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VARIATIONS ON SUMS OF SQUARES

Ivan Niven, University of Oregon

Herbert S. Zuckerman, University of Washington

One of the famous results of number theory, due to Lagrange, is that every positive integer is expressible as a sum of four squares of integers. The "four" here is best possible; not every positive integer is expressible as a sum of three squares. The conditions under which an integer is expressible as a sum of three squares are known: a positive integer n is a sum of three squares if and only if n is **not** of the form

$$4^{\alpha}(7 + 8\beta)$$

where α and β are non-negative integers.

Also well-known are the conditions that must be satisfied in order that an integer n be expressible as a sum of two squares. These conditions involve the factoring of n into prime powers,

$$n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k},$$

where $p_1, p_2, \ldots, p_k$ are distinct primes and the exponents $r_1, r_2, \ldots, r_k$ are positive integers. Then n is expressible as a sum of two squares if and only if every prime factor p_j of the form $4m+3$ has an even exponent r_j, in other words, if every prime factor of n of the form $4m+3$ divides n and even number of times.

Of these three results, the three squares theorem is the most difficult to prove. A proof is given in reference [1]. Proofs of the two squares theorem and the four squares theorem are more readily available, for example in references [3] and [4]. A full account of the historical background of these results is given in Dickson's History of the Theory of Numbers, reference [2].

One point to be noted about these results is that the squares under consideration may be zero. For example, we may say that 21 is a sum of four squares, that is, $21 = 4^2+2^2+1^2+0^2$. Similarly 13 is a sum of three squares, namely $13 = 3^2+2^2+0^2$. In Theorem 1 below we will look at the possibility of expressing integers as sums of non-zero squares of integers, i.e., as sums of squares of positive integers. In Theorem 2 below, we do not require positive squares, but we look at the possibility of replacing one of the squares in the four squares theorem by a k-th power.

To be specific, we prove two results, one based on the classical four squares theorem, the other on the classical three squares theorem, The first result is as follows:

Theorem 1. Every sufficiently large positive integer is a sum of five positive squares of integers. This result is false if 'five" is replaced by "four".

We can be specific as to what is meant here by "sufficiently large". We prove that every integer ≥ 170 is a sum of five positive squares of integers. We leave to the reader the verification of the result that every positive integer except 1, 2, 3, 4, 6, 7, 9, 10, 12, 15, 18, 33 is a sum of five positive squares.

The second result that we prove in this article is the following.

Theorem 2. Every sufficiently large positive integer n is expressible as a sum of three squares and a k-th power (i.e., $n = x^2+y^2+z^2+t^k$) if $k = 2, 4, 6$ or if k is any positive odd integer, This assertion is false if k is an even integer ≥ 8.

Here we cannot be specific about the meaning of "sufficiently large' in any absolute sense; it depends on k. If $k = 1, 2, 4$, or 6 the result holds for any positive integer; that is, the equation $n = x^2+y^2+z^2+t^k$ has a solution in integers x, y, z, t for every positive integer n. But if k is odd, $k\geq 3$, then $n = x^2+y^2+z^2+t^k$ has a solution in integers if $n\geq 3^k$. This condition, that $n\geq 3^k$, may not best possible,

Proof of Theorem 1. Let n be a positive integer >170. Then $n-169$ is a positive integer and by the classical four squares theorem the equation

$$n - 169 = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

has a solution in integers x_1, x_2, x_3, x_4, say with $x_1\geq x_2\geq x_3\geq x_4\geq 0$. If these integers are all positive then we have

$$n = 13^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2,$$

as was to be proved, If x_1, x_2, x_3 are positive, but $x_4 = 0$, then $n = 12^2 + 5^2 + x_1^2 + x_2^2 + x_3^2$. If x_1 and x_2 are positive, but $x_3 = x_4 = 0$, then $n = 12^2 + 4^2 + 3^2 + x_1^2 + x_2^2$. If x_1 is positive but $x_2 = x_3 = x_4 = 0$, then $n = 10^2 + 8^2 + 2^2 + 1^2 + x_1^2$. In every case we have expressed n as a sum of five positive squares.

To complete the proof of Theorem 1 we must show that it is false that every sufficiently large positive integer is a sum of four positive squares. To do this we first show that if r is a positive integer, then 8r is a sum of four positive squares if and only if 2r is such a sum. First if 2r is a sum of four positive squares,

say
$$2r = x_1^2 + x_2^2 + x_3^2 + x_4^2 \qquad \text{then}$$
$$8r = (2x_1)^2 + (2x_2)^2 + (2x_3)^2 + (2x_4)^2.$$

Conversely, suppose that 8r is a sum of four squares, say

$$8r = y_1^2 + y_2^2 + y_3^2 + y_4^2.$$

Now the y_i's must be all even, for if one or more were odd we see that

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 \equiv 1, 2, 3, 4, 5, 6, \text{ or } 7 \pmod 8$$

because any square $y^2 \equiv 0$, 1 or 4 (mod 8). This contradicts $8r \equiv 0 \pmod 8$. So the y_i's are all even and we have

$$2r = \left(\frac{y_1}{2}\right)^2 + \left(\frac{y_2}{2}\right)^2 + \left(\frac{y_3}{2}\right)^2 + \left(\frac{y_4}{2}\right)^2$$

It follows that if we exhibit a number 2r that is not a sum of four positive squares, then none of $8r, 32r, 128r, \ldots, 2^{2s+1}r, \ldots$ is a sum of four positive squares. Take $r = 4$: we see that 8 is not a sum of four positive squares, and thus we conclude that 2^{2s+1} is not a sum of four positive squares for $s = 1, 2, 3, \ldots$.

Proof of Theorem 2, The only integers n that need to be considered are those of the form

$$4^\alpha (7 + 8\beta),$$

with α and β non-negative integers, because all other positive integers are sums of three squares. It is clear that a sum of three squares can be regarded as a sum of three squares and a k-th power; simply take the k-th power to be zero.

Now, if $n = 7 + 8\beta$, then $n-1 = 6 + 8\beta$, so that $n-1$ is a sum of three squares, say

$$n-1 = x^2 + y^2 + z^2.$$

Thus we have $n = x^2 + y^2 + z^2 + 1^k$.

Next, if $n = 4(7+8\beta)$, then $n-1 = 27 + 32\beta = 3 + 8\gamma$ where $\gamma = 3 + 4\beta$. Once again $n-1$ is a sum of three squares, and the result follows as in the previous case.

In view of these two cases we need consider only integers of the form $4^\alpha(7 + 8\beta)$ with $\alpha\geq 2$, and no others.

Case 4. k odd. We observe that $3 \equiv 1 \pmod 8$, and so $3^k \equiv 3 \pmod 8$. Now $n = 4^\alpha(7 + 8\beta)$, so that $n \equiv 0 \pmod 8$, and we obtain

$$n - 3^k \equiv 0 - 3 \equiv 5 \pmod 8.$$

Hence $n - 3^k = x^2 + y^2 + z^2$ has a solution in integers x, y, z if $n\geq 3^k$.

Case 2. k = 2. In this case Theorem 2 is just the classical four squares theorem.

Case 3. k = 4. We treat even a and odd a separately. If a is even, say a = 2γ, then

$$n - (2^{\gamma})^4 = n - 4^{\alpha} = 4^{\alpha}(6 + 8\beta).$$

Hence $n - (2^{\gamma})^4 = x^2 + y^2 + z^2$ has a solution in integers x, y, z.

If a is odd, say a = $1 + 2\gamma$ then

$$n - (2^{\gamma})^4 = 4^{\alpha}(7 + 8\beta) - (2^{\gamma})^4 = 4^{2\gamma}(3 + 8\lambda)$$

where $\lambda = 3 + 4\beta$. Again we see that $n - (2^{\gamma})^4$ is expressible as a sum of three suqares.

Case 4. k = 6. When a-2 is divided by 3, let the quotient be γ. The remainder may be 0, 1 or 2 and so we have $a = 2 + 3\gamma$, $3 + 3\gamma$ or $4 + 3\gamma$. Then we see that in these three **subcases** we have

$$n - (2^{1+\gamma})^6 = 4^{\alpha}(3 + 8\beta),\ 4^{\alpha}(6 + 8\beta) \text{ or } 4^{\alpha-1}(3 + 8\lambda)$$

where $\lambda = 3 + 4\beta$. Hence there are integers x, y, z such that

$$n - (2^{1+\gamma})^6 = x^2 + y^2 + z^2 \text{ or } n = x^2 + y^2 + z^2 + (2^{1+\gamma})^6.$$

Notice that this completes all the cases in which n is to be shown equal to a sum of three squares and a k-th power.

Case 5, k even, $k \geq 8$. If t is any odd positive integer then $t^2 \equiv 1 \pmod 8$ and so $t^k \equiv 1 \pmod 8$. If t is any even non-negative integer then $t^k \equiv 0 \pmod{2^k}$, and this implies that $t^k = 2^7 \cdot q$. We prove that if $n = 4^2(7 + 8\beta)$ with $\beta \geq q$ then $n-t^k$ is not a sum of three squares, no matter what t is. If t is odd we see that

$$n - t^k \equiv 0 - 1 \equiv 7 \pmod 8.$$

If t is even, we observe that

$$n - t^k = 16(7 + 8\beta) - 128q = 16\{7 + 8(\beta - q)\}.$$

In neither case is $n-t^k$ equal to a sum of three squares, so that n is not representable as a sum of three squares and a k-th power.

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3. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, 4th edition, Oxford, 1960, Chapter 20.
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THE DISTRIBUTION OF n-PRIMES

George F. Grob
Georgetown University

The distribution of primes among the integers has never been satisfactorily described by mathematicians. It appears that it may be useful to consider the primes as a subset of some appropriate set of integers whose distribution can more readily be accounted for and whose properties are interesting in themselves.

Label the primes $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11$, etc. If an integer a is relatively prime to the first n primes, let us call a an n-prime.

Clearly, for any $n \in Z$, the integers, the set of primes larger than p_n is a subset of the set of n-primes.

Unlike the primes which seem to follow no pattern, there is a pattern for n-primes, as the following theorem indicates.

Theorem 1: An integer a is an n-prime if and only if $MP_n + a$ is an n-prime, for all $M \in Z$, and $P_n = p_1 p_2 \cdots p_n$.

Proofs Suppose a is an n-prime. Assume $MP + a$ is **not an** n-prime for some $M \in Z$. Then $p_i | MP_n + a$, for some $i = 1,2,\ldots,n$. But $p_i | MP_n$. So $p_i | MP_n + a - MP_n$, or $p_i | a$, a contradiction to the fact that a is an n-prime. The converse is proved in the same way.

Let $P_n^* = \{x \in Z | 1 \leq x \leq P_n\}$. Once we know the distribution of n-primes in P_n^*, then the pattern established there will be repeated every P consecutive integers. We shall now investigate the distribution of n-primes in P_n^*.

Theorem 2: An integer, $a \in P_n^*$ is an n-prime if and only if $(P_n - a)$ is an n-prime.

The proof of Theorem 2 is similar to the proof of 1 above.

Theorem 3: There are $(p_2-1)(p_3-1)\ldots(p_n-1)$ n-primes in P^*.

Proof: The number of n-primes in P_n^* is equal to the number of positive integers relatively prime to and smaller than P_n. Using Euler's ϕ function, there are $\phi(P_n) = (p_2-1)(p_3-1)\ldots(p_n-1)$ such integers.

We may ask about the distribution of n-primes almost any question asked about the distribution of primes. A famous unanswered question about primes is: 'How many twin primes are there?' By twin primes we mean a pair of primes a and a+2. We can similarly define a pair of twin n-primes to be a pair of n-primes a and a+2. The following theorem partially describes the distribution of twin n-primes among the integers.

Theorem 4: There are $(p_3-2)(p_4-2)\ldots(p_n-2)$ integers $x \in P^*$ such that $x-1$, $x+1$ is a pair of twin n-primes.

Proof: Let $A_i = \{x \in Z \mid 0 \le x \le p_i - 1\}$. Define

$$f: Z \to A_1 \times A \times \ldots \times A_n \quad \text{by:}$$

for $a \in Z$, $f(a) = (a_1, a_2, \ldots, a\ )$ where for $1 \le i \le n$, a_i is the remainder when a is divided by p_i. By the Chinese remainder theorem, f is onto. Riven the n-tuple $a' = (a_1, a_2, \ldots, a_n)$, let a be such that $f(a) = a'$. Then, since $p_i \mid MP_n$ for all $1 \le i \le n$, $f(MP_n + a) = a'$ for all $M \in Z$. Choose M_o such that $1 \le f(M_o P_n + a) \le P_n$. Then for each $a' \in (A_1 \times A_2 \times \ldots \times A_n)$, there is some $1 \le a \le P$ such that $f(a) = a'$. Let f^* be the restriction of f to P^*. Then

$$f^*: P_n^* \to A_1 \times A_2 \times \ldots \times A_n$$

is onto, and since

$$\text{card}(A_1 \times A_2 \times \ldots \times A\ ) = P_n = \text{card}\ (P_n^*),$$

the mapping f^* 3s a bijection.

It is clear that $x \in Z$ is an n-prime if and only if $f(x)$ has no zero entries. Then for $x \in P_n^*$, the pair $x-1$, $x+1$ is a pair of twin n-primes if and only if $f^*(x) = f(x)$ never has 1 or $p_i - 1$ as its i-th entry, where $1 \le i \le n$. By counting all n-tuples without such entries, the theorem is proved.

[Student Paper presented at the National Meeting, August, 1968, Madison, Wisconsin.]

UNDERGRADUATE RESEARCH PROPOSAL

Proposed by Leon Bankoff

The problem of locating a point which minimizes the sum of the distances from three fixed points to the point in question is well known. There is a nice geometric solution to the problem.

Modify the problem so that the point lies on a fixed line (or curve). An analytical solution would be straightforward (though perhaps messy). Is there a geometric solution to the location?

This would be more picturesquely stated in the following way. Where along a straight railroad line should one locate a station to serve three neighboring towns such that the sum of the distances from the three towns to the station is a minimum?

CORRECTION for the Fall, 1968 Undergraduate Research Project: In the Undergraduate Research Project in the Fall, 1968 issue part of the formula was omitted. The displayed formula should read

$$\binom{[m/2]}{2} + \binom{[(m+1)/2]}{2} = \left[\frac{m}{2}\right]\left[\frac{m-1}{2}\right]$$

AN EXTENSION OF EULER'S PHI-FUNCTION

Andrew E. Rouse
University of Mississippi

Euler's phi- function, written $\phi(M)$ for M a positive integer, gives the number of positive integers less than or equal to an integer M (>PO) and relatively prime to M.

Now consider the problem of finding the number of positive integers relatively prime to M ($M = P_1^{\alpha_1} P_2^{\alpha_2} \ldots P_N^{\alpha_N}$ where the P's are primes), $M>1$, and less than an integer Y where $Y = KP_1P_2\ldots P_N$, K a positive integer. This function will bc read Euler's phi-function with respect to Y and will be written $\phi^Y(M)$. Obviously, Euler's phi-function is the special case of Euler's phi-function with respect to Y occurring when $Y = M$.

Theorem 1: $\phi^{a_1+a_2+\ldots+a_n}(M) = \phi^{a_1}(M) + \phi^{a_2}(M) + \ldots + \phi^{a_n}(M)$, where $M = P_1^{\alpha_1} P_2^{\alpha_2} \ldots P_N^{\alpha_N}$ (the P's are primes) and $a_1 = K_1(P_1P_2\ldots P_N)$, $a_2 = K_2(P_1P_2\ldots P_N), \ldots, a_n = K_n(P_1P_2\ldots P_N)$, (the K's are positive integers). $a_1 < a_2 < \ldots < a_n$.

Proof: Let $a_2 + \ldots + a_n = Y$. So $\phi^{a_1+a_2+\ldots+a_n}(M) = \phi^{a_1+Y}(M)$. Let $X_1, X_2, X_3, \ldots, X_{\phi^Y(M)}$ be the integers less than Y and relatively prime to M.

Obviously $a_1 + X_1$, $a_1 + X_2$, $a_1 + X_3, \ldots, a_1\ \ X_{\phi^Y(M)}$ are also relatively prime to M. So there are at least $\phi^Y(M)$ intcgers between a and $a_1 + Y$ which are relativcly prime to M.

Assume L is an integer less than or equal to Y and $(L,M) \neq 1$. Then $(a_1+L, M) \neq 1$ since a_1 is a multiple of all the prime divisors of M and $(L,M) \neq 1$.

Thus there are exactly $\phi^Y(M)$ integers greater than a_1 and less than $a_1 + Y$ which are relatively prime to M.

It follows that $\phi^{a_1+Y}(M) = \phi^{a_1}(M) + \phi^Y(M)$. Hence by induction $\phi^{a_1+a_2+\ldots+a_n}(M) = \phi^{a_1}(M) + \phi^{a_2}(M) + \ldots + \phi^{a_n}(M)$.

Example: $\phi^{36}(6) = \phi^{6+12+18}(6) = 12$, where 1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35 are relatively prime to 6.

$\phi^6(6) = 2$ where 1 and 5 are relatively prime to 6. $\phi^{12}(6) = 4$ where 1, 5, 7, 11 are relatively prime to 6. $\phi^{18}(6) = 6$ where 1, 5, 7, 11, 13, 17 are relatively prime to 6. $\phi^6(6) + \phi^{12}(6) + \phi^{18}(6) = 12 = \phi^{6+12+18}(6)$.

Lemma 1: $\phi^X(M)\cdot\phi^Y(N) = \phi^{XY}(MN)$ where $M = P_1^{\alpha_1}P_2^{\alpha_2}\ldots P_m^{\alpha_m}$ (P's are primes), $N = Q_1^{\beta_1}Q_2^{\beta_2}\ldots Q_n^{\beta_n}$ (Q's are primes), $X = P_1P_2\ldots P_m$, $Y = Q_1Q_2\ldots Q_n$ and $(M,N) = 1$.

Proof: List the integers up to XY in the following manner:

$$\begin{array}{cccccc}
1 & 2 & 3\ \ldots & h\ \ldots & & X \\
X+1 & X+2 & X+3\ \ldots & X+h\ \ldots & & 2X \\
2X+1 & 2X+2 & 2X+3\ \ldots & 2X+h\ \ldots & & 3X \\
\vdots & & & & & \vdots \\
(Y-1)X+1 & (Y-1)X+2 & (Y-1)X+3\ldots & (Y-1)X+h\ldots & & XY
\end{array}$$

There are $\phi^X(M)$ integers in the first row relatively prime to It. Now examine a column with one of these $\phi^X(M)$ integers at the top, say the hth column. Obviously every integer in such a column is also relatively prime to M. Consider

$$SX + h = q_SY + r_S,\ r_S<Y$$
$$TX + h = q_TY + r_T,\ r_T<Y.$$

If $r_S = r_T$, then $(S-T)X = (q_S-q_T)Y$. This implies $Y|(S-T)X$. But $0\le S<Y$ and $0\le T<Y$ and $(X,Y) = 1$ since $(M,N) = 1$. So $Y|(S-T)X \Rightarrow S = T$. For $S \ne T$, $r_S \ne r_T$.

Thus no two remainders, upon division of the integers in the hth column by Y, are equal. Since there are Y remainders they must be 0, 1, 2,...,$(Y-1)$ in some order.

Obviously whether an integer, $SX + h$, in the column is relatively prime to N depends on whether or not the remainder, upon division by Y, is relatively prime to N. Since the remainders range from 0 to $Y-1$ there are $\phi^Y(N)$ integers in each such column relatively prime to B and so relatively prime to AB.

Since there are $\phi^X(M)$ columns, each with $\phi^Y(N)$ integers relatively prime to MN

$$\phi^X(M)\cdot\phi^Y(N) = \phi^{XY}(MN).$$

Lemma 2: $\phi^L(M)\cdot\phi^L(N) = L4^L(MN)$ where $M = P_1^{\alpha_1}P_2^{\alpha_2}\ldots P_m^{\alpha_m}$ (P's are primes), $N = Q_1^{\beta_1}Q_2$^...I)''' (Q's are primes), $L = K(P_1P_2\ldots P_m)(Q_1Q_2\ldots Q_n)$, (K is a positive integer). and $(M,N) = 1$.

Proof:

$$\phi^L(M) = \phi^{K(P_1P_2\ldots P_m)(Q_1Q_2\ldots Q_n)}(M)$$
$$= K(Q_1Q_2\ldots Q_n)\phi^{P_1P_2\ldots P_m}(M) \quad \text{[by Theorem 1].}$$
$$\phi^L(N) = \phi^{K(P_1P_2\ldots P_m)(Q_1Q_2\ldots Q_n)}(N)$$
$$= K(P_1P_2\ldots P_m)\phi^{Q_1Q_2\ldots Q_n}(N) \quad \text{[by Theorem 1].}$$
$$\phi^L(M)\cdot\phi^L(N) = K(Q_1Q_2\ldots Q_n)\phi^{P_1P_2\ldots P_m}(M)\cdot K(P_1P_2\ldots P_m)\phi^{Q_1Q_2\ldots Q_n}(N)$$

$$= K^2(P_1P_2\ldots P_m)(Q_1Q_2\ldots Q_n)[\phi^{P_1P_2\ldots P_m}(M)\cdot\phi^{Q_1Q_2\ldots Q_n}(N)]$$
$$= KL\phi^{P_1P_2\ldots P_m}(M)\cdot\phi^{Q_1Q_2\ldots Q_n}(N)$$
$$= KL\phi^{(P_1P_2\ldots P_m)(Q_1Q_2\ldots Q_n)}(MN) \quad \text{[by Theorem 1].}$$
$$= L\phi^{K(P_1P_2\ldots P_m)(Q_1Q_2\ldots Q_n)}(MN) \quad \text{[by Lemma 1].}$$
$$= L\phi^L(MN).$$

Theorem 2: $\phi^X(M) = X\left(\frac{P_1-1}{P_1}\right)\left(\frac{P_2-1}{P_2}\right)\ldots\left(\frac{P_N-1}{P_N}\right)$, $M>1$ where $M = P_1^{\alpha_1}P_2^{\alpha_2}\ldots P^{\alpha m}$ (P's are primes) and $X = KP_1P_2\ldots P_m$, K a positive integer.

Proof: (By induction on m). Let $M = P_1^{\alpha_1}$. Then $X = KP_1$ where K is a positive integer. Since all integers except multiples of P_1 will be relatively prime to $P_1^{\alpha_1}$,

$$\phi^X(P_1^{\alpha_1}) = KP_1 - K = KP_1\left(1 - \frac{1}{P_1}\right) = X\left(\frac{P_1-1}{P_1}\right).$$

Now let $M = P_1^{\alpha_1}\ldots P^{\alpha m}$. Then $X = KP_1P_2\ldots P$ where K is a positive integer. Assume

$$\phi^X(M) = X\left(\frac{P_1-1}{P_1}\right)\left(\frac{P_2-1}{P_2}\right)\ldots\left(\frac{P_m-1}{P_m}\right).$$

Let $M' = P_1^{\alpha_1}P_2^{\alpha_2}\ldots P_m^{\alpha_m}P_{m+1}^{\alpha_{m+1}}$. Then $X' = K'P_1P_2\ldots P_mP_{m+1}$ where K' is a positive integer.

$\phi^X(M) = X\left(\frac{P_1-1}{P_1}\right)\left(\frac{P_2-1}{P_2}\right)\ldots\left(\frac{P_m-1}{P_m}\right)$. Replace X by $X' = (K'P_{N+1})P_1P_2\ldots P_m$.

Then $\phi^{X'}(M) = X'\left(\frac{P_1-1}{P_1}\right)\left(\frac{P_2-1}{P_2}\right)\ldots\left(\frac{P_m-1}{P_m}\right)$. But $\phi^{X'}(P_{m+1}^{\alpha_{m+1}}) = X'\left(\frac{P_{m+1}-1}{P_{m+1}}\right)$.

Multiplying gives

$$\phi^{X'}(M)\cdot\phi^{X'}(P_{m+1}^{\alpha_{m+1}}) = X'\left[\left(\frac{P_1-1}{P_1}\right)\left(\frac{P_2-1}{P_2}\right)\quad\left(\frac{P_m-1}{P_m}\right)\right]X'\left(\frac{P_{m+1}-1}{P_{m+1}}\right).$$

By Lemma 2 $\phi^{X'}(M)\cdot\phi^{X'}(P_{m+1}^{\alpha_{m+1}}) = X'\phi^{X'}(M\cdot P_{m+1}^{\alpha_{m+1}})$. Hence

$X'\phi^{X'}(M\cdot P_{m+1}^{\alpha_{m+1}}) = X'^2\left(\frac{P_1-1}{P_1}\right)\left(\frac{P_2-1}{P_2}\right)\ldots\left(\frac{P_m-1}{P_m}\right)\left(\frac{P_{m+1}-1}{P_{m+1}}\right)$. So

$$\phi^{X'}(M') = X'\left(\frac{P_1-1}{P_1}\right)\left(\frac{P_2-1}{P_2}\right)\ldots\left(\frac{P_m-1}{P_m}\right)\left(\frac{P_{m+1}-1}{P_{m+1}}\right).$$

Example: $\phi^{10}(40) = 10[(2-1)/2][(5-1)/5] = 10(1/2)(4/5) = 4$. 1, 3, 7, 9 are relatively prime to 40.

Theorem 3: $\phi^{X}(M)\cdot\phi^{Y}(N) = \phi^{XY}(MN)$ where $(M,N) = 1$, $M = P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_m^{\alpha_m}$, $N = Q_1^{\beta_1}Q_2^{\beta_2}\cdots Q_n^{\beta_n}$, $X = K_1P_1P_2\cdots P_m$ and $Y = K_2Q_1Q_2\cdots Q_n$ (K' are positive integers, P's and P's are primes.)

Proof: The proof follows immediately from Theorem 2.

$$\phi^{X}(M)\cdot\phi^{Y}(N) = X\left[\left(\frac{P_1-1}{P_1}\right)\left(\frac{P_2-1}{P_2}\right)\cdots\left(\frac{P_m-1}{P_m}\right)\right]\cdot Y\left[\left(\frac{Q_1-1}{Q_1}\right)\left(\frac{Q_2-1}{Q_2}\right)\cdots\left(\frac{Q_n-1}{Q_n}\right)\right]$$

$$= XY\left[\left(\frac{P_1-1}{P_1}\right)\left(\frac{P_2-1}{P_2}\right)\cdots\left(\frac{P_m-1}{P_m}\right)\left(\frac{Q_1-1}{Q_1}\right)\left(\frac{Q_2-1}{Q_2}\right)\cdots\left(\frac{Q_n-1}{Q_n}\right)\right]$$

$$= \phi^{XY}(MN).$$

Example: $\phi^{6}(8)\cdot\phi^{30}(15) = 3\cdot 16 = 48.$ $\phi^{180}(120) = 48.$

Euler's phi-function with respect to Y probably can be applied to related areas in number theory. One of the possible areas to which the function might be applied is primitive roots. Let us define a concept which somewhat overlaps primitive roots called very primitive roots.

Definition: Let a and M be two relatively prime positive integers. If the exponent to which a belongs modulo M is $\phi^{L}(M)$ where $M = P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_m^{\alpha_m}$ and $L = P_1P_2\cdots P_m$, a is said to be a **very primitive root** modulo M.

For those interested, the following problem is open for further research. Prove or disprove:

Conjecture: For $M = P^{\alpha}, 2P^{\alpha}$ where P is an odd prime, there exist $\phi^{\phi^{L}(M)}[\phi^{L}(M)]$ very primitive roots modulo M, $L = P, 2P$ respectively.

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[Student Paper presented at the National Meeting, August, 1968, Madison, Wisconsin.]

RECURSIONS ASSOCIATED WITH PASCAL'S PYRAMID

Stephen Mueller

Wisconsin State University, Oshkosh

As developed previously in a paper by Raab [1], the sum of the terms on certain diagonals through a generalized Pascal triangle (Fig. 1) can be found by means of the formula

$$1$$
$$a \qquad b$$
$$a^2 \qquad 2ab \qquad b^2$$
$$a^3 \qquad 3a^2b \qquad 3ab^2 \qquad b^3$$
$$\cdots$$

Figure 1

$$(1)\quad X_{J,R} = \sum_{k=0}^{\left[\frac{J}{R+1}\right]} \binom{J-Rk}{k} a^{J-k(R+1)}\, b^{k}$$

where R is a non-negative integer.

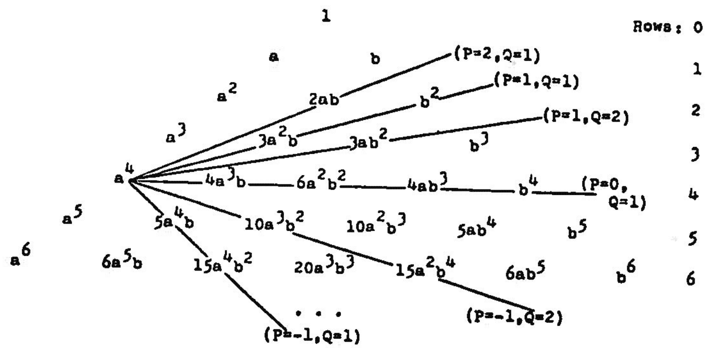
It was then shown that certain recursions exist between parallel diagonals. It is the purpose of this article to find a formula which will yield the sum of the terms on any plane through a generalized Pascal pyramid (the trinomial analogue of Pascal's triangle) and then to find a recursion formula with respect to parallel planes. First, let us define the diagonal with slope P/Q.

Definition 1. $\{J,P/Q\}_J$ represents the set of diagonals determined by the first term in the J-th row, a^J, and the (Q+1)-st term in the (J-P)-th row, $\binom{J-P}{Q} a^{J-(P+Q)} b^{Q}$, where J,P,Q are integers with $J,Q \geq 1$.

Using this definition let us find an expression which will yield the sum of the terms on the diagonal, J,P/Q, which radiates from a^J. We can see from Fig. 2 that we may have negative values for P. But since we are concerned only with finite sums, we shall consider only values of P/Q greater than -1. In Fig. 2 the diagonals shown radiate from the first term in the fourth row, a^4.

Figure 2 Generalized Pascal triangle

J=4 for all diagonals shown



In view of Fig. 2 we may obtain

$$(2)\quad X_{J,P/Q} = \sum_{m=0}^{\left[\frac{J}{P+Q}\right]} \binom{J-Pm}{Qm} a^{J-m(P+Q)}\, b^{Qm}\ , \quad P/Q>-1.$$

Notice that when P = R and Q = 1 (2) reduces to (1). Next we define Pascal's pyramid.

Definition 2. Pascal's pyramid is the three-faced pyramidal array of coefficients in the expansion of the trinomial, $(a+b+c)^J$, such that the coefficients of $(a+b+c)^J$ are systematically placed beneath those of $(a+b+c)^{J-1}$, resulting in a Pascal triangle on each of the three faces.

Analogous to the rows of Pascal's triangle are the levels of Pascal's pyramid. In Fig. 3 we have a generalized Pascal pyramid to the third level.

In the generalized Pascal triangle we sought the sum of the terms on any diagonal through the triangle. Now we are concerned with finding the sum of the terms on any plane passing through the pyramid. Clearly, there are planes passing through the pyramid which have no. terms on them, but we are interested' only in planes that do. Just as it is possible to find diagonals through the generalized Pascal triangle whose first term is not a^J, so it is also possible to find planes through the generalized Pascal Pryamid whose term closest to one of the three lateral edges is not a^J, b^J, or c^J. So here we will require that the plane contain the term, a^J, which will be the reference point for the plane.

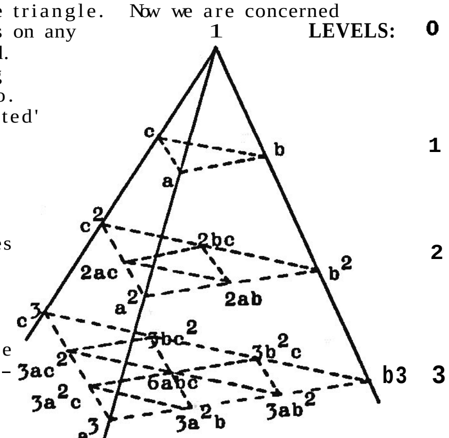
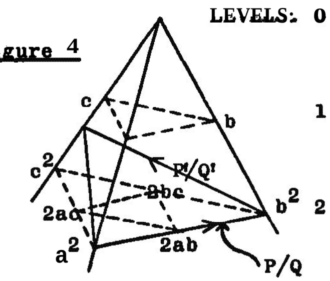


Figure 3

Although a plane passing through the pyramid creates three diagonals on the three faces (Fig. 4) only two diagonals are needed to determine the plane and the terms on the plane. From the reference point, a^J, let P/Q be the first diagonal taken from a^J towards the edge containing b and let P'/Q' be the second diagonal from the b edge toward the edge containing c. Thus we will designate planes by J,P/Q, P'/Q'.

Before we give a more concrete definition of J,P/Q, P'/Q', let us examine the terms in the expansion of $(a+b+c)^J$ a little more closely.

Figure 4



Plane of Fig. 4: $2,\frac{P}{Q},\frac{P'}{Q'}$

$$(a+b+c)^J = \sum_{m=0}^{J}\binom{J}{m} a^{J-m}(b+c)^m = \sum_{m=0}^{J}\binom{J}{m} a^{J-m}\sum_{n=0}^{m}\binom{m}{n} b^{m-n}c^n =$$

$$\sum_{m=0}^{J}\sum_{n=0}^{m}\binom{J}{m}\binom{m}{n} a^{J-m}b^{m-n}c^n.$$

This shows that each trinomial coefficient may be expressed as the product of two Binomial coefficients. This fact is used in the following definition.

Definition 3. $\{J,P/Q,\ P'/Q'\}_J$ represents the set of planes through the generalized Pascal pyramid determined by the three terms a^J, $\binom{J-Pm}{Qm} a^{J-m(P+Q)} b^{Qm}$ and $\binom{J-Pm-P'}{Qm-P'}\binom{Qm-P'}{Q'} a^{J-m(P+Q)} b^{Qm-(P'+Q')} c^{Q'}$ for some positive integer $m \le \left[\frac{J}{P+Q}\right]$ where J,P,Q,P' and Q' are integers with $J,Q,Q' \ge 1$.

For example, the terms a^4, b^2 and c (corresponding to m = 2) determine the plane 4,1/1,1/1 while a^4 and $3a^2b$ (corresponding to m = 1) do not. Furthermore, we shall designate the sum of the terms on a particular plane J, P/Q, P'/Q' by $X_{J,P/Q,\ P'/Q'}$.

$$\begin{array}{ccccc} & & c^4 & & \\ & 4ac^3 & & 4bc^3 & \\ 6a^2c^2 & & 12abc^2 & & 6b^2c^2 \\ 4a^3c & 12a^2bc & , & 12ab^2c & 4b^3c \\ a4 \quad 4a^3b & & 6a^2b^2 & & 4ab^3 \quad b^4 \end{array}$$

Figure 5

If we look at the terms in the plane 4,0/1,0/1 we find that they may be arranged as in Fig. 5. This array is the fourth level in Fig. 7. The coefficients of Fig. 5 may then be replaced by those of Fig. 6.

$$\binom{4}{4}\binom{4}{4}c^4$$

$$\binom{4}{3}\binom{3}{3}ac^3 \qquad \binom{4}{4}\binom{4}{3}bc^3$$

$$\binom{4}{2}\binom{2}{2}ac^2 \qquad \binom{4}{3}\binom{3}{2}abc^2 \qquad \binom{4}{4}\binom{4}{2}b^2c^2$$

$$\binom{4}{1}\binom{1}{1}a^3c \qquad \binom{4}{2}\binom{2}{1}a^2bc \qquad \binom{4}{3}\binom{3}{1}ab^2c \qquad \binom{4}{4}\binom{4}{1}b^3c$$

$$\binom{4}{0}\binom{0}{0}a^4 \qquad \binom{4}{1}\binom{1}{0}a^3b \qquad \binom{4}{2}\binom{2}{0}a^2b^2 \qquad \binom{4}{3}\binom{3}{0}ab^3 \qquad \binom{4}{4}\binom{4}{0}b^4$$

Figure 6

These terms are obtained, of course, from

$$X_{4,0/1,0/1} = X_{4,0,0} = (a+b+c)^4 = \sum_{m=0}^{4}\sum_{n=0}^{m}\binom{4}{m}\binom{m}{n} a^{4-m} b^{m-n} c^n .$$

That a^4, $\binom{4}{1}a^3b$ and $\binom{4}{1}\binom{1}{1}a^3c$ determine this plane can be easily verified by definition 3.

In general, the sum of the terms on any plane with diagonals $P/Q = P'/Q' = 0/1 = 0$ can be found by

$$(3)\quad X_{J,0,0} = \sum_{m=0}^{J}\sum_{n=0}^{m}\binom{J}{m}\binom{m}{n} a^{J-m} b^{m-n} c^n = (a+b+c)^J .$$

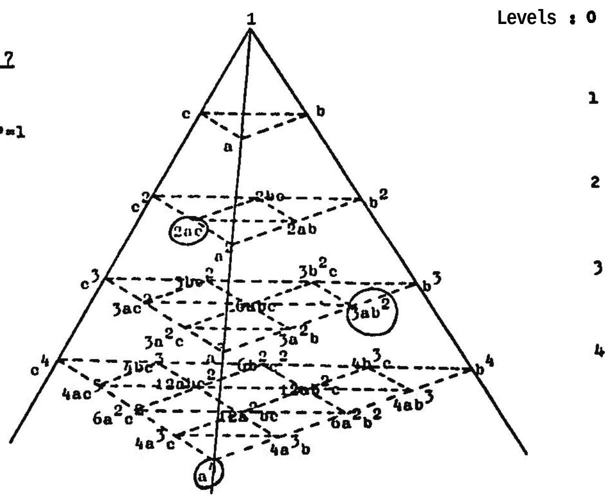
Let us look at the terms in the plane 4,1/2,1/1. These terms are circled in the four levels of the generalized Pascal pyramid in Fig. 7.

Figure 7

J=4

P=P'=Q'=1

Q=2



Expressing the coefficients of these terms as products of binomial coefficients, we have

$$X_{4,1/2,1/1} = \binom{4}{0}\binom{0}{0} a^4 + \binom{3}{2}\binom{2}{0} ab^2 + \binom{2}{1}\binom{1}{1} ac .$$

These three terms also happen to determine the plane 4,1/2,1/1. Comparing these terms with definition 3 we find that we may write this sum as

$$X_{4,1/2,1/1} = \sum_{m=0}^{1}\sum_{n=0}^{m}\binom{4-m-n}{2m-n}\binom{2m-n}{n} a^{4-3m} b^{2m-2n} c^n .$$

The term $\binom{J-Pm-P'}{Qm-P'}\binom{Qm-P'}{Q'} a^{J-m(P+Q)} b^{Qm-(P'+Q')} c^{Q'}$ in definition 3 corresponds to the case m = m, n = 1 of the general term

$$\binom{J-Pm-P'n}{Qm-P'n}\binom{Qm-P'n}{Q'n} a^{J-m(P+Q)} b^{Qm-n(P'+Q')} c^{Q'n}$$

which was found by using empirical methods. This sugests that $X_{J,P/Q,P'/Q'}$ will be a double summation over m and n. We still need to find the upper limits of the summations. Any limits we obtain must not violate these conditions: i.) both h and k of $\binom{h}{k}$ must be non-negative integers with $h \geq k$ and ii.) all exponents of a, b and c must be non-negative integers. These conditions are met when $J-Pm-P'n \geq Qm-P'n$ or $m \leq J/(P+Q)$ and when $J-Pm-P'n \geq Q'n$ or $n \leq (J-Pm)/(P'+Q')$. Since we are summing over integers we shall use $\left[\frac{J}{P+Q}\right]$ and $\left[\frac{J-Pm}{P'+Q'}\right]$ to eliminate problems with fractions.

In general, then, the sum of the terms $X_{J,P/Q,P'/Q'}$ is given by

$$(4)\quad X_{J,P/Q,P'/Q'} = \sum_{m=0}^{\left[\frac{J}{P+Q}\right]}\sum_{n=0}^{\left[\frac{J-Pm}{P'+Q'}\right]}\binom{J-Pm-P'n}{Qm-P'n}\binom{Qm-P'n}{Q'n} a^{J-m(P+Q)} b^{Qm-n(P'+Q')} c^{Q'n}$$

where $P/Q,\ P'/Q' > -1$. It is easily seen that (4) reduces to (2) when the diagonal P'/Q' aoes not exist.

In [1] it was shown that for sequences $X^{J,R,J}$ of sums of terms on parallel diagonals of the generalized Pascal triangle,

$$X_{J,R} = aX_{J-1,R} + bX_{J-(R+1),R}.$$

Likewise, if we let Q and Q' of (4) be 1, (4) becomes

$$(5)\quad X_{J,P,P'} = \sum_{m=0}^{\left[\frac{J}{P+1}\right]}\sum_{n=0}^{\left[\frac{J-Pm}{P'+1}\right]}\binom{J-Pm-P'n}{m-P'n}\binom{m-P'n}{n} a^{J-m(P+1)} b^{m-n(P'+1)} c^n$$

Using this expression we may obtain the following:

<u>Theorem</u>. For sequences of sums of terms $X_{J,P,P'}$,

$$(6)\quad X_{J,P,P'} = aX_{J-1,P,P'} + bX_{J-(P+1),P,P'} + cX_{J-(P+1)(P'+1),P,P'}$$

Proof:

$$aX_{J-1,P,P'} = \sum_{m=0}^{\left[\frac{J-1}{P+1}\right]}\sum_{n=0}^{\left[\frac{J-Pm-1}{P'+1}\right]}\binom{J-Pm-P'n-1}{m-P'n}\binom{m-P'n}{n} a^{J-m(P+1)} b^{m-n(P'+1)} c^n$$

$$= \sum_{m=0}^{\left[\frac{J}{P+1}\right]}\sum_{n=0}^{\left[\frac{J-Pm}{P'+1}\right]}\binom{J-Pm-P'n-1}{m-P'n}\binom{m-P'n}{n} a^{J-m(P+1)} b^{m-n(P'+1)} c^n$$

$$bX_{J-(P+1),P,P'} = \sum_{m=0}^{\left[\frac{J-P-1}{P+1}\right]}\sum_{n=0}^{\left[\frac{J-Pm-P-1}{P'+1}\right]}\binom{J-Pm-P'n-P-1}{m-P'n}\binom{m-P'n}{n} a^{J-m(P+1)-P-1} b^{m-n(P'+1)+1} c^n$$

$$= \sum_{m=0}^{\left[\frac{J}{P+1}\right]} \sum_{n=0}^{\left[\frac{J-Pm}{P'+1}\right]} \binom{J-Pm-P'n-1}{m-P'n-1} \binom{m-P'n-1}{n} a^{J-m(P+1)} b^{m-n(P'+1)} c^{n};$$

$$cX_{J-(P+1)(P'+1),P,P'} =$$

$$\sum_{m=0}^{\left[\frac{J-(P+1)(P'+1)}{P+1}\right]} \sum_{n=0}^{\left[\frac{J-Pm-(P+1)(P'+1)}{P'+1}\right]} \binom{J-Pm-P'n-(P+1)(P'+1)}{m-P'n} \binom{n-P'n}{n} a^{J-m(P+1)-(P+1)(P'+1)} b^{m-n(P'+1)} c^{n+}$$

$$= \sum_{m=0}^{\left[\frac{J}{P+1}\right]} \sum_{n=0}^{\left[\frac{J-Pm}{P'+1}\right]} \binom{J-Pm-P'n-1}{m-P'n-1} \binom{m-P'n-1}{n-1} a^{J-m(P+1)} b^{m-n(P'+1)} c^{n}$$

If we let $J' = J-Em-P'n$, $m' = m-P'n$, and $n' = n$, then

$$aX_{J-1,P,P'} + bX_{J-(P+1),P,P'} + cX_{J-(P+1)(P'+1),P,P'}$$

$$= \sum_{m=0}^{\left[\frac{J}{P+1}\right]} \sum_{n=0}^{\left[\frac{J-Pm}{P'+1}\right]} \left(\binom{J'-1}{m'}\binom{m'}{n'} + \binom{J'-1}{m'-1}\binom{m'-1}{n'} + \binom{J'-1}{m'-1}\binom{m'-1}{n'-1} \right) a^{J-m(P+1)} b^{m-n(P'+1)} c^{n};$$

and since $\binom{J'-1}{m'}\binom{m'}{n'} + \binom{J'-1}{m'-1}\binom{m'-1}{n'} + \binom{J'-1}{m'-1}\binom{m'-1}{n'-1} = \binom{J'-1}{m'}\binom{m'}{n'} + \binom{J'-1}{m'-1}\left\{\binom{m'-1}{n'} + \binom{m'-1}{n'-1}\right\}$

$= \binom{J'-1}{m'}\binom{m'}{n'} + \binom{J'-1}{m'-1}\binom{m'}{n'} = \left\{\binom{J'-1}{m'} + \binom{J'-1}{m'-1}\right\}\binom{m'}{n'} = \binom{J'}{m'}\binom{m'}{n'}$, we then have

$$\sum_{m=0}^{\left[\frac{J}{P+1}\right]} \sum_{n=0}^{\left[\frac{J-Pm}{P'+1}\right]} \binom{J'}{m'}\binom{m'}{n'} a^{J-m(P+1)} b^{m-n(P'+1)} c^{n} = \sum_{m=0}^{\left[\frac{J}{P+1}\right]} \sum_{n=0}^{\left[\frac{J-Pm}{P'+1}\right]} \binom{J-Pm-P'n}{m-P'n}\binom{m-P'n}{n} a^{J-m(P+1)} b^{m-n(P'+1)} c^{n}$$

$$= X_{J,P,P'}$$

As an example of the above theorem note that the elements of the sequence

$\{X_{J,1,}\}_J = \{1,\ a,\ a^2+b,\ a^3+2ab,\ a^4+3a^2b+b^2,\ a^5+4a^3b+3ab^2,\ a^6+5a^4b+6a^2b^2b^3+c\ldots\}$

do in fact satisfy the recursion formula:

$$X_{J,1,2} = aX_{J-1,1,2} + bX_{J-2,1,2} + cX_{J-6,1,2}$$

REFERENCE

1. J. A. Raab, "A Generalization of the Connection Between the Fibonacci Sequence and Pascal's Triangle", Fibonacci Quarterly, October, 1963, pp. 21-31.

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The Governing Council of Pi Mu Epsilon has approved an increase in the maximum amount per chapter allowed as a matching prize from $20.00 to $25.00. If your chapter presents awards for outstanding mathematical papers and students, you may apply to the National Office to match the amount spent by your chapter--i.e., $30.00 of awards, the National Office will reimburse the chapter for $15.00, etc.,--up to a maximum of $25.00. Chapters are urged to submit their best student papers to the Editor of the Pi Mu Epsilon Journal for possible publication.

PARTIAL SUMS OF CERTAIN INFINITE SERIES OF POLYGONAL NUMBERS

Michael Kopkas

John Carroll University

In the December, 1966 issue of the American Mathematical Monthly, these two problems were proposed by J. M. Khatri of Baroda, India:

(1) Prove or disprove: There exists an infinite series of triangular numbers such that every partial sum is a perfect square number.

(2) The same except that every partial sum shall be a triangular number. [1]

The purpose of this paper will be to solve these two problems and then to generalize them as to the numbers used in the series and the numbers arrived at with each partial sum.

INTRODUCTION

Before beginning our investigation of series of polygonal numbers, we must first see what these numbers are and how they might be generated.

Polygonal numbers are sets of numbers first classified by the mathematicians of ancient Greece. The Pythagoreans were fascinated by the mystery of numbers. In fact, Pythagoras himself went so far as to claim that numbers are in some way the cause of the form of an object [2]. One set of numbers which especially interested them was that set generated by counting the number of dots in different sized equilateral triangles (hence, the name triangular). The number 1 was included in the list even though it is not strictly generated by a triangle. Thus, the first four triangular numbers and their dot representations are as follows.

1 3 6 10

We may note that each succeeding row of the triangle is formed by adding one dot onto the previous row. Thus, the fifth row would contain five dots, and in general, the nth row would contain n dots. The triangular number is the sum of the number of dots in each row. Hence, the nth triangular number equals $1+2+3+\ldots+n$, which equals $n(n+1)/2$ by the formula for the sum of an arithmetic progression.

Now we might generalize this notion of triangular numbers to numbers which are generated by counting the number of dots in different sized equilateral four-sided figures, or squares. The concept of square numbers is indeed familiar. Again, the number 1 is included in the list by convention. Hence, the first four square numbers and their dot representations are as follows.

1 4 9 16

We may note that each square number is formed by adding one row and one column of dots to the previous square plus one dot in the corner to complete the square. Thus, the nth square number is formed by adding $2(n-1)+1$, or $2n-1$ to the $(n-1)$st square number. The nth square equals $1+(2(2)-1)+(2(3)-1)+\ldots+(2(n)-1)=1+3+5+\ldots+(2n-1)$, which by the formula for the sum of an arithmetic progression equals $n(2n)/2=n^2$.

We can continue this process for pentagonal numbers; that is, numbers which are generated by counting the number of dots in different sized equilateral five-sided figures, or regular pentagons. Again we include the number 1. Thus, the first four pentagonal numbers and their dot representations are as follows.

1 5 12 22

After a close inspection, we may note that each pentagonal number is formed by adding three lines of dots next to three sides of the previous pentagon (each line of dots containing as many dots as in one side of the previous pentagon) plus one dot to complete the new pentagon. Thus, the nth pentagonal number is formed by adding $3(n-1)+1$, or $3n-2$ to the $(n-1)$st pentagonal number. The nth pentagonal equals $1+(3(2)-2)+(3(3)-2)+\ldots+(3(n)-2)=1+4+7+\ldots+(3n-2)$, which by the formula for the sum of an arithmetic progression equals $n(3n-1)/2$.

We can continue this process for hexagonal numbers, septagonal numbers, etc., to include numbers formed by counting the dots in any regular polygon. Also, we can arrive at a formula for the nth k-gonal number, which will be hereafter designated $f_k(n)$. This formula is given here without proof to be $n(2+(k-2)(n-1))/2$ [3]. A short table of polygonal numbers is given in the Appendix.

This notion of polygonal numbers extends back to the time of Pythagoras (570-501 B.C.). In 175 B.C., Hypsicles defined polygonal numbers in the following way:

> If there are as many numbers as we please beginning with one and increasing by the same common difference, then when the common difference is 1, the sum of all terms is a triangular number; when 2, a square; when 3, a pentagonal number. (If the common difference is k-2, the sum of all terms is a k-gonal number.) [4].

A book of arithmetic by Theon of Smyrna and one by Nichomachus, both about 100 A.D., contained definitions of polygonal-numbers and several theorems about them. The great mathematician Diophantus (ca. 250 A.D.) also proved a few theorems dealing with polygonal numbers.

As the years passed and the development of additive arithmetic increased, so did interest in polygonal numbers. Men like Descartes, Euler, and Fermat devoted much time to them. Most of their work, however, dealt with the representation of an integer as the sum of a particular combination of polygonal numbers. This paper will deal instead with the nth partial sums of particular series of polygonal numbers. It is meant to be a beginning and a start toward further research, for I feel confident that there are many more generalizations that might be proved. With this in mind, let us begin.

FIRST PROBLEM

The first problem we shall examine is whether or not there exists an infinite series of triangular numbers such that each nth partial sum is another triangular number.

We begin by considering the first triangular number, 1. Can 1 be the first term of such a series? By the way triangular numbers are formed, it can easily be seen that no two triangular numbers are one unit apart. The second triangular is two units from the first, the third is three units from the second, and so on, the nth triangular being n units from the $(n-1)$st. Therefore, 1 cannot possibly start our series; for if we began with 1, we could find no other triangular number which, when added to 1, would yield another triangular number. Hence, we would have no second term for our series.

Will this hold true of every triangular number? Can the second triangular number, 3, be the first term of such a series? That is, are there two triangular numbers that are three units apart? Obviously, the third triangular is three units from the second. Therefore, our series may begin $3+3+\ldots$. We note that the second partial sum is 6, which is the third triangular. Now, are there two triangulars that are six units apart? Obviously, the fifth and sixth triangulars are six units apart; so we let the third term of our series be the fifth triangular, 15. Our series now looks like $3+3+15+\ldots$. In general, if the nth partial sum is k, then the next term of the series can be the $(k-1)$st triangular, since the $(k-1)$st triangular plus k yields the kth triangular. Since this process can be carried on indefinitely, we have shown that there exists an infinite series of triangular numbers, namely, beginning with the number 3, such that the nth partial sum is another triangular. Thus, the next term of the series we have started would be the twentieth triangular, which is 210, making our series $3+3+15+210+\ldots$. We see that the fourth partial sum is 231, which is the twenty-first triangular. The fifth term would then be the 230th triangular.

Note that since our process works whenever the nth partial sum is greater than 1, we can generalize and say that beginning with any triangular number except 1, there exists an infinite series of triangular numbers whose every nth partial sum is another triangular number. The procedure for constructing these series is as follows: given any triangular number c, greater than 1, we can always find two triangular numbers, the $(c-1)$st and the cth, that are c units apart. Therefore, the next term in the series can be the $(c-1)$st triangular number. The exception comes in when $c=1$, because the $(c-1)$st triangular would become the 0th triangular, which is undefined.

In summary, we have found that there exists an infinite number of infinite series of triangular numbers such that all nth partial sums are triangular numbers.

SECOND PROBLEM

The next problem we will examine is whether or not there exists an infinite series of triangular numbers whose nth partial sums are perfect squares. There are a number of ways of attacking this problem, and we will demonstrate two of them.

Before we begin the first proof, let us state a well-known mathematical fact. Every square can be represented as the nth partial sum of the series of odd integers, and every such nth partial sum is a square. This fact can be shown quite easily be mathematical induction.

Now let's look at the sum of the first fifteen odd integers

$$1+3+5+7+9+11+13+15+17+19+21+23+25+27+29.$$

1 and 3 are triangular numbers, so we'll use them as the first two terms of our series. Now we take the next three numbers--5,7,9. We can partition 5 (which means to factor it in an additive manner] into $1+4$. Also $7 = 2+5$ and $9 = 3+6$. Therefore

$$5+7+9 = 1+4+2+5+3+6 = 1+2+3+4+5+6$$

which is precisely the sixth triangular number 21. We use 21 as the next term of our series, yielding $1+3+21+\ldots$, Note that as we go along, each partial sum is a perfect square, since all we are really doing is rewriting nth partial sums of the series of odd integers.

Looking at the next nine numbers, we partition $11 = 1+10$, $13 = 2+11, \ldots, 27 = 9+18$. Therefore

$$11+13+\ldots+27 = 1+10+2+11+\ldots+9+18 = 1+2+\ldots+18$$

which is the eighteenth triangular number, 171. Our series looks like $1+3+21+171+\ldots$.

How do we know how many numbers are to be partitioned? Starting with the first odd number, call it a, we partition a into $1+(a-1)$. Partitioning must continue until the number $(a-1)-1$, or a-2, appears as an additive factor. Thus

$$\begin{aligned} a &= 1 &&+ a-1 \\ a+2 &= 2 &&+ a \\ a+4 &= 3 &&+ a+1 \\ &\vdots && \vdots \\ a+m &= a-2 &&+ m+2 \end{aligned}$$

where m is a positive even integer. From a to a+m there are exactly $(a-2)$ numbers, as we can see from the first column of factors. Since there are a total of $(m+2)$ factors, and (a-2) is half the number of factors, then $m+2 = 2(a-2)$. Therefore

$$a+m = (m+2)+(a-2) = 2(a-2)+(a-2) = 3(a-2).$$

So if a is our first odd integer, we take each of the odd integers from a to $3(a-2)$ and partition them as before to determine the next triangular number in the series. Thus, the next term in the series we have generated so far would be found as follows: our first odd integer, a, is 29. The last one is $3(29-2)$, which is 81. We partition the odd integers from 29 to 81 into the sum of the first $2(a-2)$ consecutive integers, or the first 54 integers. Our next triangular number, then, would be the 54th, 1485. The process we have described can be carried on indefinitely, yielding an infinite series of triangular numbers whose nth partial sums are perfect squares.

The second method of proof is that of mathematical induction. We may note that

> the sum of the first and second triangulars yields the second square.
> The sum of the first, second, and sixth triangulars yields the fifth square.
> The sum of the first, second, sixth, and eighteenth triangulars yields the fourteenth square.

A close examination will reveal a pattern forming. Using $f_3(k)$ and $f_4(k)$ to mean the kth triangular and the kth square respectively, we can summarize this pattern with the following formula.

$$1 + \sum_{j=0}^{k-1} f_3(2\cdot 3^j) = f_4\left(1 + \sum_{j=0}^{k-1} 3^j\right)$$

The proof proceeds by mathematical induction, The case where $k = 1$ yields $1 + f_3(2) = 1 + 3 = 4 = (1 + 1)^2 = (1 + 3^0)^2 = f_4(1 + 3^0)$. As the next step in the induction proof, we assume that

$$1 + \sum_{j=0}^{k-1} f_3(2\cdot 3^j) = f_4\left(1 + \sum_{j=0}^{k-1} 3^j\right)$$

and must show that

$$1 + \sum_{j=0}^{k} f_3(2\cdot 3^j) = f_4\left(1 + \sum_{j=0}^{k} 3^j\right)$$

We make use of the formula for the sum of a geometric progression

$$\sum_{j=0}^{k} 3^j = (3^{k+1}-1)/2.$$

$$\begin{aligned} \text{Therefore } f_4\left(1 + \sum_{j=0}^{k} 3^j\right) &= [1+(3^{k+1}-1)/2]^2 = [(3^{k+1}+1)/2]^2 \\ &= [(3^{k+1})^2+2(3^{k+1})+1]/4 \\ &= [(3(3^k))^2+2(3(3^k))+1]/4 \\ &= [9(3^k)^2+6(3^k)+1]/4 \\ &= [(3^k)^2+2(3^k)+1+8(3^k)^2+4(3^k)]/4 \\ &= [(3^k)^2+2(3^k)+1]/4 + 4[(2(3^k)^2+3^k)]/4 \end{aligned}$$

$$= [(3^k+1)/2]^2+3^k(2\cdot 3^k+1)$$
$$= f_4\left(1 + \sum_{j=0}^{k-1} 3^j\right) + f_3(2\cdot 3^k)$$

which, by the induction hypothesis, equals

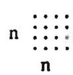
$$1 + \sum_{j=0}^{k-1} f_3(2\cdot 3^j) + f_3(2\cdot 3^k) = 1 + \sum_{j=0}^{k} f_3(2\cdot 3^j)$$

which is precisely what we were to show. Therefore, by mathematical induction,

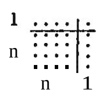
$$1 + \sum_{j=0}^{k-1} f_3(2\cdot 3^j) = f_4\left(1 + \sum_{j=0}^{k-1} 3^j\right)$$

We have found a pattern that can he generated infinitely to yield an infinite series of triangular numbers whose nth partial sums are perfect squares.

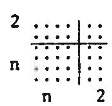
How many such series are there? Let us begin to answer the question in this way. Given any square, call it n^2, we can represent this square geometrically as n rows and n columns of dots.



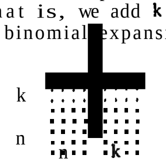
To form $(n+1)^2$ we add to n^2 a row of n dots and a column of n dots plus one dot in the corner to complete the square: that is, we add $2n+1$ dots. So $(n+1)^2 = n^2 + (2n+1)$.



We form $(n+2)^2$ by adding to n^2 two rows of n dots each and two columns of n dots each plus four dots in the corner to complete the square; that is, we add $2n+2n+4$, or $2(2n+2)$ dots. So $(n+2)^2 = n^2+2(2n+2)$.



In general, to form $(n+k)^2$, we add to n^2, k rows of n dots eachandkcolumns of n dots each plus k^2 dots in the corner to complete the square; that is, we add $k(2n+k)$ dots. This is easily verified by the binomial expansion $(n+k)^2 = n^2+2kn+k^2 = n^2+k(2n+k)$.



Now we would like to find conditions under which $k(2n+k)$ is a triangular number; that is, under which $k(2n+k) = m(m+1)/2$ for some integer m.

Let us first assume that m is an even integer. This is equivalent to saying that $m/2$ is an integer. Let $k = m/2$. Then $(2n+k) = m+1$ and $m = 2k$. From these we derive $k = 2n-1$. Thus

$$k(2n+k) = (2n-1)(2n+2n-1) = (2n-1)(4n-1) = (4n-2)(4n-1)/2$$

which is the $(4n-2)$nd triangular number.

Keeping this in mind, let us now assume that m is an odd integer. This is equivalent to saying that$(m+1)/2$is an integer. Let $k = [m+1]/2$. Then $(2n+k) = m$ and $m = 2k-1$. From these we derive $k = 2n+1$. Thus

$$k(2n+k) = (2n+1)(2n+2n+1) = (2n+1)(4n+1) = (4n+2)(4n+1)/2$$

which is the $(4n+1)$st triangular number.

What we have shown is that, given any n^2, we can always find two triangular numbers, the $(4n-2)$nd and the $(4n+1)$st, such that when either of these is added to n^2, we get another square, $(n+k)^2$.

Now if we can show that there are an infinite number of square triangular numbers to be used as the first term of a series, we will have shown that there are an infinite number of infinite series of triangular numbers whose every nth partial sum is a perfect square. In other words, we must find conditions under which $n(n+1)/2 = m^2$ for some integers m and n, and show that there are an infinite number of pairs of integers satisfying these conditions.

Let us begin by considering those squares which can be rewritten as the product of two squares; that is, $m^2 = a^2b^2$.

First we consider the case where n is even; that is, $n = 2k$. Then

$$a^2b^2 = m^2 = n(n+1)/2 = 2k(2k+1)/2 = k(2k+1).$$

Letting $a^2 = k$, we get $b^2 = 2k+1 = 2a^2+1$, or $2a^2-b^2 = -1$

Keeping this in mind, we consider next the case where n is odd; that is, $n = 2k-1$. Then

$$a^2b^2 = m^2 = n(n+1)/2 = (2k-1)(2k)/2 = k(2k-1).$$

Letting $a^2 = k$, then $b^2 = 2k-1 = 2a^2-1$, or $2a^2-b^2 = 1$.

What we have shown is that choosing a and b, such that $2a^2-b^2 = \pm 1$, enables us to find m and n (since $m^2 = a^2b^2$ and $n = 2a^2$ or $2a^2-1$) such that $m^2 = n(n+1)/2$.

To complete the proof, we need only to show that there are an infinite number of pairs of integers a and b such that $2a^2-b^2 = \pm 1$. Setting up a table of numbers, we obtain the following:

a	b	$2a^2-b^2$
1	1	1
2	3	-1
5	7	1
12	17	-1
29	41	1
⋮	⋮	⋮

We might note that adding the number in the a column with its partner in the b column gives us the next number in the a column, and that adding twice the number in the a column with its partner in the b column gives us the next number in the b column. In other words, if $2a^2-b^2 = -1$, then $2(a+b)^2-(2a+b)^2$ should equal 1; and, if $2a^2-b^2 = 1$, then $2(a+b)^2-(2a+b)^2$ should equal -1. This can be proved quite easily, for

$$2(a+b)^2-(2a+b)^2 = 2a^2+4ab+2b^2-4a^2-4ab-b^2$$
$$= -2a^2+b^2 = -(2a^2-b^2).$$

Therefore, $2a^2-b^2$ and $2(a+b)^2-(2a+b)^2$ are always of equal absolute value but of opposite sign. Thus, given the numbers $a = 1$ and $b = 1$ as a starting point, we can generate an infinite set of pairs a and b such that $2a^2-b^2 = \pm 1$, which is what we needed to show.

Summing up the entire proof, we have shown that given any square triangular number, of which there are infinitely many, we can use this number as the first term of an infinite series of triangular numbers whose every nth partial sum is a perfect square. Thus, there are infinitely many such series.

GENERALIZATION?

We demonstrated earlier a series of triangular numbers with triangulars as nth partial sums. Can we find a series of squares yielding squares as nth partial sums? Again we can immediately see the problem of beginning with the number 1. No two squares of positive integers are one unit apart. Thus, 1 must be excluded,

An examination of the way in which squares are formed might give us some hint on how to proceed. As was mentioned earlier, each square is a kth partial sum of the series of odd integers, Suppose we take two squares - one odd and one even - whose sum is also a square. For example, take 9 and 16, whose sum is 25. We can choose for the next number in the series that square which is the sum of all the odd integers up to and including the odd integer before 25; that is, the next term would be $1+3+...+23$, which equals 144. We might note that there are $(25-1)/2$, or 12, odd integers before 25 and that $144 = 12^2$. This is a direct consequence of the fact that $n^2 = 1+3+...+(2n-1)$, which can be proved quite easily by induction. The third partial sum of our series is $9+16+144 = 169 = 13^2$. Next, we take the number of odd integers less than 169, which is $(169-1)/2 = 84$. We add 84^2, or 7056 to the series, giving us $9+16+144+7056 = 7225 = 85^2$. The process we have set up is as follows.

1) Take the kth partial sum.
2) Subtract 1 from it and divide the result by 2. This will always yield an even integer, as can be shown by the following argument: as long as the kth partial sum is an odd square (and we will show that it always is), it is necessarily the square of an odd number and can be represented by $(2k+1)^2$ for some integer k. This yields $4k^2+4k+1$. Subtracting 1 and dividing by 2 yields $2k^2+2k$, which is an even integer.
3) Square this number and include the answer in the series. The square of an even integer is always even; thus, except for the first term, every term will be an even number. Since the first term is odd, every partial sum will be an odd square, and we can go back to step 1 to find the next term in the series.

Since this process can be carried on indefinitely, we have found an infinite series of square numbers whose nth partial sums are also squares.

Is there more than one such series? Choosing any odd square except 1, we can begin with step 2 of the process and easily arrive at our desired series. When we begin with 1, we obtain a series with 1 as the first term and 0 as every other term; but we are looking for series of squares of positive integers, and therefore we exclude 0. However, since there are infinitely many odd squares to be used as a starting point, we have shown that there are an infinite number of infinite series of squares whose nth partial sums are also squares.

We have already found that there exists a series of triangular numbers yielding square partial sums. Can we reverse this and find a series of squares with triangular partial sums? Examining a table of squares and triangulars, one may note that:

1 is both square and triangular and can be used to start the series.

$1^2+3^2 = 10 = f_3(4) = f_3(1+3)$

$1^2+3^2+9^2 = 91 = f_3(13) = f_3(1+3+9)$

In general, $1^2+3^2+9^2+...+(3^k)^2 = f_3(1+3+9+...+3^k)$, or

$$\sum_{k=0}^{n}(3^k)^2 = f_3\left(\sum_{k=0}^{n}3^k\right).$$

This generalization can be proved directly,

$$f_3\left(\sum_{k=0}^{n}3^k\right) = f_3\left(\frac{3^{n+1}-1}{2}\right) = \frac{\left(\frac{3^{n+1}-1}{2}\right)\left(\frac{3^{n+1}-1}{2}+1\right)}{2} = \frac{(3^2)^{n+1}-1}{8}$$
$$= \sum_{k=0}^{n}(3^2)^k = \sum_{k=0}^{n}(3^k)^2$$

Therefore, there exists an infinite series of square numbers whose nth partial sums are triangular numbers.

Let us continue in this direction. Can we demonstrate a series of pentagonal numbers yielding squares as nth partial sums? A short examination of a table of pentagonals and squares provides the following formula, which we shall prove by induction;

$$\sum_{i=0}^{n} f_5(5^i) = f_4\left(\sum_{i=0}^{n} 5^i\right)$$

where $f_5(k)$ means the kth pentagonal. Making use of the formula given earlier for finding the nth k-gonal number, we can see that the case where n = 1 gives

$$\sum_{i=0} f_5(5^i) = f_5(1)+f_5(5) = 1+35 = 36 = 6^2 = f_4(1+5) = f_4\left(\sum_{i=0}^{1} 5^i\right)$$

Next, we assume that $\sum_{i=0}^{n} f_5(5^i) = f_4\left(\sum_{i=0}^{n} 5^i\right)$ and show that

$$\sum_{i=0}^{n+1} f_5(5^i) = f_4\left(\sum_{i=0}^{n+1} 5^i\right)$$

Now

$$\sum_{i=0}^{n+1} f_5(5^i) = \left(\sum_{i=0}^{n} f_5(5^i)\right) + f_5(5^{n+1}) = f_4\left(\sum_{i=0}^{n} 5^i\right) + \frac{3(5^{n+1})^2-5^{n-1}}{2}$$

$$= \left(\frac{5^{n+1}-1}{4}\right)^2 + \frac{24(5^{2n+2})-8(5^{n+1})}{16}$$

$$= \frac{5^{2n+2}-2(5^{n+1})+1+24(5^{2n+2})-8(5^{n+1})}{16}$$

$$= \frac{25(5^{2n+2})-10(5^{n+1})+1}{16} = \frac{5^{2n+4}-2(5^{n+2})+1}{16}$$

$$= \left(\frac{5^{n+2}-1}{4}\right)^2 = f_4\left(\sum_{i=0}^{n+1} 5^i\right)$$

which is what we were to show. Therefore, we have found an infinite series of pentagonal numbers whose nth partial sums are squares.

Let us look at what we have done. We can see that in our series of square numbers with triangular nth partial sums, the numbers squared have differed by a factor of three; in our series of pentagonal numbers with square nth partial sums, the numbers "pentagonalized" have differed by a factor of five. Examination of the tables in the Appendix would indicate that in a series of hexagonal numbers with pentagonal nth partial sums, the numbers "hexagonalized" will differ by a factor of seven. We may generalize by saying that in a series of k-gonal numbers with (k-1)-gonal nth partial sums (k>3), the numbers "k-gonalized" must differ by a factor of (2k-5). Thus, letting $f_k(m)$ be the mth k-gonal number, we must prove the following

$$\sum_{i=0}^{n} f_k[(2k-5)^i] = f_{k-1}\left(\sum_{i=0}^{n} (2k-5)^i\right)$$

where k is any positive integer greater than or equal to 4.

The proof proceeds by mathematical induction on n. The case n = 0 follows trivially, since $f_k[(2k-5)^0] = f_k(1) = 1 = f_{k-1}(1) = f_{k-1}[(2k-5)^0]$. This was to be expected since 1 is the first k-gonal number for any integer k.

Since this case was so trivial, we might include the case n = 1, although it is not strictly necessary to the proof. When n = 1, we must prove the following

$$\sum_{i=0}^{1} f_k[(2k-5)^i] = f_{k-1} \sum_{i=0}^{1} (2k-5)^i$$

Applying our formula for finding the mth k-gonal number, namely $f_k(m) = m(2+(k-2)(m-1))/2$, we reduce our problem to proving that

$$1 + \frac{[2k-5]\{2+(k-2)([2k-5]-1)\}}{2} = \frac{[1+(2k-5)]\{2+([k-1]-2)([1+(2k-5)]-1)\}}{2}$$

Beginning with the left side, we get

$$1 + \frac{[2k-5]\{2+(k-2)([2k-5]-1)\}}{2} = 1 + \frac{(2k-5)(2k^2-10k+14)}{2}$$

$$= 1+(2k-5)(k^2-5k+7) = 2k^3-15k^2+39k-34 = (k-2)(2k^2-11k+17)$$

$$= \frac{(2k-4)(2+[k-3][2k-5])}{2}$$

$$= \frac{[1+(2k-5)]\{2+([k-1]-2)([1+(2k-5)]-1\}}{2}$$

which is precisely what the right side equals.

Continuing with our induction proof, we assume

$$\sum_{i=0}^{n} f_k[(2k-5)^i] = f_{k-1}\left(\sum_{i=0}^{n} (2k-5)^i\right)$$

and using this assumption, we must show that

$$\sum_{i=0}^{n+1} f_k[(2k-5)^i] = f_{k-1}\left(\sum_{i=0}^{n+1} (2k-5)^i\right)$$

Again wc make use of our formula for finding the mth k-gonal number, which makes our induction hypothesis

$$\sum_{i=0}^{n} \frac{(2k-5)^i(2+[k-2][(2k-5)^i-1])}{2} = \frac{\left(\sum_{i=0}^{n} (2k-5)^i\right)[2+(k-3)\{\left(\sum_{i=0}^{n} (2k-5)^i\right)-1\}]}{2}$$

what we must prove is

$$\sum_{i=0}^{n+1} \frac{(2k-5)^i(2+[k-2][(2k-5)^i-1])}{2} = \frac{\left(\sum_{i=0}^{n+1} (2k-5)^i\right)[2+(k-3)\{\left(\sum_{i=0}^{n+1} (2k-5)^i\right)-1\}]}{2}$$

Beginning with the left side, we have

$$\sum_{i=0}^{n+1}\frac{(2k-5)^i\{2+[k-2][(2k-5)^i-1]\}}{2}$$

$$=\sum_{i=0}^{n}\frac{(2k-5)^i\{2+[k-2][(2k-5)^i-1]\}}{2}+\frac{(2k-5)^{n+1}\{2+[k-2][(2k-5)^{n+1}-1]\}}{2}$$

which, by the induction hypothesis, equals

$$\frac{\left(\sum_{i=0}^{n}(2k-5)^i\right)[2+(k-3)\{\left(\sum_{i=0}^{n}(2k-5)^i\right)-1\}]}{2}+\frac{(2k-5)^{n+1}(4+k[2k-5]^{n+1}-k-2[2k-5]^{n+1})}{2}$$

$$=\frac{(2k-5)^{n+1}-1}{(2k-5)-1}\left\{\frac{2+(k-3)\left(\frac{[2k-5]^{n+1}-1}{2[k-3]}-1\right)}{2}\right\}$$

$$+\frac{4(2k-5)^{n+1}+k(2k-5)^{2n+2}-k(2k-5)^{n+1}-2(2k-5)^{2n+2}}{2}$$

$$=\frac{(2k-5)^{n+1}-1}{2(k-3)}\left(\frac{3-2(k-3)+(2k-5)^{n+1}}{4}\right)$$

$$\frac{4(2k-5)^{n+1}+k(2k-5)^{2n+2}-k(2k-5)^{n+1}-2(2k-5)^{2n+2}}{2}$$

$$=\frac{(2k-5)^{2n+2}-2(k-4)(2k-5)^{n+1}+2k-9}{8(k-3)}$$

$$+\frac{4k(k-3)(2k-5)^{2n+2}+16(k-3)(2k-5)^{n+1}-4k(k-3)(2k-5)^{n+1}-8(k-3)(2k-5)^{2n+2}}{8(k-3)}$$

$$=\frac{[(4k-8)(k-3)+1][2k-5]^{2n+2}+[-2(k-4)+16(k-3)-4k(k-3)][2k-5]^{n+1}+2k-9}{8(k-3)}$$

$$=\frac{(4k^2-20k+25)(2k-5)^{2n+2}+(-4k^2+26k-40)(2k-5)^{n+1}+2k-9}{8(k-3)}$$

$$=\frac{(2k-5)^{2n+4}-(2k-8)(2k-5)^{n+2}+2k-9}{8(k-3)}$$

$$=\frac{(2k-5)^{2n+4}-2k(2k-5)^{n+2}+9(2k-5)^{n+2}-(2k-5)^{n+2}+2k-9}{8(k-3)}$$

$$=\frac{[(2k-5)^{n+2}-1][(2k-5)^{n+2}-2k+9]}{8(k-3)}$$

$$=\frac{(2k-5)^{n+2}-1}{2(k-3)}\left\{\frac{4+2(k-3)\left(\frac{(2k-5)^{n+2}-1-2(k-3)}{2(k-3)}\right)}{4}\right\}$$

$$=\frac{(2k-5)^{n+2}-1}{2(k-3)}\left\{\frac{2+(k-3)\left(\frac{(2k-5)^{n+2}-1}{2(k-3)}-1\right)}{2}\right\}$$

$$=\frac{\left(\sum_{i=0}^{n+1}(2k-5)^i\right)[2+(k-3)\{\left(\sum_{i=0}^{n+1}(2k-5)^i\right)-1\}]}{2}$$

which is precisely what the right side equalled. Thus, by the principle of mathematical induction, we have established that

$$\sum_{i=0}^{n} f_k[(2k-5)^i] = f_{k-1}\left(\sum_{i=0}^{n}(2k-5)^i\right).$$

Since we did not specify k, other than saying that k is a positive integer greater than or equal to 4, our results hold for all positive integers greater than or equal to 4. Actually, the formula holds for all positive integers, but we have not defined k-gonal numbers for k = 1 or k = 2, since these have no geometric significance. Thus, for k greater than or equal to 4, we have proved that one can find an infinite series of k-gonal numbers such that the sum of the first n terms of the series is a (k-1)-gonal number.

CONCLUSION

Rather than summing up my results, I would like to conclude this article with a few comments on why and how it was written.

The initial problems upon which the article was based first appeared (as was stated in the introductory remarks) as elementary problems in the American Mathematical Monthly. They were fairly easy to prove and lent themselves quite well to more difficult generalizations. While my results will not, I am sure, shake modem mathematics to its foundations, these generalizations were good exercises in analyzing polygonal numbers and proving theorems about them.

Worthy of mention is the role of a computer in obtaining my results. While the computer furnished no proof to any theorem, it supplied me with tables of numbers from which I was able to make generalizations. Without these tables, I would have spent many long hours computing values of polygonal numbers, with a very good possibility of an error. Thus, the computer was a necessary tool for formulating the problems, even though it could not prove them.

Finally, I feel that this article was a good review of the method of proving mathematical existence theorems constructively. In this method of proof, the existence of something is proven by giving a method for constructing it. One must then show that this construction is valid, either by proving the validity of the generating process, or by a method such as mathematical induction. Both of these methods were used throughout the article.

APPENDIX

n	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}
1	1	1	1	1	1	1	1	1	1
2	3	4	5	6	7	8	9	1 0	11
3	6	9	12	15	18	21	24	27	30
4	10	16	22	28	34	40	46	52	58
5	15	25	35	45	55	65	75	85	95
6	21	36	51	66	81	96	111	126	141
7	28	49	70	91	112	133	154	175	196
8	36	64	92	120	148	176	204	232	260
9	45	81	117	153	189	225	261	297	333
10	55	100	145	190	235	280	325	370	415

[Student paper presented at the National Meeting, August, 1968, Madison, Wisconsin.]

THE FIBONACCI SEQUENCE: AN INTRODUCTION

Donald F. Reynolds
Texas Christian University

In 1202 an Italian merchant. Leonardo Pisano, known to history by his nickname Fibonacci, published a mathematical textbook, Liber Abacci, which was responsible for the introduction of Hindu-Arabic numerals to the western world. One of the problems appearing in Leonardo's book was the following:

> "Suppose we place one pair of rabbits in an enclosure in the month of January; that these rabbits will breed another pair during February; that pairs of rabbits always breed in the second month following birth, and thereafter produce one pair of rabbits monthly; and that none die. How many pairs of rabbits would we have at the end of December?"

An analysis of the number of pairs of rabbits at the end of each month yields the following sequence

1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377,....

These numbers are the terms of the Fibonacci sequence. or more simply, the Fibonacci numbers. The sequence is formally defined as follows

$$F_1 = 1,\ F_2 = 1,\ F_n = F_{n-1} + F_{n-2} \quad n \geq 3.$$

An examination of this sequence reveals some rather interesting and often surprising results.

Some elementary number theoretic properties which can be readily obtained are the following

(1) $F_1 + F_2 + \ldots + F_n = F_{n+2} - 1.$

(2) $F_1^2 + F_2^2 + \ldots + F_n^2 = F_nF_{n+1}.$

(3) $F_{n+1}^2 = F_nF_{n+2} + (-1)^n.$

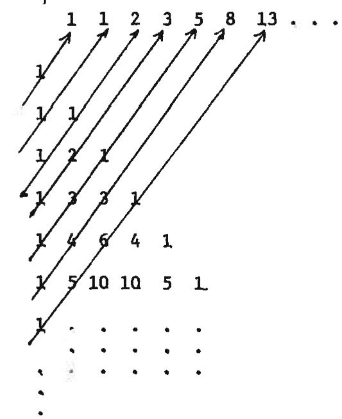
(4) Neighboring Fibonacci numbers are relatively prime.

(5) For any m, n, $(F_m, F_n) = F_{(m,n)}$, where (a,b) denotes the greatest common divisor of a and b.

(6) F_m divides F_n if and only if m divides n.

In addition to these properties, which relate only to the Fibonacci numbers themselves, we find that the Fibonacci sequence is related to other areas of mathematics and. in fact, even pervades a number of non-mathematical disciplines.

For example. if we write the binomial coefficients in the familiar triangular arrangement due to Pascal and consider the sums of the numbers on the "rising diagonals" of the triangle, we note that we obtain the Fibonacci sequence.



Another concept to which Fibonacci numbers are intimately related is that of continued fractions. A finite continued fraction is an expression of the form

$$q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cdots + \cfrac{1}{q_n}}}$$

where $q_1, \ldots, q_n$ are positive integers and q_0 is a non-negative integer. The integers $q_0, q_1, \ldots, q_n$ are called the partial denominators of the continued fraction. Infinite continued fractions are defined in the obvious way.

Theorem: If a finite continued fraction has n partial denominators $q_0, q_1, \ldots, q_{n-1}$ and $q_0 = q_1 = \ldots = q_{n-1} = 1$, then the fraction equals

$$\frac{F_{n+1}}{F_n}.$$

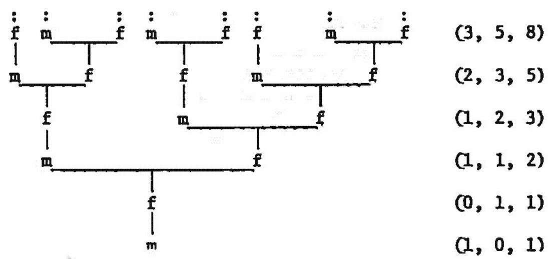
This result can easily he extended to the infinite case to obtain the following theorem.

Theorem: If an infinite continued fraction has all its partial denominators equal to unity, then the value of the fraction is given by

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n}.$$

Thus the sequence formed by taking ratios of successive Fibonacci numbers converges to a number, which we shall call ϕ, which is also the value of an infinite continued fraction. The number ϕ is interesting for still another reason, for if we construct a rectangle with sides in the ratio of ϕ and remove from it the largest possible square, the remaining rectangle has sides whose ratio is again equal to ϕ. Such a rectangle is known as a golden rectangle and ϕ is known as the golden ratio.

Fibonacci numbers also appear in many non-mathematical disciplines such as optics, botany, and genetics. Consider, for example, the ancestry of the male bee. Noting the fact that the male bee is hatched from an unfertilized egg, and hence has only a female parent, we construct the following family tree.



In the above figure, m represents male, f represents female, and the numbers at the right indicate the number of males, females, and total ancestors at that level. All three follow the pattern determined by the Fibonacci sequence.

The Fibonacci numbers are also related to leaf arrangements, the chambered nautilus, the number of ways which light can reflect within to plates of glass, and many other physical phenomena.

[Student Paper presented at the National Meeting, August, 1968, Madison, Wisconsin..]

MEETING ANNOUNCEMENT

Pi Mu Epsilon will meet August 25-27, 1969, at the University of Oregon, Eugene, Oregon, in conjunction with the Mathematical Association of America. Chapters should start planning NOW to send delegates or speakers to this meeting, and to attend as many of the lectures by other mathematical groups as possible.

The National Office of Pi Mu Epsilon will help with expenses of a speaker OR delegate (one per chapter) who is a member of Pi Mu Epsilon and who has not received a Master's Degree by April 15, 1969, as follows: SPEAKERS will receive 5¢ per mile or lowest cost, confirmed air travel fare; DELEGATES will receive 2 1/2¢ per mile or lowest cost, confirmed air travel fare.

Select the best talk of the year given at one of your meetings by a member of Pi Mu Epsilon who meets the above requirement and have him or her apply to the National Office. Nominations should be in our office by April 15, 1969. The following information should be included: Your Name; Chapter of Pi Mu Epsilon; school; topic of talk; what degree you are working on; if you are a delegate or a speaker; when you expect to receive your degree; current mailing address; summer mailing address; who recommended by; and a 50-75 word summary of talk, if you are a speaker. MAIL TO: Pi Mu Epsilon, 1000 Asp Ave., Room 215, Norman, Oklahoma 73069.

PROBLEM DEPARTMENT

Edited by

Leon Bankoff, Los Angeles, California

This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity, but occasionally we shall publish problems that should challenge the ability of the advanced undergraduate or candidate for the Master's Degree. Solutions should be submitted on separate, signed sheets and mailed before July 31, 1969,

Address all communications concerning problems to Leon Bankoff, 6360 Wilshire Boulevard, Los Angeles, California 90048.

PROBLEMS FOR SOLUTION

213. Proposed, by Gregory Wulczyn, Bucknell University.

Prove that a triangle is isosceles if and only if it has a pair of equal ex-symmedians. (Editorial note: See Mathematics Magazine , Problem 637, November 1966, May 1967 and January 1968, for the corresponding problem involving symmedians.)

214. Proposed by Charles W. Trigg, San Diego, California.

Find the unique 9-digit triangular number A which has distinct digits and for which n has the form abbbb.

215. Proposed by Leon Bankoff, Los Angeles, California.

In an acute triangle ABC whose circumcenter is 0, let D, E, F denote the midpoints of sides BC, CA, AB and let P, Q, R denote the midpoints of the minor arcs BC, CA, AB of the circumcircle. Show that

$$\frac{DP + EQ + FR}{OB + OD + OC + OE + OA + OF} = \frac{\sin^2(A/2) + \sin^2(B/2) + \sin^2(C/2)}{\cos^2(A/2) + \cos^2(B/2) + \cos^2(C/2).}$$

216, Proposed by Erwin Just, Bronx Community College.

Prove that the Diophantine equation

$$x^9 + 2y^9 + 3z^9 + 4w^9 = k$$

has no solution if $k \varepsilon \{11, 12, 13, 14, 15, 16\}$.

217, Proposed by C. S. Venkataraman, Sree Kerala Varma College, Trichur, South India.

A transverse common tangent of two circles meets the two direct common tangents in B and C. Prove that the feet of the perpendiculars from B and C on the line of centers are a pair of common inverse points of both the circles.

218. Proposed by Charles W. Trigg, San Diego, California.

Find the three 3-digit numbers each of which is equal to the product of the sum of its digits by the sum of the squares of its digits,

219. Proposed by Stanley **Rabinowitz,** Polytechnic Institute of Brooklyn,

Consider the following method of solving $x^3 - 11x^2 + 36x - 36 = 0$. Since $(x^3 - 11x^2 + 36x)/36 = 1$, we may substitute this value for 1 back in the first eauation to obtain

$$x^3 - 11x^2 + 36x(x^3 - 11x^2 + 36x)/36 - 36 = 0,$$

or $x^4 - 10x^3 + 25x^2 - 36 = 0$, with roots -1, 2, 3 and 6. We find that $x = -1$ is an extraneous root.

Generalize the method and determine what extraneous roots are generated.

220. Proposed by Daniel Pedoe, University of Minnesota,

a) Show that there is no solution of the Apollonius problem of drawing circles to touch three given circles which has only seven solutions.

b) What specializations of the three circles will produce 0, 1, 2, 3, 4, 5 and 6 distinct solutions?

221. Proposed by Murray S. **Klamkin,** Ford Scientific **Laboratory.**

Determine 8 vertices of an inscribed rectangular **parallelopiped** in the sphere

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2) = 0.$$

Editorial Note: The previous issue of this Journal **(Fall, 1968)** contained a re-statement of twelve problems for which solutions have not yet been published, Omitted from this collection was **problem 50,** which was first proposed in the Fall, 1952 issue and later repeated in the Spring, 1955 issue, Another omission is problem **111,** proposed in Spring, 1959 and corrected in Spring, 1960, Readers are invited to offer solutions to these problems,

50. (Fall, 1952) Proposed by Pedro **Piza,** San Juan, Puerto **Rico.**

Prove that the integer $2n+1$ is a prime if and only if, for every value of $r = 1, 2, 3, \ldots, [\sqrt{n/2}]$, the binomial coefficient $\binom{n+r}{n-r}$ is divisible by $2r+1$.

111. (Spring* 1960) Proposed by **M.** S. **Klamkin,** AVCO **RAD,** and D. J. **Newman,** Brown University.

It is conjectured that at most N-2 super-queens can be placed on an N x N $(N>2)$ chessboard so that none can take each other, A superqueen can move like an ordinary queen or a knight,

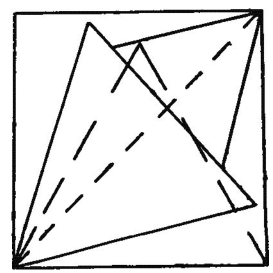
It should have been stipulated that N is even. For N = 5, Michael **J.** Pascual shows that one can place 4 **super-queens.**

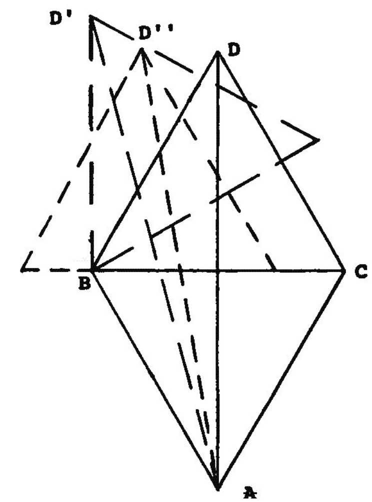
SOLUTIONS

102. (Fall, **1958).** Proposed by Leo **Moser,** University of Alberta, Edmonton, Canada.

Give a complete proof that two equilateral triangles of edge 1 cannot be placed, without overlap, in the interior of a square of edge **1.**

Solution by Charles W. **Trigg,** San **Diego,** California.





In order that the shortest distance between the most remote vertices of two non-overlapping congruent equilateral triangles in the plane may be achieved, the triangles must be in contact, Any relative positions they may assume may be reached by translation and/or rotation from the position where two sides are coincident, as in the figure. In that position the most remote vertices of two triangles with side 1 are $\sqrt{3}$ apart. Any motion of translation or rotation increases the distance between A and D until, when they become closer than $\sqrt{3}$, another pair will have become the most remote vertices,

The two most remote points on a unit square are opposite vertices whose distance apart is $\sqrt{2}(<\sqrt{3})$. **Consequently,** the two triangles cannot be placed in a unit square without overlap,

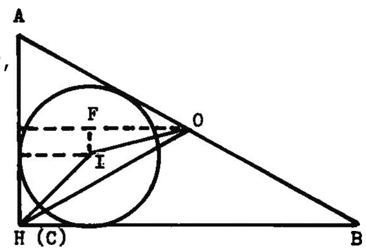
This conclusion can be **confirmed** by considering a triangle inside the square, In order to provide a maximum area in which to place another triangle, one of the unit triangle's vertices must coincide with a vertex of the square. The side opposite that vertex must be perpendicular to the diagonal of the square from that vertex, in order that the altitude of the maximum second triangle will be $(\sqrt{2} - \sqrt{3}/2) < \sqrt{3}/2$ and its side will be $(2\sqrt{6}/3 - 1) < 1$.

Following the same reasoning, the maximum pair of congruent triangles which can be placed in a unit square without overlap will have altitudes of $\sqrt{2}/2$ and sides of $\sqrt{2/3}$.

202. (Spring, 1968). Proposed by Leon Bankoff, Los Angeles, California,

Let I, O, H denote the incenter, circumcenter and orthocenter, respectively, of a right triangle. Find angle HIO given that ΔHIO is isosceles,

Solution by Charles W. Trigg, San Diego, California,



In the right triangle ABC, C and H coincide, Let F be a foot of the perpendicular from O to AC. Then $OF = a/2 - r$, $IF = b/2 - r$, and $IO = IH = r\sqrt{2}$. Then in the right triangle IOF,

$$(r\sqrt{2})^2 = (a/2 - r)^2 + (b/2 - r)^2,$$

whereupon

$$r = (a^2 + b^2)/4(a + b) = (a + b - c)/2.$$

When this equation is simplified and the substitution $a^2 + b^2 = c^2$ is made, we have

$$c^2 - 2(a + b)c + 4ab = 0.$$

So $c = (a + b) \pm (a - b)$; that is, $c = 2a$ or $2b$, and ABC is a 30°-60° right triangle. Then angle OHA = angle OAH = 60°, angle IHA = 45°, and angle HIO = 180° - 2(60° - 45°) or 150°.

Also solved by Joe Konhauser, Macalester College; Andrew E. Rouse, University of Mississippi; Gregory Wulczyn, Bucknell University; and the proposer.

Editorial Note: Konhauser and Wulczyn applied the cosine law in triangle OIH, in which $(OI)^2 = R(R - 2r)$, $OH = R$, and $(IH)^2 = 2r^2$, to obtain $\cos OIH = -\sqrt{3}/2$, or angle OIH = 150°.

203. (Spring, 1968). Proposed by Stanley Rabinowitz, Polytechnic Institute of Brooklyn,

Let P denote any point on the median AD of ABC. If BP meets AC at E and CP meets AB at F, prove that AB = AC, if and only if BE = CF.

Almost identical solutions by the proposer and by Charles W. Trigg, San Diego, California,

By Ceva's Theorem, (AF)(BD)(CE) = (FB)(DC)(EA), and since BD = DC, we have AF/FB = AE/EC. Consequently EF is parallel to BC, and EFBC is a trapezoid,

If BE = CF: the trapezoid is isosceles, angles ECB and FBC are equal, and the triangle ABC is isosceles, with AB = AC.

If AB = AC: angles ACB and ABC are equal, so the trapezoid is isosceles and its diagonals BF and CF are equal.

Also solved by Bruce W. King, Burnt-Hills-Ballston Lake High School; Joe Konhauser, Macalester College; Graham Lord; John McNear, Lexington High School; Andrew E. Rouse, University of Mississippi; and Gregory Wulczyn, Bucknell University.

204. (Spring, 1968) Proposed by M. S. Klamkin, Ford Scientific Laboratory.

If $a_{n+1} = \sqrt{2 + a_n}$, $n = 0,1,2,\ldots$; $a_0 = \sqrt{x}$, find

$$\operatorname*{limit}_{x \to 4} \frac{a_r - 2}{x - 4}.$$

Editorial Note: Special cases of this problem occur in R. E. Johnson, F. L. Kiokemeister, Calculus with Analytic Geometry, 3rd Edition, Allyn and Bacon, Boston, p. 74.

Solution I by Robert J. Herbold, Proctor & Gamble Company, Cincinnati, Ohio.

By L'Hospital's rule,

$$\lim_{x \to 4} \frac{a_0 - 2}{x - 4} = \frac{1}{4},$$

and

$$\lim_{x \to 4} \frac{a_1 - 2}{x - 4} = \left(\frac{1}{4}\right)^2.$$

Hence, we are led to showing by induction that

$$\lim_{x \to 4} \frac{a_r - 2}{x - 4} = \left(\frac{1}{4}\right)^{r+1}.$$

From above, we know this holds for r = 0 and r = 1. Suppose it is true for r = k. Consider

$$\lim_{x \to 4} \frac{a_{k+1} - 2}{x - 4} = \lim_{x \to 4} \frac{\sqrt{2 + a_k} - 2}{x - 4}.$$

By hypothesis,

$$\lim_{x \to 4} \frac{a_k - 2}{x - 4} = \left(\frac{1}{4}\right)^{k+1}.$$

Therefore,

$$\lim_{x \to 4} a_k - 2 = \lim_{x \to 4} (x - 4)\left(\frac{1}{4}\right)^{k+1}$$

$$\lim_{x \to 4} \sqrt{a_k + 2} = \lim_{x \to 4} \sqrt{(x - 4)\left(\frac{1}{4}\right)^{k+1} + 4},$$

and

$$\lim_{x \to 4} \frac{\sqrt{a_k + 2} - 2}{x - 4} = \lim_{x \to 4} \frac{\sqrt{(x - 4)\left(\frac{1}{4}\right)^{k+1} + 4} - 2}{x - 4}$$

But by L'Hospital's rule

$$\lim_{x\to 4} \frac{\sqrt{(x-4)(\frac{1}{4})^{k+1} + 4} - 2}{x-4} = (\tfrac{1}{4})^{k+2}$$

Therefore,

$$\lim_{x\to 4} \frac{a_{k+1} - 2}{x - 4} = (\tfrac{1}{4})^{k+2}$$

and so by induction,

$$\lim_{x\to 4} \frac{a_r - 2}{x - 4} = (\tfrac{1}{4})^{r+1}, \quad r \geq 0.$$

Solution II by the Proposer.

We consider the more general problem of finding

$$\operatorname*{limit}_{x\to\bar{L}} \frac{a_r - L}{x - \bar{L}}$$

where the sequence $\{a_n\}$ is defined by

$$a_{n+1} = F(a_n),\ a_0 = G(x),\ a \leq x \leq b$$

F', G', exist, and where

$$G(\bar{L}) = L,\ \operatorname*{limit}_{n\to\infty} a_n = L \text{ (independent of } x).$$

Let

$$L_{r+1} = \operatorname*{limit}_{x\to\bar{L}} \frac{a_{r+1} - L}{x - \bar{L}} = \operatorname*{limit}_{x\to\bar{L}} \frac{F(a_r) - L}{x - \bar{L}}.$$

Since $L = F(L)$ and $a_r = L$ for $x = \bar{L}$,

$$L_{r+1} = \operatorname*{limit}_{x\to\bar{L}} \frac{F(a_r) - L}{a_r - L}\,\frac{a_r - L}{x - \bar{L}} = F'(L)L_r.$$

Whence, $L_r = \{F'(L)\}^r L_0 = \{F'(L)\}^r G'(\bar{L})$.

For the given problem

$$F(x) = \sqrt{2 + x},\ G(x) = \sqrt{x},\ \bar{L} = 4, \text{ and } L = 2.$$

Thus,

$$L_r = (\tfrac{1}{4})^{r+1}$$

Also solved by Richard Enison, New York; Keith Giles, University of Oklahoma; Michael R. Gorelick, Adelphi University; Rick Johnson, East Carolina University; Bruce W. King, Burnt-Hills-Ballston Lake High School; Graham Lord, Philadelphia; Andrew E. Rouse, University of Mississippi; David Thomas, Southeastern Louisiana College; and Gregory Wulczyn, Bucknell University.

Late solutions were received from Edgar Karst (problem 200); David Thomas (problems 200 and 201); and Dan Deignan (problem 201).

BOOK REVIEWS

Edited by

Roy B. Deal, University of Oklahoma Medical Center

1. The Art of Philosophizing by Bertrand Russell, Philosophical Library, New York. New York, 1968, 119 pp., $3.95.

   A collection of essays concerning the art of rational conjecture. the art of drawing inferences, and the art of reckoning. Written by Bertrand Russell while he was teaching philosophy at American Universities during the Second World War.

2. Evolution of Mathematical Concepts by R. L. Wilder, John Wiley and Sons, Inc., New York, New York, 1968, xiii t 224 pp., $8.00.

   Highly recommended to all Pi Mu Epsilon readers and their friends who have a lay interest in mathematics or the "New Math." This little hook looks at the history of some important mathematical concepts from the point of view of their evolution and the "forces" of various cultures on these developments.

3. Comhinatorial Theory hy Marshall Hall, Jr., Blaisdell Publishing Company, Waltham, Massachusetts, 1967, x t 310 up.

   This is another book of wide general interest to Pi Mu Epsilon readers. It is so well written and organized that it can be self-studied and it has a wide variety of results and techniques which have application to statistics, operations research, and modern physics, as well as having much of the same type of intrinsic fascination that number theory has,

4. Outline of General To ology by R. Engelking, John Wiley and Sons, Inc., New York, Nw ~~98~~ 388 pp., $17.50.

   This introduction to general topology has much classical material, some modern material and some material with a modern view of the classical. It contains more than some of the comparable books (some of which would he excellent prerequisites for advanced calculus), but because of its depth and rather formal style of presentation, even though it provides many excellent examples, it probably should follow some introduction to modern analysis.

5. Algebraic Geometry by I. G. Macdonald, W. A. Benjamin, Inc., New York, 1968, vii + 113 pp.

   A brief, formal, well-organized presentation of some of the important concepts, such as sheaves and schemes, of modern algebraic approaches to geometry and other aspects of the homological algebra of today. Although, in principle, nothing is assumed of the reader beyond elementary notions of algebra and topology, he must be prepared for the formalism and abstract nature of the subject,

6. Convergence of Probability Measures by P. Billingsley, John Wiley and Sons, Inc., New York, New York, 1968, xii + 253 pp., $12.50.

For the reader with a background of a year of real analysis and some probability theory and/or statistics, this book presents a unified treatment of many of the convergence problems in measure theory.

7. Battelle Rencontres by Cecile M. DeWitt and John A. Wheeler, W. A. Benjamin, New York, New York, 1968, xvii + 557 pp., $14.50.

A collection of most of the expositions on a wide spectrum of topics in modern physics and mathematics which were presented at a meeting of some of the world's most outstanding physicists and mathematicians at the Battelle-Seattle Center in the summer of 1967, where it was hoped to consolidate the experiences of many on some of the complex problems of our time. Needless to say, much maturity is required for some of the articles and each reader will need to judge his readiness from his own experiences.

8. Boundary Value Problems of Mathematical Physics, Volume II by Ivar Stakgold, The MacMillan Company, New York, New York, 1968, viii + 408 pp.

A continuation of the excellent first volume. It mainly emphasizes solutions of partial differential equations with boundary values, using distributions, Green's functions and variation techniques.

9. Mathematical Models of Arms Control and Disarmament by Thomas L. Saaty, John Wiley and Sons, Inc., New York, New York, ix t 190 pp., $10.95.

This book represents a sound initial effort to bring to bear some of the organizational advantages of mathematical modeling on complex political problems and to provide, at least in some cases, a partially unifying language. A reader with a little knowledge of matrix algebra and differential equations will have no difficulty with the mathematics, but even a read who is familiar with most of the mathematical models will find the organizational problems of relating the models to the real world, as presented here, very informative,

10. Mathematical Programming in Practice by E. M. L. Beale, John Wiley and Sons, New York, New York, 1968, xi t 195 pp., $5.50.

The author lists five catagories for the recent books in mathematical programming. He lists this book under "methods of organizing real problems so they can be solved numerically using standard computer codes,'' He says, however, this cannot be discuessed intelligently without reference to other aspects of the subject. Although he avoids introducing unnecessary mathematics, this book is not for anyone who takes pride in his ignorance of mathematics. In addition to standard topics in linear programming the book includes quadratic programming, separable programming, integer programming, decomposition techniques, and a brief (but interesting) chapter on stochastic programming.

11. Nonlinear Programming by Anthony V. Fiacco and Garth P. MacCormick, John Wiley and Sons, New York, New York, 1968 xiv + 210 pp., $9.95.

"The primary prupose of this hook is to provide a unified body of theory on methods of transforming a constrained minimization problem into a sequence of unconstrained minimizations of an appropriate auxiliary function." It is a rather comprehensive exposition of nonlinear programming, including the historical remarks and bibliography.

12. Filtering for Stochastic Processes with Applications to Guidance by R. S. Bucy and P. D. Joseph, John Wiley and Sons, Inc., New York New York, 1968 xviii t 195 pp., $12.95.

Although the mathematical prerequisites are listed as probability theory (Chapters 4 and 5 of Feller) and differential equations (the first few chapters of Coddington and Levinson), the book provides such a comprehensive treatment of both the theory and practice in the subject that some maturity in the area should be a prerequisite,

13. An Introduction to Harmonic Analysis by Y. Katznelson, John Wiley and Sons, Inc., New York, New York, 1968, xiv + 264 pp., $12.95.

Although the book is aimed at Fourier analysis on locally compact Abelian groups, the first six (of eight) chapters deal with ordinary Fourier series and Fourier transforms with the general case in mind. The last chapter, on commutative Banach algebras, also emphasizes those parts related to the same subject.

BOOKS RECEIVED FOR REVIEW

1. The Elements of Complex Analysis by J. Duncan, John Wiley and Sons, Inc., New York, New York, 1968, ix + 313 pp., $11.50. Cloth, also available in paper at $5.75.

2. Mathematics of Finance by Cissell and Cissell, Houghton Mifflin Company, Boston, Massachusetts, 1968, xviii + 346 + 89 pp., $7.50.

3. Basic Concepts of Calculus by John M. H. Olmsted, Appleton-Century-Crofts, New York, New York, 1968, xiv t 403 pp.

4. A Second Course in Calculus by John M. H. Olmsted, Appleton-Century-Crofts, New York, New York, 1968, xv + 389 pp.

5. Prelude to Calculus and Linear Algebra by John M. H. Olmsted, Appleton-Century-Crofts, New York, New York, 1968, xix + 332 pp.

Note: All correspondence concerning reviews and all books for review should be sent to PROFESSOR ROY B. DEAL, UNIVERSITY OF OKLAHOMA MEDICAL CENTER. 800 NE 13th STREET, OKLAHOMA CITY, OKLAHOMA 73104.

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