On the Subharmonicity of Separately Subharmonic Functions

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Abstract: We recall some of the existing subharmonicity results of separately subharmonic functions, and state the corresponding generalized counterparts for separately quasi-nearly subharmonic functions, thus giving partial generalizations of certain results of Arsove and of Cegrell and Sadullaev. Moreover, we improve a result of Kołodziej and Thornbiörnson concerning the subharmonicity of a function subharmonic in the first variable and harmonic in the second.

Key–Words: separately subharmonic, harmonic, quasi-nearly subharmonic, Harnack, integrability condition

1 Introduction

1.1 Separately subharmonic functions

Wiegerinck [27], see also [28], Theorem 1, p. 246, has shown that a separately subharmonic function need not be subharmonic. On the other hand, Armitage and Gardiner [1], Theorem 1, p. 256, showed that a separately subharmonic function u on a domain Ω of $\mathbb{R}^{m+n}$, $m \geq n \geq 2$, is subharmonic provided $\phi(\log^+ u^+)$ is locally integrable, where $\phi : [0,+\infty) \to [0,+\infty)$ is an increasing function such that

$$\int_1^{+\infty} s^{(n-1)/(m-1)}(\phi(s))^{-1/(m-1)}\, ds < +\infty. \qquad (1)$$

For related previous results of Lelong, Avanissian, Arsove and Riihentaus, see e.g. [10], [11], [12], [4], [3], [8], [16] and the references therein. One of these previous results was ours:

Theorem 1 *([16], Theorem 1, p. 69) Let Ω be a domain in $\mathbb{R}^{m+n}$, $m,n \geq 2$. Let $u : \Omega \to [-\infty,+\infty)$ be such that*

(a) for each $y \in \mathbb{R}^n$ the function

$$\Omega(y) \ni x \mapsto u(x,y) \in [-\infty,+\infty)$$

is subharmonic,

(b) for each $x \in \mathbb{R}^m$ the function

$$\Omega(x) \ni y \mapsto u(x,y) \in [-\infty,+\infty)$$

is subharmonic,

(c) for some $p > 0$ there is a function $v \in \mathcal{L}^p_{loc}(\Omega)$ such that $u \leq v$.

Then u is subharmonic.

Though the cited result of Armitage and Gardiner includes our Theorem 1, and in fact their result is even "almost" sharp, we present below in Theorem 4 a generalization to Theorem 1. This is justified because of two reasons. First, our $\mathcal{L}^p_{\mathrm{loc}}$ integrability condition, $p > 0$, is, unlike the condition of Armitage and Gardiner (1), very simple, and second, our generalization to Theorem 1 is stated for quasi-nearly subharmonic functions, and as such, it is very general, see **2.1.** below.

1.2 Functions subharmonic in one variable and harmonic in the other

An open problem is, whether a function, which is subharmonic in one variable and harmonic in the other, is subharmonic. For results on this area, see e.g. [3], [28], [7] and [9] and the references therein. We consider here two results. First Theorem 2 below, a result of Arsove [3], Theorem 2, p. 622, and again, but with a different proof, of Cegrell and Sadullaev [7], Theorem 3.1, p. 82, and second, Theorem 3 below, a result of Kołodziej and Thornbiörson [9], Theorem 1, p. 463.

Theorem 2 *Let Ω be a domain in $\mathbb{R}^{m+n}$, $m,n \geq 2$. Let $u : \Omega \to \mathbb{R}$ be such that*

(a) for each $y \in \mathbb{R}^n$ the function

$$\Omega(y) \ni x \mapsto u(x,y) \in \mathbb{R}$$

is subharmonic,

(b) for each $x \in \mathbb{R}^m$ the function

$$\Omega(x) \ni y \mapsto u(x,y) \in \mathbb{R}$$

is harmonic,

(c) there is a nonnegative function $\varphi \in \mathcal{L}^1_{loc}(\Omega)$ such that $-\varphi \leq u$.

Then u is subharmonic.

Arsove's proof is brief and it is based on mean value operators, [3], p. 625. Cegrell and Sadullaev use Poisson modification in their proof. Unawere of Arsove's result (and proof), we gave in [20] a detailed proof, based on mean value operators, which now, because of [3], p. 625, should, more or less, be considered just an elaboration of Arsove's argument. On the other hand, we give below in Theorem 5 a concise counterpart to a corollary of Arsove and of Cegrell and Sadullaev, [7], Corollary, p. 82, that is, Theorem 2 in the case $\varphi = 0$.

Kołodziej and Thorbiörnson gave the following result. Their proof uses the above result of Arsove and of Cegrell and Sadullaev, see [7], proof of Theorem 3.2, p. 83.

Theorem 3 *([9], Theorem 1, p. 463) Let Ω be a domain in $\mathbb{R}^{m+n}$, $m,n \geq 2$. Let $u : \Omega \to \mathbb{R}$ be such that*

(a) for each $y \in \mathbb{R}^n$ the function

$$\Omega(y) \ni x \mapsto u(x,y) \in \mathbb{R}$$

is subharmonic and $\mathcal{C}^2$,

(b) for each $x \in \mathbb{R}^m$ the function

$$\Omega(x) \ni y \mapsto u(x,y) \in \mathbb{R}$$

is harmonic.

Then u is subharmonic and continuous.

Below in Theorem 6 we give a generalization to the above result of Kołodziej and Thornbiörnson. Instead of the standard Laplacians of $\mathcal{C}^2$ functions we use generalized Laplacians, that is the Blaschke-Privalov operators.

2 Definitions and Notation

2.1 Quasi-nearly subharmonic functions

Our notation is rather standard, see e.g. [17], [18], [19], [20] and [8].

Let D be a subdomain of the Euclidean space $\mathbb{R}^N$, $N \geq 2$. A Lebesgue measurable function $u : D \to [0,+\infty)$ is *quasi-nearly subharmonic*, if $u \in \mathcal{L}^1_{\rm loc}(D)$ and if there is a constant $K = K(N,u,D) > 0$ such that

$$u(x) \leq \frac{K}{r^N} \int\limits_{B(x,r)} u(y)\, dm_N(y) \qquad (2)$$

for any ball $B^N(x,r) \subset D$. For the Lebesgue measure in $\mathbb{R}^N$, $N \geq 2$, we use m_N. (Below m will be used also for the dimension of the Euclidean space $\mathbb{R}^m$, but this will surely cause no confusion.) We write ν_N for the Lebesgue measure of the unit ball $B^N(0,1)$ in $\mathbb{R}^N$, thus $\nu_N = m_N(B^N(0,1))$. This function class of quasi-nearly subharmonic functions is natural, it has important and interesting properties and, at the same time, it is large, see e.g. [13], [17], [14], [18] and [19]. We recall here only that it includes, among others, nonnegative subharmonic functions, nonnegative nearly subharmonic functions (see e.g. [8]), functions satisfying certain natural growth conditions, especially certain eigenfunctions, and polyharmonic functions. Also, any Lebesgue measurable function $u : D \to [m,M]$, where $0 < m \leq M < +\infty$, is quasi-nearly subharmonic.

Constants will be denoted by C and K. They will be nonnegative and may vary from line to line.

2.2 Harnack functions

As a counterpart to nonnegative harmonic functions, we recall the definition of Harnack functions, see [26], p. 259. A continuous function $u : D \to [0,+\infty)$ is a *Harnack function*, if there are constants $\lambda \in (0,1)$ and $C = C(\lambda) \geq 1$ such that

$$\max_{z \in B(x,\lambda r)} u(z) \leq C \min_{z \in B(x,\lambda r)} u(z)$$

whenever $B(x,r) \subset D$. It is well-known that for each compact set F in D there exists a smallest constant $C(F) \geq C$ depending only on N, λ, C and F such that for all u satisfying the above condition,

$$\max_{z \in F} u(z) \leq C(F) \min_{z \in F} u(z).$$

One sees easily that Harnack functions are quasi-nearly subharmonic. Also the class of Harnack functions is very wide. It includes, among others, nonnegative harmonic functions as well as nonnegative solutions of some elliptic equations. Also, any continuous function $u : D \to [m,M]$, where $0 < m \leq M < +\infty$, is a Harnack function. See [26], pp. 259, 263.

2.3 Permissible functions

A function $\psi : [0,+\infty) \to [0,+\infty)$ is *permissible*, if there exists an increasing (strictly or not), convex function $\psi_1 : [0,+\infty) \to [0,+\infty)$ and a strictly increasing surjection $\psi_2 : [0,+\infty) \to [0,+\infty)$ such that $\psi = \psi_2 \circ \psi_1$ and such that the following conditions are satisfied:

(a) ψ_1 satisfies the Δ_2-condition.

(b) ψ_2^{-1} satisfies the Δ_2-condition.

(c) The function $t \mapsto \frac{\psi_2(t)}{t}$ is *quasi-decreasing*, i.e. there is a constant $C = C(\psi_2) > 0$ such that

$$\frac{\psi_2(s)}{s} \geq C\,\frac{\psi_2(t)}{t}$$

for all $0 \leq s \leq t$.

See also [14], Lemma 1 and Remark 1. Recall that a function $\varphi : [0,+\infty) \to [0,+\infty)$ satisfies the Δ_2-condition, if there is a constant $C = C(\varphi) \geq 1$ such that $\varphi(2t) \leq C\,\varphi(t)$ for all $t \in [0,+\infty)$.

3 Separately Subharmonic Functions

The following gives a counterpart to Theorem 1, see also [20], Theorem 1. Observe that, as pointed out in [20], Remark 3.2, the measurability assumption is now necessary, unlike in Theorem 1 above.

Theorem 4 *Let Ω be a domain in $\mathbb{R}^{m+n}$, $m,n \geq 2$. Let $u : \Omega \to [0,+\infty)$ be a Lebesgue measurable function such that*

(a) for each $y \in \mathbb{R}^n$ the function

$$\Omega(y) \ni x \mapsto u(x,y) \in [0,+\infty)$$

is quasi-nearly subharmonic,

(b) for each $x \in \mathbb{R}^m$ the function

$$\Omega(x) \ni y \mapsto u(x,y) \in [0,+\infty)$$

is quasi-nearly subharmonic,

(c) there exists a non-constant permissible function $\psi : [0,+\infty) \to [0,+\infty)$ such that $\psi \circ u \in \mathcal{L}^1_{loc}(\Omega)$.

Then u is quasi-nearly subharmonic.

Proof: Using the generalized mean value inequality ((2) above), first in the first variable and then in the second, one sees that $\psi \circ u$ is locally bounded in Ω. Since ψ is permissible, it follows that also u is locally bounded in Ω. With the aid of Fubini's Theorem one then sees that u is quasi-nearly subharmonic. See [20], proof of Theorem 1, for details. □

4 The Result of Arsove and of Cegrell and Sadullaev

Then a counterpart to Arsove's and Cegrell's and Sadullaev's Corollary of their result, Theorem 2 above, see [7], Corollary, p. 82. See also [20], Theorem 2.

Theorem 5 *Let Ω be a domain in $\mathbb{R}^{m+n}$, $m,n \geq 2$. Let $u : \Omega \to [0,+\infty)$ be such that*

(a) for each $y \in \mathbb{R}^n$ the function

$$\Omega(y) \ni x \mapsto u(x,y) \in [0,+\infty)$$

is quasi-nearly subharmonic,

(b) for each $x \in \mathbb{R}^m$ the function

$$\Omega(x) \ni y \mapsto u(x,y) \in [0,+\infty)$$

is a Harnack function.

Then u is quasi-nearly subharmonic.

Proof: It is well-known that u is Lebesgue measurable. Let $(a,b) \in \Omega$ and $R > 0$ be such that $B^{m+n}((a,b),R) \subset \Omega$. Choose $(x_0,y_0) \in B^m(a,\frac{R}{4}) \times B^n(b,\frac{R}{4})$ arbitrarily. Since $u(\cdot,y_0)$ is quasi-nearly subharmonic, one has

$$u(x_0,y_0) \leq \frac{K}{(\frac{R}{4})^m} \int\limits_{B^m(x_0,\frac{R}{4})} u(x,y_0)\,dm_m(x).$$

On the other hand, since the functions $u(x,\cdot)$, $x \in B^m(a,\frac{R}{2})$, are Harnack functions in $B^n(b,\frac{R}{2})$, there is a constant $C = C(n,\lambda,C_\lambda,R)$ (here λ and C_λ are the constants in **2.2**) such that

$$\frac{1}{C} \leq \frac{u(x,y_0)}{u(x,b)} \leq C$$

for all $x \in B^m(a,\frac{R}{4})$. See e.g. [5], proof of 3.6, pp. 48–49. Therefore

$$\begin{aligned} u(x_0,y_0) &\leq \frac{K}{(\frac{R}{4})^m} \int\limits_{B^m(x_0,\frac{R}{4})} C\,u(x,b)\,dm_m(x) \\ &\leq \frac{C\cdot K}{(\frac{R}{4})^m} \int\limits_{B^m(a,\frac{R}{2})} u(x,b)\,dm_m(x) \\ &\leq \frac{K}{R^m} \int\limits_{B^m(a,\frac{R}{2})} u(x,b)\,dm_m(x) < \infty. \end{aligned}$$

Thus u is locally bounded above in $B^m(a,\frac{R}{4}) \times B^n(b,\frac{R}{4})$, and therefore the result follows from Theorem 1 above. □

5 The Result of Kołodziej and Thornbiörnson

5.1 Generalized Laplacians

In our generalization to the cited result of Kołodziej and Thorbiörnson, we use the generalized Laplacian, defined with the aid of the Blaschke-Privalov operators, see e.g. [22], [21], [15], [23], [24] and [25]. Let D be a domain in $\mathbb{R}^N$, $N \geq 2$, and $f : D \to \mathbb{R}$, $f \in \mathcal{L}^1_{\mathrm{loc}}(D)$. We write

$$\Delta^* f(x) :=$$
$$= \limsup_{r \to 0} \frac{2(N+2)}{r^2} \Big[\frac{1}{\nu_N r^N} \int_{B^N(x,r)} f(x') dm_N(x') - f(x) \Big],$$

$$\Delta_* f(x) :=$$
$$= \liminf_{r \to 0} \frac{2(N+2)}{r^2} \Big[\frac{1}{\nu_N r^N} \int_{B^N(x,r)} f(x') dm_N(x') - f(x) \Big].$$

If $\Delta^* f(x) = \Delta_* f(x)$, then write $\Delta f(x) := \Delta^* f(x) = \Delta_* f(x)$. If $f \in \mathcal{C}^2(D)$, then

$$\Delta f(x) = (\sum_{j=1}^{N} \frac{\partial^2 f}{\partial x_j^2})(x),$$

the standard Laplacian with respect to the variable $x = (x_1, \ldots, x_N)$. More generally, if $x \in D$ and $f \in t_2^1(x)$, i.e. f has an $\mathcal{L}^1$ total differential at x of order 2, then $\Delta f(x)$ equals with the pointwise Laplacian of f at x, i.e.

$$\Delta f(x) = \sum_{j=1}^{N} D_{jj} f(x).$$

Here $D_{jj} f$ represents a generalization of the usual $\frac{\partial^2 f}{\partial x_j^2}$, $j = 1, \ldots, N$. See e.g. [6], p. 172, [24], p. 369, and [25], p. 29.

Recall that there are functions which are not $\mathcal{C}^2$ but for which the generalized Laplacian is nevertheless continuous. The following function gives a simple example:

$$f(x) = \begin{cases} -1, & \text{when } x_N < 0, \\ 0, & \text{when } x_N = 0, \\ 1, & \text{when } x_N > 0. \end{cases}$$

If f is subharmonic on D, it follows from [22], p. 451 (see also [21], Lemma 2.2, p. 280, and [15], Theorem 2.26, p. 52) that $\Delta^* f(x) = \Delta_* f(x)$ for almost all $x \in D$.

Below the following notation is used. Let Ω be a domain in $\mathbb{R}^{m+n}$, $m, n \geq 2$, and $u : \Omega \to \mathbb{R}$. If $y \in \mathbb{R}^n$ is such that the function

$$\Omega(y) \ni x \mapsto f(x) := u(x,y) \in \mathbb{R}$$

is in $\mathcal{L}^1_{\mathrm{loc}}(\Omega(y))$, then we write $\Delta_1^* u(x,y) := \Delta^* f(x)$, $\Delta_{1*} u(x,y) := \Delta_* f(x)$, and $\Delta_1 u(x,y) := \Delta f(x)$.

5.2 A generalization to Kołodziej's and Thornbiörnson's result

In [20], Theorem 3, we presented a generalization to the cited result of Kołodziej and Thornbiörnson, Theorem 3 above. Our result, and also its proof, was perhaps rather technical and long. Therefore we prefer to give here just a corollary. Our corollary will be concise and natural and already it contains the result of Kołodziej and Thornbiörnson. Also the proof, namely the latter part of it, will now be shorter than in [20], proof of Theorem 3.

Theorem 6 *Let Ω be a domain in $\mathbb{R}^{m+n}$, $m, n \geq 2$. Let $u : \Omega \to \mathbb{R}$ be such that*

(a) for each $y \in \mathbb{R}^n$ the function

$$\Omega(y) \ni x \mapsto u(x,y) \in \mathbb{R}$$

is continuous and subharmonic,

(b) for each $x \in \mathbb{R}^m$ the function

$$\Omega(x) \ni y \mapsto u(x,y) \in \mathbb{R}$$

is harmonic,

(c) for each $y \in \mathbb{R}^n$ the function

$$\Omega(y) \ni x \mapsto \Delta_1 u(x,y) \in \mathbb{R}$$

is defined and continuous.

Then u is subharmonic.

Proof: Let $(x_0, y_0) \in \Omega$ and let $r_0 > 0$ be such that $\overline{B^m(x_0, r_0) \times B^n(y_0, r_0)} \subset \Omega$. It is sufficient to show that $u \mid B^m(x_0, r_0) \times B^n(y_0, r_0)$ is subharmonic. We divide the proof into several steps.

Step 1. *Construction of an auxiliar set G.*

For each $k \in \mathbb{N}$ write

$$A_k := \{ x \in \overline{B^m(x_0, r_0)} : -k \leq u(x,y) \leq k \; \forall y \in \overline{B^n(y_0, r_0)} \}.$$

Clearly A_k is closed, and

$$\overline{B^m(x_0, r_0)} = \bigcup_{k=1}^{+\infty} A_k.$$

Write

$$G := \bigcup_{k=1}^{+\infty} \mathrm{int} A_k.$$

It follows from Baire's Theorem that G is dense in $B^m(x_0, r_0)$.

Step 2. *The functions $\Delta_{1r}u(x,\cdot)$ (see the definition below), $x \in G$, $0 < r < r_x := d(x, B^m(x_0,r_0) \setminus G)$, are nonnegative and harmonic on $B^n(y_0,r_0)$.*

For each $(x,y) \in G \times B^n(y_0,r_0)$ and each $0 < r < r_x$, write

$$\begin{aligned}\Delta_{1r}u(x,y) &:= \\ &= \frac{2(m+2)}{r^2}\Big[\frac{1}{\nu_m r^m}\int\limits_{B^m(x,r)} u(x',y)dm_m(x') - u(x,y)\Big] \\ &= \frac{2(m+2)}{r^2}\cdot\frac{1}{\nu_m r^m}\int\limits_{B^m(0,r)} \big[u(x+x',y)-u(x,y)\big]\,dm_m(x').\end{aligned}$$

Since $u(\cdot,y)$ is subharmonic, $\Delta_{1r}u(x,y)$ is defined and nonnegative. Since $\overline{B^m(x,r)} \subset G$ and $A_k \subset A_{k+1}$ for all $k = 1,2,\ldots$, $\overline{B^m(x,r)} \subset \mathrm{int}A_N$ for some $N \in \mathbb{N}$. Therefore

$$-N \leq u(x',y) \leq N$$

for all $x' \in B^m(x,r)$ and $y \in B^n(y_0,r_0)$, hence

$$-2N \leq u(x+x',y) - u(x,y) \leq 2N \qquad (3)$$

for all $x' \in B^m(0,r)$ and $y \in B^n(y_0,r_0)$. To show that $\Delta_{1r}u(x,\cdot)$ is continuous, pick an arbitrary sequence $y_j \to \tilde{y}_0$, $y_j, \tilde{y}_0 \in B^n(y_0,r_0)$, $j = 1,2,\ldots$. Using then (3), Lebesgue Dominated Convergence Theorem and the continuity of $u(x,\cdot)$, one gets

$$\begin{aligned}\lim_{j\to\infty}\Delta_{1r}u(x,y_j) &= \\ &= \lim_{j\to\infty}\frac{2(m+2)}{\nu_m r^{m+2}}\int\limits_{B^m(x,r)}\big[u(x',y_j)-u(x,y_j)\big]dm_m(x') \\ &= \frac{2(m+2)}{\nu_m r^{m+2}}\int\limits_{B^m(0,r)}\lim_{j\to\infty}\big[u(x+x',y_j)-u(x,y_j)\big]dm_m(x') \\ &= \frac{2(m+2)}{\nu_m r^{m+2}}\int\limits_{B^m(0,r)}\big[u(x+x',\tilde{y}_0)-u(x,\tilde{y}_0)\big]dm_m(x') \\ &= \Delta_{1r}u(x,\tilde{y}_0).\end{aligned}$$

It remains to show that $\Delta_{1r}u(x,\cdot)$ satisfies the mean value equality. For that purpose take $B^n(\tilde{y}_0,\rho)$ such that $\overline{B^n(\tilde{y}_0,\rho)} \subset B^n(y_0,r_0)$. Because of (3) we can use Fubini's Theorem. Thus

$$\begin{aligned}&\frac{1}{\nu_n\rho^n}\int\limits_{B^n(\tilde{y}_0,\rho)}\Delta_{1r}u(x,y)dm_n(y) = \\ &= \frac{1}{\nu_n\rho^n}\int\limits_{B^n(\tilde{y}_0,\rho)}\Big\{\frac{2(m+2)}{r^2}\cdot\frac{1}{\nu_m r^m}\times \\ &\times\int\limits_{B^m(0,r)}\big[u(x+x',y)-u(x,y)\big]dm_m(x')\Big\}dm_n(y) \\ &= \frac{2(m+2)}{r^2}\cdot\frac{1}{\nu_m r^m}\times \\ &\times\int\limits_{B^m(0,r)}\Big\{\frac{1}{\nu_n\rho^n}\int\limits_{B^n(\tilde{y}_0,\rho)}\big[u(x+x',y)-u(x,y)\big]dm_n(y)\Big\}dm_m(x') \\ &= \frac{2(m+2)}{\nu_m r^{m+2}}\int\limits_{B^m(0,r)}\big[u(x+x',\tilde{y}_0)-u(x,\tilde{y}_0)\big]dm_m(x') \\ &= \Delta_{1r}u(x,\tilde{y}_0).\end{aligned}$$

Step 3. *The functions $\Delta_1u(x,\cdot) : B^n(y_0,r_0) \to \mathbb{R}$, $x \in B^m(x_0,r_0)$, are defined, nonnegative and harmonic.*

By definition

$$\Delta_1u(x,y) := \lim_{r\to 0}\Delta_{1r}u(x,y).$$

By Step 2 the functions $\Delta_{1r}u(x,\cdot)$, $x \in G$, $0 < r < r_x$, are nonnegative and harmonic in $B^n(y_0,r_0)$. Using then e.g. [2], Lemma 1.5.6 and Theorem 1.5.8, pp. 16-17, one sees that the functions $\Delta_1u(x,\cdot)$, $x \in G$, are nonnegative and harmonic. From this it follows, because of the assumption (c), and again with the aid of [2], Lemma 1.5.6 and Theorem 1.5.8, pp. 16-17, that also the functions $\Delta_1u(x,\cdot)$, $x \in B^m(x_0,r_0) \setminus G$, are nonnegative and harmonic.

Step 4. *For each $x \in B^m(x_0,r_0)$ the functions*

$$B^n(y_0,r_0) \ni y \mapsto v(x,y) := \int G_{B^m(x_0,r_0)}(x,z)\Delta_1u(z,y)dm_m(z) \in \mathbb{R}$$

and

$$B^n(y_0,r_0) \ni y \mapsto h(x,y) := u(x,y) + v(x,y) \in \mathbb{R}$$

are harmonic. Above and below $G_{B^m(x_0,r_0)}(x,z)$ is the Green function of the ball $B^m(x_0,r_0)$, with x as a pole.

With the aid of Lebesgue Dominated Convergence Theorem, say, one sees easily that for each $x \in B^m(x_0,r_0)$ the function $v(x,\cdot)$ is continuous. Using then Fubini's Theorem one sees easily that for each $x \in B^m(x_0,r_0)$ the function $v(x,\cdot)$ satisfies also the mean value equality, and thus is harmonic. That also the function $h(x,\cdot)$ is harmonic, follows then from the assumption (b).

Step 5. *For each $y \in B^n(y_0, r_0)$ the function*

$$B^m(x_0,r_0) \ni x \mapsto h(x,y) := u(x,y) + v(x,y) \in \mathbb{R}$$

is harmonic.

With the aid of the version of Riesz's Decomposition Theorem, given in [21], 1.3, Theorem II, p. 279, and p. 278, too (see also [23], Theorem 1, p. 499), for each $y \in B^n(y_0, r_0)$ one can write

$$u(x,y) = h(x,y) - v(x,y)$$

where

$$v(x,y) := \int G_{B^m(x_0,r_0)}(x,z)\Delta_1 u(z,y)dm_m(z)$$

and $h(\cdot,y)$ is the least harmonic majorant of $u(\cdot,y) \mid B^m(x_0,r_0)$. Here $v(\cdot,y)$ is continuous and superharmonic on $B^m(x_0,r_0)$.

Step 6. *The use of the results of Lelong and of Avanissian.*

By Steps 4 and 5 we know that $h(\cdot,\cdot)$ is separately harmonic on $B^m(x_0,r_0) \times B^n(y_0,r_0)$. By Lelong's result [11], Théorème 11, p. 554, $h(\cdot,\cdot)$ is harmonic and thus locally bounded on $B^m(x_0,r_0) \times B^n(y_0,r_0)$. Therefore also $u(\cdot,\cdot)$ is locally bounded above on $B^m(x_0,r_0) \times B^n(y_0,r_0)$. But then it follows from Avanissian's result [4], Théorème 9, p. 140, that $u(\cdot,\cdot)$ is subharmonic on $B^m(x_0,r_0) \times B^n(y_0,r_0)$. $\square$

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