Charles University in Prague
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Master's thesis

Products of Fréchet spaces

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Declaration

I declare that I carried out this bachelor (master, doctoral) thesis independently, and only with the cited sources, literature and other professional sources.

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In Prague, 30. 7. 2015

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Přehled

Název práce: Součiny Fréchetovských prostorů
Autor: Miroslav Olšák
Katedra: Katedra algebry
Vedoucí práce: prof. RNDr. Simon Petr, DrSc.
Abstrakt: Práce se zabývá konstrukcemi příkladů k-tice prostorů, jejichž součin nemá Fréchet-Urysohnovu vlastnost, ale všechny menší podsoučiny ji mají. Pro tyto konstrukce jsou použity skoro disjunktní systémy. V práci je zopakována konstrukce Petra Simona dvou kompaktních prostorů s touto vlastností. Pro příklad s více prostory práce zobecňuje pojmy skoro disjunktních systémů do více dimenzí a předvádí konstrukci obecného takového příkladu za pomocí silně úplně separabilního maximálního skoro disjunktního systému. Ten je sestrojen za předpokladu $\mathfrak{s} \leq \mathfrak{b}$, kde $\mathfrak{s}$ je splitting number a $\mathfrak{b}$ je bounding number.
Klíčová slova: topologie, Fréchet Urysohnova vlastnost, součin, AD systém, úplná separabilita

Summary

Title: Products of Fréchet spaces
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Abstract: The article gives a constructions of k-tuples of topological spaces such that the product of the k-tuple is not Fréchet-Urysohn but all smaller subproducts are. The construction uses almost disjoint systems. The article repeats the construction by Petr Simon of two such compact spaces. To achieve more dimensional example there are generalized terms of AD systems. The example is constructed under the assumption of existence of a strong completely separable MAD system. It is constructed under the assumption $\mathfrak{s} \leq \mathfrak{b}$ where $\mathfrak{s}$ is the splitting number and $\mathfrak{b}$ is the bounding number.
Keywords: topology, Fréchet Urysohn property, product, AD system, complete separability

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Introduction

A Fréchet topological space (also Fréchet-Urysohn) is one with the following property: For every subset X of the domain of the space and for any point x in the closure of X there is a sequence of points from X converging to x. For example, spaces with countable characters have this property, namely metric spaces do.

A lot of properties are preserved by making product. The product of Hausdorff spaces is a Hausdorff space, the product of compacts is a compact, the countable product of metrizable spaces is metrizable. However it is not the case of the Fréchet property. We will show a simple counter-example.

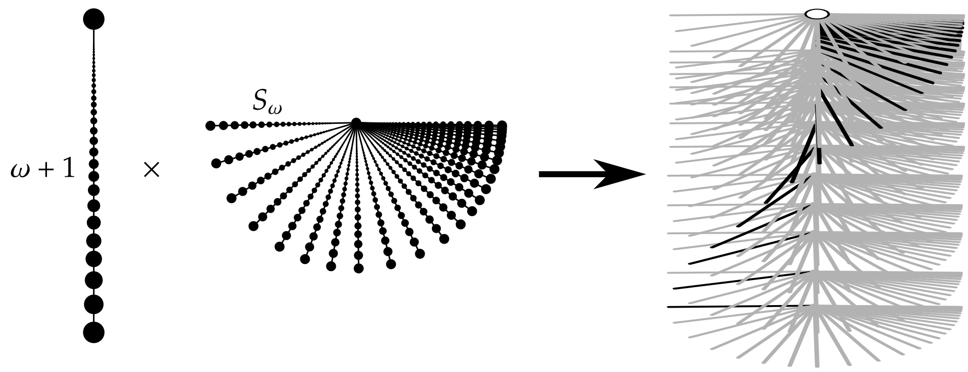

Figure 0.1. A non-Fréchet product of two Fréchet spaces.

Consider the space $\omega+1$, i. e. an infinite sequence of isolated points which converges to a point ∞. Further take the space S_ω, i. e. a space with domain $(\omega \times \omega) \cup \{\infty\}$ and with the weekest possible topology such that for each n the sequence (n, i) converges to ∞ when i goes to ∞.

Both these spaces has Fréchet property. Yet their product does not. Consider a subset $X = \{(n, (n, i)) : n, i \in \omega\}$, thus one converging sequence in S_ω is taken in each floor. The closure of X contains all points (n, ∞), so also the point (∞, ∞) is there. On the other hand there is no sequence of points from X converging to (∞, ∞). If a sequence converges to ∞ within the projection to S_ω it intersects one sequence from S_ω at infinitely many points. Therefore the sequence can not converge to ∞ within the projection to $\omega + 1$.

This work investigates under what assumptions such counter-examples can be constructed. Article [1] shows all these counter-examples under the assumption of Martin's axiom 2.17. As usually we use almost disjoint systems for the construction. They are handy for use and the resulting space will be compact as a bonus.

The aim of the work was to examine following questions without Martin axiom:

a) Is there an n-tuple of (compact) Fréchet spaces such that their product is not Fréchet but all products of subtuples of the length $n-1$ are Fréchet?
b) The question a) under the assumption of existence of an infinite completely separable MAD system.
c) Is there a countable system of Fréchet spaces such that their product is not Fréchet but all finite products of spaces from the system are Fréchet?

The question c) in this form is answered by remark 1.3 but it is rather a special case without an opportunity of generalization. The main contribution of the work consists of the examination of the question b). Even a stronger form of the question c) is involved in it. Exact specification of the question is described in definition 1.2. The construction is not found under the assumption of only infinite completely separable MAD system. Despite this it is found under a similar slightly stronger assumption of the existence of infinite strong completely separable MAD system.

Paper [5] describes a construction of an infinite completely separable MAD system under the assumption $\mathfrak{s} \leq \mathfrak{a}$. In the chapter 5 we will show a generalization of this result for the construction of an infinite strong completely separable MAD system under a similar assumption $\mathfrak{s} \leq \mathfrak{a}_\omega$. It is the weekest assumption which is already known.

Paper [2] uses a slightly different approach. It showes a counter-example with two compact spaces without additional set theoretical assumptions. We will repeat this result in the section 3.3 but we are not able to generalize it.

Chapter 1

Basic notation

As usual in set theory, zero is considered as a natural number and each natural n is identified with the set $\{0, 1, \ldots, n-1\}$. Symbol ω denotes the set of all naturals, or equivalently the supremum of all naturals in ordinal numbers. Natural numbers, exceptionally including ω, will be denoted by small letters $i, j, k, n, \ldots$.

Sets of numbers or points will be usually denoted by capital letters. For any set S one can consider some systems of subsets. Systems of sets will be denoted by calligraphic letters. The system of all subsets of S is denoted by $\mathcal{P}(S)$. For example: $\mathcal{A} \subset \mathcal{P}(S)$.

Natural numbers are extended to ordinal numbers. We usually denote them by small Greek letters $\alpha, \gamma, \ldots$. Ordinal ω_1 denotes the first uncountable ordinal and $\mathfrak{c}$ denotes continuum, i. e. the cardinality of $\mathcal{P}(\omega)$.

As usual in set theory, a map $\varphi\colon A \to B$ is identified with its graph, i. e. a subset of the Cartesian product $A \times B$ containing pairs $(a, \varphi(a))$. Namely notation $\psi \subset \varphi$ for map $\psi\colon A_0 \to B_0$ is equivalent to $\varphi \restriction (A \cap A_0) = \psi$. Further we say that maps φ and ψ are compatible if $\varphi \cup \psi$ is still a map, equivalently $\varphi \restriction (A \cap A_0) = \psi \restriction (A \cap A_0)$. In the opposite case we call them incompatible.

A map φ followed by square brackets instead of parentheses denotes the pointwise image. To be precise $\varphi[X] = \{\varphi(x) : x \in X\}$.

For sets or spaces S_0, ..., S_{k-1} consider the Cartesian product

$$S_0 \times \cdots \times S_{k-1} = \prod_{i \in k} S_i.$$

In case of all sets S_i equal we will simply write S^k. In every such case we define simple projections $\pi_0, \ldots, \pi_{k-1}$ as follows.

$$\pi_i\colon \prod_{j \in k} S_j \to S_i, \qquad \pi_i(x_0, \ldots, x_{k-1}) = x_i.$$

The indexing set of simple projections is the same as the indexing set of the product.

Next to simple projections, we also define projections to a set: Assume indexing set I and $J \subset I$. We define a projection

$$\pi_J\colon \prod_{i \in I} S_i \to \prod_{i \in J} S_i$$

preserving all simple projections π_j, where $j \in J$. Finally we define a special case of a projection to the set: For $i \in I$ let $\pi_{\neg i}$ denote $\pi_{I \setminus \{i\}}$.

Topological spaces, as pairs (domain, topology), will be denoted by bold capital letters. For example $\boldsymbol{X} = (S, \tau)$.

Topological spaces will usually contain a special point denoted by ∞. If some such spaces are multiplied there is also a special point $(\infty, \ldots, \infty)$ in the product. For brevity, it will be denoted still just by ∞.

1.1 Fréchet property

We will repeat and specify definitions from the introduction.

Definition 1.1. A topological space $\boldsymbol{X}$ is said to be Fréchet if for any $X \subset \boldsymbol{X}$ and $x \in \overline{X}$ there is a sequence $x_0, x_1, \ldots \in X$ converging to x. More precisely, each neighborhood of x contains all points x_i up to finitely many of them.

Definition 1.2. Let $k > 1$ be a natural number. Then a k-counter-example is defined to be a k-tuple of compact spaces $\boldsymbol{X}_0, \ldots, \boldsymbol{X}_{k-1}$ such that the product $\prod_{i \in k} \boldsymbol{X}_i$ is not a Fréchet space but for any map $\sigma\colon (k-1) \to k$ the product $\prod_{i \in (k-1)} \boldsymbol{X}_{\sigma(i)}$ is a Fréchet space.

Similarly ω-counter-example is a infinite sequence of compact spaces $\boldsymbol{X}_0, \boldsymbol{X}_1, \ldots$ such that the product $\prod_{i \in \omega} \boldsymbol{X}_i$ is not a Fréchet space but for any $k \in \omega$ and any map $\sigma\colon k \to \omega$ the product $\prod_{i \in k} \boldsymbol{X}_{\sigma(i)}$ is a Fréchet space.

Remark 1.3. Assume a weeker ω-counter-example such that

- spaces $\boldsymbol{X}_0, \boldsymbol{X}_1, \ldots$ are not required to be compact,
- the map σ is just the identity on k.

Then the following example would work: Set $\boldsymbol{X}_0$ to be the space S_ω described in introduction. All other spaces $\boldsymbol{X}_i$ will be two element discrete spaces. Then all partial products has the Fréchet property because they are just discrete disjoint unions of Fréchet spaces. Yet the whole product is not Fréchet since $\omega + 1$ is a subspace of $\prod_{0<i<\omega} \boldsymbol{X}_i$.

One may require a special case of a k-counter-example such that all its spaces are equal. Namely a space $\boldsymbol{X}$ such that $\boldsymbol{X}^k$ ($\boldsymbol{X}^\omega$) is not Fréchet but $\boldsymbol{X}^{k-1}$ (all finite powers) is. We will show that the existence of such a spesific counter-example is equivalent to the existence of a general one.

Proposition 1.4. Fix $k \leq \omega$. Assume that a k-counter-example exists. Then there is a k-counter-example in which all spaces are identical.

Proof: Take a k-counter-example $\boldsymbol{X}_0, \boldsymbol{X}_1, \ldots$ There is no lost of generality in assuming pairwise disjoint domains of these spaces. At first, we prove the case $k < \omega$. Consider the sum of these spaces $\boldsymbol{X} = \bigcup_{i \in k} \boldsymbol{X}_i$. Then one can write the $(k-1)$-th power as follows

$$\boldsymbol{X}^{k-1} = \bigcup_{\sigma \in S} \prod_{i \in (k-1)} \boldsymbol{X}_{\sigma(i)},$$

where S denotes the set of all maps from $(k-1)$ to k.

Spaces $\boldsymbol{X}_i$ form a k-counter-example so each summand is a Fréchet space. Also a finite sum of them is a Fréchet space. Yet the power $\boldsymbol{X}^k$ is not a Fréchet space since it contains a non-Fréchet subspace $\prod_{i \in k} \boldsymbol{X}_i$.

It remains to prove the case $k = \omega$. In that case we can again consider the sum of spaces $\boldsymbol{X}_i$ but it is not compact anymore. Therefore we add a point ∞ to the sum such that each neighborhood of ∞ contains all but finitely many $\boldsymbol{X}_i$. It is easily seen that such modification of $\boldsymbol{X}$ forms a compact space and that the power $\boldsymbol{X}^\omega$ is not a Fréchet space. What is left is to show that all finite powers are Fréchet spaces.

Consider a set $X \subset \boldsymbol{X}^n$, where $n \in \omega$. Pick $x \in \overline{X}$. For each coordinate there can be two cases.

(i) $\pi_i(x) \neq \infty$. Then $\pi_i(x) \in \boldsymbol{X}_{\sigma(i)}$ for some $\sigma(i)$. So it suffices to restrict X to an open subset $\pi_i^{-1}[\boldsymbol{X}_{\sigma(i)}]$.

(ii) $\pi_i(x) = \infty$. Then consider the space $\omega + 1$, its domain is $\omega \cup \{\infty\}$, elements of ω are isolated and converging to ∞. Further take a map $f\colon \boldsymbol{X} \to \omega \cup \{\infty\}$ defined as follows:

$$f(y) = \begin{cases} i & \text{if } y \in \boldsymbol{X}_i \\ \infty & \text{if } y = \infty \end{cases}.$$

The sequence $x_0, x_1, \ldots \in \boldsymbol{X}$ converges to $\infty \in \boldsymbol{X}$ if and only if the sequence $f(x_0), f(x_1), \ldots$ converges to $f(\pi_i(x)) = \infty$. So we factorize the space $\boldsymbol{X}$ by the mapping

f obtaining $\omega + 1$. Since the product of spaces $\boldsymbol{X}_i$ is not Fréchet, it is not possible to be isolated for all $\boldsymbol{X}_i$. Therefore we can embed $\omega + 1$ to some $\boldsymbol{X}_{\sigma(i)}$.

By applying this process to all coordinates we transform the original problem to the Fréchet property of a product $\prod_{i \in n} \boldsymbol{X}_{\sigma(n)}$. ∎

Chapter 2

Ideals

In this chapter, we will not consider the requirement of compactness. The property $x \in \overline{X}$ and the existence of a sequence converging to x clearly depends only on the system of neighborhoods of the point x. So we transfer from the language of open sets and topological spaces to the language of ideals. We will explain the relation between these views in this chapter.

Definition 2.1. An ideal on a set S is a system $\mathcal{I} \subset \mathcal{P}(S)$ satisfying:

- For each $I, J \in \mathcal{I}$ it holds $I \cup J \in \mathcal{I}$,
- if $I \in \mathcal{I}$ and $J \subset I$ than also $J \in \mathcal{I}$,
- for any $x \in S$ we have $\{x\} \in \mathcal{I}$.

Remark 2.2. Usually the third condition is not required for ideals. Nevertheless it is useful and not restricting here.

Now we will describe the conversion from ideals to spaces and vice versa.

Definition 2.3. Let $\mathcal{I}$ be an ideal on S, suppose $\infty \notin S$. Then we define $\boldsymbol{X}(\mathcal{I})$ to be a space on domain $S \cup \{\infty\}$ with the following topology: A set U is open if and only if it does not contain ∞ or if $S \setminus U \in \mathcal{I}$.

Definition 2.4. Consider a T_1 space $\boldsymbol{X} = (S, \tau)$ and a point $x \in \boldsymbol{X}$. We define a system $\mathcal{I}(\boldsymbol{X}, x)$ of subsets of $S \setminus \{x\}$ such that $I \in \mathcal{I}(\boldsymbol{X}, x)$ if and only if there is a $U \in \tau$ containing x and being disjoint with I.

Observation 2.5.

- $\boldsymbol{X}(\mathcal{I})$ defines a Hausdorff topological space, the Hausdorff property yields from the third property of ideals,
- $\mathcal{I}(\boldsymbol{X}, x)$ is an ideal, third property follows from the T_1 property of the space,
- for any ideal $\mathcal{I}$ it holds $\mathcal{I}(\boldsymbol{X}(\mathcal{I}), \infty) = \mathcal{I}$,
- let $\boldsymbol{X}$ be a T_1 space with a point ∞. Then $\boldsymbol{X}(\mathcal{I}(\boldsymbol{X}, \infty))$ has identical domain to the one of $\boldsymbol{X}$. Moreover neighborhoods of ∞ are equal in both spaces. Hence following conditions are preserved.
  - The closure of a given set contains point ∞.
  - A sequence $x_0, x_1, \ldots$ converges to ∞.

We see that all key properties for the Fréchet property are preserved. Further we will describe more of ideal terminology.

Definition 2.6. Let $\mathcal{A}$ be a system of subsets of S. The ideal generated by $\mathcal{A}$, denoted by $\langle \mathcal{A} \rangle$, is defined to be the smallest ideal containing $\mathcal{A}$. In other words $\langle \mathcal{A} \rangle$ is defined to be the system of all finite unions of elements of $\mathcal{A}$ together with finitely many extra points and all subsets of such unions.

Definition 2.7. Let $\mathcal{A}$ be a system of subsets of S. The orthogonal complement, denoted by $\mathcal{A}^{\perp}$, is defined to be the system of all such sets which intersects all $A \in \mathcal{A}$ at only finitely many points.

Observation 2.8.

- $\mathcal{A}^\perp$ forms an ideal,
- $\mathcal{A}^\perp = \langle\mathcal{A}\rangle^\perp$.

Moreover the orthogonal complement forms a Galois connection with itself via relation "have finite intersection". So for instance $(\mathcal{A}^\perp)^\perp \supset \mathcal{A}$. Equality occurs if and only if $\mathcal{A}$ is an orthogonal complement.

Definition 2.9. Let $\mathcal{A}$ be a system of sets. We define orthogonal closure of $\mathcal{A}$, $\overline{\mathcal{A}} = (\mathcal{A}^\perp)^\perp$. An ideal $\mathcal{I}$ is said to be orthogonally closed if $\mathcal{I} = \overline{\mathcal{I}}$.

It remains to realize the correspondence of these concepts with properties of topological spaces.

Observation 2.10. Let $\mathcal{I}$ be an ideal on a set S and $\boldsymbol{X} = \boldsymbol{X}(\mathcal{I})$. Furthermore pick any basis $\mathcal{B}$ of neighborhoods of point ∞ and set $\mathcal{A} = \{S \setminus B : B \in \mathcal{B}\}$. Then following holds:

- $\mathcal{I} = \langle\mathcal{A}\rangle$.
- A sequence $x_0, x_1, \ldots \in S$ converges to ∞ if and only if each element of S occurs only finitely many times in it and if $\{x_i : i \in \omega\} \in \mathcal{I}^\perp$. One can use $\mathcal{A}^\perp$ instead of $\mathcal{I}^\perp$ since $\mathcal{I}^\perp = \mathcal{A}^\perp$.
- The closure of a set $X \subset S$ contains the point ∞ if and only if $X \notin \mathcal{I}$.
- The non-existence of a sequence of points from a set X converging to ∞ is equivalent to the fact that each sequence converging to ∞ intersects X at only finitely many points. Equivalently $X \in \overline{\mathcal{I}}$, or $X \in \overline{\mathcal{A}}$.
- The space $\boldsymbol{X}$ has the Fréchet property if and only if each set X such that $X \notin \mathcal{I}$ satisfy $X \notin \overline{\mathcal{I}}$. Equivalently if $\mathcal{I}$ is orthogonally closed.

So the orthogonal complement provides a nice operation making a Fréchet space from a non-Fréchet one. Unfortunately it is not compatible with products well so we are not going to use it further.

Example 2.11. The spaces mentioned in the introduction can be actually simply constructed by ideals. Let $\mathcal{I}$ be the ideal of finite subsets of ω. Let $\mathcal{B}$ be a disjoint decomposition of a countable domain to infinitely many infinite sets. Then $\omega + 1 \simeq \boldsymbol{X}(\mathcal{I})$ and $S_\omega \simeq \boldsymbol{X}(\mathcal{B}^\perp)$. Obviously, $\mathcal{I}$ is orthogonally closed. The ideal $\mathcal{B}^\perp$ is orthogonally closed just from the fact that the ideal $B^\perp$ is an orthogonal complement. This illustrates that both these spaces are Fréchet.

2.1 Spaces with small character

As mentioned in the introduction, spaces with countable character has the Fréchet property. We will prove this simple observation.

Observation 2.12. Take a space $\boldsymbol{X}$, a subset $X \subset \boldsymbol{X}$ and a point $x \in \overline{X}$. Under the assumption that $\boldsymbol{X}$ has a finite local basis, there exists a sequence $x_0, x_1, \ldots \in X$ converging to x.

Proof: We may choose the local basis of the point x in the form $B_0 \supset B_1 \supset \cdots$. Then it suffices to pick $x_i \in X \cap B_i$. ∎

Accordingly, we may ask about the minimal cardinality of a local basis in a non-Fréchet space. In such general case the first uncountable cardinal will suffice.

Example 2.13. Take the ideal of all countable subsets of ω_1. Its orthogonal complement contains no infinite set. Yet it is generated by the system

$$\{A_\alpha : \alpha \in \omega_1\}, \text{where } A_\alpha = \{\gamma : \gamma < \alpha\}.$$

Note that by the set theoretical view $A_\alpha = \alpha$.

The point ω_1 has equal system of neighborhoods as in the space ω_1+1 with the usual ordinal topology. Hereby we constructed even a compact space with weight ω_1 which is not Fréchet.

We can continue with this example: The space $\omega_1 + 1$ can be embedded to the uncountable power $\{0,1\}^{\omega_1}$. Therefore no uncountable product of at least two-element spaces have the Fréchet property.

So we are interested just in countable spaces and countable products.

Definition 2.14. The symbol $\mathfrak{p}$ (pseudo-intersection number) denotes the smallest weight of a countable non-Fréchet space.

Usual definition of the cardinal $\mathfrak{p}$ is slightly different, yet equivalent.

Definition 2.15. Let $\mathcal{A}$ be a system of sets. A set P is said to be its pseudo-intersection if P is infinite and the set difference $P \setminus A$ is finite for each $A \in \mathcal{A}$.

Observation 2.16. One can define the cardinality $\mathfrak{p}$ by the following way. The number $\mathfrak{p}$ is the smallest possible cardinality of a system $\mathcal{A}$ of subsets of ω such that the intersection of any finite sets of it is non-empty but there is no pseudo-intersection of whole $\mathcal{A}$.

Axiom 2.17. (Martin's) Let P be a partially ordered set such that for any uncountable set $C \subset P$ one can find a triple $x, y, z \in P$ such that $x \neq y$, $x, y \in C$, $z \leq x$, $z \leq y$ (the c.c.c. condition).

Furthermore assume $\mathcal{D} \subset \mathcal{P}(P)$ satisfying $|\mathcal{D}| < \mathfrak{c}$ and for any $D \in \mathcal{D}$ and $p \in P$ there is a $q \in D$, $q \leq p$ (the condition of density). Then there is a set $F \subset P$, such that:

(1) For any $p, q \in F$ there is an $r \in F$, $r \leq p$, $r \leq q$.
(2) F intersects every $D \in \mathcal{D}$.

Proposition 2.18. Under the assumption of Martin's axiom it holds $\mathfrak{p} = \mathfrak{c}$. In other words for every countable set S and a system of sets $\mathcal{A} \subset \mathcal{P}(S)$ satisfying $|\mathcal{A}| < \mathfrak{c}$ and $S \notin \langle \mathcal{A} \rangle$ there is an infinite set $X \in \mathcal{A}^\perp$.

Proof: We are going to use Martin's axiom. So we construct an appropriate partially ordered set P as follows.

$$P = \{(n, \sigma, \mathcal{K}) : n \in \omega,\ \sigma \text{ is an injective map } n \to S, \mathcal{K} \text{ is a finite subset of } \mathcal{A}\}$$

$$(n, \sigma, \mathcal{K}) \geq (m, \tau, \mathcal{L}) \Leftrightarrow n \leq m,\ \sigma \subset \tau,\ \mathcal{K} \subset \mathcal{L},\ \forall i \in (m \setminus n)\, \forall K \in \mathcal{K}\colon \tau(i) \notin K.$$

We verify the condition c.c.c.: Take uncountable set $C \subset P$. Since S is countable, there are only finitely many functions of a type $n \to S$. Thus there are two elements of the following form in C.

$$x = (n, \sigma, \mathcal{K}), y = (n, \sigma, \mathcal{L}).$$

Then it suffices to use $z = (n, \sigma, \mathcal{K} \cup \mathcal{L})$.

Now we are going to construct the system of dense sets $\mathcal{D}$. It will consist of sets $D_m = \{(n, \sigma, \mathcal{K}) \in P : n \geq m\}$ for all $m \in \omega$ and $D_A = \{(n, \sigma, \mathcal{K}) \in P : A \in \mathcal{K}\}$ for all $A \in \mathcal{A}$. Obviously the number of such sets is less than continuum and all of them are dense. According to Martin's axiom there is a set $F \subset P$ satisfying conditions (1), (2).

Due to the condition (2) the set F contains triples $(n, \sigma, \mathcal{K})$ where n is arbitrarily large. Due to the condition (1) for any pair of elements $(n, \sigma, \mathcal{K}), (m, \tau, \mathcal{L}) \in F$ functions σ and τ are compatible. Therefore the union all such functions from F is an injective map $f\colon \omega \to S$. We will show that the infinite set $X = f[\omega]$ is a member of $\mathcal{A}^\perp$.

Assume $A \in \mathcal{A}$. Due to the condition (2) the set F intersects D_A at a point $(n, \sigma, \mathcal{K})$. Then due to the condition (1) for every i, $(m, \tau, \mathcal{L})$, where $m > n$ and $i \in (m \setminus n)$, elements $\tau(i)$ have to be outside of set A. Therefore the intersection $X \cap A$ is allowed to contain only elements of the set $\sigma[n]$ so there are only finitely many of them. ∎

2.2 Products of ideals

Definition 2.19. Let I be an indexing set and $\mathcal{I}_i$ form a system of ideals on domains S_i, where $i \in I$. We define the product of ideals $\prod_{i\in I} \mathcal{I}_i$ on the domain $\prod_{i\in I} S_i$ to be the ideal generated by sets of the form $\pi_i^{-1}[A]$, where $i \in I$ and $A \in \mathcal{I}_i$.

Observation 2.20. Under assumptions of the previous definition consider $X \subset \prod_{i\in I} S_i$. Then following statements are equivalent.

- $X \in (\prod_{i\in I} \mathcal{I}_i)^\perp$,
- For each $i \in I$ following holds.

  (1) $\pi_i[X] \in \mathcal{I}_i^\perp$,
  (2) no set of the form $\pi_i[S_i]$ intersects X at infinitely many points.

Observation 2.21. The space $\boldsymbol{X}(\prod_{i\in I} \mathcal{I}_i)$ is a subspace of the product of spaces $\prod_{i\in I} \boldsymbol{X}(\mathcal{I}_i)$.

The preceding observation gives a subspace. One may ask if remaining elements may spoil something. The following proposition shows that for the purpose of this work one do not have to worry about them.

Proposition 2.22. Assume an n-tuple of ideals $\mathcal{I}_0, \dots, \mathcal{I}_{n-1}$ on domains $S_0, \dots, S_{n-1}$ respectively. Further for each $I \subset n$ suppose the ideal $\prod_{i\in I} \mathcal{I}_i$ to be orthogonally closed. Then the product of spaces $\boldsymbol{X} = \prod_{i\in n} \boldsymbol{X}_i$ has the Fréchet property.

Proof: The proof is by induction on n. Observation 2.10 gives the validity of the case $n = 1$. Further suppose $n \geq 2$. Consider $X \subset \boldsymbol{X}$ and a point $z \in \overline{X}$. If $\pi_i(z) \neq \infty$ for some $i \in n$ it suffices to restrict everything to an open subset $\pi_i^{-1}(\pi_i(z))$ and the claim follows from the induction assumption. So assume it is not the case and $z = \infty$.

If there is an i such that the closure of the set $X \cap \pi_i^{-1}(\infty)$ contains the point $\infty \in \boldsymbol{X}$ it suffices to restrict everything to the subspace $\pi_i^{-1}(\infty)$ and the induction assumption proves the proposition. So assume the opposite.

Then for
$$X_2 = X \cap \prod_{i\in I} S_i \text{ it holds } \infty \in \overline{X_2}.$$

Thus $X_2 \notin \mathcal{I} = \prod_{i\in I} \mathcal{I}_i$. So by the assumption $X_2 \notin \overline{\mathcal{I}}$. Hence there is an $Y' \in \mathcal{I}^\perp$ having an infinite intersection with X_2. Finally, it gives the desired countable sequence $Y \in \mathcal{P}(X_2) \cap \mathcal{I}^\perp$. ∎

One may ask why the previous proposition demands the orthogonal closedness of all partial products. In fact, the only reason is a degenerated ideal containing the whole domain. Such ideal causes that the product will be also degenerated so it will be orthogonally closed. On the other hand whenever there is a non-Fréchet factor, the product is also non-Fréchet.

The following proposition shows that if we suppressed such ideals it would suffice to check the condition just for the whole product of all ideals.

Proposition 2.23. Assume an orthogonally closed product of ideals $\mathcal{I} \times \mathcal{J}$ on domains $S \times T$. If $T \notin \mathcal{J}$ then $\mathcal{I}$ is orthogonally closed.

Proof: Consider a set $X \subset S$, $X \notin I$. We need to find a set $Y \subset S$ satisfying $Y \in \mathcal{I}^\perp$. The proposition will be proven by it.

Since $T \notin \mathcal{J}$ also $X \times T$ is not a member of the orthogonally closed product $\mathcal{I} \times \mathcal{J}$. Therefore we find an $Y_0 \in (\mathcal{I} \times \mathcal{J})^\perp$ intersecting $X \times T$ at infinitely many points. The desired Y is $\pi_0[Y_0]$. ∎

We have described a tool how to recognize whether a product of spaces $\boldsymbol{X}_i$ is a Fréchet space or not. However this tool is quite week yet, it is actually just a reformulation. A stronger tool for some specific spaces will be described in chapter 4. Now we will show how to achieve a non-Fréchet product of spaces.

Observation 2.24. Let $\mathcal{I}_i$ be ideals on one domain S, where $i \in I$. Then the space $\boldsymbol{X}(\langle \bigcup_{i \in I} I_i \rangle)$ is embeddable into the space $\prod_{i \in I} \boldsymbol{X}(\mathcal{I}_i)$. Namely one can map it on the diagonal i. e. an element $x \in \boldsymbol{X}(\langle \bigcup_{i \in I} I_i \rangle)$ is mapped to the corresponding element $(x, x, \ldots, x)$ on the diagonal.

Corollary 2.25. Let I be an indexing set and $\mathcal{I}_i$ be an ideal on one domain S, where $i \in I$. Suppose

$$S \notin \left\langle \bigcup_{i \in I} I_i \right\rangle$$

and at the same time

$$S \in \overline{\bigcup_{i \in I} I_i}.$$

Then $\prod_{i \in I} \boldsymbol{X}(\mathcal{I}_i)$ is not a Fréchet space.

Chapter 3

AD systems

Ideals describe local properties of a space in general. Further we are going to investigate a specific type of ideals. There is a lost of generality but there are lots of useful properties. For instance it allows to construct compact spaces.

Definition 3.1. Almost disjoint system, briefly AD system, on an infinite domain S is a system of infinite sets $\mathcal{A} \subset \mathcal{P}(S)$ such that each pair of elements of the system has a finite intersection.

Observation 3.2. There is an AD system on a countable set of the cardinality of continuum.

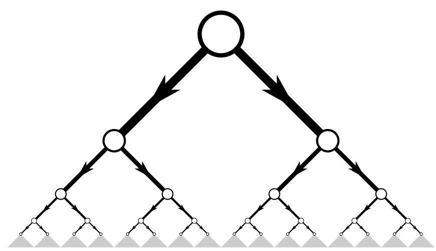

Figure 3.1. AD system of the cardinality $\mathfrak{c}$

Proof: Consider an infinite countable rooted binary tree. There are continuum many paths begining at the root in the tree. Yet each two such paths intersect at only finitely many nodes. So they form an AD system. ∎

3.1 The space

Definition 3.3. Let $\mathcal{A}$ be an AD system on a domain S. Then the space constructed from $\mathcal{A}$, denoted $\boldsymbol{Y}(\mathcal{A})$, is defined as follows. Its domain is the disjoint union $S \cup \mathcal{A} \cup \{\infty\}$ and the topology is the weekest possible such that:

- For every $x \in S$ the set $\{x\}$ is clopen,
- for every $A \in \mathcal{A}$ the set $\{A\} \cup A$ is clopen.

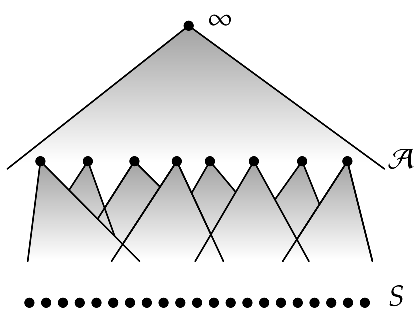

Figure 3.2. The space $\boldsymbol{Y}(\mathcal{A})$.

Proposition 3.4. For any AD system $\mathcal{A}$ the space $\boldsymbol{Y}(\mathcal{A})$ is a compact Hausdorff space.

Proof: The Hausdorff property follows from the fact that each two points can be separated by a clopen set. To verify the compactness it suffices to check it for elements of a subbasis. Take the subbasis consisted of

- sets $\{x\}$, where $x \in S$,
- sets $\{A\} \cup A$, where $A \in \mathcal{A}$,
- complements of sets above.

Consider a coverage by elements of the subbasis. The point ∞ has to be covered by either a complement of a singleton or by a complement of a set of the form $\{A\} \cup A$, where $A \in \mathcal{A}$. The first case is trivial. Let us focus on the second one. There are three possible forms of a set covering the point A.

a) The set $\{A\} \cup A$. Thus the whole space is covered by two sets.
b) A complement of a set $\{x\}$, where $x \in S$. Then it remains to cover a single point x. One more set will suffice.
c) A complement of a set $\{B\} \cup B$, where $B \in \mathcal{A}$ is different from A. Then it remains to cover finitely many points $A \cap B$. Finitely many extra sets will suffice. ∎

Observation 3.5. Let $\mathcal{A}$ be an AD system. Then the space $\boldsymbol{X}(\langle\mathcal{A}\rangle)$ is a subspace of the space $\boldsymbol{Y}(\mathcal{A})$.

Yet there are points $A \in \mathcal{A}$ in $\boldsymbol{Y}(\mathcal{A})$ outside of $\boldsymbol{X}(\langle\mathcal{A}\rangle)$. The following observations will show that these points does not interfere in investigation of the Fréchet property.

Observation 3.6. Assume $\mathcal{A}$ to be an AD system and X be a subset of the space $\boldsymbol{Y}(\mathcal{A})$.

- If a point $A \in \mathcal{A}$ is an element of $\overline{X} \setminus X$ then the intersection $X \cap A$ contains infinitely many points. Hence any sequence of different points of the intersection converges to A.
- If the closure of $X \cap \mathcal{A}$ contains the point ∞ then $X \cap \mathcal{A}$ is infinite and any sequence of different points of $X \cap \mathcal{A}$ converges to ∞.
- The space $\boldsymbol{Y}(\mathcal{A})$ has the Fréchet property if and only if the space $\boldsymbol{X}(\langle\mathcal{A}\rangle)$ has the Fréchet property.

3.2 Terminology

Definition 3.7. Maximal AD system, briefly MAD system, is defined to be such AD system that there is no AD system on the same domain which is a proper super-system of the MAD system.

Observation 3.8. By Zorn's lemma, each AD system may be extended to a MAD system.

Definition 3.9. Let $\mathcal{A}$ be an AD system on a set S and let S_0 be a subset of S. We define the restriction $\mathcal{A} \restriction S_0$ of $\mathcal{A}$ to S_0 to be the AD system $\{S_0 \cap A : A \in \mathcal{A} \setminus \{S_0\}^{\perp}\}$. An AD system is called nowhere MAD if $\mathcal{A} \restriction S_0$ is not an infinite MAD system for any $S_0 \subset S$.

Remark 3.10. There is a rather degenerated case of a finite AD system $\mathcal{A}$. Such an AD system is always nowhere MAD but it may form a MAD system either, in the case when $\bigcup \mathcal{A}$ covers whole domain up to finitely many points. Though this is the only case of a MAD which is also nowhere MAD. The definition allows finite AD systems but they are rather marginal. We usually consider infinite AD systems.

Types of sets in AD systems are called differently in different papers. This work introduces its own complex terminology in the area.

Definition 3.11. Consider an AD system $\mathcal{A}$ on S. We talk about following subsets of $X \subset S$.

- We call X basic if $X \in \mathcal{A}$.
- We call X small if $X \in \langle\mathcal{A}\rangle$.
- We call X non-small if $X \notin \langle\mathcal{A}\rangle$, well if it is not small.
- We call X missing if $X \notin \mathcal{A}$ and $\mathcal{A} \cup \{X\}$ forms an AD system. Equivalently if X is infinite and $X \in \mathcal{A}^{\perp}$.
- We call X large if it intersects infinitely many elements of $\mathcal{A}$ at infinitely points. Formally if $|\{A \in \mathcal{A} : |A \cap X| = \omega\}| = \omega$
- We call X non-large if it is not large.
- We call X MAD-ish if it is finite or $\mathcal{A} \restriction X$ forms an AD system. Equivalently if $X \in \overline{\mathcal{A}}$.

We will use notation $\mathcal{A}$-basic, $\mathcal{A}$-small, . . . to specify the corresponding AD system.

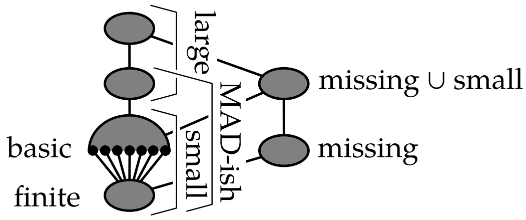

Figure 3.3. Illustration of terminology of sets corresponding to AD system in the lattice $\mathcal{P}(S)$

Observation 3.12.

- Small sets form an ideal.
- MAD-ish sets form an ideal.
- Non-large sets form an ideal.
- Missing and finite sets form together an ideal.
- Each missing set is both non-small and non-large.
- Each set which is both non-small and non-large is an union of a missing set and a small set.
- In MAD system, large sets are identical to non-small sets.
- A set is small if and only if it is both non-large and MAD-ish.

Observation 3.13. Let $\mathcal{A}$ be an infinite MAD system $\mathcal{A}$ on S. Then $S \notin \langle\mathcal{A}\rangle$ but $S \in \overline{\mathcal{A}}$. So an AD system $\mathcal{A}$ is nowhere MAD if and only if $\boldsymbol{Y}(\mathcal{A})$ is a Fréchet space.

Definition 3.14. We call an AD system completely separable if each large set contains a basic subset.

Observation 3.15. Following statements about an AD system are equivalent.

- Is it a completely separable MAD system.
- Each non-small set contains a basic subset.

3.3 Decomposition of a MAD system

In this section, we will show a construction based on [2] enhanced by E. K. van Douwen. Due to observation 2.24 a decomposition of an infinite MAD system into two nowhere MAD systems gives a 2-counter-example. The idea of the construction is to try to dissect the MAD system somehow. In the case of failure, we will focus to the corresponding subset. At the end, we will get a contradiction by the following lemma about a non-small pseudo-intersection.

In general, we can construct a decomposition of MAD system up to countably many nowhere MAD systems. Unfortunately, this does not yield a k-counter-example. The best what one can derive from it is a tuple of compact spaces such that the diagonal in the product of all of them is not a Fréchet space but diagonals of products of proper subtuples are.

Lemma 3.16. Consider an AD system and an infinite decreasing sequence of non-small sets $X_0 \supset X_1 \supset \cdots$. Then there is a non-small pseudo-intersection of them i. e. a non-small set Y such that $Y \setminus X_i$ is finite for each i.

One could prove it just elementarily. Nevertheless we will skip the proof now and wait for a more general version as a corollary of proposition 4.7.

Lemma 3.17. Let $\mathcal{M}$ be a MAD system and $\mathcal{A} \subset \mathcal{M}$ be a subsystem. Then following statements are equivalent.

- Each set X is $\mathcal{M}$-large if and only if it is $\mathcal{A}$-large.
- $\mathcal{M} \setminus \mathcal{A}$ is nowhere MAD.

Proof:

$\Rightarrow$ Consider a $(\mathcal{M}\setminus\mathcal{A})$-non-small set X. We will find $A \in \mathcal{A}$ having an infinite intersect with X. If X is $\mathcal{M}$-small the existence of such set follows from the fact that is not $(\mathcal{M}\setminus\mathcal{A})$-small. Otherwise X is $\mathcal{M}$-large, by assumption also $\mathcal{A}$-large, so there is such set $A \in \mathcal{A}$. Finally, $A \cap X$ is the required $(\mathcal{M}\setminus\mathcal{A})$-missing set.

$\Leftarrow$ Each $\mathcal{A}$-large X is clearly $\mathcal{M}$-large either. Reversely, consider for contradiction a $\mathcal{A}$-non-large, $\mathcal{M}$-non-small set X. Then X is $\mathcal{A}$-non-small so there is a $\mathcal{A}$-missing set $Y = X \setminus Z$, where Z is $\mathcal{A}$-small. Therefore $(\mathcal{M} \setminus \mathcal{A}) \restriction Y = \mathcal{M} \restriction Y$. So Y is a $(\mathcal{M}\setminus\mathcal{A})$-non-small MAD-ish set. ■

Lemma 3.18. Let $\mathcal{M}$ be an infinite MAD system on countable domain S. Further assume a disjoint decomposition $\mathcal{M} = \mathcal{A} \cup \mathcal{B}$. Suppose each $\mathcal{M}$-large set is also a $\mathcal{B}$-large set. Then there is a disjoint decomposition $\mathcal{B} = \mathcal{C} \cup \mathcal{D}$ and an $\mathcal{M}$-large set $X \subset S$ such that each $\mathcal{M}$-large set $Y \subset X$ is both $\mathcal{C}$-large and $\mathcal{D}$-large.

Proof: Consider an arbitrary countable set of decompositions $\mathcal{C}_i$, $\mathcal{D}_i$, where $i \in \omega$, such that: $\mathcal{C}_i \cup \mathcal{D}_i = \mathcal{B}$, $\mathcal{C}_i \cap \mathcal{D}_i = \emptyset$ and moreover for any $I \subset \omega$ it holds

$$\left| \bigcap_{i \in I} \mathcal{C}_i \cap \bigcap_{i \notin I} \mathcal{D}_i \right| \leq 1.$$

If $I = \omega$ or $I = \emptyset$, the formula in the absolute value means $\bigcap_{i\in\omega} \mathcal{C}_i$ or $\bigcap_{i\in\omega} \mathcal{D}_i$ respectively.

One can find such countable system of decompositions according to the fact that the cardinality of $\mathcal{B}$ is at most continuum. The set $\{0,1\}^\omega$ can be divided into singletons by cuts along particular projections.

We will show that there is the required decomposition in one of these decompositions. Suppose the contrary. Then we begin with a $\mathcal{M}$-large set $X_0 = S$ and perform the following process for $i = 0, 1, \ldots$.

The set X_i is $\mathcal{M}$-non-small. Thus there is an $\mathcal{M}$-large subset $X'_{i+1} \subset X_i$ which is either $\mathcal{C}_i$-non-large or $\mathcal{D}_i$-non-large. Let $\mathcal{E}_i$ denote the system $\mathcal{C}_i$ or $\mathcal{D}_i$ such that X'_{i+1} is $\mathcal{E}_i$-non-large. So there are only finitely many elements in $\mathcal{E}_i$ intersecting X'_{i+1} at an infinite set. By subtracting of these elements we construct the $\mathcal{M}$-large set

$$X_{i+1} = X'_{i+1} \setminus \bigcup \left\{ E \in \mathcal{E}_i : \left| E \cap X'_{i+1} \right| = \omega \right\}.$$

Futhermore $X_{i+1} \in \mathcal{E}_i^\perp$.

After that process, lemma 3.16 gives a $\mathcal{M}$-non-small pseudo-intersection Y of all X_i. Nevertheless Y can not have an infinite intersection with any element of $\bigcup\{\mathcal{E}_i : i \in \omega\}$.

Yet the complement of such union to $\mathcal{B}$ has only one element due to the choice of decompositions. It contradicts the fact that Y is a $\mathcal{B}$-large set. ■

Moreover an analogical observation to 2.24 holds for AD systems too.

Observation 3.19. Let $\mathcal{A}_i$, where $i \in I$, be a system of AD systems on one domain S. In additional suppose that $\mathcal{A} = \bigcup_{i \in I} \mathcal{A}_i$ is again an AD system. Then $\boldsymbol{Y}(\mathcal{A})$ can be embedded into $\prod_{i \in I} \boldsymbol{Y}(\mathcal{A}_i)$. Points $x \in S$ and ∞ may be mapped to the corresponding element of the diagonal and points $A \in \mathcal{A}_i$ may be mapped to a point in the product such that i-th coordinate is A and remaining coordinates are equal to ∞.

Theorem 3.20. For any $2 \leq n \in \omega$ there is an infinite MAD system $\mathcal{M}_n$ on a countable domain and its decomposition into disjoint parts $\mathcal{A}_0, \ldots, \mathcal{A}_{n-1}$ such that the union of any $n-1$ of them is nowhere MAD. Moreover there exists an infinite MAD system $\mathcal{M}_\omega$ on a countable domain and its decomposition into countably many parts $\mathcal{A}_0, \mathcal{A}_1, \ldots$ such that for every $i \in \omega$ the union $\bigcup_{j \neq i} \mathcal{A}_j$ is nowhere MAD.

Proof: We begin by taking an arbitrary infinite MAD system $\mathcal{M}$ on a set S_0. We use the preceding lemma for $\mathcal{A} = \emptyset, \mathcal{B} = \mathcal{M}$. There is a set $S_1 \subset S_0$ and a decomposition of $\mathcal{M} \restriction S_1$ to two AD systems $\mathcal{C}_0 \restriction S_1$, $\mathcal{D}_0 \restriction S_1$ with equal large sets. We continue recursively by applying the lemma for

$$\mathcal{A} = \bigcup_{j \leq i} \mathcal{C}_j \restriction S_{i+1}, \quad \mathcal{B} = \mathcal{D}_i \restriction S_{i+1}$$

and getting the decomposition of $\mathcal{D}_i \restriction S_{i+2}$ to systems $\mathcal{C}_{i+1}$ and $\mathcal{D}_{i+1}$ on domain $S_{i+2} \subset S_{i+1}$. Yet S_{i+2} is a $\mathcal{M}$-non-small set and both parts of the decomposition have identical large sets. The obtained decomposition of $\mathcal{M} \restriction S_{n-1}$ to

$$\mathcal{C}_0 \restriction S_{n-1} \cup \ldots \cup \mathcal{C}_{n-2} \restriction S_{n-1} \cup \mathcal{D}_{n-2} \restriction S_{n-1}.$$

is the required one by lemma 3.17.

For the countable case we use lemma 3.16 again and obtain a $\mathcal{M}$-non-small pseudo-intersection S of sets $S_0 \supset S_1 \supset \cdots$. Then $\mathcal{M} \restriction S$ is an infinite MAD system having the same large sets as each $\mathcal{C}_i \restriction S$. Hence we build the decomposition from systems $\mathcal{C}_i$. Yet there may be missing elements from $\bigcap_{i \in \omega} \mathcal{D}_i$. For completing the construction we add these elements to $\mathcal{C}_0$. ■

Corollary 3.21. A 2-counter-example exists. Moreover for any $3 \leq n \leq \omega$ there is a system of compact spaces $\boldsymbol{X}_i$, where $i \in n$, and $X \subset \prod_{i \in n} \boldsymbol{X}_i$, $x \in \overline{X}$ such that:

- There is no sequence of elements of X converging to x.
- For each $i \in n$ the subspace on the set $\pi_{\neg i}\left[\overline{X}\right]$ is a Fréchet space.

Chapter 4

AD systems in finite dimensions

In this chapter finite products of AD systems are introduced. In contrast with the case of topology spaces or ideals it is not possible to understand a product of AD systems as a new AD system. That is why the product will be understood just formally. The terminology of small, large, missing, ... sets will be extended to such products.

Definition 4.1. We call a set $X \subset \prod_{i\in n} S_i$ injective if all simple projections of it are injective. Equivalently if the following inequation holds for each $x \in S_i, i \in n$

$$|\pi_i^{-1}(x) \cap X| \leq 1.$$

Definition 4.2. Let $\mathcal{A}$ be a formal product of AD systems $\prod_{i\in n} \mathcal{A}_i$ on a domain $S = \prod_{i\in n} S_i$. A set $X \subset S$ is said to be $\mathcal{A}$-basic if for some $i \in n$, X is a preimage of some $A \in \mathcal{A}_i$ in the projection π_i. Further a set is called $\mathcal{A}$-small if it is contained by the ideal $\prod_{i\in n}\langle\mathcal{A}_i\rangle$. Otherwise it is called $\mathcal{A}$-non-small.

Observation 4.3. An injective set is $\mathcal{A}$-small if and only if it is contained by the ideal generated by $\mathcal{A}$-basic sets.

Definition 4.4. Let $\mathcal{A}$ be a formal product of AD systems $\prod_{i\in n} \mathcal{A}_i$ on a domain $S = \prod_{i\in n} S_i$. A set $X \subset S$ is called $\mathcal{A}$-missing if it is injective and for each $i \in n$ the projection $\pi_i[X]$ is $\mathcal{A}_i$-missing.

Observation 4.5. An infinite injective set is $\mathcal{A}$-missing if and only if it is in the orthogonal complement of the ideal $\mathcal{A}$-small sets. Reversely each infinite set in the orthogonal complement of a system is a super-set of a missing set.

Observation 4.6. For any infinite injective set $X \subset S$ one can find an infinite subset $Y \subset X$ such that for each $i \in n$ the projection $\pi_i[Y]$ is either an $\mathcal{A}_i$-missing set or a subset of an $\mathcal{A}_i$-basic set.

In the previous chapter, it was shown that a non-small set in AD system is either large or a union of a missing set and a small set. The following technical proposition gives a similar characterization of non-small sets in a product of AD systems.

Proposition 4.7. Let $\mathcal{A}_0, \dots, \mathcal{A}_{n-1}$ be AD systems of infinite sets $S_0, \dots, S_{n-1}$ respectively. Further let $\mathcal{A}$ denote the formal product of the AD systems. Assume a set $X \subset \prod_{i\in n} S_i$. Then following are equivalent.

- X is $\mathcal{A}$-non-small.
- There is an infinite sequence of infinite disjoint sets $X_0, X_1, \dots \subset X$ such that:

  (1) The union $\bigcup_{k\in\omega} X_k$ is a countable injective set.
  (2) For each $i \in n$, $k \in \omega$ the projection $\pi_i[X_k]$ is either an $\mathcal{A}_i$-missing set or a subset of an $\mathcal{A}_i$-basic set.
  (3) For any fixed $i \in n$ and any different $k, l \in \omega$ the projections $\pi_i[X_k], \pi_i[X_l]$ are never subsets of one $\mathcal{A}_i$-basic set.

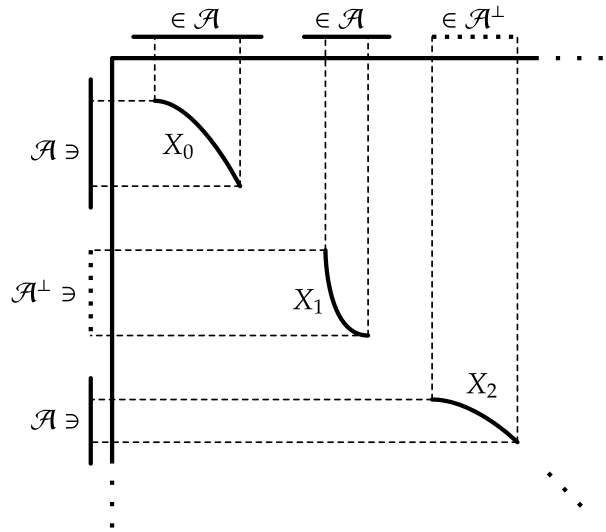

Figure 4.1. A general example of a non-small set in ω^2.

Proof:

$\Rightarrow$ We will construct infinite injective sets Y_i, initially without the requirement of disjointness and that the union should be injective. Yet it will satisfy requirements (2), (3).

Next to Y_k we are going to construct auxiliary non-small sets Z_k. We begin with $Z_0 = X$ and continue by following inductive process. At k-th step, pick any infinite countable injective subset $Y_k \subset Z_k$. We can do that – if we got stuck during choosing of points the Z_k would be covered by finitely many preimages of points in projections. It would contradict the fact that Z_k is non-small. Due to the previous observation we make Y_k to meet the condition (2). On the end of the cycle we set

$$Z_{k+1} = Z_k \setminus \bigcup \left\{ \pi_i^{-1}[A] : i \in n, \pi_i[Y_k] \subset A, A \in \mathcal{A}_i \right\}.$$

We have taken just finitely many basic sets from Z_k so Z_{k+1} is still non-small. Moreover the condition (3) will be forced by the choice of Z_{k+1}.

So there are injective infinite sets satisfying (2), (3). Whenever we take infinite subsets $X_i \subset Y_i$, they will also satisfy the conditions (2) and (3). Hence it suffices to arrange particular projections $\pi_i[X_k]$ for fixed i to be different. Let k_t be a sequence of naturals such that each number has infinitely many occurrences in it. We will construct a sequence x_t such that for each $t \in \omega$ following holds.

- $x_t \in Y_{k_t}$,
- for any $t' < t$ and $i \in n$ the projection $\pi_i(x_t) \neq \pi_i(x_{t'})$.

In each step there are infinitely many elements of Y_{k_t}. Since Y_{k_t} is injective, additional conditions forbid only finitely many of them. So we can find an element satysfying them. Finally setting $X_k = \{x_t : k_t = k\}$ completes the construction.

$\Leftarrow$ Acording to the condition (3) and the fact that $\mathcal{A}_i$ form AD systems, each $\mathcal{A}$-basic set intersects only finitely many sets X_i at an infinite set. Consequently $X \supset \bigcup_{k\in\omega} X_k$ can not be small. ∎

Remark 4.8. The previous proposition offers to two possibilities on each projection. Even in total there are just finitely many of such possibilities so one can choose a sequence X_k such that all its elements behave in the same way. One could ask whether even simpler generalization could be possible: "For each simple projection π_i either the set $\pi_i[\bigcup_{k\in\omega} X_k]$ is missing or projections $\pi_i[X_k]$ are subsets of different basic sets." But such claim is refuted by theorem 3.18. Suppose an infinite MAD system decomposed to two AD systems with identical large sets. Further consider the diagonal of the product of these two AD systems and its subsets X_k. It is not possible for projections of fixed X_k to be both missing neither both under a basic set. But even the case that the union of all X_k is large via one projection and missing via the other is impossible.

Corollary 4.9. In each non-small set one can find an injective non-small subset.

Proof: It is the union $\bigcup_{k\in\omega} X_k$ from proposition cite[adp-charnem]. ∎

Let us note that this is not a general property of topological space. The example $(\omega+1)\times S_\omega$ in the introduction refutes it. Now we show how to easily use it for proving lemma 3.16. The previous corollary did not need results of the section 3.3.

Corollary 4.10. Assume an infinite decreasing sequence of $\mathcal{A}$-non-small sets $S \supset X_0 \supset X_1 \supset \cdots$, where $\mathcal{A}$ is a formal product of finitely many AD systems and S is the Cartesian product of their domains. Then there is a non-small pseudo-intersection of them i. e. a non-small set Y such that for each $i \in \omega$ the set $Y \setminus X_i$ is finite.

Proof: Let $\mathcal{B}$ be an empty AD system on the domain ω. Consider a set $X \subset \omega \times S$ defined as follows.

$$X = \{(i,x) : i \in \omega,\ x \in X_i\}.$$

The set X is $(\mathcal{B}\times\mathcal{A})$-non-small so there is a non-small injective subset Y' of it. Then the projection $Y = \pi_{\neg 0}(Y')$ is still a non-small set satisfying $|Y \setminus X_i| \le i$ for each $i \in \omega$. ∎

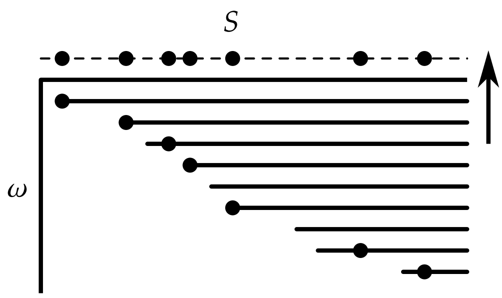

Figure 4.2. Illustration of the construction of a pseudo-intersection using an injective set.

Now we prepared vehicles for proving a rather technical lemma which is going to be the tool for construction of spaces such that the product of them has the Fréchet property. It is not just a variant of the proposition 2.22. In the following lemma, the verification is restricted just to injective sets which are non-small in a fixed super-system.

Lemma 4.11. Consider n triples as follows: $(S_0, \mathcal{A}_0, \mathcal{B}_0), \ldots, (S_{n-1}, \mathcal{A}_{n-1}, \mathcal{B}_{n-1})$. Further suppose that for each $i \in n$ the AD system $\mathcal{A}_i$ is a subsystem of $\mathcal{B}_i$ and they are AD systems on the domain S_i. Then following are equivalent.

- for every subset $I \subset n$ and for every injective $\mathcal{B}$-non-small set $X \subset \prod_{i\in I} S_i$ there is an $\mathcal{A}$-missing $Y \subset X$, where $\mathcal{A}$, $\mathcal{B}$ denotes formal products

$$\mathcal{A} = \prod_{i\in I} \mathcal{A}_i, \quad \mathcal{B} = \prod_{i\in I} \mathcal{B}_i$$

- The space

$$\boldsymbol{X} = \boldsymbol{Y}(\mathcal{A}_0) \times \cdots \times \boldsymbol{Y}(\mathcal{A}_{n-1})$$

  has the Fréchet property.

Proof: The reverse implication is obvious. We will prove the forward one.

The proof is by induction primarily on the number of non-empty sets $\mathcal{B}_i$, secondarily on n. If $n = 0$ the space of a single point has the Fréchet property.

Let us first note that deletion of some triple $(S_i, \mathcal{A}_i, \mathcal{B}_i)$ or adding of a triple $(S, \emptyset, \emptyset)$ preserves the assumption. The case of deletion is obvious. In the case of addition empty systems, it suffices to examine projections outside this AD system because each infinite set is missing in empty AD system.

Consider $X \subset \boldsymbol{X}$ and $z \in \overline{X}$. We need to find a countable set $Y \subset X$ converging to z. We first analyze some trivial cases:

a) The projection $\pi_i(z) \in S_i$ for some i. We will denote $\pi_i(z)$ briefly by z_i. The point z_i is isolated in the space $\boldsymbol{Y}(\mathcal{A}_i)$. So the subspace $\boldsymbol{X'} \subset \boldsymbol{X}$ on the domain $\pi_i^{-1}(z_i)$ forms an open subspace. Hence $z \in \overline{X \cap \boldsymbol{X'}}$ and it suffices to use the Fréchet property of the space $\boldsymbol{X'}$ homeomorphic to the product $\prod_{j \neq i} \boldsymbol{Y}(\mathcal{A}_i)$. Such product has the Fréchet property by the induction assumption upon deleting the triple $(S_i, \mathcal{A}_i, \mathcal{B}_i)$.

b) Some projection π_i satisfies $\pi_i(z) \in \mathcal{A}_i$. Let us denote by A the point $\pi_i(z)$. Consider a subspace $\boldsymbol{X}'_i \subset \boldsymbol{Y}(\mathcal{A}_i)$ on the domain $\{A\} \cup A$. Such set is open so again we can restrict the problem to $X' = X \cap \pi_i^{-1}[\boldsymbol{X}'_i]$. Moreover $\boldsymbol{X}'_i$ is homeoporphic to the space constructed from empty AD system on the domain A. Therefore after replacing S_i by A and $\mathcal{A}_i, \mathcal{B}_i$ by empty systems the induction assumption proves the case.

What is left is to resolve the case $z = \infty$. Consider a map

$$\mathrm{type}_i\colon \boldsymbol{Y}(\mathcal{A}_i) \to \{0,1,2\}, \quad \mathrm{type}_i(x) = \begin{cases} 0, & \text{where } x \in S_i \\ 1, & \text{where } x \in \mathcal{A}_i \\ 2, & \text{where } x = \infty \end{cases},$$

$$\mathrm{type}\colon \boldsymbol{X} \to \{0,1,2\}^n, \quad \mathrm{type}((x_0, \dots, x_{n-1})) = (\mathrm{type}_0(x_0), \dots, \mathrm{type}_{n-1}(x_{n-1}))$$

There are just finitely many possible values of the map so there is some $t \in \{0,1,2\}^n$ such that $\infty \in \overline{X \cap \mathrm{type}^{-1}(t)}$. Let us denote by X_2 the set $X \cap \mathrm{type}^{-1}(t)$.

We again analyze some trivial cases.

a) For some $i \in n$ it occurs $\pi_i(t) = 2$. Assume deletion of $(S_i, \mathcal{A}_i, \mathcal{B}_i)$. By the induction assumption on the set $\pi_{\neg i}[X_2]$ there is a sequence $x'_0, x'_1, \ldots \in \pi_{\neg i}[X_2]$ converging to ∞. We build the requested sequence $x_0, x_1, \ldots \in X_2$ in a such way that $\pi_{\neg i}(x_k) = x'_k$. Such sequence will converge to the infty even along the projection π_i because it is the constant ∞ there.

b) For some $i \in n$ it occurs $\pi_i(t) = 1$. Let us denote by $\boldsymbol{X}'_i$ the subset of $\boldsymbol{Y}(\mathcal{A}_i)$ on the domain $\{\infty\} \cup \mathcal{A}_i$. The whole set $X_2 \cup \{\infty\}$ is a subset of $\pi_i^{-1}[\boldsymbol{X}'_i]$. Hence it suffices to restrict everything to such product where $\boldsymbol{Y}(\mathcal{A}_i)$ is replaced by $\boldsymbol{X}'_i$. The space $\boldsymbol{X}'_i$ is homeomorphic to the space constructed from the empty AD system on the domain $\mathcal{A}_i$. Induction assumption completes the case.

The only remaining case is $t = (0, 0, \ldots, 0)$ and $z = \infty$. Denote

$$\mathcal{A} = \prod_{i \in n} \mathcal{A}_i, \quad \mathcal{B} = \prod_{i \in n} \mathcal{B}_i.$$

The set X_2 is $\mathcal{A}$-non-small. It remains to explore following two cases.

a) X_2 is $\mathcal{B}$-small. Then it is possible to divide X_2 to finitely many subsets of $\mathcal{B}$-basic sets. At least one of these parts in $\mathcal{A}$-non-small, denote it $X_3 \subset \pi_i^{-1}(B)$, where $i \in n$, $B \in \mathcal{B}_i$. Therefore B is $\mathcal{A}_i$-missing. Again the subspace $\boldsymbol{X}'_i \subset \boldsymbol{Y}(\mathcal{A}_i)$ on the domain $\{\infty\} \cup B$ is homeomorphic to a space constructed from the empty AD system on the set B. The induction assumption completes the case.

b) X_2 is $\mathcal{B}$-non-small. By corollary 4.9 there is an injective $\mathcal{B}$-non-small $X_3 \subset X_2$. Finally, we use the assumption of the proposition and get a missing subset. The missing set is the required sequence. ■

Remark 4.12. Similarly as in the case of ideals, due to proposition 2.23 if all $\mathcal{B}_i$ are infinite it suffices to test the condition of the previous lemma only where $I = n$.

We complete definitions of remaining terms.

Definition 4.13. Let $\mathcal{A} = \prod_{i\in n} \mathcal{A}_i$ be a formal product of AD systems on a product of domains $S = \prod_{i\in n} S_i$. Assume $X \subset S$. Then:

- We call X $\mathcal{A}$-MAD-ish if there is no missing set in it.
- We say that $\mathcal{A}$ is nowhere MAD if all $\mathcal{A}$-MAD-ish sets are $\mathcal{A}$-small.
- We call X $\mathcal{A}$-large if there is an infinite sequence of disjoint infinite injective subsets $X_k \subset X$ such that:
  (1) The union $\bigcup_{k\in\omega} X_k$ is a countable injective set.
  (2) For each $i \in n$, $k \in \omega$ the projection $\pi_i[X_k]$ is a subset of an $\mathcal{A}_i$-basic set.
  (3) For fixed $i \in n$ and different $k, l \in \omega$ never happens that $\pi_i[X_k]$ and $\pi_i[X_l]$ are a subset of a common $\mathcal{A}_i$-basic set.
- We call X $\mathcal{A}$-non-large if it is not $\mathcal{A}$-large.

Observation 4.14. Assume a formal product of AD systems $\mathcal{A} = \prod_{i\in n} \mathcal{A}_i$.

- If $n = 1$ the terminologies for the product $\mathcal{A}$ and for the AD system $\mathcal{A}_0$ are identical.
- By proposition 4.7, all $\mathcal{A}$-large sets are $\mathcal{A}$-non-small. In addition, if all AD systems $\mathcal{A}_i$ are maximal also the reverse holds.
- By lemma 4.11, all products $\prod_{i\in I} \mathcal{A}_i$ are nowhere MAD, where $I \subset n$, if and only if the product $\prod_{i\in n} \boldsymbol{Y}(\mathcal{A}_i)$ has the Fréchet property.

4.1 Strong complete separability

In this section, a construction of k-counter-example will be shown under the assumption of existence of an infinite $(k + 1)$-completely separable MAD system.

Definition 4.15. An AD system $\mathcal{A}$ on a domain S is said to be completely k-separable if in every $\mathcal{A}^k$-large set $X \subset S_k$ there is a subset $Y \subset X$ such that all simple projection of Y are $\mathcal{A}$-basic. An AD system which is completely k-separable for each $k \in \omega$ is said to be strongly completely separable.

Observation 4.16.

- Complete 1-separability is equivalent to an ordinary complete separability from definition 3.14.
- If an AD system is completely k-separable it is also completely k'-separable for any $1 \le k' \le k$.
- Let $\mathcal{A}$ be an AD system. Then following are equivalent:
  - the system $\mathcal{A}$ is a completely k-separable MAD system,
  - in each $\mathcal{A}^k$-non-small set there is a subset such that all its projections are $\mathcal{A}$-basic.

Remark 4.17. The terminology slightly differs from the one used in [4]. That paper uses $[\omega]^k$ instead of ω^k so repeating coordinates is forbidden there. On the other hand a statement "$\mathcal{A}$ is a completely k-separable AD system for all $k < k_0$" has the same meaning in both publications.

4.1.1 An equivalent condition

The existence of a strongly completely separable MAD system may sound as a bold assumption. Thus we will show an equivalent condition to it at first. It is enough to assume an existence of an infinite MAD system such that each its subsystem $\mathcal{B} \subset \mathcal{A}$ of a cardinality lesser than continuum satisfies that $\mathcal{B}^n$ is nowhere MAD for all $n \in \omega$.

The assumption obviously holds in the case of $\mathfrak{p} = \mathfrak{c}$, namely under the assumption of Martin's axiom. Even weeker condition will be specified in next chapter.

Lemma 4.18. Let $\mathcal{A}$ be a completely n-separable AD system on S. Then in every $\mathcal{A}^n$-large set one can find continuum many different subsets $Y \subset X$ such that all projections $\pi_i[Y]$ are $\mathcal{A}$-basic sets.

Proof: The set X is large so there are relevant sets $X_0, X_1, \dots$ Choose some infinite proper subsets $Z_k \subset X_k$. Further pick arbitrary AD system C of the cardinality of continuum. The set $X(C) = \bigcup_{k \in C} Z_k$ is still large for each $C \in \mathcal{C}$. By the definition of completely n-separable system there are subsets $Y(C) \subset X(C)$ having $\mathcal{A}$-basic projections.

What is left is to show that these $Y(C)$ are different. Conversely, suppose that $Y = Y(C_0) = Y(C_1)$ but $|C_0 \cap C_1| < \omega$. Then $Y \subset \bigcup_{k \in C_0 \cap C_1} Z_k$, so it has an infinite intersection with some of these finitely many Z_k. Hence even $\pi_0[Y]$ has an infinite intersection with this $\pi_0[Z_k]$. So $\pi_0[Y]$ is equal to this $\mathcal{A}$-basic super-set $\pi_0[X_k]$. But it is impossible since Y is disjoint with $X_k \setminus Z_k$. ■

Proposition 4.19. Let $\mathcal{A}$ be an completely k-separable MAD system and $\mathcal{B} \subset \mathcal{A}$ be its subsystem of a cardinality less than continuum. Then $\mathcal{B}^k$ is nowhere MAD.

Proof: By lemma 4.11 it suffices to show that for each $k' \leq k$ and for every $\mathcal{A}^{k'}$-non-small set X there is its $\mathcal{B}^{k'}$-missing subset. Consider such a set X. Let us denote by $\mathcal{Y} \subset \mathcal{P}(X)$ the system of all subsets of X having all projections $\mathcal{A}$-basic. According to lemma 4.18 $|\mathcal{Y}| \geq \mathfrak{c}$. Since X is injective, different $Y_0, Y_1 \in \mathcal{Y}$ have different projections $\pi_i[Y_0], \pi_i[Y_1]$, where i is fixed. So for each $i \in n$ there are less than continuum $Y \in \mathcal{Y}$ such that $Y \in B$. Hence there is an $Y \in \mathcal{Y}$ such that all projections satisfy $\pi_i(Y) \in \mathcal{A} \setminus \mathcal{B}$. Such Y is the required $\mathcal{B}$-missing set. ■

Lemma 4.20. For any infinite injective set $X \subset S^n$ $(n \in \omega)$ there is an infinite subset $Y \subset X$ such that sets $\pi_i[Y], \pi_j[Y]$ are disjoint or identical for any $i, j < n$.

Proof: The proof is by induction on n. If there are two indices $i, j \in n$ such that the set

$$X_1 = \{x \in X : \pi_i(x) = \pi_j(x)\}$$

is infinite then consider the projection $\pi_{\neg j}[X_1] \subset S^{k-1}$. The induction assumption gives a set $Y_1 \subset \pi_{\neg j}[X_1]$. Finally $Y = \pi_{\neg j}^{-1}[X_1] \cap X_1$ satisfies the requirements since $\pi_j[Y] = \pi_i[Y]$.

Now suppose there is no such pair of indices. Then following set is infinite.

$$X_2 = \{x \in X : \forall i, j < n : i \neq j \Rightarrow \pi_i(x) \neq \pi_j(x)\}$$

So we pick elements $x_0, x_1, x_2 \dots \in X_2$ successively in a such way that for each $i, j \in n$, $k' < k$ there inequality $\pi_i(x_{k'}) \neq \pi_j(x_k)$ holds. In each step there are just finitely many of such condition so some elements will always remain available. ■

Proposition 4.21. Let $\mathcal{A}$ be a MAD system on ω. Then there is a MAD system $\mathcal{B}$ such that:

(1) The system $\mathcal{B}$ is completely k-separable for all $0 < k \in \omega$ such that each subsystem $\mathcal{A}_0 \subset \mathcal{A}$ of a cardinality less that continuum is nowhere MAD.
(2) $\langle \mathcal{A} \rangle = \langle \mathcal{B} \rangle$

Proof: The case of finite $\mathcal{A}$ is trivial. Assume that $\mathcal{A}$ is infinite.

Let I be the set of all k satisfying the condition at item (1). We want to ensure complete k-separability of the system $\mathcal{B}$ for them. Consider a set

$$\{(k \in I, X \subset \omega^k) : X \text{ is } \mathcal{A}^k\text{-large}\}.$$

The size of the set equals continuum. Hence we may index its elements by continuum as (k_α, X_α), where $\alpha \in \mathfrak{c}$. By the transfinite recursion on continuum we construct pairs

$$(Y_\alpha \subset X_\alpha, (A_{\alpha,0}, \dots, A_{\alpha,k_\alpha - 1})),$$

such that Y_α are infinite and $\pi_i[Y_\alpha] \subset A_{\alpha,i} \in \mathcal{A}$ holds for each $i \in n$. Moreover sets $\pi_i[Y_\alpha], \pi_j[Y_\alpha]$ will be equal or disjoint for any pair $i, j \in n$ and sets $A_{i,\alpha}$, $A_{j,\gamma}$ will never be equal if $\gamma < \alpha$. We will ensure that by the following process.

Set $\mathcal{A}_\alpha = \{A_{\gamma,i} : \gamma < \alpha, i < k_\gamma\}$. The cardinality of the system $\mathcal{A}_\alpha$ is less than continuum, hence $\mathcal{A}_\alpha^k$ is nowhere MAD. We may find an $\mathcal{A}_\alpha^k$-missing set $X'_\alpha \subset X_\alpha$. Finally we choose $Y_\alpha \subset X'_\alpha$ according to observation 4.6 and the previous lemma 4.20.

It remains to build the required system $\mathcal{B}$ from such pairs. Each $A \in \mathcal{A}$ occurs in at most one step so only finitely many times. In that case we replace the element $A = A_{\alpha,i} \in \mathcal{A}$ by elements $\pi_i[Y_\alpha]$ for all i for which $A = A_{\alpha,i}$. Furthermore if the complement $A \setminus \bigcup\{\pi_i[Y_\alpha] : A = A_{\alpha,i}\}$ is infinite we add it to $\mathcal{A}$ also.

Since we have just replaced each element of the AD system by its finite decomposition, the ideal generated by it is not changed. Yet we get the condition of complete k-separability by adding of $\pi_i[Y_\alpha]$. ■

Corollary 4.22. Assume $1 < n \leq \omega$. Then following are equivalent.

- There is an infinite MAD system which is completely k-separable for all $1 \leq k < n$.
- There is an infinite MAD system $\mathcal{M}$ such that for each $1 \leq k < n$ and for any subsystem $\mathcal{A} \subset \mathcal{M}$ of cardinality less than continuum the power $\mathcal{A}^k$ is nowhere MAD.

4.1.2 The construction of k-counter-example

Theorem 4.23. Assume a fixed $1 < k' \leq \omega$. If there is an infinite MAD system on ω which is completely k-separable for each $k \in k'$ then for any $1 < n \leq \omega$ there are compact spaces $\boldsymbol{Y}_i$ $(i \in n)$ such that the product $\prod_{i \in n} \boldsymbol{Y}_i$ has not the Fréchet property but for any non-surjective function $\sigma : k \to n$, where $k \in k'$, the product $\prod_{j \in k} \boldsymbol{Y}_{\sigma(j)}$ has the Fréchet property.

Proof: Let $\mathcal{M}$ be the given MAD system. We will construct disjoint AD systems $\mathcal{A}_i$ $(i \in n)$ such that $\bigcup_{i \in n} \mathcal{A}_i = \mathcal{M}$ a we will set $\boldsymbol{Y}_i = \boldsymbol{Y}(\mathcal{A}_i)$. Observation 3.19 provides that the product $\prod_{i \in n} \boldsymbol{Y}_i$ will not satisfy the Fréchet property. It remains to ensure the validity of the second condition. Lemma 4.11 will manage it. It suffices to assure that for each $k \in k'$, each function $\sigma : k \to n$ and each $\mathcal{M}^k$-non-small injective set $X \subset \omega^k$ there will be a $\prod_{j \in k} \mathcal{A}_{\sigma(j)}$-missing subset $Y \subset X$.

We index the set

$$\{(k, i, X) : k \in k',\ i \in n,\ X \subset \omega^k \text{ is injective and } \mathcal{M}^k\text{-large}\}$$

by $\mathfrak{c}$ as $\{(k_\alpha, i_\alpha, X_\alpha)\}$. By transfinite recursion, we will construct systems $\mathcal{A}_{i,\alpha}$ $(i \in n, \alpha \leq \mathfrak{c})$. Let us denote $\mathcal{A}'_\alpha = \bigcup_{i \in n} \mathcal{A}_{i,\alpha}$. All $\mathcal{A}_{i,0}$ are empty on the beginning. The limit step is union, as usual. It remains to describe how to get from a step α to the step $\alpha + 1$.

The set X_α is $\mathcal{M}^{k_\alpha}$-large. Hence by lemma 4.18 there are continuum many subsets of X_α such that all simple projections are $\mathcal{M}$-basic. Since $|\mathcal{A}'_\alpha| < \mathfrak{c}$, we find $Y_\alpha \subset X_\alpha$ such that the system $\mathcal{M} \setminus \mathcal{A}'_\alpha$ contains all projections $\pi_j[Y_\alpha]$, where $j \in k_\alpha$. We put

$$\mathcal{A}_{i_\alpha,\alpha+1} = \mathcal{A}_{i_\alpha,\alpha} \cup \{\pi_j[Y_\alpha] : j \in k_\alpha\},$$

$$\mathcal{A}_{i,\alpha+1} = \mathcal{A}_{i,\alpha} \text{ for all } i \in n, i \neq i_\alpha.$$

Finally, we set $\mathcal{A}_0 = \mathcal{A}_{0,\mathfrak{c}} \cup (\mathcal{M} \setminus \mathcal{A}'_\mathfrak{c})$ and for the remaining $0 < i \in n$ set $\mathcal{A}_i = \mathcal{A}_{i,\mathfrak{c}}$. What is left is to verify that these constructed AD systems satisfies the assumptions of lemma 4.11.

Consider a non-surjective map $\sigma\colon k \to n$ and a $\mathcal{A}^k$-non-small set $X \subset \omega^k$. We find $i \in n$, $i \notin \sigma[k]$ and α such that $(k, i, X) = (k_\alpha, i_\alpha, X_\alpha)$. Then Y_α is a subset of X such that all its projections are $\mathcal{A}_i$-basic. Hence it is a $\left(\prod_{j\in k} \mathcal{A}_{\sigma(j)}\right)$-missing set. ∎

Corollary 4.24. Under the assumption of existence of an infinite completely k-separable MAD system there is a $(k+1)$-counter-example. If there is even strongly completely separable MAD system then there exists a k-counter-example for each $k \leq \omega$.

Chapter 5

The construction of an infinite completely k-separable MAD system

Infinite complete separable MAD systems are quite investigated objects. The paper [5] constructs one under the assumption $\mathfrak{s} \leq \mathfrak{a}$, where $\mathfrak{s}$ is the minimal cardinality of a splitting system and $\mathfrak{a}$ is the minimal cardinality of an infinite MAD system.

According to the previous chapter, just infinite completely MAD system is not probably an object strong enough for a construction of k-counter-example. We need a completely k-separable MAD system. Fortunately the properties of completely separable MAD systems and completely k-separable MAD systems are similar. So statements about completely separable MAD systems can often be generalized.

It is even the case of a Shellah's construction of a completely k-separable AD system using ω, ω-splitting system, as it is shown in this chapter.

Convention 5.1. All AD systems in this chapter will have the domain ω.

5.1 Small cardinals

Definition 5.2. We say that a set S splits a set X if sets $X \cap S$, $X \setminus S$ are both infinite. A system of sets $\mathcal{S} \subset \mathcal{P}(\omega)$ is called splitting if for any infinite $X \subset \omega$ there is a set $S \in \mathcal{S}$ splitting X. A system $\mathcal{S}$ is called ω, ω-splitting if for any countable sequence of infinite sets $X_0, X_1, \ldots \subset \omega$ there is a set $S \in \mathcal{S}$ such that:

- There are infinitely many indices $i \in \omega$ such that $|X_i \cap S| = \omega$.
- There are infinitely many indices $i \in \omega$ such that $|X_i \setminus S| = \omega$.

The splitting number, denoted by $\mathfrak{s}$, is defined to be the minimal cardinality of a splitting system. Similarly, the ω, ω-splitting number, denoted by $\mathfrak{s}_{\omega,\omega}$, is the minimal cardinality of an ω, ω-splitting system.

Observation 5.3. Any ω, ω-splitting system is also splitting. One may choose constantly X_i as the given X. Thus $\mathfrak{s} \leq \mathfrak{s}_{\omega,\omega}$.

Definition 5.4. The cardinality $\mathfrak{a}$ denotes the minimal cardinality of an infinite MAD system. For a non-zero $n \in \omega$ let $\mathfrak{a}_n$ denote the minimal cardinality such that there are AD systems $\mathcal{A}_i$ $(i \in n)$, $|\mathcal{A}_i| \leq \mathfrak{a}_n$ such that $\prod_{i \in n} \mathcal{A}_i$ is not nowhere MAD. Finally let $\mathfrak{a}_\omega$ denote the minimum of all a_n over $n \in \omega$.

Observation 5.5. It holds $\mathfrak{a} = \mathfrak{a}_1$.

One could generalize $\mathfrak{a}_n$ to $\mathfrak{p}$ described by definition 2.14 as the minimal weight of a non-Fréchet countable space. But obviously $\mathfrak{p} \leq \mathfrak{s}$. It is the reverse inequality than we need. So it is suitable to choose a tighter estimate.

Definition 5.6. Assume a system of functions $\mathcal{F}$ from ω to ω. A function $f\colon \omega \to \omega$ is said to be an upper bound of $\mathcal{F}$ if the set $\{k : f(k) < \varphi(k)\}$ is finite for any $\varphi \in \mathcal{F}$. The bounding number, denoted by $\mathfrak{b}$, is the minimal cardinality of a system $\mathcal{F}$ which does not have any upper bound.

Proposition 5.7. It holds $\mathfrak{b} \leq \mathfrak{a}_n$ for any non-zero $n \in \omega$. Thus also $\mathfrak{b} \leq \mathfrak{a}_\omega$.

Proof: Consider n AD systems $\mathcal{A}_0, \ldots, \mathcal{A}_{n-1}$. Suppose that

$$\left| \bigcup_{i \in n} \mathcal{A}_i \right| < \mathfrak{b}.$$

Set $\mathcal{A} = \prod_{i \in n} \mathcal{A}_i$ and take an $\mathcal{A}$-non-small set $X \subset \omega^n$. By the characterization in proposition 4.7, we find its disjoint subsets $X_0, X_1, \ldots$. Let φ_k be bijections $X_k \to \omega$.

We construct the function f_A for each $\mathcal{A}$-basic set A as follows.

$$f_A(k) = \begin{cases} \max\{\varphi_k(x) : x \in X_k \cap A\} + 1 & \text{if } X_k \cap A \text{ is finite,} \\ 0 & \text{otherwise.} \end{cases}$$

There are less than $\mathfrak{b}$ of such functions so there exists its upper bound f. Then there is a required missing set $\{\varphi^{-1}(f(k)) : k \in \omega\}$. ∎

To be complete, we mention following propositions.

Proposition 5.8. It holds

- $\mathfrak{p} \leq \mathfrak{b}$,
- $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$.

For proofs we refer reader to [??]. As shown by proposition 2.18, under the assumption of Martin's axiom it holds $\mathfrak{p} = \mathfrak{c}$ so all mentioned cardinal numbers are equal. Well, it is a boring case.

$$\begin{tikzcd}
& \mathfrak{b} \arrow[r] & \mathfrak{a}_n \arrow[r] & \cdots \arrow[r] & \mathfrak{a}_1 \\
\mathfrak{p} \arrow[ur] \arrow[dr] & & & & \\
& \mathfrak{s} = \mathfrak{s}_{\omega,\omega} & & &
\end{tikzcd}$$

Figure 5.1. Inequalities between small cardinals

5.2 The construction

Let us fix an ω,ω-splitting system $\mathcal{S} = \{S_\alpha : \alpha \in \mathfrak{s}_{\omega,\omega}\}$.

Definition 5.9. Let $\alpha \in \mathfrak{s}_{\omega,\omega}$. We denote $S_\alpha^0 = S_\alpha$, $S_\alpha^1 = \omega \setminus S_\alpha$. Further we assume an infinite set $X \subset \omega$ and define α_X to be the smallest index such that S_{α_X} splits X. Moreover we define the function $\sigma_X \colon \alpha_X \to \{0,1\}$ as follows

- $\left|X \cap S_\alpha^{\sigma_X(\alpha)}\right| = \omega$,
- $\left|X \cap S_\alpha^{1-\sigma_X(\alpha)}\right| < \omega$.

Observation 5.10. If σ_X, σ_Y are incompatible functions then the intersection $X \cap Y$ is finite.

Our aim is to construct the strongly complete separable MAD $\mathcal{M}$ system stepwise in such a way that particular σ_A will be different for different $A \in \mathcal{M}$. It will help with the construction of a new element by restricting to a part of size at most $\mathfrak{s}_{\omega,\omega}$. But first we prove some lemmas.

Lemma 5.11. Consider an AD system $\mathcal{A}$, an $\mathcal{A}^n$-non-small set $X \subset \omega^n$ and $i \in n$. Then there is an index $\alpha \in \mathfrak{s}_{\omega,\omega}$ such that both sets $X \cap \pi_i^{-1}[S_\alpha^0]$, $X \cap \pi_i^{-1}[S_\alpha^1]$ are non-small.

Proof: Consider sets X_k from the characterization of non-small sets 4.7. Then it suffices to use the ω,ω-splitting property to the system of sets $\{\pi_i[X_k] : k \in \omega\}$. The fact that both parts $\pi_i^{-1}[S_\alpha^0]$, $\pi_i^{-1}[S_\alpha^1]$ are still non-small follows from the reverse implication of 4.7. ∎

According to this lemma we may generalize the definition of the function σ.

Definition 5.12. Let $\mathcal{A}$ be an AD system, $i \in n$ and $X \subset \omega^n$ be a non-small set. We define $\alpha_X^{\mathcal{A},i}$ to be the smallest possible α given by the previous lemma. We further define the function $\sigma_X^{\mathcal{A},i} : \alpha_X^{\mathcal{A},i} \to \{0,1\}$ as follows

- $X \cap \pi_i^{-1}\left[S_\alpha^{\sigma_X^{\mathcal{A},i}(\alpha)}\right]$ is $\mathcal{A}^n$-non-small,
- $X \cap \pi_i^{-1}\left[S_\alpha^{1-\sigma_X^{\mathcal{A},i}(\alpha)}\right]$ is $\mathcal{A}^n$-small.

Observation 5.13. If $X \subset Y$, or just $|X \setminus Y| < \omega$, then $\sigma_X^{\mathcal{A},i} \supset \sigma_Y^{\mathcal{A},i}$.

Observation 5.14. If the set X is $\mathcal{A}^n$-missing then $\sigma_X^{\mathcal{A},i} = \sigma_{\pi_i[X]}$.

Lemma 5.15. Let AD system $\mathcal{A}$ have a cardinality less that continuum. Consider an $\mathcal{A}^n$-non-small set $X \subset \omega^n$ and a coordinate $i \in n$. Then there is an $\mathcal{A}^n$-non-small set $Y \subset X$ such that for no $A \in \mathcal{A}$ it happens $\sigma_Y^{\mathcal{A},i} \subset \sigma_A$.

Proof: By lemma 5.11, we divide X to two disjoint non-small sets $X = X_0 \cup X_1$ such that functions $\sigma_{X_0}^{\mathcal{A},i}$, $\sigma_{X_1}^{\mathcal{A},i}$ are incompatible. We continue by dividing $X_0 = X_{00} \cup X_{01}$, $X_1 = X_{10} \cup X_{11}$, $X_{00} = X_{000} \cup X_{001}$, and so on. Formally, we are indexing such sets by functions $k \to \{0,1\}$, where $k \in \omega$. Whenever indexing functions are incompatible also corresponding sets are incompatible.

Take a function $f : \omega \to \{0,1\}$. By corollary 4.10, one can find a non-small pseudo-intersection of all $X_{f \restriction k}$. Let us choose one an denote it by X_f. If two functions $f, g : \omega \to \{0,1\}$ are different they are incompatible. Therefore functions $\sigma_{X_f}^{\mathcal{A},i}$, $\sigma_{X_g}^{\mathcal{A},i}$ are also incompatible. So there are continuum many non-small sets X_f while sets of possible functions

$$\{\tau \supset \sigma_{X_f}^{\mathcal{A},i} : \tau : \{0,1\} \to \alpha \in \mathfrak{s}_{\omega,\omega}\}$$

are disjoint. Consequently one may pick a set X_f such that the corresponding set of functions does not contain any σ_A, where $A \in \mathcal{A}$. It is the required $Y = X_f$. ∎

Lemma 5.16. Consider an AD system $\mathcal{A}$ of cardinality less than continuum. Suppose that for each function σ there is only countably many $A \in \mathcal{A}$ satisfying $\sigma_A = \sigma$. Further consider an $\mathcal{A}^n$-non-small set $X \subset \omega^n$. In addition suppose $\mathfrak{s}_{\omega,\omega} \leq \mathfrak{a}_n$. Then there is an $\mathcal{A}^n$-missing set $Y \subset X$ such that for each $i \in n$ the function $\sigma_{\pi_i[Y]}$ differs from original sets σ_A, where $A \in \mathcal{A}$. Moreover in each pair of sets of $\pi_0[Y], \dots, \pi_{n-1}[Y]$ they are equal or disjoint.

Proof: We begin by applying the previous lemma to X successively for all $i \in n$. By that we get a non-small set Z such that extensions $\tau \supset \sigma_Z^{\mathcal{A},i}$ differs from current σ_A, where $i \in n$, $A \in \mathcal{A}$. Moreover by corollary 4.9 we may assume Z to be injective. We need to construct an $\mathcal{A}^n$-missing subset $Y \subset Z$. By that and observation 5.14, we will assure the requirement to $\sigma_{\pi_i[Y]}$. The additional requirement for disjoint projections will be easily get by lemma 4.20.

For the construction of Y we will use AD systems $\mathcal{A}_i$ of the cardinality less than $\mathfrak{s}_{\omega,\omega}$ chosen as follows. Consider any $i \in n$ and $\alpha < \alpha_Z^{\mathcal{A},i}$.

(1) Up to finitely many elements it is possible to cover the set

$$Z \cap \pi_i^{-1}\left[S_\alpha^{1-\sigma_Z^{\mathcal{A},i}(\alpha)}\right]$$

by finitely many $\mathcal{A}^n$-basic sets. We pick one such coverage and for each basic set $\pi_j^{-1}[A]$ we put A into the system $\mathcal{A}_j$.

(2) The system $\mathcal{A}_i$ also contain all $A \in \mathcal{A}$ such that $\sigma_A = \sigma_Z^{\mathcal{A},i} \restriction \alpha$.

The cardinality of these $\mathcal{A}_i$ is less than or equal to $\left|\max\left\{\alpha_Z^{\mathcal{A},i} : i \in n\right\}\right| < \mathfrak{s}_{\omega,\omega}$. Hence we may use the assumption $\mathfrak{s}_{\omega,\omega} \leq \mathfrak{a}_k$ and find a $\prod_{i\in n} \mathcal{A}_i$-missing set $Y \subset Z$. Lemma 4.20 additionally ensures that simple projections of Y will be equal or disjoint. The proof is completed by showing that Y is also $\mathcal{A}^n$-missing.

Consider an $A \in \mathcal{A}$ and $i \in n$. Set $Y_1 = Y \cap \pi_i^{-1}[A]$. We need to show $|Y_1| < \omega$. If functions $\sigma_Y^{\mathcal{A},i}$, σ_A are incompatible then $A \in \mathcal{A}_i$ by the item (2). Then the fact that Y is $\prod_{i\in n} \mathcal{A}_i$-missing establishes the assertion. Assume conversely that they are incompatible. Then there is an index α such that $\sigma_Y^{\mathcal{A},i}(\alpha) = 1 - \sigma_A(\alpha)$. The set

$$Y_2 = Y \cap \pi_i^{-1}\left[S_\alpha^{\sigma_A(\alpha)}\right]$$

is $\prod_{i\in n} \mathcal{A}_i$-small due to sets added at the item (1). Since Y is $\prod_{i\in n} \mathcal{A}_i$-missing, the set Y_2 is finite. Yet $|Y_1 \setminus Y_2| < \omega$ by the definition of σ_A. So Y_1 is finite. ■

Theorem 5.17. Assume $\mathfrak{s}_{\omega,\omega} \leq \mathfrak{a}_k$, where $1 \leq k \leq \omega$. Then there is an infinite MAD system being completely n-separable for all naturals $1 \leq n \leq k$.

Proof: Take all subsets of finite powers of ω with the exponent k or less and index them by continuum. So for $\alpha \in \mathfrak{c}$ there is a set $X_\alpha \subset \omega^{n_\alpha}$, where $n_\alpha \leq k$. Let us begin with any infinite countable AD system $\mathcal{A}_0$. We proceed by transfinite recursion up to $\mathfrak{c}$. In each isolated step $\alpha + 1 \in \mathfrak{c}$ there are two possibilities. If the set X_α is $\mathcal{A}_\alpha^{n_\alpha}$-small we keep $\mathcal{A}_{\alpha+1} = \mathcal{A}_\alpha$. Otherwise the previous lemma finds an $\mathcal{A}_\alpha^{n_\alpha}$-missing subset $Y \subset X$. Then $\mathcal{A}_\alpha \cup \{\pi_i[Y] : i \in n_\alpha\}$ is an AD system. We set $\mathcal{A}_{\alpha+1}$ to be that. The limit step is the union, as usual. The property "Each function occurs among σ_A, where $A \in \mathcal{A}$ at most countably many times." remains valid since each such function may be added at at most one step. Finally we set $\mathcal{A}$ to be the union of all $\mathcal{A}_\alpha$.

The step α, where X_α is an $\mathcal{A}^{n_\alpha}$-non-small set, assured the property of completely n_α-separable MAD system for the set X_α. Therefore the final AD system is an n_α-separable MAD system for all $n_\alpha \leq k$. ■

Together with the construction 4.23 and the proposition 5.8 we obtain following results.

Corollary 5.18. If inequality $\mathfrak{s} \leq \mathfrak{a}_n$ holds then there is a k-counter-example for all $k \leq n+1$. If even $\mathfrak{s} \leq \mathfrak{a}_\omega$ then there is a strongly completely separable MAD system, hence there is an ω-counter-example and all k-counter-examples.

Corollary 5.19. If $\mathfrak{s} \leq \mathfrak{b}$ there is an ω-counterexample and all k-counterexamples.

Conclusion

The work showed that assumption $\mathfrak{s} \leq \mathfrak{b}$ or $\mathfrak{s} \leq \mathfrak{a}_{k-1}$ suffices for the existence of a k-counter-example. Products of AD systems and terminology for them was introduced. It led to cardinalities $\mathfrak{a}_k$. By intuition, one would expect that individual cardinals $\mathfrak{a}_i$ are closer to each other than to $\mathfrak{b}$. Thus there is following open problem:

Open problem. Does each $\mathfrak{a}_k$ equal to $\mathfrak{a}$?

Other investigation may be made among products of AD systems. AD systems in one dimension are already quite known so some knowledge about them could be generalized.

Yet even previous chapters leaves a space for further investigation. For example it is not clear at all whether the restriction to spaces constructed by AD systems can cause a lost of k-counter-example. In general, it is tricky to characterize an ideal generated by AD system.

Thus the work offers more ways for continuations. It is just the choice of readers which way they pick if they decide to devote their time and talent to AD systems and their products.

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Appendix A

Notation

A.1 Table of terminology

A table translating terms between three used terminologies follows.

topological space $\boldsymbol{X}$	ideal $\mathcal{I}$	AD system $\mathcal{A}$
S is bounced from ∞	$S \in \mathcal{I}$	S is $\mathcal{A}$-small
S converges to ∞	$S \in \mathcal{I}^{\perp}$	S is $\mathcal{A}$-missing
	orthogonal complement (1D)	MAD-ish sets (1D)
$\boldsymbol{X}$ is a Fréchet space	$\mathcal{I}$ is orthogonally closed	$\mathcal{A}$ is nowhere MAD

A.2 Symbols

The list of used symbols follows. Each symbol is followed by short description and the number of the corresponding definition.

ω ... the set of all natural numbers, see 1.0.
$\mathcal{P}(S)$ ... the set of all subsets of S, see 1.0.
ω_1 ... the first uncountable ordinal, see 1.0.
$\mathfrak{c}$ ... the cardinality of reals, see 1.0.
$\psi \subset \varphi$ ... the map ψ is a restriction of the map φ, see 1.0.
$\varphi[X]$ ... the pointwise image of the set X, see 1.0.
π_i ... projection to the i-th coordinate, see 1.0.
π_I ... projection to coordinates from the set I, see 1.0.
$\pi_{\neg i}$ ... projection to all coordinates with the exception of the i-th one, see 1.0.
∞ ... a special point in a topological space, see 1.0.
$\boldsymbol{X}(\mathcal{I})$ ... the space constructed from an ideal $\mathcal{I}$, see 2.3.
$\mathcal{I}(\boldsymbol{X}, x)$ ... the ideal of sets bounced from x in the space $\boldsymbol{X}$, see 2.4.
$\langle A \rangle$ ... the ideal generated by $\mathcal{A}$, see 2.6.
$\mathcal{A}^{\perp}$ ... the orthogonal complement $\mathcal{A}$, see 2.7.
$\overline{\mathcal{A}}$ ... the orthogonal closure of a system $\mathcal{A}$, see 2.9.
$\mathfrak{p}$ ... pseudo-intersection number, see 2.14.
$\boldsymbol{Y}(\mathcal{A})$ ... the space constructed from AD system $\mathcal{A}$, see 3.3.
$\prod_{i \in n} \mathcal{A}_i$ ... a formal product of AD systems, see 4.0.
$\mathfrak{s}$ ... splitting number, see 5.2.
$\mathfrak{s}_{\omega,\omega}$ ... ω,ω-splitting number, see 5.2.
$\mathfrak{a}, \mathfrak{a}_n$ ... first, n-th MAD number, see 5.4.
$\mathfrak{a}_\omega$ ... the minimum of all $\mathfrak{a}_n$, see 5.4.
$\mathfrak{b}$ ... bounding number, see 5.6.
S^0_α, S^1_α ... α-th splitting set, see 5.9.
α_A ... the splitting position of a set A, see 5.9.
σ_A ... function describing the passage of A through splitting sets, see 5.9.
$\alpha_X^{\mathcal{A},i}, \sigma_X^{\mathcal{A},i}$ ... generalization of the previous according to the AD system $\mathcal{A}$, see 5.12.

A.3 Index of terms

The index references to numbers of definitions, in parentheses follows numbers of pages.