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SOME NORM INEQUALITIES IN EUCLIDEAN JOURDAN ALGEBRAS

Qingmin Meng* J. Tao†, Guoqiang Wang‡ and Xiaoni Chi§

Abstract: In this article, we prove two norm inequalities in the setting of Euclidean Jordan algebras via majorization theory.

Key words: *Euclidean Jordan algebra, majorization*

Mathematics Subject Classification: *15A33, 17C20, 17C55*

1 Introduction

In a most recent paper [7], Wang, Tao, and Kong proved the following norm inequality in the setting of Euclidean Jordan algebras,

$$||x \circ y||_1 \leq ||x||_F \, ||y||_F, \tag{1.1}$$

where $||x||_1 = \sum_{i=1}^r |\lambda_i(x)|$ with the spectral decomposition $x = \sum_{i=1}^r \lambda_i(x) e_i$ and $||x||_F := \sqrt{\sum_{i=1}^r \lambda_i^2(x)}$ (see Section 2.1).

One objective of this paper is to give an alternative proof of (1.1) by using a case-by-case analysis via majorization theory.

Recently, Zangiabadi, Gu, and Roos [8] showed the following norm inequality in the Jordan spin algebra (see Section 2.1),

$$\|(x \circ s)^{-\frac{1}{2}}\|_F \geq \|(P_{x^{1/2}}(s))^{-\frac{1}{2}}\|_F. \tag{1.2}$$

Our second objective in this paper is to extend (1.2) to the setting of Euclidean Jordan algebras.

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2 Preliminaries

2.1 Euclidean Jordan Algebras

We assume that the reader is familiar with the basic Euclidean Jordan algebra theory and recall some concepts used in this paper from Euclidean Jordan algebras. Most of these can be found in [2].

Throughout this paper, we let $(V, \circ, \langle \cdot, \cdot \rangle)$ denote a Euclidean Jordan algebra: V is a finite dimensional vector space over R (the field of real numbers) with inner product $\langle x, y\rangle$ and Jordan product $x \circ y$. The symmetric cone of V is the cone of squares $K := \{x \circ x : x \in V\}$. We use the notation $x \geq 0$ ($x > 0$) when $x \in K$ (respectively, $x \in K^o$ (=interior (K))) and $x \leq 0$ ($x < 0$) when $-x \geq 0$ ($-x > 0$).

An element $c \in V$ such that $c^2 = c$ is called an *idempotent* in V; it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. We say that a finite set $\{e_1, e_2, \ldots, e_r\}$ of primitive idempotents in V is a *Jordan frame* if

$$e_i \circ e_j = 0 \text{ if } i \neq j, \text{ and } \sum_{1}^{r} e_i = e,$$

where e is the unit element of V.

A Euclidean Jordan algebra is said to be *simple* if it is not a direct sum of two (non-trivial) Euclidean Jordan algebras. It is well known that any nonzero Euclidean Jordan algebra is a product of simple Euclidean Jordan algebras and every simple algebra is isomorphic to one of the algebras given below:

(i) The algebra $\mathcal{S}^n$ of $n \times n$ real symmetric matrices with trace inner product and the Jordan product $X \circ Y = \frac{1}{2}(XY + YX)$;

(ii) The Jordan spin algebra $\mathcal{L}^n$ ($n \geq 3$) quadratic forms in R^n with standard inner product and the Jordan product

$$x \circ y := (x^T y, x_1 y_2 + y_1 x_2, \cdots, x_1 y_n + y_1 x_n)^T;$$

(iii) The algebra $\mathcal{H}^n$ of all $n \times n$ complex Hermitian matrices with trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$;

(iv) The algebra $\mathcal{Q}^n$ of all $n \times n$ quaternion Hermitian matrices with (real) trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$;

(v) The algebra $\mathcal{O}^3$ of all 3×3 octonion Hermitian matrices with (real) trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$.

The spectral decomposition Let V be a Euclidean Jordan algebra with rank r. Then, for every $x \in V$, there exist a Jordan frame $\{e_1, \ldots, e_r\}$ and real numbers $\lambda_1, \ldots, \lambda_r$ such that

$$x = \lambda_1 e_1 + \cdots + \lambda_r e_r. \tag{2.1}$$

The numbers λ_i are called the eigenvalues of x.

Given (2.1), $|x| = \sum_{i=1}^r |\lambda_i(x)| e_i$, the trace of x is defined by $trace(x) := \sum_{i=1}^r \lambda_i(x)$, $||x||_1 := \sum_{i=1}^r |\lambda_i(x)|$, and $||x||_F := \sqrt{\langle x, x\rangle}$. Since the inner product is defined by $\langle x, y \rangle = trace(x \circ y)$, we have $||x||_F = \sqrt{trace(x \circ x)} = \sqrt{\sum_{i=1}^r \lambda_i^2(x)}$.

The Peirce decomposition Fix a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in V. For $i, j \in \{1, 2, \ldots, r\}$, define the eigenspaces

$$V_{ii} := \{x \in V : x \circ e_i = x\} = R\, e_i$$

and when $i \neq j$,

$$V_{ij} := \{x \in V : x \circ e_i = \frac{1}{2}x = x \circ e_j\}.$$

Then we have the following theorem.

Theorem 2.1 ([2], Theorem IV.2.1)**.** *The space V is the orthogonal direct sum of spaces V_{ij} $(i \leq j)$. Furthermore,*

$$\begin{aligned}
&V_{ij} \circ V_{ij} \subset V_{ii} + V_{jj},\\
&V_{ij} \circ V_{jk} \subset V_{ik} \text{ if } i \neq k, \text{ and}\\
&V_{ij} \circ V_{kl} = \{0\} \text{ if } \{i, j\} \cap \{k, l\} = \emptyset.
\end{aligned}$$

Thus, given a Jordan frame $\{e_1, e_2, \ldots, e_r\}$, we can write any element $x \in V$ as

$$x = \sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij}$$

where $x_i \in R$ and $x_{ij} \in V_{ij}$. This expression is the **Peirce decomposition** of x with respect to $\{e_1, e_2, \ldots, e_r\}$.

For any given idempotent $c \in V$, we have the Peirce decomposition

$$V = V(c, 1) \oplus V(c, \frac{1}{2}) \oplus V(c, 0),$$

where

$$V(c, \gamma) := \{x \in V : x \circ c = \gamma x\},$$

for $\gamma = 0, \frac{1}{2}, 1$. Thus, given any element $x \in V$, we write the decomposition $x = u + v + w$, where $u \in V(c, 1)$, $v \in V(c, \frac{1}{2})$ and $w \in V(c, 0)$.

The quadratic representation For a given $a \in V$, the *quadratic representation* $P_a : V \to V$ are defined respectively by

$$P_a(x) := 2a \circ (a \circ x) - a^2 \circ x.$$

$\boxed{2.2}$ Majorization

Given a vector $x = (x_1, x_2, \ldots, x_r)$ in $\mathbb{R}^r$, we write $x^{\downarrow} := (x_1^{\downarrow}, x_2^{\downarrow}, \ldots, x_r^{\downarrow})$ for the vector obtained by rearranging the components of x in the decreasing order. For two vectors $x = (x_1, x_2, \ldots, x_r)$ and $y = (y_1, y_2, \ldots, y_r)$ in $\mathbb{R}^r$, we say that x is *majorized* by y and write $x \prec y$ if

$$\sum_1^k x_i^{\downarrow} \leq \sum_1^k y_i^{\downarrow} \quad (k = 1, 2, \ldots, r-1)$$

and

$$\sum_1^r x_i^{\downarrow} = \sum_1^r y_i^{\downarrow}.$$

Theorem 2.2 ([1], Theorem II. 3.1)**.** *For* $x = (x_1^{\downarrow}, x_2^{\downarrow}, \dots, x_r^{\downarrow})$ *and* $y = (y_1^{\downarrow}, y_2^{\downarrow}, \dots, y_r^{\downarrow})$ *in* $\mathbb{R}^r$ *and for any convex function* $\phi : \mathbb{R} \to \mathbb{R}$*, the following two conditions are equivalent:*

$$x \prec y. \tag{2.2}$$

$$\sum_1^r \phi(x_i) \le \sum_1^r \phi(y_i). \tag{2.3}$$

Lemma 2.3 (Theorem 3.3.14, [4])**.** *Let* $V = \mathcal{S}^n$ *or* $V = \mathcal{H}^n$*. Let* $X, Y \in V$*. Then*

$$\sum_1^k \sigma_i^{\downarrow}(XY) \le \sum_1^k \sigma_i^{\downarrow}(X)\sigma_i^{\downarrow}(Y) \quad (k = 1, 2, \dots, n),$$

where $\sigma_i^{\downarrow}(X)$, $i = 1, 2, \dots, n$ *are singular values of* X *written in the decreasing order.*

Lemma 2.4. *(Corollary 3.4.3, [4]) Let* $V = \mathcal{S}^n$ *or* $V = \mathcal{H}^n$*. Let* $X, Y \in V$*. Then*

$$\sum_1^k \sigma_i^{\downarrow}(X + Y) \le \sum_1^k \sigma_i^{\downarrow}(X) + \sum_1^k \sigma_i^{\downarrow}(Y) \quad (k = 1, 2, \dots, n).$$

3 Main Results

First we give a proof of (1.1) by using a case-by-case analysis.

Theorem 3.1. *Let* V *be any Euclidean Jordan algebra. Then for* $x, y \in V$*.*

$$||x \circ y||_1 = \sum_1^r |\lambda_i(x \circ y)| \le \frac{1}{2}(||x||_F^2 + ||y||_F^2).$$

Given an idempotent $c \in V$, let $\gamma \in \{0, \frac{1}{2}, 1\}$ and define the eigenspaces

$$V(c, \gamma) := \{x \in V : x \circ c = \gamma x\}.$$

Theorem 3.2. *Let* V *be a simple Euclidean Jordan algebra. For any* $x \in V$*,* $x = u + v + w$*, where* $u \in V(c, 1)$*,* $v \in V(c, \frac{1}{2})$ *and* $w \in V(c, 0)$*. Then*

$$||u||_F^2 + ||w||_F^2 \le ||x||_F^2.$$

Proof. Since $\lambda^{\downarrow}(u+w) \prec \lambda^{\downarrow}(x)$ (see Theorem 6.1, [6]), $\lambda(u+w) = \lambda(u) \cup \lambda(w)$, and $f(t) = x^2$ is a convex function, by Theorem 2.2, we have $||u||_F^2 + ||w||_F^2 \le ||x||_F^2$. $\square$

Proof of Theorem 3.1. First suppose that V is a simple Euclidean Jordan algebra. We prove this by case-by-case analysis. We note that if $x \circ y \ge 0$, then $tr(|x \circ y|) = tr(x \circ y)$. Since

$$x \circ y \le \frac{1}{2}(x^2 + y^2) \Rightarrow \lambda_i^{\downarrow}(x \circ y) \le \frac{1}{2}\lambda_i^{\downarrow}(x^2 + y^2),$$

we have $tr(|x \circ y|) = tr(x \circ y) \le \frac{1}{2}(tr(x^2) + tr(y^2)) = \frac{1}{2}(||x||_F^2 + ||y||_F^2)$. If $x \circ y \le 0$, then $tr(|x \circ y|) = tr(-x \circ y)$. Since

$$-x \circ y \le \frac{1}{2}(x^2 + y^2) \Rightarrow \lambda_i^{\downarrow}(-x \circ y) \le \frac{1}{2}\lambda_i^{\downarrow}(x^2 + y^2),$$

we have $tr(|x \circ y|) = tr(-x \circ y) \leq \frac{1}{2}(tr(x^2) + tr(y^2)) = \frac{1}{2}(||x||_F^2 + ||y||_F^2)$. Therefore, we consider the case of $x \circ y \ngeq 0$ and $x \circ y \nleq 0$.

(i) When $V = \mathcal{S}^n$ or $V = \mathcal{H}^n$, $X \circ Y = \frac{1}{2}(XY + YX)$. Thus, by Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned}
\sum_1^r |\lambda_i(X \circ Y)| = ||X \circ Y||_1 = \frac{1}{2}\sum_1^n \sigma_i^{\downarrow}(XY + YX) &\leq \frac{1}{2}\left[\sum_1^n \sigma_i^{\downarrow}(XY) + \sum_1^n \sigma_i^{\downarrow}(YX)\right] \\
&\leq \sum_1^n \sigma_i^{\downarrow}(X)\sigma_i^{\downarrow}(Y) \\
&\leq \left[\sum_1^n (\sigma_i^{\downarrow}(X))^2\right]^{1/2} \left[\sum_1^n (\sigma_i^{\downarrow}(Y))^2\right]^{1/2} \\
&= ||X||_F||Y||_F \\
&\leq \frac{1}{2}(||X||_F^2 + ||Y||_F^2).
\end{aligned}$$

(ii) When $V = \mathcal{Q}^n$, $X \circ Y = \frac{1}{2}(XY + YX)$.

For an $n \times n$ quaternion matrix A, we write $A = A_1 + A_2 j$, where A_1, A_2 are $n \times n$ complex matrices. The complex adjoint matrix of A is defined by $\chi_A := \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix}$. From Theorem 4.2 and Corollary 6.2 in [9], it is easy to verify that

$$\chi_{X \circ Y} = \frac{1}{2}(\chi_X \chi_Y + \chi_Y \chi_X) = \chi_X \circ \chi_Y, \ ||\chi_X||_1 = 2||X||_1, \text{ and } ||\chi_X||_F^2 = 2||X||_F^2.$$

Thus, by (i),

$$||\chi_{X \circ Y}||_1 \leq \frac{1}{2}(||\chi_X)||_F^2 + ||\chi_X)||_F^2) \Rightarrow ||X \circ Y||_1 \leq \frac{1}{2}(||X||_F^2 + ||Y||_F^2).$$

(iii) $V = \mathcal{L}^n$,

Let $x \circ y = \lambda_1(x \circ y)f_1 + \lambda_2(x \circ y)f_2$. Without loss of generality, we assume that $\lambda_1(x \circ y) \geq 0$ and $\lambda_2(x \circ y) < 0$. Now, let $w = f_1 - f_2$. Writing the Peirce decomposition of y as $y = y_1 f_1 + y_2 f_2 + y_{12}$, we have $y \circ w = y_1 f_1 - y_2 f_2$. Thus,

$$||y \circ w||_F^2 = y_1^2 + y_2^2 \leq (\lambda_1(y))^2 + (\lambda_2(y))^2 = ||y||_F^2 \Rightarrow ||y \circ w||_F \leq ||y||_F.$$

Note that the first inequality follows by Corollary 4.6 in [3] and Theorem 2.2. Now,

$$\begin{aligned}
\sum_1^2 |\lambda_i(x \circ y)| = tr(|x \circ y|) &= \langle |x \circ y|, e\rangle \\
&= \langle (x \circ y) \circ w, e\rangle \\
&= \langle x, y \circ w\rangle \\
&\leq ||x||_F \, ||y \circ w||_F \\
&\leq ||x||_F \, ||y||_F \leq \frac{1}{2}(||x||_F^2 + ||y||_F^2).
\end{aligned}$$

(iv) $V = \mathcal{O}^3$.

Without loss of generality, we write the spectral decomposition of $x \circ y$ as $x \circ y = \lambda_1(x \circ y)e_1 + \lambda_2(x \circ y)e_2 + \lambda_3(x \circ y)e_3$, where $\lambda_1(x \circ y) := \mu_1 \geq 0$, $\lambda_2(x \circ y) := \mu_2 \geq 0$, and

$\lambda_3(x \circ y) := -\mu_3 \leq 0$. Let $w := e_1 + e_2 - e_3$. Then we have $|x \circ y| = w \circ (x \circ y)$. Since $\{E_1, E_2, E_3\}$ is a Jordan frame in $\mathcal{O}_3$, where E_i is the matrix with one in the (i,i) slot and zeros elsewhere, there exists an algebra automorphism Θ such that $\Theta(e_i) = E_i$, $i = 1, 2, 3$. Thus, $\Theta(w) = E_1 + E_2 - E_3$. Now, let

$$W := \Theta(w) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ and } \mathrm{Y} := \Theta(\mathrm{y}) = \begin{bmatrix} f & a & b \\ \bar{a} & g & c \\ \bar{b} & \bar{c} & r \end{bmatrix}.$$

Then we have

$$\Theta(w \circ y) = \Theta(w) \circ \Theta(y) = \frac{1}{2}(WY + YW) = \begin{bmatrix} f & a & 0 \\ \bar{a} & g & 0 \\ 0 & 0 & -r \end{bmatrix}.$$

Thus, the eigenvalues of $w \circ y$ are $\lambda_1^{\downarrow}(Y_{11}), \lambda_2^{\downarrow}(Y_{11})$, and $-r$, where $Y_{11} = \begin{bmatrix} f & a \\ \bar{a} & g \end{bmatrix}$. Hence,

$$||w \circ y||_F^2 = (\lambda_1^{\downarrow}(Y_{11}))^2 + (\lambda_2^{\downarrow}(Y_{11}))^2 + |-r| = ||Y_{11}||_F^2 + |r|^2 \leq ||\Theta(y)||_F^2 = ||y||_F^2.$$

Note that the inequality follows by Theorem 3.2. Hence, $||y \circ w||_F \leq ||y||_F$. Now,

$$\begin{aligned} \sum_1^3 |\lambda_i(x \circ y)| = tr(|x \circ y|) &= \langle |x \circ y|, e \rangle \\ &= \langle (x \circ y) \circ w, e \rangle \\ &= \langle x, y \circ w \rangle \\ &\leq ||x||_F \, ||y \circ w||_F \\ &\leq ||x||_F \, ||y||_F \leq \frac{1}{2}(||x||_F^2 + ||y||_F^2). \end{aligned}$$

Now suppose that V is not simple, i.e., $V = V_1 \times V_2 \times \ldots \times V_k$, where each V_i is a simple algebra. Since for $x = (x_1, x_2, \ldots, x_k)$, $y = (y_1, y_2, \ldots, y_k) \in V$, x_i, $y_i \in V_i$,

$$x \circ y = (x_1 \circ y_1, x_2 \circ y_2, \ldots, x_k \circ y_k),$$
$$r = rank(x) = \sum_1^k rank(x_i), ||x||_F^2 + ||y||_F^2 = \sum_1^k (||x_i||_F^2 + ||y_i||_F^2),$$

and

$$trace(|x \circ y|) = \sum_1^r \lambda_i(|x \circ y|) = \sum_{i=1}^k \sum \lambda_j(|x_i \circ y_i|) = \sum_1^k trace(|x_i \circ y_i|),$$

we have $||x \circ y||_1 = \sum_1^r \lambda_i |x \circ y| \leq \frac{1}{2}(||x||_F^2 + ||y||_F^2)$. □

In what follows, we extend (1.2) to the setting of Euclidean Jordan algebras.

Theorem 3.3. *Let V be any Euclidean Jordan algebra. Let $x, s \in \mathrm{int}\mathcal{K}$ and $x \circ s \in \mathrm{int}\mathcal{K}$. Then*

$$\|(x \circ s)^{-\frac{1}{2}}\|_F \geq \|(P_{x^{1/2}}(s))^{-\frac{1}{2}}\|_F.$$

The proof of the above result is based on several lemmas and theorems. First we recall some results used in this paper.

Lemma 3.4 (see Proposition III 5.3, [1]). *Let A be an $n \times n$ complex matrix. Then*

$$Re(\lambda(A)) \prec \lambda(Re(A)),$$

where $Re(A) = \frac{A+A^*}{2}$ and $\lambda(A)$ denotes the vector of eigenvalues of A.

Lemma 3.5 (see proof in Lemma 30, [5]). *Let $x, s \in \text{int}\mathcal{K}$. Then*

$$\lambda_{max}(x \circ s) \geq \lambda_{max}(P_{x^{1/2}}(s)) \text{ and } \lambda_{min}(x \circ s) \leq \lambda_{min}(P_{x^{1/2}}(s)).$$

Lemma 3.6. *Let $x, s \in V$. Then*

$$tr(P_{x^{1/2}}(s)) = tr(x \circ s).$$

Proof. Since $P_x(e) = x^2$ for $x \in V$, we have

$$tr(P_{x^{1/2}}(s)) = \langle P_{x^{1/2}}(s), e\rangle = \langle s, P_{x^{1/2}}(e)\rangle = \langle s, x\rangle = tr(x \circ s).$$

This completes the proof. □

Theorem 3.7. *Let V be a simple Euclidean Jordan algebra. Let $x, s \in \text{int}\mathcal{K}$ and $x \circ s \in \text{int}\mathcal{K}$. Then*

$$\lambda(P_{x^{1/2}}(s)) \prec \lambda(x \circ s),$$

where $\lambda(x)$ denotes the vector of eigenvalues of x.

Proof. We prove this by case-by-case analysis.

(i) $V = \mathcal{L}^n$.

By Lemma 3.5 and Lemma 3.6, we have $\lambda(P_{x^{1/2}}(s)) \prec \lambda(x \circ s)$.

(ii) $V = \mathcal{S}^n$ and $V = \mathcal{H}^n$.

Since $X \circ S = \frac{XS+SX}{2}$, by Lemma 3.4, we have $Re(\lambda(XS)) \prec \lambda(X \circ S)$. Since $X, S \in \mathcal{H}^n_+$, $\lambda_i(XS) = \lambda_i(X^{1/2}X^{1/2}S) = \lambda_i(X^{1/2}SX^{1/2}) = \lambda_i(P_{X^{1/2}}(S))$, where $P_{X^{1/2}}(S) = X^{1/2}SX^{1/2}$. Thus, $\lambda(P_{X^{1/2}}(S)) \prec \lambda(X \circ S)$.

(iii) $V = \mathcal{Q}_n$.

For an $n \times n$ quaternion matrix A, we write $A = A_1 + A_2 j$, where A_1, A_2 are $n \times n$ complex matrices. The complex adjoint matrix of A is defined by $\chi_A := \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix}$. It is well known (e.g., Theorem 4.2, [9]) that χ_A is Hermitian if and only if A is Hermitian and the eigenvalues of A coincide with the eigenvalues of χ_A (see Theorem 5.4 and Corollary 5.1, [9]) when A is Hermitian. Now, $P_{X^{1/2}}(S) = X^{1/2}SX^{1/2}$. By Theorem 4.2, [9], we have $\chi_{X^{1/2}SX^{1/2}} = \chi_{X^{1/2}}\chi_S\chi_{X^{1/2}} = (\chi_X)^{1/2}\chi_S(\chi_X)^{1/2}$ and $\chi_{(XS+SX)/2} = (\chi_X\chi_S + \chi_S\chi_X)/2$. Therefore, the result follows by Case (ii).

(iv) $V = \mathcal{O}^3$.

By Lemma 3.5 and Lemma 3.6, we have $\lambda(P_{x^{1/2}}(s)) \prec \lambda(x \circ s)$. This completes the proof. □

Theorem 3.8. *Let V be any Euclidean Jordan algebra. Let $x, s \in \text{int}\mathcal{K}$ and $x \circ s \in \text{int}\mathcal{K}$. Then*

$$\text{tr}(x^{-1} \circ s^{-1}) \leq \text{tr}((x \circ s)^{-1})).$$

Proof. First suppose that V is a simple Euclidean Jordan algebra.

Taking $f(t) = \frac{1}{t}$ on $(0, \infty)$, by Theorem 2.2 and Theorem 3.7, we have

$$\sum_{i=1}^{r} \frac{1}{\lambda_i(P_{x^{1/2}}(s))} \leq \sum_{i=1}^{r} \frac{1}{\lambda_i(x \circ s)}.$$

Since $\frac{1}{\lambda_i\left(P_{x^{1/2}}(s)\right)} = \lambda_i(P_{x^{1/2}}(s))^{-1} = \lambda_i(P_{x^{-1/2}}(s^{-1}))$ and $\frac{1}{\lambda_i(x \circ s)} = \lambda_i(x \circ s)^{-1}$, we have

$$\operatorname{tr}(x^{-1} \circ s^{-1}) \leq \operatorname{tr}((x \circ s)^{-1})).$$

Now suppose that V is not simple, i.e., $V = V_1 \times V_2 \times \ldots \times V_k$, where each V_i is a simple algebra. Since for $x = (x_1, x_2, \ldots, x_k) \in V$, $x_i \in V_i$,

$$trace(x) = \sum \lambda_i(x) = \sum_{i=1}^{k} \sum \lambda_j(x_i) = \sum_{1}^{k} trace(x_i),$$

we have

$$\operatorname{tr}(x^{-1} \circ s^{-1}) \leq \operatorname{tr}((x \circ s)^{-1})).$$

This completes the proof. □

Remark 3.9. When $V = \mathcal{L}^n$, another proof of Theorem 3.8 was given in [8] (see the proof of Lemma 2.6).

Proof of Theorem 3.3. Since

$$\begin{aligned} \|(x \circ s)^{-\frac{1}{2}}\|_F^2 &= \langle (x \circ s)^{-\frac{1}{2}}, (x \circ s)^{-\frac{1}{2}} \rangle \\ &= \langle (x \circ s)^{-1}, e \rangle \\ &= \operatorname{tr}((x \circ s)^{-1})), \end{aligned}$$

and

$$\begin{aligned} \|(P_{x^{1/2}}(s))^{-\frac{1}{2}}\|_F^2 &= \langle (P_{x^{1/2}}(s))^{-\frac{1}{2}}, (P_{x^{1/2}}(s))^{-\frac{1}{2}} \rangle \\ &= \langle (P_{x^{1/2}}(s))^{-1}, e \rangle \\ &= \langle P_{x^{-1/2}}(s^{-1}), e \rangle \\ &= \langle s^{-1}, P_{x^{-1/2}}(e) \rangle \\ &= \langle s^{-1}, x^{-1} \rangle \\ &= \operatorname{tr}(s^{-1} \circ x^{-1}). \end{aligned}$$

By Theorem 3.8, we have

$$\begin{aligned} \|(x \circ s)^{-\frac{1}{2}}\|_F^2 &= \operatorname{tr}((x \circ s)^{-1})) \\ &\geq \operatorname{tr}(s^{-1} \circ x^{-1}) \\ &= \|(P(x)^{\frac{1}{2}} s)^{-\frac{1}{2}}\|_F^2 \Rightarrow \|(x \circ s)^{-\frac{1}{2}}\|_F \\ &\geq \|(P_{x^{1/2}}(s))^{-\frac{1}{2}}\|_F. \end{aligned}$$

□

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References

[1] R. Bhatia, *Matrix Analysis*, Springer Graduate Texts in Mathematics, Springer-Verlag, New York, 1997.

[2] J. Faraut and A. Koranyi, *Analysis on Symmetric Cones*, Oxford University Press, Oxford, 1994.

[3] M.S. Gowda and J. Tao, The Cauchy interlacing theorem in simple Euclidean Jordan algebras and some consequences, Linear and Multilinear Algebra 59 (2011) 65–86.

[4] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, New York, 1991.

[5] S.H. Schmieta and F. Alizadeh, Extension of primal-dual interior point algorithms to symmetric cones, Math. Prog. A 96 (2003) 409–438.

[6] J. Tao, L. Kong, Z. Luo and N. Xiu, Some majorization inequalities in Euclidean Jordan algebras, Linear Alg. Appl. 461 (2014) 92–122.

[7] G. Wang, J. Tao and L. Kong, A note on an inequality involving Jordan product in Euclidean Jordan algebras, Optim. Lett. 10 (2016) 731–736.

[8] M. Zangiabadi, G. Gu and C. Roos, Full Nesterov-Todd step primal-dual interior-point mathods for second-order cone optimization, J. Optim. Theory Appl. 158 (2013) 816–858.

[9] F. Zhang, Quaternions and matrices of quaternions, Linear Alg. Appl. 251 (1997) 21–27.

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Qingmin Meng
The Central Hospital of Taian
Shandong 271000, P. R. China
E–mail address: mengqm71@sohu.com

J. Tao
Department of Mathematics and Statistics
Loyola University Maryland, Baltimore
Maryland 21210, U.S.A.
E–mail address: jtao@loyola.edu

Guoqiang Wang
College of Fundamental Studies
Shanghai University of Engineering Science
Shanghai 201620, P.R. China
E–mail address: guoq_wang@hotmail.com

Xiaoni Chi
School of Mathematics and Computing Science
Guangxi Key Laboratory of Automatic Detecting
Technology and Instruments Guilin University of
Electronic Technology Guilin 541004, Guangxi, P. R. China
E–mail address: chixiaoni@126.com