Section 13 – Implicit Differentiation

Recall notation $f'(x) \equiv \dfrac{d\,f(x)}{dx}$ since $y' \equiv \dfrac{dy}{dx}$

The portion of the above expression $\dfrac{d}{dx}$ is an <u>operator</u>. It tells us to take the derivative of whatever is next to d in the numerator.

So $\dfrac{d(e^x)}{dx} = e^x$

$$\frac{d}{dx}(4x^3 - x + 5) = 12x^2 - 1$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(6^{-x}) = 6^{-x} \cdot \ln 6 \cdot (-1) = -6^{-x}\ln 6$$

(better order: $6^{-x} \cdot (-1) \cdot \ln 6$

Another notation to know is $f'(a)$, which means ~~take~~ find the <u>value</u> of $f'(x)$ at $x = a$.

Equivalently:

$$\left.\frac{d\,f(x)}{dx}\right|_{x=a} \equiv f'(a) \quad \text{or} \quad \left.\frac{dy}{dx}\right|_{x=a} \equiv f'(a)$$

"d-dx of f of x evaluated at x=a"

When we use the $\frac{dy}{dx}$ notation, we revert to thinking of $f(x)$ as y. It's much neater to write

$$\frac{dy}{dx} \quad \text{rather than} \quad \frac{d\,f(x)}{dx}$$

but both are good.

<u>Ex</u> $f(x) = e^{-4x}$

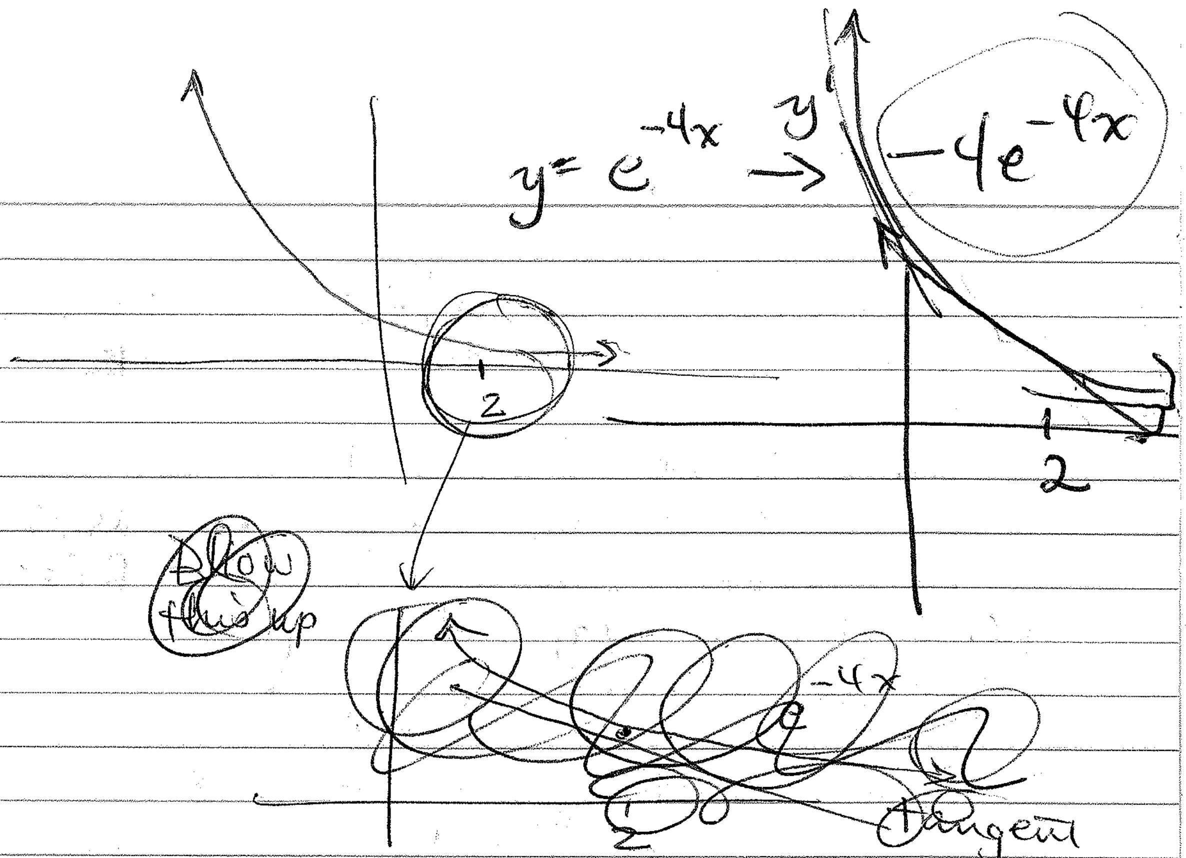
What is $\left.\frac{d\,f(x)}{dx}\right|_{x=2}$?

$\frac{d\,f(x)}{dx} = f'(x) = -4e^{-4x}$, ~~so~~ $\cancel{\left.\frac{d(-4e^{-4x})}{dx}\right|_{x=2}}$

~~So~~ $\cancel{\frac{d}{dx}(-4e}$

Thus $\left.\frac{d}{dx}\left(e^{-4x}\right)\right|_{x=2} = -4e^{-4(2)} = -4e^{-8} = \frac{-4}{e^{8}}$

Notice it's a negative value. Look at the graph of $y = e^{-4x}$ to see why



See the tangent has a negative slope here.

We're going to be more interested again in $\left.\frac{dy}{dx}\right|_{x=a}$ and what it means graphically.

This is because we will have to look at where a function of interest is at its maximum or minimum value. As usual, before resorting to derivatives to tell us this, we want to grasp the reality of the graph for a fcn. of interest.

Deviate

Turns out $\frac{dy}{dx}$ notation expresses the chain rule in a nice, clear way if we use the "u" notation introduced earlier.

So where $f(g(x))$ has derivative $f'(g(x))\,g'(x)$

if we let $\boxed{g(x) = u}$ (leave off x), then

We can write

chain part

$$\frac{d}{dx}\left[f(g(x))\right] = \frac{d}{dx}\left[f(u)\right] = \frac{df}{du} \cdot \frac{du}{dx}$$

and if we further trim this down notationally by letting ~~g = f(x), then~~ $y = f(g(x)) = f(u)$

then $$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

This is what I was getting at by calling y (of f) the "outer function" an u the "inner" fcn.

By the chain rule, it's then somewhat intuitive (in a rhythmic sort of way) that

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$ "y is a fcn of u, which is a fcn. of x"

when y is a fcn. of u rather than just a fcn. of x, i.e. $f \circ u$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \text{ is Leibniz notation}}$$

Now, u itself may by composed of another fcn v, so the chain rule must continue to be employed until the entire derivative is dealt with:

Ex from book p. 104:

$$y = \ln(2x+41)^{15} \qquad \text{Let } (2x+41)^{15} = u$$

Then $\dfrac{dy}{du} = \dfrac{d \ln u}{du} = \dfrac{1}{u}$ or $\boxed{y' = \dfrac{1}{u} = \dfrac{1}{(2x+41)^{15}}}$

Now since $u = (2x+41)^{15}$, $\dfrac{du}{dx} = 15\underbrace{(2x+41)^{14}\cdot 2}$ by chain rule

So, $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$ Breaks down more

However, since u is composed of a fcn. raised to a power, it makes sense to break it down as

$$u = v^n \qquad \text{where } v = 2x+41, \; n = 15$$

so $\dfrac{du}{dx} = \dfrac{du}{dv} \cdot \dfrac{dv}{dx}$

and $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \underbrace{\dfrac{du}{dv}}_{\text{outer}} \cdot \underbrace{\dfrac{dv}{dx}}_{\text{inner}}$

on $(2x+41)^{15}$

Hence, for $y = \ln(2x+41)^{15}$

$$\boxed{\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} \\ \frac{dy}{dx} &= \frac{1}{(2x+41)^{15}} \cdot 15(2x+41)^{14} \cdot (2) \end{aligned}}$$

So make sure you understand where each of these parts came from.

Try an example where you label the u & v so $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$

You see by now that the parts of these rate-of-change ratios appear to cancel.

While these are not fractions, they can be manipulated as such.

It's as if $\frac{dy}{dx} = \frac{dy}{\cancel{du}} \cdot \frac{\cancel{du}}{\cancel{dv}} \cdot \frac{\cancel{dv}}{dx}$

All this notation gives us the terminology we need to take on <u>implicit differentiation</u>

~~Put simply, we differential a term that has y alone as if it were a variable a fcn. of x (which it is, if we were to isolate it.~~

Implicit Differentiation – continued

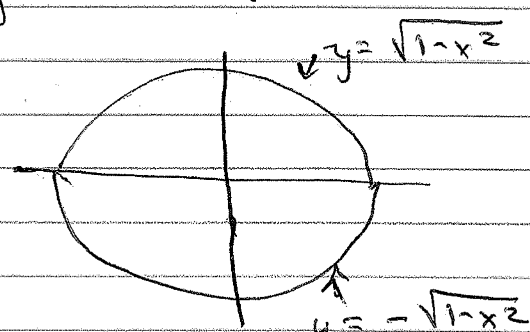
Suppose you have the equation of a circle

$$x^2 + y^2 = 1$$

You know you could isolate y ~~and~~ and take the derivative of what has only x on one side.

Ex $x^2 + y^2 = 1 \rightarrow y^2 = 1 - x^2$

$$\rightarrow y = \pm\sqrt{1-x^2}$$



Then $\dfrac{dy}{dx} = \begin{cases} \frac{1}{2}(1-x^2)^{-1/2}(-2x), & y \geq 0 \\ -(1-x^2)^{-1/2}(-2x), & y < 0 \end{cases} \rightarrow \dfrac{dy}{dx} = \begin{cases} -x(1-x^2)^{-1/2} \\ +x(1-x^2)^{-1/2} \end{cases}$

The analysis of this is a little dodgy, so here's a cleaner approach:

Keep it as $x^2 + y^2 = 1$

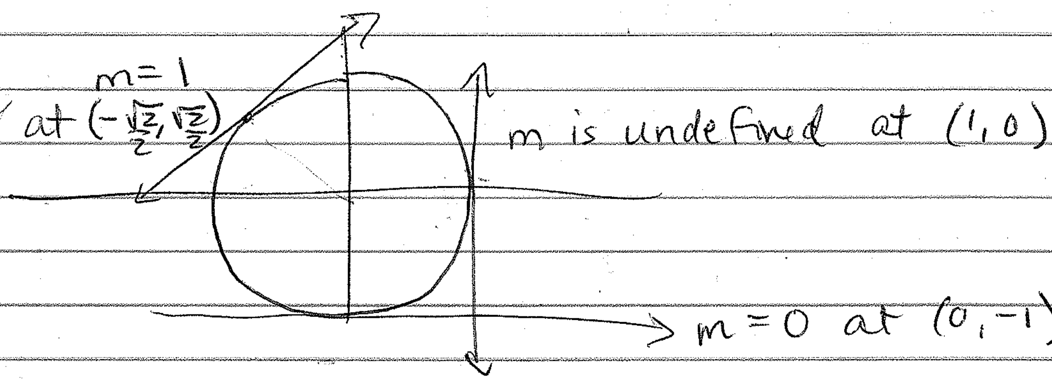
and knowing that y is ultimatly a fcn. of x piecewise, since a circle is not a fcn but a half circle ⌒ or ◡ is, perform what we call an "implicit differentiation with respect to x. It goes like this.

$$x^2 + y^2 = 1 \longrightarrow 2x + 2y \cdot \frac{dy}{dx} = 0$$

Solve for $\frac{dy}{dx}$: $\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$

$$\boxed{\text{So } \frac{dy}{dx} \text{ for } x^2+y^2=1 \text{ is } -\frac{x}{y}}$$

which when evaluated at various x, shows us the behavior of the tangent lines to the circle all around.



An important precalculus problem asks what the coordinates on the unit circle are for a given angle. It used trigonometry.

At 45°, $x = y = \frac{\sqrt{2}}{2}$

At 135°, which is 45° off the axis also, $x = -\frac{\sqrt{2}}{2}$ and $y = \frac{\sqrt{2}}{2}$

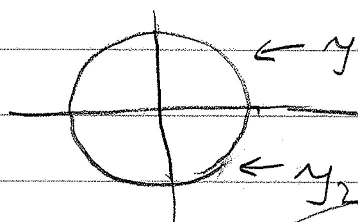
The slope there is

$$\left.\frac{dy}{dx}\right|_{\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)} = \frac{-x}{y} = \frac{+\sqrt{2}/2}{\sqrt{2}/2} = 1$$

Sec 13 HW – Implicit Differentiation

1. $x^2 + y^2 = 1$ is not a function, but it can be written as two fcns:

$$y^2 = 1 - x^2 \longrightarrow y_1 = \sqrt{1-x^2}, \quad y_2 = -\sqrt{1-x^2}$$



$\leftarrow y_1 = \sqrt{1-x^2} \rightarrow y_1' = \frac{1}{2}(1-x^2)^{-1/2}(\cancel{-2}x)$

$$y_1' = \frac{-x}{\sqrt{1-x^2}}$$

$\leftarrow y_2 = -\sqrt{1-x^2}$

$$y_2' = \frac{+x}{\sqrt{1-x^2}}$$

Are these the same as what implicit differentiation give?

$$x^2 + y^2 = 1$$

$$2x + 2y\frac{dy}{dx} = 0 \longrightarrow \frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}y} =$$

$$\boxed{\frac{dy}{dx} = \frac{-x}{y}} = \frac{-x}{\pm\sqrt{1-x^2}} = \frac{\mp x}{\sqrt{1-x^2}} = y_1' \text{ or } y_2'$$

2a) $x^2 + 3y^2 = 6$

$$2x + 6y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{6y} = \frac{-x}{3y}$$

$$\boxed{\frac{dy}{dx} = \frac{-x}{3y}}$$

b) $9x - x^2y^2 = 2xy$

Requires product rule:

$$9 - 2xy^2 - x^2 \cdot 2y\frac{dy}{dx} = 2y + 2x\frac{dy}{dx}$$

$\longrightarrow$

2b) $9-2xy^2-2y = 2x^2y\frac{dy}{dx} + 2x\frac{dy}{dx}$

$$9-2xy^2-2y = (2x^2y+2x)\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{9-2xy^2-2y}{2x^2y+2x}}$$

c) $3xy - \frac{y}{3} = 2x^{-1}$

$$3y + 3x\frac{dy}{dx} - \frac{1}{3}\frac{dy}{dx} = -2x^{-2}$$

$$\left(3x - \frac{1}{3}\right)\frac{dy}{dx} = -2x^{-2} - 3y$$

$$\frac{dy}{dx} = \frac{-2x^{-2}-3y}{3x-1/3} = \frac{\frac{-2}{x^2}-3y}{3x-\frac{1}{3}}$$ (LCD $= 3x^2$)

$$\frac{dy}{dx} = \frac{-3x^2}{-3x^2}\cdot\frac{\frac{-2}{x^2}-3y}{3x-\frac{1}{3}} = \frac{6+9x^2y}{-9x^3+x^2}$$ (book has negative ~~factored~~ out)

f. $3x^2 - 4y^3 + 3 = \sqrt{5x+y}$

$$6x - 12y^2\frac{dy}{dx} = \frac{1}{2}(5x+y)^{-1/2}\cdot\left(5+\frac{dy}{dx}\right)$$

$$6x - 12y^2\frac{dy}{dx} = \frac{5}{2}\left(\frac{1}{\sqrt{5x+y}}\right) + \frac{1}{\sqrt{5x+y}}\frac{dy}{dx}$$

$$6x - \frac{5}{2\sqrt{5x+y}} = \left(12y^2 + \frac{1}{\sqrt{5x+y}}\right)\frac{dy}{dx}$$

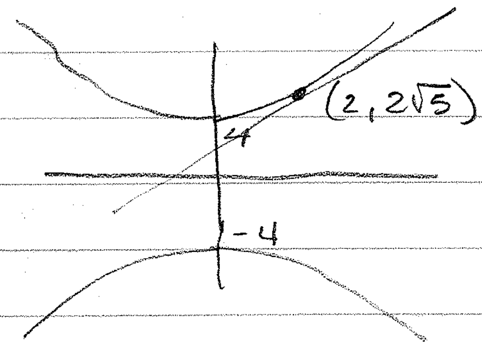
Sec 13 con'd

#2f

$$\frac{dy}{dx} = \frac{6x - \frac{5}{2\sqrt{5x+y}}}{12y^2 + \frac{1}{\sqrt{5x+y}}}$$

(same as book, bit different form)

#3. $y^2 - x^2 = 16$ hyperbola



Eqn. of tangent line at $(2, 2\sqrt{5})$ is found:

$$2y\frac{dy}{dx} - 2x = 0$$

$$\left.\frac{dy}{dx}\right|_{(2,2\sqrt{5})} = \left.\frac{x}{y}\right|_{(2,2\sqrt{5})} = \frac{2}{2\sqrt{5}} = \frac{1}{\sqrt{5}} = \text{slope}$$

$$\boxed{y - 2\sqrt{5} = \frac{1}{\sqrt{5}}(x-2)}$$

6. $\boxed{e^{xy} = x}$ Find slope at $x=3$, that is, $\left.\frac{dy}{dx}\right|_{x=3}$ (You'll need y value, so plug $x=3$ into $e^{xy}=x$ first)

$e^{3y} = 3 \longrightarrow \ln e^{3y} = \ln 3 \rightarrow 3y = \ln 3$

$\longrightarrow \boxed{y = \frac{\ln 3}{3} \text{ when } x = 3}$

Now find dy/dx: $\underbrace{e^{xy}}_{e^u}\underbrace{\left(1y + x\frac{dy}{dx}\right)}_{du} = 1 \longrightarrow e^{xy}y + xe^{xy}\frac{dy}{dx} = 1$

$$xe^{xy}\frac{dy}{dx} = 1 - ye^{xy} \longrightarrow \boxed{\frac{dy}{dx} = \frac{1-ye^{xy}}{xe^{xy}}}$$

So $$\left.\frac{dy}{dx}\right|_{(3,\ln 3)} = \frac{1-\ln 3\, e^{3\ln 3}}{3e^{3\ln 3}}$$

book is wrong

$$= \frac{1-\ln 3\cdot e^{\ln 9}}{3e^{\ln 9}} = \frac{1-\ln 3\cdot 9}{3\cdot 9} = \boxed{\frac{1-9\ln 3}{27}}$$

10. $$y^2 = x \longrightarrow 2y\frac{dy}{dx} = 1 \longrightarrow \boxed{\frac{dy}{dx} = \frac{1}{2y}}$$

a) So $$\frac{d^2y}{dx^2} = -1(2y)^{-2}\cdot 2\frac{dy}{dx}$$ ← Chain rule + implicit

~~thus~~ $$\frac{d^2y}{dx^2} = \frac{-\cancel{2}}{\cancel{2}4y^2}\frac{dy}{dx}$$ ← Now substitute

$$\frac{d^2y}{dx^2} = \frac{-1}{2y^2}\cdot\frac{1}{2y} = \boxed{\frac{-1}{4y^3}}$$

b) Find d^3y/dx^3 by quotient or power rule:

Quotient rule: $$\frac{d^3y}{dx^3} = \frac{0(4y^3)-(-1)(12y^2\,dy/dx)}{(4y^3)^2} = \frac{12y^2\,dy/dx}{16y^6}$$

$$= \frac{3\,dy/dx}{4y^4} \quad \underset{dy/dx}{\text{substitute}} \quad = \frac{3\left(\frac{1}{2y}\right)}{4y^4} = \boxed{\frac{3}{8y^5}}$$

Power rule

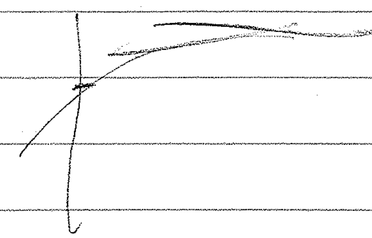
$$\frac{d^2y}{dx^2} = -(4y^3)^{-1} \longrightarrow \frac{d^3y}{dx^3} = +(4y^3)^{-2}\left(12y^2\frac{dy}{dx}\right)$$

$$= \frac{12y^2}{(4y^3)^2}\frac{dy}{dx} = \frac{12y^2}{16y^6}\frac{dy}{dx} = \frac{3}{4y^4}\frac{dy}{dx}$$

Substituting dy/dx

$$\frac{d^3y}{dx^2} = \frac{3}{4y^4}\cdot\frac{1}{2y} = \boxed{\frac{3}{8y^5}}$$

#8 For a certain product we have the following fcns implicit in q, the first because it is not explicitly solved for C, the second, not explicitly solved for R.



Cost $\quad C^2 = q^2 + 100\sqrt{q} + 100$

Revenue $\quad 900(q-4)^2 + R^2 = 25{,}500$

Marg. Cost $C'(q)$

$$2C\frac{dC}{dq} = 2q + \frac{1}{2}\cdot 100q^{-1/2} + 0$$

$$2C\frac{dC}{dq} = 2q + \frac{50}{\sqrt{q}} \longrightarrow \frac{dC}{dq} = \frac{2q}{2C} + \frac{50}{2C\sqrt{q}} = \boxed{\frac{q}{C} + \frac{25}{C\sqrt{q}}}$$

marginal cost $C'(q)$

Marg. Rev $R'(q)$

$$1800(q-4)^1 + 2R\frac{dR}{dq} = 0$$

$$\frac{dR}{dq} = \frac{-1800(q-4)}{2R} = \boxed{\frac{-900(q-4)}{R}}$$

marginal revenue fcn $R'(q)$.

To find $C'(5)$ + $R'(5)$ we need to find what C + R would be at $q=5$, since the marg fcns are not explicit.

① Cost at $q=5$: $C^2 = 5^2 + 100\sqrt{5} + 100$

$$C = \sqrt{125 + 100\sqrt{5}}$$
$$= \sqrt{25(5+4\sqrt{5})} \quad C = 5\sqrt{5+4\sqrt{5}}$$

$$C'(5) = \frac{q\sqrt{q} + 25}{C\sqrt{q}} = \frac{5\sqrt{5} + 25}{5\sqrt{5+4\sqrt{5}}\sqrt{5}} = \frac{\sqrt{5}+5}{\sqrt{5+4\sqrt{5}}\sqrt{5}}$$

Multiply top + bottom by $\sqrt{5}$:

$$C'(q) = \frac{\sqrt{5}+5 \cdot \sqrt{5}}{\sqrt{5+4\sqrt{5}}\sqrt{5} \cdot \sqrt{5}} = \frac{5+5\sqrt{5}}{\sqrt{5+4\sqrt{5}} \cdot 5} = \frac{1+\sqrt{5}}{\sqrt{5+4\sqrt{5}}}$$

A calculator tells $\approx$ .87, or a rate of $\frac{.87}{100 \text{ items}}$

② Revenue at $q=5$: $900(5-4)^{\frac{1}{2}} + R^2 = 25,500$

$$R^2 = 25,500 - 900 = 24,600$$
$$R = \sqrt{24,600} = 10\sqrt{246}$$

Then $R'(5) = \frac{-900(5-4)}{10\sqrt{246}} = \frac{-90}{\sqrt{246}}$ or $\frac{-5.74}{100 \text{ items}}$ rate

Meaning: When production is 500 units, costs are increasing at a rate of 87¢/next 100 items

When 500 units are sold, revenue for next 100 decreases by \$5.74