UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XLVII
2009

DEFORMATIONS OF THE HEISENBERG ALGEBRA INSIDE $\mathrm{gl}(3, \mathbb{K})$

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Abstract. We study non-trivial deformations of the natural embedding of the Lie algebra $\mathfrak{h}_1$ of lower triangular matrices (the Heisenberg Lie algebra) into $\mathrm{gl}(3, \mathbb{K})$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Our first result is the calculation of the first cohomology space $H^1(\mathfrak{h}_1; \mathrm{gl}(3, \mathbb{K}))$. We prove that there are no obstructions for integrability of infinitesimal deformations and, furthermore, give an explicit formula for the most general deformation.

1. Introduction. The Heisenberg algebra $\mathfrak{h}_1$ is the Lie algebra of three by three lower triangular matrices. It is a very important algebra in Physics since it encodes the commutation relations of the momentum and position operators on which Quantum Physics is based. One striking property of $\mathfrak{h}_1$ is the fact that there is only one way to represent it (up to unitary equivalence) via self-adjoint operators on a separable Hilbert space. This is the content of Stone-Von Neumann theorem. As a corollary, one gets the equivalence between Heisenberg and Schroedinger pictures (matrix algebra versus wave Mechanics) of Quantum Physics.

We would like to know more about this algebra. There is a standard embedding

$$\rho : \mathfrak{h}_1 \hookrightarrow \mathrm{gl}(3, \mathbb{K}), \tag{1.1}$$

where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Since many interesting features of objects can be discovered by deforming them (see for example Quantum Groups), it is natural to want to determine all the possible deformations of this embedding. The theory of deformations of Lie structures (algebras and morphisms) is now a classical subject (see e.g. [**4, 8, 11**]). However, a new concept of miniversal deformations of Lie algebras has been introduced in [**1**]. It is of course inspired by the notion of universal unfolding in singularity theory. While Fialowski and Fuchs developed this notion for Lie algebras, the case of Lie algebra homomorphisms has been recently considered in [**9, 10**].

In this paper we will completely describe the miniversal deformation of the embedding (1.1).

2. Deformations of homomorphisms. Let $\rho : \mathfrak{h} \to \mathfrak{g}$ be a homomorphism of Lie algebras. A deformation of ρ is an expression of the type

$$\rho(t) = \rho_0 + \sum_{m=1}^{\infty} \rho_m(t), \tag{2.1}$$

where $t = (t_1, \ldots, t_r)$ are the parameters of the deformation and each term $\rho_m(t)$ is a linear map from $\mathfrak{h}$ to $\mathfrak{g}$ homogeneous in t of degree m. Note that $t_1, \ldots, t_r$ can be considered as real (or complex) parameters, or as generators of a commutative associative algebra (cf. [**1**]), upon the context.

The deformation must be a Lie homomorphism for every value of the parameter t, i.e. satisfy:

$$\rho(t)([X,Y]) = [\rho(t)(X), \rho(t)(Y)] \tag{2.2}$$

for every $X, Y \in \mathfrak{h}$.

Equivalent deformations and the first cohomology. The standard Chevalley–Eilenberg differential is given, in the case of a linear map m from $\mathfrak{h}$ to $\mathfrak{g}$, by the following formula:

$$\delta^1 m(X,Y) = m([X,Y]) - [\rho(X), m(Y)] + [\rho(Y), m(X)]. \tag{2.3}$$

Let us expand formula (2.2) as a series in t, the first order term is of the form

$$\rho_1(t) = \sum_{i=1}^{r} t_i\, \rho_1^i. \tag{2.4}$$

From (2.2) one obtains $\delta \rho_1^i = 0$, that is, each map ρ_1^i is a one-cocycle.

Two deformations $\rho(t)$ and $\rho'(t)$ are equivalent if there exists an inner automorphism $I(t) : \mathfrak{g} \otimes \mathbb{K}[t_1, \ldots, t_r] \longrightarrow \mathfrak{g} \otimes \mathbb{K}[t_1, \ldots, t_r]$ of the form

$$I(t) = \exp\left(\sum_{1 \le i \le r} t_i\, adA_i + \sum_{1 \le i,j \le r} t_i t_j\, adA_{ij} + \ldots \right),$$

where $A_i, A_{ij}, \ldots$ are some elements of $\mathfrak{g}$, such that the relation $I(t) \circ \rho(t) = \rho'(t)$ is satisfied.

Moreover, one can check that the first order terms ρ_1^i and ${\rho'}_1^i$ differ by a coboundary, i.e. ${\rho'}_1^i = \rho_1^i + \delta A_i$. It follows that infinitesimal deformation of the homomorphism ρ are classified by the first cohomology space $H^1(\mathfrak{h}; \mathfrak{g})$.

Cup-product and Maurer–Cartan equation. The standard cup-product (or Nijenhuis–Richardson product) of linear maps $a, b : \mathfrak{h} \to \mathfrak{g}$ is the linear map $[\![a, b]\!] : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{g}$ defined by

$$[\![a, b]\!](X, Y) = [a(X), b(Y)] - [a(Y), b(X)].$$

Put $\phi(t) = \rho(t) - \rho_0$, the morphism equation (2.2) reads

$$\delta\phi(t) - \frac{1}{2}[\![\phi(t), \phi(t)]\!] = 0 \tag{2.5}$$

(see [**8, 11**]). This equation is called the Maurer–Cartan equation (or the deformation equation).

Developing the Maurer–Cartan equation (2.5), one gets

$$\delta\rho_m(t) = \frac{1}{2}\sum_{i+j=m}[\![\rho_i(t), \rho_j(t)]\!] \tag{2.6}$$

for each m. The right hand side of this equation is always a 2-cocycle for any m (cf., e.g., [**3**]). The equation admits a solution if and only if it is a coboundary. The cohomology class of the 2-cocycle in the right hand side of (2.6) is an obstruction for prolongation of the deformation to the order m.

Construction of the miniversal deformation. We are interested in deformations up to equivalence, hence, we will set $r = \dim H^1(\mathfrak{h}; \mathfrak{g})$ and choose the basis $[c_1], \ldots, [c_r]$ of $H^1(\mathfrak{h}; \mathfrak{g})$, where $c_1, \ldots, c_r$ are non-trivial 1-cocycles on $\mathfrak{h}$ with coefficients in $\mathfrak{g}$. We then put

$$\rho_1^i = c_i.$$

The construction of the miniversal deformation goes as follows. Assume, by induction, that we constructed the deformation (2.1) to the order $m-1$. To construct the m-th order term, one has to solve the equation (2.6). The right hand side of (2.6) is an element of $Z^2(\mathfrak{h}; \mathfrak{g}) \otimes \mathbb{K}_m[t_1, \ldots, t_r]$; if this is a coboundary then there exists a solution of (2.6).

The solution of (2.6) can be chosen arbitrarily up to the equivalence and reparametrization. Indeed, if $\rho_m(t)$ and $\rho'_m(t)$ are two solutions, then their difference is a 1-cocycle

3. The main results. We formulate here the main results of this paper, all proofs will be given in Section 4.

The first group of cohomology. One determines $H^1(\mathfrak{h}_1, \mathrm{gl}(3, \mathbb{K}))$ in order to know the dimension of the parameter space, i.e. the number of infinitesimal generators of the deformation. We then give an expression for these generators.

THEOREM 3.1. *Dim* $H^1(\mathfrak{h}_1, \mathrm{gl}(3, \mathbb{K}))$=*4.*

We give a basis of this cohomology space. Let $e_{ij} \in \mathrm{gl}(3,\mathbb{K})$ be the standard basis of $\mathrm{gl}(3,\mathbb{K})$, namely $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Denote $\mathfrak{B}$ the natural basis of $\mathfrak{h}_1$:

$$X = e_{21}, \qquad Y = e_{31}, \qquad Z = e_{32}$$

and let X^*, Y^*, Z^* be the dual basis in $\mathfrak{h}_1^*$. The basis of $H^1(\mathfrak{h}_1, \mathrm{gl}(3,\mathbb{K}))$ is given by the classes of the following 1-cocycles.

$$\begin{aligned}
\rho^1 &= X^* \otimes e_{32},\\
\rho^2 &= Z^* \otimes e_{21},\\
\rho^3 &= Z^* \otimes (e_{11} + e_{22} + e_{33}),\\
\rho^4 &= X^* \otimes (e_{11} + e_{22} + e_{33}).
\end{aligned} \tag{3.1}$$

An expression for the miniversal deformation. We will apply the algorithm described in Section 2 to the 1-cocycles (3.1). The result is an explicit formula for the miniversal deformation. Its expression involves the 2-cocycle $\rho^{12} := X^* \otimes e_{21} + Z^* \otimes e_{32}$ and the function $\theta(t_1, t_2) := \sum_{n=1}^{\infty} \theta_n t_1^n t_2^n$, with $\theta_1 := 1$ and $\theta_n := -\frac{1}{2}(\sum_{i+j=n, i<j} \theta_i\theta_j + \frac{1}{2}\theta_{\frac{n}{2}}^2)$.

Theorem 3.2. *Up to equivalence and reparametrisation, the miniversal deformation of ρ is given by the formula*

$$\widetilde{\rho}(t_1, t_2, t_3, t_4) := \rho + t_1\,\rho^1 + t_2\,\rho^2 + t_3\,\rho^3 + t_4\,\rho^4 + \theta(t_1, t_2)\,\rho^{12}, \tag{3.2}$$

in other terms:

$$\begin{aligned}
\widetilde{\rho}(t_1, t_2, t_3, t_4) &\begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ c & b & 0 \end{pmatrix} \\
&= \begin{pmatrix} bt_3 + at_4 & 0 & 0 \\ a + bt_2 + a\theta(t_1, t_2) & bt_3 + at_4 & 0 \\ c & b + at_1 + b\theta(t_1, t_2) & bt_3 + at_4 \end{pmatrix}.
\end{aligned}$$

Note that no obstruction to the integrability appears, and hence the parameter space is the free commutative algebra $\mathbb{K}[t_1, t_2, t_3, t_4]$. In other words, there are no relations on the parameters of the deformation.

Towards higher dimensional generalizations. The symmetry and simplicity of the result (3.2) gives the hope to generalize it to higher dimensions. It should be possible to rewrite (3.2) in terms of root systems, and hence give a conceptual formulation more likely to be generalized.

4. Proofs of the main results. The proofs consist in applying the methods described in Section 2. When choices are to be made, we always try to make choices that preserve the symmetry of the problem.

4.1. Proof of Theorem 3.1. The proof consists in two steps. We first calculate the space of 1-cocycles and then determine its subspace of coboundaries.

Computation of $Z^1(\mathfrak{h}_1, \mathrm{gl}(3,\mathbb{K}))$.

Let us calculate explicitly the expression of the differential δ^1 defined by the formula (2.3). We will then determine its kernel.

LEMMA 4.1. *A basis of* Z^1 *is given by the vectors:*

$$\begin{aligned}
e_1 &= X^* \otimes e_{32},\\
e_2 &= Z^* \otimes e_{21},\\
e_3 &= X^* \otimes e_{11} + X^* \otimes e_{22} + X^* \otimes e_{33},\\
e_4 &= Z^* \otimes e_{11} + Z^* \otimes e_{22} + Z^* \otimes e_{33},\\
e_5 &= Z^* \otimes e_{31},\\
e_6 &= Y^* \otimes e_{31} + X^* \otimes e_{21},\\
e_7 &= Y^* \otimes e_{32} + X^* \otimes e_{22} - X^* \otimes e_{11},\\
e_8 &= X^* \otimes e_{31},\\
e_9 &= Z^* \otimes e_{32} + Y^* \otimes e_{31},\\
e_{10} &= X^* \otimes e_{23} + Y^* \otimes (e_{33} - e_{11}) - Z^* \otimes e_{12},\\
e_{11} &= Z^* \otimes e_{33} + \frac{1}{2} Z^* \otimes e_{11} - \frac{1}{2} Y^* \otimes e_{21}.
\end{aligned}$$

Let Φ be a one cochain on $\mathfrak{h}_1$ with coefficients in $\mathrm{gl}(3,\mathbb{K})$. Let us denote its coordinates in the basis $\mathcal{B} := \{X^* \otimes e_{ij}, Y^* \otimes e_{kl}, Z^* \otimes e_{mn}\}$ by:

$$(\underbrace{\Phi_X^{11}, \dots, \Phi_X^{33}}_{\Phi(X)}, \underbrace{\Phi_Y^{11}, \dots, \Phi_Y^{33}}_{\Phi(Y)}, \underbrace{\Phi_Z^{11}, \dots, \Phi_Z^{33}}_{\Phi(Z)}).$$

Using these coordinates, one can rewrite the coboundary operator δ^1. One has

$$\begin{aligned}
\delta^1\Phi(X,Y) &= \Phi([e_{21}, e_{31}]) - [e_{21}, \Phi(Y)] + [e_{31}, \Phi(X)]\\
&= -[e_{21}, \sum_{i,j=1}^{3} \Phi_Y^{ij} e_{ij}] + [e_{31}, \sum_{i,j=1}^{3} \Phi_X^{ij} e_{ij}]\\
&= -\sum_{i,j=1}^{3} \Phi_Y^{ij} [e_{21}, e_{ij}] + \sum_{i,j=1}^{3} \Phi_X^{ij} [e_{31}, e_{ij}]
\end{aligned}$$

which gives

$$\delta^1\Phi(X,Y) = -\sum_{j=1}^{3} \Phi_Y^{1j} e_{2j} + \sum_{i=1}^{3} \Phi_Y^{i2} e_{i1} + \sum_{j=1}^{3} \Phi_X^{1j} e_{3j} - \sum_{i=1}^{3} \Phi_X^{i3} e_{i1}.$$

Similarly,

$$\delta^1\Phi(X,Z) = -\sum_{i,j=1}^{3}\Phi_Y^{ij}e_{ij} - \sum_{j=1}^{3}\Phi_Z^{1j}e_{2j} + \sum_{i=1}^{3}\Phi_Z^{i2}e_{i1} + \sum_{j=1}^{3}\Phi_X^{2j}e_{3j} - \sum_{i=1}^{3}\Phi_X^{i3}e_{i2}$$

and

$$\delta^1\Phi(Y,Z) = -\sum_{j=1}^{3}\Phi_Z^{1j}e_{3j} + \sum_{i=1}^{3}\Phi_Z^{i3}e_{i1} + \sum_{j=1}^{3}\Phi_Y^{2j}e_{3j} - \sum_{i=1}^{3}\Phi_Y^{i3}e_{i2}.$$

We now want to express the linear operator δ^1 in a matrix form. Since $\delta^1\Phi$ is a 2 cochain, it can be decomposed in the basis

$$\mathcal{B}' := \{X^\star \wedge Y^\star \otimes e_{ij}, \quad X^\star \wedge Z^\star \otimes e_{ij}, \quad Y^\star \wedge Z^\star \otimes e_{ij}\}.$$

More precisely,

$$\delta^1\Phi = \sum_{i,j=1}^{3}\Big((\delta^1\Phi)_{X,Y}^{ij}\, X^\star \wedge Y^\star + (\delta^1\Phi)_{X,Z}^{ij}\, X^\star \wedge Z^\star + (\delta^1\Phi)_{Y,Z}^{ij}\, Y^\star \wedge Z^\star\Big) \otimes e_{ij}.$$

Applying $\delta^1\Phi$ to (X,Y) one gets

$$\delta^1\Phi(X,Y) = \sum_{i,j=1}^{3}(\delta^1\Phi)_{X,Y}^{ij}\, e_{ij}.$$

One can then identify the first nine coefficients $(\delta^1\Phi)_{X,Y}^{ij}, 1 \leq i,j \leq 3$ which correspond to the first nine rows of matrix of δ^1 in the bases $\mathcal{B}$ and $\mathcal{B}'$. Applying the same procedure to $(\delta^1\Phi)_{X,Z}^{ij}$ and $(\delta^1\Phi)_{Y,Z}^{ij}$, one finally gets a (27×27)-matrix, see Appendix 5. In order to determine the kernel of δ^1, one has to find a maximal free subfamily among the column vectors of the matrix of δ^1. Dependence relations among remaining vectors will then give the kernel. Details of these computations can also be found in Appendix 5. This completes the proof of Lemma 4.1.

Computation of $B^1(\mathfrak{h}_1, \mathrm{gl}(3,\mathbb{K}))$.

The space of coboundaries $B^1(\mathfrak{h}_1, \mathrm{gl}(3,\mathbb{K}))$ is the image of $\mathrm{gl}(3,\mathbb{K})$ by the operator $\delta^0 : \bigwedge^0 \longrightarrow \bigwedge^1$ defined by

$$\delta^0(A)(a) = [A,a],$$

where $a \in \mathfrak{h}_1$ and $A \in \mathrm{gl}(3,\mathbb{K})$.

Proceeding as above, one has for $A = (A^{ij})$ with $i, j = 1, 2, 3$

$$\begin{aligned}
\delta^0 A(X) &= -[e_{21}, \sum_{i,j=1}^{3} A^{ij} e_{ij}] \\
&= -\sum_{j=1}^{3} A^{1j} e_{2j} + \sum_{i=1}^{3} A^{i2} e_{i1}, \\
\delta^0 A(Y) &= -\sum_{j=1}^{3} A^{1j} e_{3j} + \sum_{i=1}^{3} A^{i3} e_{i1}, \\
\delta^0 A(Z) &= -\sum_{j=1}^{3} A^{2j} e_{3j} + \sum_{i=1}^{3} A^{i3} e_{i2}.
\end{aligned}$$

The matrix of this operator is given in Appendix 5. A basis of the image is as follows.

$$\begin{aligned}
\delta_{11} &= -X^\star \otimes 21 - Y^\star \otimes 31, \\
\delta_{12} &= X^\star \otimes (11 - 22) - Y^\star \otimes 32, \\
\delta_{13} &= -X^\star \otimes 23 + Y^\star \otimes (11 - 33) + Z^\star \otimes 12, \\
\delta_{21} &= -Z^\star \otimes 31, \\
\delta_{22} &= X^\star \otimes 21 - Z^\star \otimes 32, \\
\delta_{23} &= Y^\star \otimes 21 + Z^\star \otimes (22 - 33), \\
\delta_{32} &= X^\star \otimes 31.
\end{aligned}$$

Computation of $H^1(\mathfrak{h}_1, \mathrm{gl}(3, \mathbb{K}))$.

The dimension of $Z^1(\mathfrak{h}_1, \mathrm{gl}(3, \mathbb{K}))$ is 11, the one of $B^1(\mathfrak{h}_1, \mathrm{gl}(3, \mathbb{K}))$ is 7. Hence the quotient space $H^1(\mathfrak{h}_1, \mathrm{gl}(3, \mathbb{K}))$ has dimension $11 - 7 = 4$. One can check, see Appendices 5-5, that the first four elements e_1, e_2, e_3, e_4 are independent modulo $B^1(\mathfrak{h}_1, \mathrm{gl}(3, \mathbb{K}))$. Their classes form a basis of H^1. Theorem 3.1 is proved.

4.2. Proof of Theorem 3.2. We will show that every infinitesimal deformation is integrable. In other words, there are no obstructions to integrability.

Integrability at order 2.

One needs to evaluate the cup-products $[\![\rho^i, \rho^j]\!], 1 \le i, j \le 4$ of the cocycles (3.1). It turns out that the only non vanishing term is the coboundary:

$$[\![\rho^1, \rho^2]\!] = X^* \wedge Z^* \otimes e_{31} = \delta^1(\frac{1}{2}(X^\star \otimes e_{21} + Z^\star \otimes e_{32})).$$

Let us choose

$$\rho^{12} = \frac{1}{2}(X^\star \otimes e_{21} + Z^\star \otimes e_{32})$$

and $\rho^{ij} = 0$ otherwise. One extends the deformation to the order 2. The equation (2.6) is satisfied to the second order.

Integrability at any order.

The key point, allowing an induction, is the fact that all the non vanishing cup-products are of the form $[\![\rho^{12}, \rho^{12}]\!]$, hence the deformation can be extended by means of ρ^{12}, and one is again in the same situation, enabling to pursue the induction.

More precisely: let us suppose (induction hypothesis) that up to order m the deformation is given by $\widetilde{\rho}(t_1, t_2, t_3, t_4) := \rho + t_1\,\rho^1 + t_2\,\rho^2 + t_3\,\rho^3 + t_4\,\rho^4 + \sum_{k=1}^{m} \theta_k t_1^k t_2^k \rho^{12}$. Solving the deformation equation (2.6) at order m is equivalent to find a 1-cocycle ρ^{m+1} such that

$$\delta^1 \rho^{m+1} = \Big(\sum_{i+j=m+1, i<j} \theta_i \theta_j + \frac{1}{2} \theta^2_{\frac{m+1}{2}} \Big) [\![\rho^{12}, \rho^{12}]\!]$$

($\theta_{\frac{m+1}{2}}$ meaning zero when not defined i.e. when m even). An easy computation shows that $[\![\rho^{12}, \rho^{12}]\!] = -\frac{1}{2}[\![\rho^1, \rho^2]\!]$, hence, by the previous computation, it suffices to set $\rho^{m+1} := \theta_{m+1}\rho^{12}$ with $\theta_{m+1} := -\frac{1}{2}(\sum_{i+j=m+1, i<j} \theta_i\theta_j + \frac{1}{2}\theta^2_{\frac{m+1}{2}})$. Theorem 3.2 is proved.

5. Appendix. In this appendix we have moved the details of the determination of the expression of the boundary operator, of its kernel and of its image.

Computations appearing in proof of Theorem 3.1. We first need an expression of the boundary operator with which we can compute, i.e. as a matrix.

Computing δ^1.

The columns of the matrix of δ^1 in the bases $\mathcal{B}$ and $\mathcal{B}'$ are given by the decomposition in $\mathcal{B}'$ of the vectors

$$\delta^X_{ij} = \delta^1(X^\star \otimes e_{ij}), \delta^Y_{ij} = \delta^1(Y^\star \otimes e_{ij}), \delta^Z_{ij} = \delta^1(Z^\star \otimes e_{ij}),$$

we can, doing the identification explained in Section 4.1, give their explicit description. The first fourteen elements of this family write:

$$\begin{aligned}
\delta^X_{11} &= \underline{X^\star \wedge Y^\star \otimes e_{31}},\\
\delta^X_{12} &= \underline{X^\star \wedge Y^\star \otimes e_{32}},\\
\delta^X_{13} &= X^\star \wedge Y^\star \otimes (\underline{e_{33}} - e_{11}) - X^\star \wedge Z^\star \otimes e_{12},
\end{aligned}$$

$$\begin{aligned}
\delta_{21}^X &= \underline{X^\star \wedge Z^\star \otimes e_{31}}, \\
\delta_{22}^X &= \underline{X^\star \wedge Z^\star \otimes e_{32}}, \\
\delta_{23}^X &= -X^\star \wedge Y^\star \otimes e_{21} + X^\star \wedge Z^\star \otimes (\underline{e_{33}} - e_{22}), \\
\delta_{11}^Y &= -X^\star \wedge Y^\star \otimes e_{21} - \underline{X^\star \wedge Z^\star \otimes e_{11}}, \\
\delta_{12}^Y &= X^\star \wedge Y^\star \otimes (e_{11} - \underline{e_{22}}) - X^\star \wedge Z^\star \otimes e_{12}, \\
\delta_{13}^Y &= -\underline{X^\star \wedge Y^\star \otimes e_{23}} - X^\star \wedge Z^\star \otimes e_{13} - Y^\star \wedge Z^\star \otimes e_{12}, \\
\delta_{21}^Y &= -\underline{X^\star \wedge Z^\star \otimes e_{21}} + Y^\star \wedge Z^\star \otimes e_{31}, \\
\delta_{22}^Y &= X^\star \wedge Y^\star \otimes e_{21} - X^\star \wedge Z^\star \otimes e_{22} + \underline{Y^\star \wedge Z^\star \otimes e_{32}}, \\
\delta_{23}^Y &= -X^\star \wedge Z^\star \otimes e_{23} + Y^\star \wedge Z^\star \otimes (e_{33} - \underline{e_{22}}), \\
\delta_{13}^Z &= -X^\star \wedge Z^\star \otimes e_{23} - Y^\star \wedge Z^\star \otimes (e_{33} - \underline{e_{11}}), \\
\delta_{23}^Z &= \underline{Y^\star \wedge Z^\star \otimes e_{21}}.
\end{aligned}$$

They are independent since each these vectors has an *underlined* non vanishing component which vanishes for the other vectors. One can check that the two following vectors are independent from the preceding ones.

$$\begin{aligned}
\delta_{33}^Y &= -X^\star \wedge Z^\star \otimes e_{33} - Y^\star \wedge Z^\star \otimes e_{32}. \\
\delta_{11}^Z &= -X^\star \wedge Z^\star \otimes e_{21} - Y^\star \wedge Z^\star \otimes e_{31},
\end{aligned}$$

The remaining vectors

$$\begin{aligned}
\delta_{31}^X &= 0, \\
\delta_{32}^X &= 0, \\
\delta_{33}^X &= -X^\star \wedge Y^\star \otimes e_{31} - X^\star \wedge Z^\star \otimes e_{32}. \\
\delta_{31}^Y &= -X^\star \wedge Z^\star \otimes e_{31}, \\
\delta_{32}^Y &= X^\star \wedge Y^\star \otimes e_{31} - X^\star \wedge Z^\star \otimes e_{32}, \\
\delta_{12}^Z &= -X^\star \wedge Z^\star \otimes (e_{22} - e_{11}) - Y^\star \wedge Z^\star \otimes e_{32}, \\
\delta_{21}^Z &= 0, \\
\delta_{22}^Z &= X^\star \wedge Z^\star \otimes e_{21}, \\
\delta_{31}^Z &= 0, \\
\delta_{32}^Z &= X^\star \wedge Z^\star \otimes e_{31}, \\
\delta_{33}^Z &= Y^\star \wedge Z^\star \otimes e_{31}
\end{aligned}$$

are linear combinations of the previous ones. More precisely, δ_{31}^X,δ_{32}^X,δ_{21}^Z and δ_{31}^Z vanish and

$$\begin{aligned}
\delta_{33}^X &= -\delta_{22}^X - \delta_{11}^X, \\
\delta_{31}^Y &= -\delta_{21}^X, \\
\delta_{32}^Y &= -\delta_{22}^X + \delta_{11}^X, \\
\delta_{22}^Z &= -\frac{1}{2}\delta_{11}^Z - \frac{1}{2}\delta_{21}^Y, \\
\delta_{32}^Z &= -\delta_{31}^Y, \\
\delta_{12}^Z &= -\delta_{11}^Y + \delta_{23}^X + \delta_{33}^Y, \\
\delta_{33}^Z &= -\frac{1}{2}\delta_{11}^Z + \frac{1}{2}\delta_{21}^Y.
\end{aligned}$$

These relations are important since they give us the basis of Z^1:

$$\begin{aligned}
e_1 &= \underline{X^* \otimes e_{32}}, \\
e_2 &= \underline{Z^* \otimes e_{21}}, \\
e_3 &= \underline{X^* \otimes e_{33}} + X^* \otimes e_{22} + X^* \otimes e_{11}, \\
e_4' &= Z^* \otimes e_{22} + \frac{1}{2}\underline{Z^* \otimes e_{11}} + \frac{1}{2}Y^* \otimes e_{21}, \\
e_5 &= Z^* \otimes e_{31}, \\
e_6 &= Y^* \otimes e_{31} + X^* \otimes e_{21}, \\
e_7 &= Y^* \otimes e_{32} + X^* \otimes e_{22} - X^* \otimes e_{11}, \\
e_8 &= X^* \otimes e_{31}, \\
e_9 &= Z^* \otimes e_{32} + Y^* \otimes e_{31}, \\
e_{10} &= X^* \otimes e_{23} + Y^* \otimes (e_{33} - e_{11}) - Z^* \otimes e_{12}, \\
e_{11} &= Z^* \otimes e_{33} + \frac{1}{2}Z^* \otimes e_{11} - \frac{1}{2}Y^* \otimes e_{21}.
\end{aligned}$$

These elements are in the kernel because of the preceding relations. In order to obtain a symmetric formula (3.2), one replaces e_4' by $e_4 := e_4' + e_{11} = \underline{Z^* \otimes e_{11}} + Z^* \otimes e_{22} + Z^* \otimes e_{33}$.

Computing δ^0.

The matrix of δ^0 is given by the following set of vectors:

$$\begin{aligned}
\delta_{11} &= -X^\star \otimes 21 - \underline{Y^\star \otimes 31},\\
\delta_{12} &= X^\star \otimes (\underline{11} - 22) - Y^\star \otimes 32,\\
\delta_{13} &= -\underline{X^\star \otimes 23} + Y^\star \otimes (11-33) + Z^\star \otimes 12,\\
\delta_{21} &= -\underline{Z^\star \otimes 31},\\
\delta_{22} &= X^\star \otimes 21 - \underline{Z^\star \otimes 32},\\
\delta_{23} &= \underline{Y^\star \otimes 21} + Z^\star \otimes (22-33),\\
\delta_{32} &= \underline{X^\star \otimes 31}
\end{aligned}$$

which are independent and

$$\begin{aligned}
\delta_{31} &= 0,\\
\delta_{33} &= Y^\star \otimes 31 + Z^\star \otimes 32,
\end{aligned}$$

which are linear combinations of the above ones. Indeed, $\delta_{31} = 0$ and $\delta_{33} = -\delta_{11} + \delta_{22}$.

Computing the basis of $H^1(\mathfrak{h}_1, \mathrm{gl}(3, \mathbb{K}))$.

The first four elements e_1, e_2, e_3, e_4 of the basis of the space of cocycles $Z^1(\mathfrak{h}_1, \mathrm{gl}(3, \mathbb{K}))$ are linearly independent modulo the coboundaries, since each of them has an *underlined* component which does not appear in the elements of the basis of the space of coboundaries $B^1(\mathfrak{h}_1, \mathrm{gl}(3, \mathbb{K}))$.

Acknowledgements. The problem was stated to me by B. Kostant and V. Ovsienko; I am grateful to V. Ovsienko for his help. I would also like to thank C. Duval for enlightening discussions, C. Roger and F. Pellegrini for their interest in this work. Finally I thank the Luxembourgian Ministry of Research for financial support via a BFR grant.

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Received February 25, 2009

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