Initial boundary value problems on half-line $x > 0$.

Problem for homogeneous diffusion equation with homogeneous Dirichlet boundary condition

$$u_t(x,t) - ku_{xx}(x,t) = 0\,, \quad x > 0,\ t > 0$$
$$u(0,t) = 0\,, \quad t > 0$$
$$u(x,0) = \varphi(x)\,, \quad x > 0$$

The method to solve this problem is as follows:

We extend the initial state to the whole axis, i.e. introduce some

$$\widetilde{\varphi}(x) \quad \text{for} \quad x \in \mathbb{R}$$

such that

$$\widetilde{\varphi}(x) = \varphi(x) \quad \text{in case} \quad x > 0$$

and replace the original problem by the Cauchy problem

$$v_t(x,t) - kv_{xx}(x,t) = 0\,, \quad x \in \mathbb{R},\ t > 0$$
$$v(x,0) = \widetilde{\varphi}(x)\,, \quad x \in \mathbb{R}$$

We specify the extension $\widetilde{\varphi}$ so that the corresponding solution of the Cauchy problem $v(x,t)$ has the following property:

$$v(0,t) = 0\,, \quad t > 0$$

Then the restriction of $v(x,t)$ to the right half-axis $x > 0$, i.e.

$$u(x,t) = v(x,t)\,, \quad x > 0\,, \ \ t \in \mathbb{R}$$

satisfies the original initial boundary value problem.

A suitable extension is the *odd reflection*

$$\widetilde{\varphi}(x) \;=\; \begin{cases} \varphi(x) & \text{if } \; x > 0 \\ -\varphi(-x) & \text{if } \; x < 0 \end{cases}$$

To become convinced in it, we take the solution formula of the Cauchy problem:

$$v(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\widetilde{\varphi}(y)dy\,, \quad G(x,t) = \frac{1}{\sqrt{4\pi kt}}\, e^{-\frac{x^2}{4kt}}$$

and compute

$$v(0,t) = \int_{-\infty}^{\infty} G(y,t)\widetilde{\varphi}(y)dy$$

We obtain the desired relation $v(0,t) = 0$.

Consequently, the solution of the original initial boundary value problem is

$$u(x,t) = v(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\widetilde{\varphi}(y)dy =$$

$$= \int_{-\infty}^{0} G(x-y,t)[-\varphi(-y)]dy + \int_{0}^{\infty} G(x-y,t)\varphi(y)dy\,,$$

$$x > 0\,, \quad t \in \mathbb{R}$$

After simplifications we obtain the solution formula

$$u(x,t) = \int_{0}^{\infty} \left[G(x-y,t) - G(x+y,t)\right]\varphi(y)dy$$

Problem for homogeneous wave equation with homogeneous Dirichlet boundary condition

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) \; = 0\,, \quad x > 0,\; t > 0$$
$$u(0,t) = 0\,, \quad t > 0$$
$$u(x,0) = \varphi(x)\,, \quad u_t(x,0) = \psi(x)\,, \quad x > 0$$

Defining the extensions of φ and ψ by means of *odd reflection*

$$\widetilde{\varphi}(x) \; = \begin{cases} \varphi(x) & \text{if } \; x > 0 \\ -\varphi(-x) & \text{if } \; x < 0 \end{cases} \qquad \widetilde{\psi}(x) \; = \begin{cases} \psi(x) & \text{if } \; x > 0 \\ -\psi(-x) & \text{if } \; x < 0 \end{cases}$$

the solution of the Cauchy problem

$$v_{tt}(x,t) - c^2 v_{xx}(x,t) \; = 0\,, \quad x \in \mathbb{R},\; t > 0$$
$$v(x,0) = \widetilde{\varphi}(x)\,, \quad v_t(x,0) = \widetilde{\psi}(x)\,, \quad x \in \mathbb{R}$$

possesses the property

$$v(0,t) = 0\,, \quad t > 0$$

Consequently,

$$u(x,t) = v(x,t)\,, \quad x > 0\,, \;\; t \in \mathbb{R}$$

is expressed by d'Alembert's formula

$$u(x,t) = v(x,t) = \frac{1}{2}\left[\widetilde{\varphi}(x+c\,t) + \widetilde{\varphi}(x-c\,t)\right] \,+\, \frac{1}{2c}\int_{x-c\,t}^{x+c\,t} \widetilde{\psi}(\tau)d\tau\,,$$

$x > 0$, $t \in \mathbb{R}$

After simplifications we obtain the solution formula

$$u(x,t) = \begin{cases} \frac{1}{2}\left[\varphi(x+c\,t) + \varphi(x-c\,t)\right] \,+\, \frac{1}{2c}\int\limits_{x-c\,t}^{x+c\,t} \psi(\tau)d\tau & \text{if } \; x > c\,t \\ \frac{1}{2}\left[\varphi(x+c\,t) - \varphi(c\,t-x)\right] \,+\, \frac{1}{2c}\int\limits_{c\,t-x}^{x+c\,t} \psi(\tau)d\tau & \text{if } \; 0 < x < c\,t \end{cases}$$

Other problems.

Problems with the homogeneous Neumann boundary condition

$$
\begin{aligned}
& u_t(x,t) - k u_{xx}(x,t) = 0\,, \quad x > 0,\ t > 0 \\
& u_x(0,t) = 0\,, \quad t > 0 \\
& u(x,0) = \varphi(x)\,, \quad x > 0
\end{aligned}
$$

$$
\begin{aligned}
& u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0\,, \quad x > 0,\ t > 0 \\
& u_x(0,t) = 0\,, \quad t > 0 \\
& u(x,0) = \varphi(x)\,, \quad u_t(x,0) = \psi(x)\,, \quad x > 0
\end{aligned}
$$

Method is the same.

Only instead of the odd reflection,
an *even reflection* of the initial conditions is used.

This means that in the case of the diffusion equation we define

$$\widetilde{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } x > 0 \\ \varphi(-x) & \text{if } x < 0 \end{cases}$$

and in the case of the wave equation we define

$$\widetilde{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } x > 0 \\ \varphi(-x) & \text{if } x < 0 \end{cases} \qquad \widetilde{\psi}(x) = \begin{cases} \psi(x) & \text{if } x > 0 \\ \psi(-x) & \text{if } x < 0 \end{cases}$$

Problems with nonhomogeneous equations and homogeneous boundary conditions.

Method is the same.

In addition to initial conditions, the function $f(x,t)$ is reflected.

For example, let us consider the diffusion problem with Dirichlet boundary condition

$$\begin{aligned} & u_t(x,t) - ku_{xx}(x,t) \; = \; f(x,t)\,, \quad x > 0,\; t > 0 \\ & u(0,t) = 0\,, \quad t > 0 \\ & u(x,0) = \varphi(x)\,, \quad x > 0 \end{aligned}$$

We define the extensions of φ and f by means of odd reflection

$$\widetilde{\varphi}(x) \; = \; \begin{cases} \varphi(x) & \text{if } \; x > 0 \\ -\varphi(-x) & \text{if } \; x < 0 \end{cases} \qquad \widetilde{f}(x,t) \; = \; \begin{cases} f(x,t) & \text{if } \; x > 0 \\ -f(-x,t) & \text{if } \; x < 0 \end{cases}$$

As a result, the solution of the Cauchy problem

$$v_t(x,t) - kv_{xx}(x,t) \;=\; \widetilde{f}(x,t)\,, \quad x \in \mathbb{R},\; t > 0$$
$$v(x,0) = \widetilde{\varphi}(x)\,, \quad x \in \mathbb{R}$$

possesses the property

$$v(0,t) = 0, \quad t > 0$$

and the solution $u(x,t)$ of the original initial boundary value problem can be expressed as the restriction of $v(x,t)$ to $x > 0$.

Problems with nonhomogeneous boundary conditions.

They can be transformed to problems with homogeneous boundary conditions by means of changes of variables.

For example, let we be given the problem

$$u_t(x,t) - ku_{xx}(x,t) \;=\; 0\,, \quad x > 0,\; t > 0$$
$$u(0,t) = h(t)\,, \quad t > 0$$
$$u(x,0) = \varphi(x)\,, \quad x > 0$$

Then we introduce the new unknown function $\widehat{u}$:

$$\widehat{u}(x,t) = u(x,t) - h(t)$$

The function $\widehat{u}$ is a solution of the following problem with homogeneous boundary condition:

$$\widehat{u}_t(x,t) - k\widehat{u}_{xx}(x,t) \;=\; -h'(t)\,, \quad x > 0,\; t > 0$$
$$\widehat{u}(0,t) = 0\,, \quad t > 0$$
$$\widehat{u}(x,0) = \varphi(x) - h(0)\,, \quad x > 0$$