Revista Colombiana de Matemáticas
Volumen 51(2017)1, páginas 15-19

On analytic families of conformal maps

Sobre familias analíticas de mapeos conformes

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Abstract. Let Λ be a domain in $\mathbb{C}$ and let $f_\lambda(z) = z + a_0(\lambda) + a_1(\lambda)z^{-1} + ...$ be meromorphic in $\mathbb{D}_* := \{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$. We assume that $f_\lambda(z)$ is holomorphic in $\lambda \in \Lambda$ for fixed z.

The main theorem states: Let Λ_0 be a subdomain of Λ such that f_λ is univalent in $\mathbb{D}_*$ for $\lambda \in \Lambda_0$. If f_{λ_0} has a quasiconformal extension to the closure of $\mathbb{D}_*$ for one $\lambda_0 \in \Lambda_0$ then f_λ has a quasiconformal extension for all $\lambda \in \Lambda_0$.

This result is related to a theorem of Mañé, Sad and Sullivan (1983) where the assumptions are however different. The main tool of our proof is the Grunsky inequality for univalent functions.

Key words and phrases. Univalent function, quasiconformal extension, analytic parameter, Grunsky inequality.

2010 Mathematics Subject Classification. 30C55, 30C62, 32A10.

Resumen. Sea Λ a dominio en $\mathbb{C}$ y sea $f_\lambda(z) = z + a_0(\lambda) + a_1(\lambda)z^{-1} + ...$ meromorfa en $\mathbb{D}_* := \{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$. Suponemos que $f_\lambda(z)$ es holomorfa en $\lambda \in \Lambda$ para z fijo.

El teorema principal dice: Sea Λ_0 un subdominio de Λ tal que f_λ es univalente en $\mathbb{D}_*$ para $\lambda \in \Lambda_0$. Si f_{λ_0} tiene una extensión cuasiconforme a la clausura de $\mathbb{D}_*$ para un $\lambda_0 \in \Lambda_0$ entonces f_λ tiene una extensión cuasiconforme para todo $\lambda \in \Lambda_0$.

Este resultado está relacionado a un teorema de Mañé, Sad y Sullivan (1983) donde sin embargo las hipótesis son diferentes. Para nuestra demostración la herramienta principal es la desigualdad de Grunsky para funciones univalentes.

Palabras y frases clave. Funciones univalentes, extensión cuasiconforme, parámetro analítico, desigualdad de Grunsky.

1. Introduction

In 1983, Mañé, Sad and Sullivan proved the following surprising result. Suppose that

$$\begin{aligned}&\text{(a) } f_\lambda : \mathbb{D} \to \mathbb{C} \text{ is injective for each fixed } \lambda \in \mathbb{D},\\ &\text{(b) } f(z,\lambda) = f_\lambda(z) \text{ is holomorphic in } \lambda \in \mathbb{D} \text{ for each fixed } z \in \mathbb{D},\\ &\text{(c) } f_0(z) = z \text{ for } z \in \mathbb{D}.\end{aligned} \tag{1}$$

Then each f_λ can be extended to a quasiconformal homeomorphism of $\bar{\mathbb{D}}$ into $\mathbb{C}$. See [6] and see [10] and [1] for further results.

We shall consider a different but related set of assumptions. Let Λ be a domain in $\mathbb{C}$. We write $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ and $\mathbb{D}_* := \{z : |z| > 1\} \cup \{\infty\}$. Let the function $f(z,\lambda)$ be defined for $(z,\lambda) \in \mathbb{D}_* \times \Lambda$. We assume that

$$\begin{aligned}&\text{the function } f(\cdot,\lambda) : \mathbb{D}_* \to \hat{\mathbb{C}} \text{ is holomorphic for fixed } \lambda \in \Lambda\\ &\text{except that } \; f(z,\lambda) = z + a_0(\lambda) + a_1(\lambda) z^{-1} + ...,\end{aligned} \tag{2}$$

$$f(z,\cdot) : \Lambda \to \hat{\mathbb{C}} \text{ is holomorphic for fixed } z \text{ with } 1 < |z| < \infty. \tag{3}$$

We will often write $f_\lambda(z)$ instead of $f(z,\lambda)$. The assumption (3) corresponds to (b) in (1) whereas assumption (2) is quite different from (a). The initial condition (c) has no counterpart.

We need the Hartogs theorem of the theory of several complex variables, see e.g. [2, p.140]. We write it in a form adapted to our present context.

Proposition 1.1. *Let* (2) *and* (3) *be satisfied. Then the function*

$$f(\cdot,\cdot) : \mathbb{D}_* \times \Lambda \to \hat{\mathbb{C}}$$

is holomorphic except in $z = \infty$ and therefore continuous in every compact subset of $\mathbb{D}_* \times \Lambda$.

2. Univalence

A complex-valued function is called univalent if it is injective and meromorphic in a domain in $\hat{\mathbb{C}}$. We define

$$U := \{\lambda \in \Lambda : f_\lambda \text{ is univalent in } \mathbb{D}_*\}. \tag{4}$$

Since $f_\lambda(z) = z + ...$ by assumption (2) we can therefore write

$$\log \frac{f_\lambda(z) - f_\lambda(\zeta)}{z - \zeta} = -\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} b_{k,l}(\lambda) z^{-k} \zeta^{-l} \quad \text{for } |z| > 1, |\zeta| > 1\,. \tag{5}$$

Since the coefficients a_k in (2) are holomorphic in Λ it follows that the coefficients $b_{k,l}$ are defined and holomorphic for $\lambda \in \Lambda$, see [7, p.58]. The Grunsky inequality states that

$$\Big|\sum_k \sum_l b_{k,l}(\lambda) x_k x_l\Big| \leq 1 \quad \text{for} \quad x_k \in \mathbb{C}, \sum_{k=1}^{\infty} \frac{1}{k}|x_k|^2 \leq 1,\ \lambda \in U, \tag{6}$$

see [4] [7, p.60] [3, p.122]. It easily follows from (5) and (6) that

$$f_\lambda \text{ is univalent if and only if } |b_{k,l}| \leq 1 \text{ for } k, l \in \mathbb{N}, \tag{7}$$

see e.g. [7, p.59].

Theorem 2.1. *Let* (2) *and* (3) *be satisfied. Then U is relatively closed. For every component C of the open set $\Lambda \setminus U$ we have $\partial C \cap \partial\Lambda \neq \varnothing$.*

Proof. (i) Let $\lambda_0 \in \Lambda \cap \partial U$. Then there are $\lambda_n \in U$ such that $\lambda_n \to \lambda_0$ as $n \to \infty$. It follows from Proposition 1.1 that

$$f_{\lambda_n} \to f_{\lambda_0}\ (n \to \infty) \quad \text{locally uniformly in } \mathbb{D}_*.$$

Now f_{λ_n} is univalent by (4). Since f_{λ_0} is non-constant by assumption (2) it follows that f_{λ_0} is univalent in $\mathbb{D}_*$ so that $\lambda_0 \in U$.

(ii) If C is unbounded then obviously $\infty \in \partial C \cap \partial\Lambda$. Now let C be a bounded component and suppose that $\partial C \cap \partial\Lambda = \varnothing$ so that $\partial C \subset \Lambda$. Since C is a component of $\Lambda \setminus U$ we have $\partial C \subset \bar{U} \cap \Lambda = U$. It follows, by (4) and (7), that $|b_{k,l}(\lambda)| \leq 1$ for $\lambda \in \partial C$. Since the $b_{k,l}(\lambda)$ are holomorphic in C we conclude by the maximum principle that the inequality $|b_{k,l}(\lambda)| \leq 1$ also holds for $\lambda \in C$. Hence we obtain from (7) that f_λ is univalent in D_* so that $\lambda \in U$ by (4). This contradicts $C \subset \Lambda \setminus U$. ☑

***Remark* 2.2.** . If Λ is simply connected then Theorem 2.1 implies that every component of the open kernel U° is simply connected. If in addition $\partial U \subset \Lambda$ then $\Lambda \setminus U$ is a domain.

Example 2.3. Let $p(\lambda)$ be a non-constant entire function and let $f_\lambda(z) = z + p(\lambda)z^{-1}$. Then (2) is satisfied with $\Lambda = \mathbb{C}$. We have

$$f'_\lambda(z) = 1 - \frac{p(\lambda)}{z^2}, \quad \frac{f_\lambda(z) - f_\lambda(\zeta)}{z - \zeta} = 1 - \frac{p(\lambda)}{z\zeta} \quad \text{for } |z| > 1, |\zeta| > 1. \tag{8}$$

If $|p(\lambda)| > 1$ then f'_λ has the zero $\sqrt{p(\lambda)}$ in $\mathbb{D}_*$ so that f_λ is not univalent. If $|p(\lambda)| \leq 1$ then $|1 - p(\lambda)/(z\zeta)| > 0$ so that f_λ is univalent. Hence we have $U = \{\lambda \in \mathbb{C} : |p(\lambda)| \leq 1\}$. Since p is arbitrary this provides us with a huge variety of closed sets where f_λ is univalent. If p is a non-constant polynomial then U is compact.

If we choose $p(\lambda) = \lambda^2 - 1$ then U is the classical lemniscate $|\lambda^2 - 1| \leq 1$. If we choose $p(\lambda) = \exp(\lambda^2)$ then U is the unbounded closed set $\pi/4 \leq |\arg\lambda| \leq 3\pi/4$ which consists of two quarter-planes that meet at 0.

3. Quasiconformality

We assume that our conditions (2) and (3) are satisfied. Let U be defined by (4) and $b_{k,l}(\lambda)$ by (5).

Proposition 3.1. (Schiffer and Springer). *Let $\lambda \in U$. The function f_λ has a quasiconformal extension to $\bar{D}_*$ if and only if there exists $\kappa < 1$ such that*

$$\Big|\sum_k \sum_l b_{k,l}(\lambda) x_k x_l\Big| \le \kappa \text{ for all } x_k \in \mathbb{C} \text{ with } \sum_k \frac{1}{k}|x_k|^2 \le 1. \tag{9}$$

See [8] [9] [5] [7, Th.9.12,Th.9.13].

Theorem 3.2. *Let Λ_0 be a subdomain of Λ and let f_λ be univalent in $\mathbb{D}_*$ for $\lambda \in \Lambda_0$. If there exists $\lambda_0 \in \Lambda_0$ such that f_{λ_0} has a quasiconformal extension to $\bar{\mathbb{D}}_*$ then f_λ has a quasiconformal extension to $\bar{\mathbb{D}}_*$ for every $\lambda \in \Lambda_0$.*

Let V be any component of U°. An obvious consequence of this theorem is that f_λ has a quasiconformal extension either for all or for no $\lambda \in V$. This raises an interesting question: If f_λ has a quasiconformal extension for some component V, does it follow that f_λ has quasiconformal extension for all $\lambda \in U^\circ$?

Proof. Let $\lambda_1 \in \Lambda_0$. Then there is a simply connected domain G with $\lambda_0, \lambda_1 \in G$ and $\bar{G} \subset \Lambda_0$. Let g map $\mathbb{D}$ conformally onto G and let $g(\zeta_0) = \lambda_0, g(\zeta_1) = \lambda_1$.

Now let $x_k \in \mathbb{C}\,(k \in \mathbb{N})$ with $\sum \frac{1}{k}|x_k|^2 \le 1$ and

$$\varphi(\lambda) := \sum_k \sum_l b_{k,l}(\lambda) x_k x_l \ (\lambda \in \Lambda), \quad h(\zeta) := \varphi(g(\zeta)) \ (\zeta \in \mathbb{D}).$$

Since f_λ is univalent in $\mathbb{D}_*$ we obtain from (6) that $|\varphi(\lambda)| \le 1$ for $\lambda \in \Lambda_0$. Furthermore, since $\bar{G} \subset \Lambda_0$ and φ is holomorphic in Λ_0 it follows that $|h(\zeta)| < 1$ for $\zeta \in \mathbb{D}$. Finally we have

$$|h(\zeta_0)| = |\varphi(g(\zeta_0))| = |\varphi(\lambda_0)| \le \kappa_0 < 1$$

by our assumption and by Proposition 3.1.

The hyperbolic metric in $\mathbb{D}$ is defined by

$$d(a,b) = \tfrac{1}{2}\log\Big((1 + |\frac{a-b}{1-\bar{a}b}|)/(1 - |\frac{a-b}{1-\bar{a}b}|)\Big) \ (a, b \in \mathbb{D}).$$

We have $d(\varphi(\lambda_1), \varphi(\lambda_0)) = d(h(\zeta_1), h(\zeta_0)) \le d(\zeta_1, \zeta_0))$ because $h(\mathbb{D}) \subset \mathbb{D}$ and therefore

$$\begin{aligned} d(\varphi(\lambda_1), 0) &\le d(\varphi(\lambda_1), \varphi(\lambda_0)) + d(\varphi(\lambda_0), 0) \\ &\le d(\zeta_1, \zeta_0) + \tfrac{1}{2}\log\frac{1+\kappa_0}{1-\kappa_0} =: \tfrac{1}{2}\log\frac{1+\kappa_1}{1-\kappa_1}. \end{aligned}$$

It follows that $|\varphi(\lambda_1)| \le \kappa_1 < 1$. ☑

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(Recibido en julio de 2016. Aceptado en noviembre de 2016)

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