant d - 1$, π: $X \to \mathbb{P}^m$ is a generic projection, and $X' = \pi(X) \subset \mathbb{P}^m$, then $\operatorname{Sing}(X') = \pi(\operatorname{Sing} X) \cup V$, where:*

(i) *V is closed in X' and of pure dimension $2r - m$;*

(ii) *X' is transversally biplanar in the analytic sense at every closed point of some dense open subset of V.*

REMARK: Since $d \geqslant r + 2$, the assumption that X is not $(r-1)$-strange follows from the other hypotheses, by Lemma 2.1. It is also interesting to note that if $\operatorname{char}(k) = 0$, then d is the dimension of $\operatorname{Sec}(X)$, the variety of secant lines of X. (See [4, Lemma 2.1] or [16]). Thus, if $\operatorname{char}(k) = 0$, it follows that $\operatorname{Sing}(X') = \pi(\operatorname{Sing} X)$ for $m \geqslant d$. In particular, if $\operatorname{char}(k) = 0$ and X is nonsingular, then X' is either nonsingular or else transversally biplanar at every point of some dense open subset of $\operatorname{Sing}(X')$.

Before proving the proposition, we will use it to prove Theorem 1.2.

PROOF OF THEOREM 1.2: A normal variety is nonsingular in codimension 1, so Lemma 2.1 implies that X is not $(r-1)$-strange. By the same Lemma, we also conclude that $\operatorname{Span}(t_{X,x}, t_{X,y})$ has dimension $\geqslant r + 2$ for (x, y) in a dense open subset of $\operatorname{Reg}(X) \times \operatorname{Reg}(X) - \Delta$, where $\operatorname{Reg}(X)$ is the set of nonsingular points of X and Δ is the diagonal. Taking $m = r + 1$ in Proposition 3.1, we immediately deduce the conclusions of Theorem 1.2.

PROOF OF PROPOSITION 3.1: There is a dense open subset $U \subset \operatorname{Reg}(X) \times \operatorname{Reg}(X) - \Delta$ such that if $(x, y) \in U$, then

(a) $\operatorname{Span}(t_{X,x}, t_{X,y})$ is d-dimensional;

(b) $\overline{xy} \cap X = \{x, y\}$, and $\overline{xy}$ is not tangent to X.

Here, $\overline{xy}$ is the line joining x and y. We set $Y = X \times X - (\Delta \cup U)$. This notation will be used throughout the proof.

It is known that the set of $(n - m - 1)$-subspaces $L \subset \mathbb{P}^n$, with $L \cap X$

$=\emptyset$, is open in the Grassmann variety $G = G(n, n-m-1)$. It is also fairly well known that each of the following additional properties $P_0, \ldots, P_4$ of the projection $\pi = \pi_L : X \to \mathbb{P}^m$ holds for all L in some dense open subset of G.

(P_0) $\pi_L : X \to X' = \pi(X) \subset \mathbb{P}^m$ is birational.

(P_1) $D_\pi = \{(x, y) \in X \times X - \Delta \mid \pi(x) = \pi(y)\}$ has pure dimension $2r - m$, or is empty.

(P_2) $D'_\pi = \{(x, y) \in Y \mid \pi(x) = \pi(y)\}$ has dimension $< 2r - m$.

(P_3) $S_\pi = \{x \in \mathrm{Reg}(X) \mid L \cap t_{X,x} \neq \emptyset\}$ has pure dimension $2r - m - 1$, or is empty.

(P_4) If T_π consists of all non-collinear triples (x, y, z) such that $\pi(x) = \pi(y) = \pi(z)$, then T_π has pure dimension $3r - 2m$, or is empty.

For (P_0), see [12, Proposition 3]. For (P_1) and (P_2) we consider the closed subsets Z, Z' of $(X \times X - \Delta) \times G$, where $Z = \{(x, y, L) \mid L \cap \overline{xy} \neq \emptyset\}$, and $Z' = Z \cap (Y \times G)$. The methods of [12] can be used to show that Z has pure dimension $= 2r - m + \dim(G)$, while $\dim(Z') < 2r - m + \dim(G)$. So if q: $Z \to G$ and q': $Z' \to G$ are induced by the projection $X \times X \times G \to G$, then $\dim q^{-1}(\lambda) = 2r - m$ (or $q^{-1}(\lambda) = \emptyset$) and $\dim q'^{-1}(\lambda) < 2r - m$ for all λ in a dense open subset of G, which is what we need for (P_1) and (P_2). For (P_3) we consider the general fiber of the projection $\Sigma \to G$, where $\Sigma = \{(x, L) \subset \mathrm{Reg}(X) \times G \mid L \cap t_{X,x} \neq \emptyset\}$. In particular, the methods of [12] imply that Σ is a closed subset of $\mathrm{Reg}(X) \times G$, and that $\dim(\Sigma) = 2r - m - 1 + \dim(G)$. Finally, if U_3 consists of all $(x, y, z) \in X \times X \times X$ such that x, y, and z are not collinear, and

$$Z_3 = \{((x, y, z), L) \in U_3 \times G \mid \dim(L \cap \mathrm{Span}(x, y, z)) \geqslant 1\}$$

then one can use the methods of [12] to show that Z_3 is closed in $U_3 \times G$ and of pure dimension $= 3r - 2m + \dim(G)$. So we study the general fiber of the projection $Z_3 \to G$ to complete the verification.

Thus, it follows that if L is chosen from a dense open subset of G, then π_L: $X \to X' \subset \mathbb{P}^m$ is finite and birational, that D_π has pure dimension $2r - m$ (or is empty), and that S_π, T_π and D'_π have strictly smaller dimension. Let $M_2 = M_{2,\pi} = p_1(D_\pi)$, $M'_2 = p_1(D'_\pi)$, and $M_3 = p_1(T_\pi)$ where p_1 is the projection of $X \times X$, or $X \times X \times X$, to the first factor. It is clear that $x \in M_2$ if and only if there exists $y \in X$, with $y \neq x$ and $\pi(y) = \pi(x)$, and that $x \in M_3$ if and only if there exist $y, z \in X$ such that x, y, and z are not collinear but $\pi(x) = \pi(y) = \pi(z)$. Therefore $M_3 \subset M_2$. Similarly $M'_2 \subset M_2$. We can also show that $S_\pi \subset \overline{M}_2$, the closure of M_2 in X. In fact, if Z is defined as before and "cl( )" denotes closure in $X \times X \times G$, then $(x, x, L) \in \mathrm{cl}(Z)$ if and only if $L \cap t_{X,x} \neq \emptyset$, by [12, Proposition 5]. This gives $\delta(S_\pi) \times \{L\} \subset q^{-1}(\{L\})$, where δ sends $x \to (x, x)$ and q: $\mathrm{cl}(Z) \to G$ extends q: $Z \to G$. By considering dimensions, we find $\delta(S_\pi) \subset \overline{D}_\pi$, so that $S_\pi \subset \overline{M}_{2,\pi}$. Therefore, if L is chosen as above, we conclude that

$\mathrm{Sing}(X') = \pi(\mathrm{Sing}\ X) \cup V$, where $V = \pi(\overline{M}_2)$, and that:

(i) $V \subset X'$ is a closed subset of pure dimension $2r - m$;

(ii) for each closed point x' in some dense open subset of V, $\pi^{-1}(x')$ consists of exactly 2 closed points of X, and both of the corresponding analytic branches of X' at x' are simple.

(Recall that the *analytic branches of X' at x'* are the minimal prime ideals $\mathfrak{p} \subset \hat{\mathcal{O}}_{X',x'}$ and that if $\pi^{-1}(x')$ consists of normal points of X, then there is a bijection $\pi^{-1}(x') \leftrightarrow \{$analytic branches of X' at $x'\}$ under which $x \in \pi^{-1}(x')$ corresponds to the ideal $\mathrm{Ker}(\hat{\mathcal{O}}_{X',x'} \to \hat{\mathcal{O}}_{X,x})$; see [12, §3] or [10, pp. 394–5]. We say that $\mathfrak{p}$ is *simple* when $\hat{\mathcal{O}}_{X'/x'}/\mathfrak{p}$ is a regular local ring. We know that $\mathfrak{p}$ is simple if and only if $L \cap t_{X,x} = \emptyset$, by [12, Proposition 3].)

To complete the proof, we will show that if L is chosen from a (possibly smaller) dense open subset of $G = G(n, n-m-1)$, then $X' = \pi(X)$ will be transversally biplanar in the analytic sense at every closed point x' of some dense open subset of V. If $\pi_L^{-1}(x') = \{x_1, x_2\}$, then this condition will be satisfied if:

(a) x_1 and x_2 are nonsingular points of X,

(b) $L \cap t_{X,x_i} = \emptyset$ for $i = 1, 2$, and

(c) $\mathrm{Span}(\pi_L(t_{X,x_1}), \pi_L(t_{X,x_2})) = \mathbb{P}^m$.

(For (a) and (b), see [12, Proposition 3 and Lemma 3]. For (c) one considers the natural homomorphism $j^{\#} : \hat{\mathcal{O}}_{\mathbb{P}^m,x'} \to \hat{\mathcal{O}}_{X',x'}$ coming from j: $X' \hookrightarrow \mathbb{P}^m$. Then (c) says that $(j^{\#})^{-1}(\mathfrak{p}_1)$ and $(j^{\#})^{-1}(\mathfrak{p}_2)$ are generated by *disjoint* subsets of a system of parameters). It is also easy to check the geometric statement:

$$\pi_L\big(\mathrm{Span}(t_{X,x_1}, t_{X,x_2})\big) = \mathrm{Span}\big(\pi_L(t_{X,x_1}), \pi_L(t_{X,x_2})\big). \qquad (*)$$

We recall that $Z \subset (X \times X - \Delta) \times G$ consists of all (x, y, L) such that $L \cap \overline{xy} \neq \emptyset$, and we set $Z_0 = Z \cap (U \times G)$ where U is the open set mentioned at the beginning of the proof. Let Z_1 consist of all $(x, y, L) \in Z_0$ such that $\dim(L \cap M_{xy}) > d - m - 1$, where $M_{xy} = \mathrm{Span}(t_{X,x}, t_{X,y})$. Lemma 8 of [12] implies that Z_1 is a closed subset of Z_0. Moreover, if p: $Z_0 \to U$ sends $(x, y, L) \to (x, y)$, then $p^{-1}(x, y) \cap Z_1 \subsetneq p^{-1}(x, y)$ for each $(x, y) \in U$. To see this, observe that if $(x, y) \in U$ then there exists an $(n-m-1)$-space L with $L \cap \overline{xy} \neq \emptyset$ and $\dim(L \cap M_{xy}) = d - m - 1$, because $m + 1 \leqslant d$. For such an L we have $(x, y, L) \in p^{-1}(x, y) - Z_1$. We can also observe that $p^{-1}(x, y)$ is irreducible and of dimension $=$ $\dim(G) - m$. (In fact, $p^{-1}(x, y)$ is isomorphic to the Schubert variety $S = \{L | L \cap \overline{xy} \neq \emptyset\}$.) Thus, $\dim(p^{-1}(x, y) \cap Z_1) < \dim(p^{-1}(x, y))$ for all (x, y), and we conclude that $\dim(Z_1) < \dim(Z) = \dim(G) + 2r - m$. Therefore, the following property holds for all L in some dense open subset of G.

(P_5) $D_\pi'' = \{(x, y) \in U | \pi(x) = \pi(y)$ and $\dim(L \cap M_{xy}) > d - m - 1\}$ has dimension $< 2r - m$.

Consequently, L can be chosen so that $(P_0),\ldots,(P_5)$ all hold. With such a choice of L we have, for all (x_1, x_2) in a dense open subset of D_π:

$$\dim\big(\operatorname{Span}(t_{X,x_1}, t_{X,x_2})\big) = d, \quad \text{and}$$

$$\dim\big(L \cap \operatorname{Span}(t_{X,x_1}, t_{X,x_2})\big) = d - m - 1.$$

Because of $(*)$ these conditions imply condition (c), above. But conditions (a) and (b) also hold for all (x_1, x_2) in a dense open subset of D_π. Therefore, X' is transversally biplanar (in the analytic sense) at every point of some dense open subset of V. Q.E.D.

4. Seminormality of hypersurfaces with nonsingular normalization

Let X be a complete nonsingular variety of dimension r. Let π: $X \to X'$ be a finite birational morphism, where X' is a hypersurface in $\mathbb{P}_k^{r+1}$. Thus X is the normalization of X'. Moreover there is an exact sequence of sheaves on X':

$$0 \to \mathcal{O}_{X'} \to \pi_*\mathcal{O}_X \to (\pi_*\mathcal{O}_X)/\mathcal{O}_{X'} \to 0.$$

Let $\mathscr{C}$ be the conductor of $\mathcal{O}_{X'}$ in $\pi_*\mathcal{O}_X$. By definition $\mathscr{C}$ is the largest sheaf of ideals in $\mathcal{O}_{X'}$ which annihilates $(\pi_*\mathcal{O}_X)/\mathcal{O}_{X'}$. The *double locus* of X' is the closed subscheme $D \subset X'$ whose structure sheaf is $\mathcal{O}_{X'}/\mathscr{C}$. Then $x \in D$ if and only if $\mathcal{O}_{X',x}$ is not normal. Since X' has nonsingular normalization, $x \in D$ if and only if x is a singular point of X'. D is a Cohen-Macaulay scheme of pure dimension $r-1$ [13, Theorem 3.1]. Since π is an affine morphism, the conductor $\mathscr{C} \subset \mathcal{O}_{X'}$ lifts to a sheaf of ideals in $\mathcal{O}_X$ [6, page 163]. Let Δ be the closed subscheme of X whose structure sheaf is $\mathcal{O}_X/\mathscr{C}$. Then $\Delta = \pi^{-1}(D)$. Δ is called *the inverse image of the double locus of X' by π, or, the double locus of π.*

PROPOSITION 4.1: *With the notation as above we have*:

(a) *X' is a seminormal variety if and only if Δ is a reduced subscheme of X.*

(b) *If Δ is seminormal, then D is seminormal.*

PROOF: Being reduced and being seminormal both are local properties. Let $V = \operatorname{Spec}(A)$ be an affine open subset of X', and let $U = \pi^{-1}(V) = \operatorname{Spec}(\bar{A})$, where $\bar{A}$ is the integral closure of A. Let C be the conductor of A. Then (a) amounts to saying that A is seminormal if and only if $\bar{A}/C$ is a reduced ring. Since A is S_2, the assertion of (a) follows by [14, Lemma 1.3] and [5, Corollary 2.7].

To prove (b), we show that D is covered by seminormal affine open sets. The scheme structures of $D \cap V$ and $\Delta \cap U$ are given by C taken as an ideal of A and $\bar{A}$ respectively. By (a), A is seminormal. Thus A/C is reduced. A/C is a finitely generated k-algebra, hence A/C is a Mori ring [17, Vol. I, Page 267, Theorem 9]. $\bar{A}$ is a finite A-module, thus $\bar{A}/C$ is an overring of A/C which is a finite A/C-module. Therefore by [5, Proposition 2.5] A/C is seminormal in $\bar{A}/C$. By assumption $\bar{A}/C$ is seminormal. Thus if we show that the integral closure of A/C is a subset of the integral closure of $\bar{A}/C$, A/C would be seminormal, and hence the seminormality of D will follow. Let $C = Q_1 \cap \ldots \cap Q_n$ be the minimal prime decomposition of C in $\bar{A}$. Then $C = (Q_1 \cap A) \cap \ldots \cap (Q_n \cap A) = P_1 \cap \ldots \cap P_m$ is the minimal prime decomposition of C in A for some $m \leqslant n$. The map $A/P \to \bar{A}/Q$ is injective whenever Q lies over P. Thus we have the following commutative diagram:

$$\begin{array}{ccccc}
 & & A/C & \hookrightarrow & \bar{A}/C \\
 & & \downarrow & & \downarrow \\
 & & \prod_{i=1}^{m}(A/P_i) & \hookrightarrow & \prod_{i=1}^{n}(\bar{A}/Q_i) \\
 & & \downarrow & & \downarrow \\
(A/C)^{-} & = & \prod_{i=1}^{m}(A/P_i)^{-} & \hookrightarrow & \prod_{i=1}^{n}(\bar{A}/Q_i)^{-} = (\bar{A}/C)^{-} \\
 & & \downarrow & & \downarrow \\
 & & \prod_{i=1}^{m}k(P_i) & \hookrightarrow & \prod_{i=1}^{n}k(Q_i),
\end{array}$$

where $k(P_i)$ is the residue field $A_{P_i}/P_iA_{P_i}$, and so on. If $x \in \prod_{i=1}^{m} k(P_i)$ and x is integral over $\prod_{i=1}^{m}(A/P_i)$, then $x \in \prod_{i=1}^{n} k(Q_i)$ and it is integral over $\prod_{i=1}^{n}(\bar{A}/Q_i)$. But since $(A/C)^{-} = \prod_{i=1}^{m}(A/P_i)^{-}$ and $(\bar{A}/C)^{-} = \prod_{i=1}^{n}(\bar{A}/Q_i)^{-}$, we have $(A/C)^{-} \subset (\bar{A}/C)^{-}$, as required.

COROLLARY 4.2: *In Proposition* 4.1 *let* $r = 2$. *if* Δ *is seminormal, then* $D \subset \mathbb{P}_k^3$ *is a curve with only nodes and triple points with three independent tangent lines.*

PROOF: This follows by [3, Corollary 4]. Observe also that the only singularities of Δ can be nodes, because X is a nonsingular surface, and thus at every point there is a unique tangent plane, and the tangent lines to the curve Δ lie on this plane, and are linearly independent.

EXAMPLE 4.3: It is well known that if in Corollary 4.2 π is a generic projection, then D is seminormal and its singularities are only ordinary triple points. However by the following simple example D may have nodes when π is just a finite birational morphism.

Let X' be the surface given by the following polynomial over the field of complex numbers,

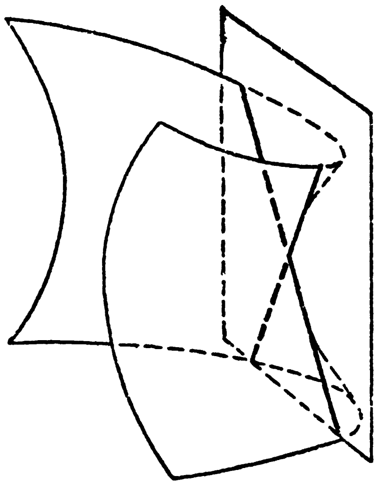

Figure 4.3.1.

$$F(x, y, z) = (1 - y)(1 - y^2 + z^2 - x^2) + (z^2 - x^2)^4.$$

F is irreducible in $\mathbb{C}[x, y, z]$. At $P = (0, 1, 0)$, X' has two analytic branches, and it locally looks like the union of the hyperboloid $1 - y^2 + z^2 - x^2 = 0$ and the plane $1 - y = 0$ (Fig. 4.3.1). A direct calculation of $\partial F/\partial x$, $\partial F/\partial y$, $\partial F/\partial z$ shows that $\mathrm{Sing}(X')$ is the union of the two lines $\{x \pm z = 0, 1 - y = 0\}$. Since the normalization of X' is nonsingular, D is the union of these two lines, and hence D has a node at P. Let $F = F_1F_2$ be the factorization of F in $\mathbb{C}[[x, 1 - y, z]]$, then $\hat{\mathcal{O}}_{X',P} \cong \mathbb{C}[[x, 1 - y, z]]/(F_1F_2)$. The conductor of $\hat{\mathcal{O}}_{X',P}$ as an ideal of $\hat{\mathcal{O}}_{X',P}$ is $(F_1, F_2)/(F_1F_2)$. Thus as an ideal of $\mathbb{C}[[x, 1 - y, z]]/(F_1) \times \mathbb{C}[[x, 1 - y, z]]/(F_2)$, the normalization of $\hat{\mathcal{O}}_{X',P}$, the conductor is $(F_1, F_2)/(F_1) \times (F_1, F_2)/(F_2)$. By isomorphism of power series rings, we have $\mathbb{C}[[x, 1 - y, z]]/(F_1, F_2) \cong \mathbb{C}[[x, z]]/(z^2 - x^2)$. This means that if we let $\pi^{-1}(P) = \{Q_1, Q_2\}$, then Δ has a node at Q_1 and another node at Q_2. In particular Δ is seminormal.

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