

# Commensurabilities between ETNOs: a Monte Carlo survey

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## ABSTRACT

Many asteroids in the main and trans-Neptunian belts are trapped in mean motion resonances with Jupiter and Neptune, respectively. As a side effect, they experience accidental commensurabilities among themselves. These commensurabilities define characteristic patterns that can be used to trace the source of the observed resonant behaviour. Here, we explore systematically the existence of commensurabilities between the known ETNOs using their heliocentric and barycentric semimajor axes, their uncertainties, and Monte Carlo techniques. We find that the commensurability patterns present in the known ETNO population resemble those found in the main and trans-Neptunian belts. Although based on small number statistics, such patterns can only be properly explained if most, if not all, of the known ETNOs are subjected to the resonant gravitational perturbations of yet undetected trans-Plutonian planets. We show explicitly that some of the statistically significant commensurabilities are compatible with the Planet Nine hypothesis; in particular, a number of objects may be trapped in the 5:3 and 3:1 mean motion resonances with a putative Planet Nine with semimajor axis  $\sim 700$  au.

**Key words:** methods: statistical – celestial mechanics – Kuiper belt: general – minor planets, asteroids: general – Oort Cloud – planets and satellites: general.

## 1 INTRODUCTION

In Trujillo & Sheppard (2014), the extreme trans-Neptunian objects or ETNOs are defined as asteroids with heliocentric semimajor axis greater than 150 au and perihelion greater than 30 au. There are at present 16 known ETNOs (see Tables 1 and 2 for relevant data); they exhibit clustering in the values of their argument of perihelion (Trujillo & Sheppard 2014), longitude of the ascending node (Batygin & Brown 2016), eccentricity and inclination (de la Fuente Marcos & de la Fuente Marcos 2014, 2016). The analysis in de la Fuente Marcos & de la Fuente Marcos (2014) has shown that the clustering in eccentricity ( $e$  about 0.81) can be due to a selection effect, but this cannot be the case of the ones found in inclination ( $i$  about  $20^\circ$ ), longitude of the ascending node ( $\Omega$  about  $134^\circ$ ), and argument of perihelion ( $\omega$  about  $-26^\circ$ ). These patterns could be induced by the gravitational perturbation from one (Trujillo & Sheppard 2014; Gomes, Soares & Brasser 2015; Batygin & Brown 2016) or more (de la Fuente Marcos & de la Fuente Marcos 2014; de la Fuente Marcos, de la Fuente Marcos & Aarseth 2015), yet to be discovered, trans-Plutonian planets.

Malhotra, Volk & Wang (2016) have pointed out that the four ETNOs with the longest orbital periods (see Table 2) have simple numerical relationships between periods. This implies that the ETNOs could be a highly structured, dynamically speaking, population. The ETNOs are not massive enough to induce orbit–orbit coupling by themselves, but a hypothetical planet or planets could be the source of these accidental commensurabilities. Malhotra et al.

(2016) used their analysis to predict the existence of a planet with semimajor axis  $\sim 665$  au that is only marginally compatible with the one discussed by Batygin & Brown (2016) within the framework of their Planet Nine hypothesis; even if their proposed value for the semimajor axis is similar,  $\sim 665$  au versus  $\sim 700$  au, the preferred values of  $i$  and  $\Omega$  in Malhotra et al. (2016) — ( $i \sim 18^\circ$ ,  $\Omega \sim 281^\circ$ ) or ( $i \sim 48^\circ$ ,  $\Omega \sim 355^\circ$ ) — are rather different from those in Batygin & Brown (2016) — ( $i \sim 30^\circ$ ,  $\Omega \sim 113^\circ$ ).

Accidental commensurabilities are a natural by-product of the existence of mean motion resonances in the Solar system (for the theory of orbital resonance, see e.g. Murray & Dermott 1999). The orbital architecture of the main asteroid belt is mainly the result of interior mean motion resonances with Jupiter (see e.g. Holman & Murray 1996; Nesvorný & Morbidelli 1999); conversely, the one observed in the trans-Neptunian belt is induced by exterior mean motion resonances with Neptune (see e.g. Gladman et al. 2012). In the main asteroid belt, the dominant ones are located at 2.06 au (4:1), 2.5 au (3:1), 2.82 au (5:2), 2.95 au (7:3) and 3.27 au (2:1); therefore, any asteroids trapped in these resonances are also nominally resonant among themselves even if these objects are not massive enough to induce orbit–orbit coupling by themselves. For instance, the Alinda asteroids which are subjected to the 3:1 mean motion resonance with Jupiter and the Griqua asteroids that occupy the 2:1 resonance are also in a mutual 3:2 near mean motion resonance. For objects trapped in the dominant resonances, the values of the ratios of orbital periods are

1.0699, 1.1670, 1.2172, 1.2487, 1.2818, 1.3369, 1.4959,  
1.6017, 1.7137, 2.0000.

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**Table 1.** Heliocentric semimajor axes and their uncertainties for the 16 objects discussed in this Letter. (Epoch: 2457400.5, 2016-January-13.0 00:00:00.0 TDB. J2000.0 ecliptic and equinox. Source: JPL Small-Body Database. Data retrieved on 2016 March 13.)

Object	$a$ (au)
(82158) 2001 FP <sub>185</sub>	226.344 77 ± 0.322 05
(90377) Sedna	507.560 34 ± 0.808 66
(148209) 2000 CR <sub>105</sub>	227.951 33 ± 0.538 37
(445473) 2010 VZ <sub>98</sub>	152.779 38 ± 0.151 08
2002 GB <sub>32</sub>	215.762 14 ± 0.665 75
2003 HB <sub>57</sub>	164.618 11 ± 0.569 22
2003 SS <sub>422</sub>	193.832 80 ± 48.282 00
2004 VN <sub>112</sub>	321.019 87 ± 1.083 50
2005 RH <sub>52</sub>	151.137 60 ± 0.187 08
2007 TG <sub>422</sub>	492.727 74 ± 3.600 60
2007 VJ <sub>305</sub>	188.337 26 ± 0.550 10
2010 GB <sub>174</sub>	371.118 31 ± 28.601 00
2012 VP <sub>113</sub>	259.300 17 ± 7.179 10
2013 GP <sub>136</sub>	152.496 83 ± 0.562 14
2013 RF <sub>98</sub>	309.073 83 ± 37.057 00
2015 SO <sub>20</sub>	162.703 54 ± 0.130 24

This sequence represents a characteristic sample of accidental commensurabilities in the main asteroid belt. In the trans-Neptunian belt, the dominant mean motion resonances with Neptune are located at 39.4 au (2:3), 42.3 au (3:5), 43.7 au (4:7), 47.8 au (1:2) and 55.4 au (2:5); the associated ratios of orbital periods are

1.0501, 1.1124, 1.1440, 1.1681, 1.2012, 1.2477, 1.3363, 1.4274, 1.4988, 1.6673.

They define accidental commensurabilities beyond Neptune. Out of 10 values, as many as five could be common to both sequences.

The commensurability analysis carried out in Malhotra et al. (2016) provides some support for the Planet Nine hypothesis and singles out some resonant orbital relations, but it does not explore the

subject of commensurabilities between ETNOs in depth. Here, we use the seminal idea presented in Malhotra et al. (2016) and the fact that very eccentric orbits are affected by mean motion resonances no matter the value of the semimajor axis (see e.g. Gomes et al. 2005) to investigate this subject extensively. We use their heliocentric and barycentric semimajor axes, their uncertainties, and Monte Carlo techniques to find out if the values of the ratios of orbital periods of the ETNOs match those in the main and trans-Neptunian belts. This Letter is organized as follows. Section 2 presents the data used in this study. Our Monte Carlo methodology is explained and applied in Section 3. Results are discussed in Section 4 and conclusions are summarized in Section 5.

## 2 DATA: HELIOCENTRIC VERSUS BARYCENTRIC ORBITAL ELEMENTS

Ignoring the mass of the object, expressing distances in astronomical units, times in sidereal years, and masses in Solar masses, Kepler's Third Law applied to an object orbiting the Sun states that  $P^2 = a^3$ , where  $P$  is the period and  $a$  is the semimajor axis of the orbit of the object. Given two objects – the first one with  $a_i$  and  $P_i$ , and the second one with  $a_j$  and  $P_j$ ,  $i \neq j$  – Kepler's Third Law is given by  $(P_j/P_i)^2 = (a_j/a_i)^3$ . If  $P_j/P_i$  can be written in the form of a ratio of small integers then we have a mean motion resonance. In order to investigate the distribution of ratios of orbital periods,  $P_j/P_i$ , for the ETNOs the values of the semimajor axes are needed.

The orbital solutions of the ETNOs are far less robust than those available for the best-studied members of the main asteroid belt or the trans-Neptunian belt; the values of the relative error in semimajor axis are several orders of magnitude larger in the case of the ETNOs. Any study of commensurabilities between ETNOs must account for this fact; a probabilistic approach based on the values of their uncertainties can be used to address this important issue. The values of the heliocentric semimajor axes and their uncertainties

**Table 2.** Barycentric orbital elements and parameters –  $q = a(1 - e)$ ,  $Q = a(1 + e)$ ,  $\varpi = \Omega + \omega$ ,  $P$  is the orbital period,  $\Omega^*$  and  $\omega^*$  are  $\Omega$  and  $\omega$  in the interval  $(-\pi, \pi)$  instead of the regular  $(0, 2\pi)$  – for the 16 objects discussed in this Letter. The statistical parameters are  $Q_1$ , first quartile,  $Q_3$ , third quartile, IQR, interquartile range, OL, lower outlier limit ( $Q_1 - 1.5\text{IQR}$ ), and OU, upper outlier limit ( $Q_3 + 1.5\text{IQR}$ ); see the text for additional details. (Epoch: 2457400.5, 2016-January-13.0 00:00:00.0 TDB. J2000.0 ecliptic and equinox. Source: JPL Small-Body Database. Data retrieved on 2016 March 13.)

Object	$a$ (au)	$e$	$i$ (°)	$\Omega$ (°)	$\omega$ (°)	$\varpi$ (°)	$q$ (au)	$Q$ (au)	$P$ (yr)	$\Omega^*$ (°)	$\omega^*$ (°)
82158	215.981 90	0.841 41	30.801 36	179.358 94	6.884 45	186.243 39	34.252 43	397.711 38	3172.072 25	179.358 94	6.884 45
Sedna	506.088 71	0.849 45	11.928 56	144.402 52	311.285 68	95.688 20	76.190 98	935.986 45	11 377.766 05	144.402 52	-48.714 32
148209	221.972 03	0.801 22	22.755 99	128.285 90	316.689 20	84.975 10	44.122 66	399.821 40	3304.946 30	128.285 90	-43.310 80
445473	153.362 47	0.776 02	4.510 49	117.398 68	313.725 36	71.124 04	34.350 49	272.374 44	1897.997 58	117.398 68	-46.274 64
2002 GB <sub>32</sub>	206.510 87	0.828 87	14.192 46	177.043 98	37.047 06	214.091 04	35.339 79	377.681 94	2965.728 47	177.043 98	37.047 06
2003 HB <sub>57</sub>	159.666 36	0.761 38	15.500 28	197.871 07	10.829 68	208.700 76	38.098 95	281.233 78	2016.216 50	-162.128 93	10.829 68
2003 SS <sub>422</sub>	197.898 78	0.800 78	16.785 99	151.046 96	209.928 50	0.975 47	39.424 54	356.373 02	2782.157 48	151.046 96	-150.071 50
2004 VN <sub>112</sub>	327.436 83	0.855 48	25.547 60	66.022 81	326.996 97	33.019 78	47.322 01	607.551 65	5921.180 77	66.022 81	-33.003 03
2005 RH <sub>52</sub>	153.679 65	0.746 24	20.445 81	306.110 82	32.538 36	338.649 18	38.997 09	268.362 21	1903.888 74	-53.889 18	32.538 36
2007 TG <sub>422</sub>	502.046 19	0.929 16	18.595 29	112.910 74	285.685 12	38.595 86	35.562 66	968.529 73	11 241.714 13	112.910 74	-74.314 88
2007 VJ <sub>305</sub>	192.102 99	0.816 84	11.983 75	24.382 39	338.334 83	2.717 22	35.184 69	349.021 28	2660.836 51	24.382 39	-21.665 17
2010 GB <sub>174</sub>	351.127 87	0.861 69	21.562 45	130.714 45	347.225 10	117.959 55	48.562 88	653.692 86	6575.290 94	130.714 45	-12.754 90
2012 VP <sub>113</sub>	263.165 71	0.694 36	24.051 55	90.803 92	293.549 65	24.353 57	80.435 15	445.896 27	4266.394 13	90.803 92	-66.450 35
2013 GP <sub>136</sub>	149.786 79	0.725 87	33.539 03	210.727 29	42.478 11	253.205 40	41.060 80	258.512 77	1832.007 61	-149.272 71	42.478 11
2013 RF <sub>98</sub>	317.069 11	0.885 57	29.600 62	67.533 85	316.375 20	23.909 06	36.282 42	597.855 79	5642.192 69	67.533 85	-43.624 80
2015 SO <sub>20</sub>	164.905 27	0.798 85	23.411 02	33.633 85	354.829 95	28.463 79	33.170 08	296.640 46	2116.259 00	33.633 85	-5.170 05
Mean	255.175 10	0.810 83	20.325 77	133.640 51	221.526 45	107.666 96	43.647 35	466.702 84	4354.790 57	66.140 51	-25.973 55
Std. dev.	116.486 40	0.061 05	7.724 94	71.950 64	140.157 71	102.408 44	14.310 74	226.780 35	3103.348 74	105.712 29	49.062 86
Median	211.246 39	0.809 03	21.004 13	129.500 18	302.417 67	78.049 57	38.548 02	387.696 66	3068.900 36	101.857 33	-27.334 10
$Q_1$	163.595 54	0.772 36	15.173 33	84.986 41	41.120 35	27.436 24	35.301 01	292.788 79	2091.248 38	31.320 98	-46.884 56
$Q_3$	319.661 04	0.850 96	24.425 56	177.622 72	319.266 14	191.857 73	44.922 50	600.279 76	5711.939 71	134.136 47	7.870 76
IQR	156.065 50	0.078 60	9.252 24	92.636 31	278.145 80	164.421 50	9.621 49	307.490 97	3620.691 33	102.815 48	54.755 32
OL	-70.502 70	0.654 46	1.294 97	-53.968 06	-376.098 34	-219.196 01	20.868 79	-168.447 67	-3339.788 61	-122.902 24	-129.017 53
OU	553.759 28	0.968 86	38.303 91	316.577 18	736.484 84	438.489 98	59.354 73	1061.516 22	11 142.976 70	288.359 69	90.003 73

from the Jet Propulsion Laboratory, JPL, Small-Body Database<sup>1</sup> and HORIZONS On-Line Ephemeris System (Giorgini et al. 1996) for the 16 objects discussed in this Letter are given in Table 1. Three objects – 2003 SS<sub>422</sub>, 2010 GB<sub>174</sub> and 2013 RF<sub>98</sub> – have relative uncertainties in  $a$  greater than five per cent as their orbits are still in need of further improvement. However, the heliocentric orbital elements may not be adequate for this study as the objects discussed here are too distant from the Sun. In cases like this one, the use of barycentric orbital elements is perhaps more appropriate (Todorovic-Juchnicwicz 1981; Malhotra et al. 2016).

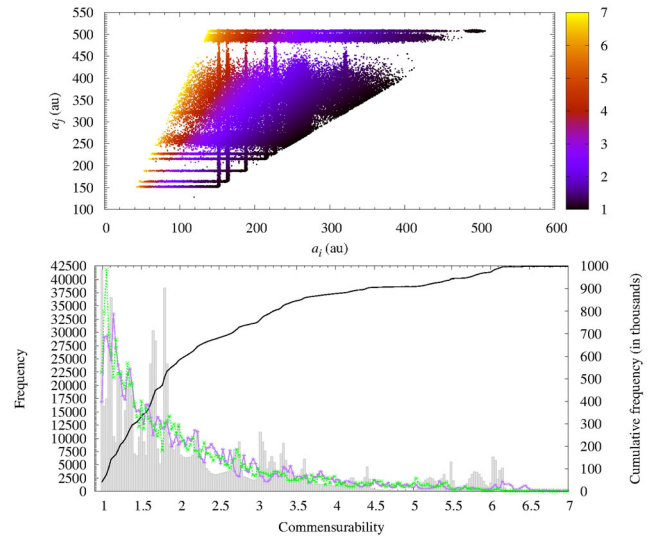
Table 2 shows the values of various orbital parameters referred to the barycentre of the Solar system for the known ETNOs as well as relevant descriptive statistics; in this table, unphysical values are displayed for completeness. The statistics is only slightly different from that of the heliocentric orbital elements. The barycentric mean value of  $e$  of the known ETNOs amounts to  $0.81 \pm 0.06$ , the one of  $i$  is  $20^\circ \pm 8^\circ$ , that of  $\Omega$  is  $134^\circ \pm 72^\circ$ , and for  $\omega$  is  $-26^\circ \pm 49^\circ$ ; these barycentric values coincide with the heliocentric ones. Considering the conventional limits to detect statistical outliers – OL, lower outlier limit ( $Q_1 - 1.5IQR$ ), and OU, upper outlier limit ( $Q_3 + 1.5IQR$ ) with  $Q_1$ , first quartile,  $Q_3$ , third quartile,  $IQR$ , interquartile range – we observe that both (90377) Sedna and 2012 VP<sub>113</sub> are statistical outliers in terms of perihelion distance,  $q$ , when barycentric orbits are considered. Sedna and 2007 TG<sub>422</sub> are also outliers in orbital period, and 2003 SS<sub>422</sub> is an outlier in terms of  $\omega$  (see the  $\omega^*$  column in Table 2).

### 3 COMMENSURABILITY MAPS

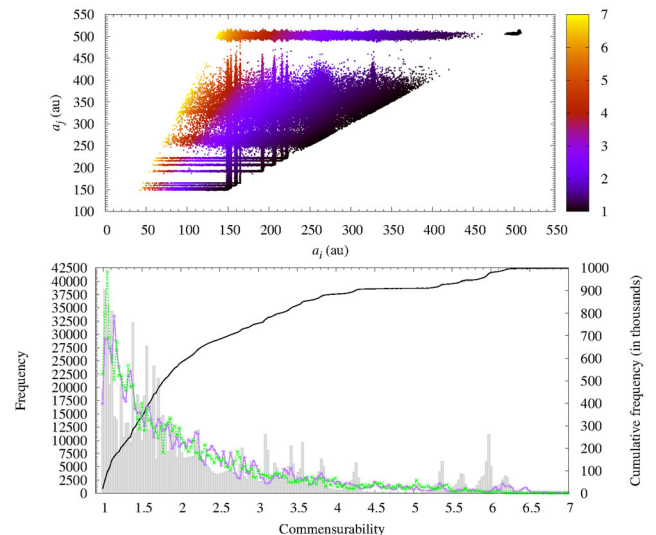
In this section we construct commensurability maps by generating pairs of virtual ETNOs from the data in Tables 1 and 2 via Monte Carlo techniques (Metropolis & Ulam 1949; Press et al. 2007) and evaluating their commensurability parameter or ratio of orbital periods,  $(a_j/a_i)^{3/2}$ , enforcing that  $a_j > a_i$  by swapping data if necessary. The value of the semimajor axis of a virtual ETNO is computed using the expression  $a_v = \langle a \rangle + \sigma_a r_i$ , where  $a_v$  is the semimajor axis of the virtual ETNO,  $\langle a \rangle$  is the mean value of the semimajor axis from the available orbit (Tables 1 and 2),  $\sigma_a$  is the standard deviation of  $a$  (Table 1), and  $r_i$  is a (pseudo) random number with normal distribution. In our calculations, the Box–Muller method (Box & Muller 1958; Press et al. 2007) was used to generate random numbers from the standard normal distribution with mean 0 and standard deviation 1. When computers are used to produce a uniform random variable (to seed the Box–Muller method) it will inevitably have some inaccuracies because there is a lower bound on how close numbers can be to 0. For a 64 bits computer the smallest non-zero number is  $2^{-64}$  which means that the Box–Muller method will not produce random variables more than 9.42 standard deviations from the mean; however, unphysical values have not been used in the calculations. The number of pairs of virtual ETNOs tested per Monte Carlo experiment is  $n = 10^6$ .

#### 3.1 Uniform and biased expectations

A random (uniform) distribution of semimajor axes should exhibit a degree of commensurability compatible with zero. We have generated ten sets of 16 values of the semimajor axis in the range that appears in Table 1 or 2 and assigned the uncertainties in Table 1 to



**Figure 1.** Commensurability map (top panel) for the values of the heliocentric semimajor axes and their uncertainties in Table 1. Frequency distribution (bottom panel) in  $(a_j/a_i)^{3/2}$  from the commensurability map. The results from a uniform spread in semimajor axis are plotted in purple and those from a biased uniform sample in green. The number of bins in the frequency distribution plot is  $2n^{1/3}$ , where  $n$  is the number of pairs of virtual ETNOs tested,  $n = 10^6$ . The black curve shows the cumulative distribution.



**Figure 2.** Same as Fig. 1 but for the barycentric values in Table 2.

them. We have obtained the frequency distribution of commensurability parameters for each set and computed the average frequency distribution. The results appear in purple in Figs 1 and 2. However, the ETNOs have been discovered from the ground and this fact induces a bias in the observed distribution of semimajor axes even if the ETNOs are not subjected to any external perturbation and just follow heliocentric, or barycentric, orbits. Following the analysis in de la Fuente Marcos & de la Fuente Marcos (2014) and, in order to reproduce this bias, we generate ten sets of 16 values of the semimajor axis according to the distribution in the top panel of fig. 3 in de la Fuente Marcos & de la Fuente Marcos (2014); i.e. objects with perigee  $< 90$  au and declination at perigee in the range  $(-24^\circ, 24^\circ)$ , see fig. 2 in de la Fuente Marcos & de la Fuente Marcos (2014). The average frequency distribution of this biased

<sup>1</sup> <http://ssd.jpl.nasa.gov/sbdb.cgi>

sample appears in green in Figs 1 and 2. The error bars have been computed assuming Poisson statistics,  $\sigma = \sqrt{n}$ , and using the approximation given by Gehrels (1986) when  $n < 21$ :  $\sigma \sim 1 + \sqrt{0.75 + n}$ , where  $n$  is the number of pairs. In both cases the results are compatible with a null degree of commensurability. Any statistically significant deviation from this null result should be interpreted as resulting from a present-day gravitational perturbation located within the region travelled by the ETNOs.

### 3.2 Heliocentric orbits

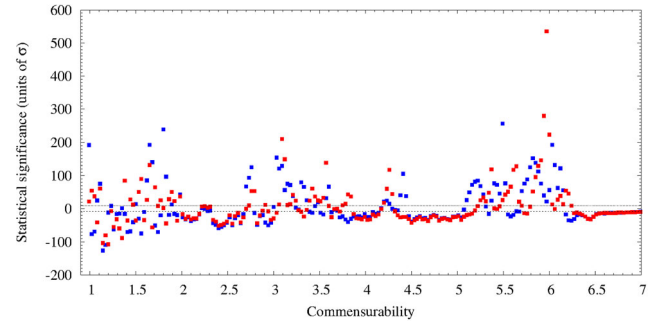
Using the data in Table 1, we created  $10^6$  pairs of virtual ETNOs following the procedure outlined above and computed the commensurability parameter. The commensurability map in terms of colour is shown in Fig. 1 (the value of the commensurability parameter or ratio of orbital periods is plotted in colour, top panel), where each point represents a pair of virtual ETNOs; the associated frequency distribution is plotted in Fig. 1 (bottom panel). There are obvious and statistically significant deviations from the uniform or biased expectations explored above. The most conspicuous commensurabilities in Fig. 1 (bottom panel) are 1.0, 1.11, 1.65 and 1.8.

### 3.3 Barycentric orbits

Using the data in Table 2, we have obtained Fig. 2. As for the heliocentric orbits, there are significant deviations from the unperturbed scenario. Considering the values of the barycentric semimajor axes, the most obvious commensurabilities in Fig. 2 are 1.02, 1.11, 1.38, 1.65 and 1.71. Three of the statistically significant commensurabilities coincide with those found in the analysis of heliocentric orbits. In addition, several of these values are also present in the sequences associated with the main and trans-Neptunian belts. This is unlikely to be the result of chance alone.

## 4 DISCUSSION

The analysis presented in the previous section shows that the frequency distribution of the ratio of orbital periods of known ETNOs is statistically incompatible with that of an unperturbed asteroid population following heliocentric/barycentric orbits. This result is unlikely to be a statistical artefact as the known ETNOs have been discovered by several independent surveys with, presumably, uncorrelated biases. If the frequency distribution of the ratio of periods is incompatible with an unperturbed scenario, how significant are the favoured commensurabilities? In order to answer this rather critical question, we have computed the difference between the frequency obtained from the observational values and that from the biased expectation (green curve in Figs 1 and 2) and divided by the value of the standard deviation in the biased case calculated as described above. Our results are plotted in Fig. 3. The number of statistically significant commensurabilities is larger than the few indicated above and found by simple visual inspection. The outcome is not dependent on the bin size; we have experimented with  $n = 10^4$  and  $n = 10^5$  and the results are fully consistent. The statistical significance analysis in Fig. 3 shows that our results are robust. Probabilistically speaking, there are simply too many ETNOs in commensurabilities compatible with the action of massive perturbers. Regarding the extremely significant value of the ratio of periods  $\sim 6$  in Fig. 3, we would like to downplay this issue as two objects have values of the semimajor axis of  $\sim 500$  au and several others have semimajor axes close to 150 au. These two values are the abrupt cutoffs in the distribution of semimajor axes.



**Figure 3.** Statistical significance in terms of  $\sigma$  of the deviations of the frequency distribution in the heliocentric (blue) and barycentric (red) cases (data from Figs 1 and 2) with respect to the biased expectations. The  $\pm 9\sigma$  limits are also indicated.

One of the most surprising properties of the sample studied is the unusually large fraction of objects with ratio of periods  $\sim 1$ . This issue was already noticed in de la Fuente Marcos & de la Fuente Marcos (2014). In particular, the orbits of 2004 VN<sub>112</sub> and 2013 RF<sub>98</sub> are alike, and several objects have values of the barycentric semimajor axis within narrow ranges of each other. This could mean that some of these objects move co-orbital, i.e. in a 1:1 mean motion resonance, with an unseen planet (or, more likely, several of them). Co-orbital motion is possible at high eccentricity (Namouni 1999; Namouni, Christou & Murray 1999; Namouni & Murray 2000). However, a more likely scenario is in the presence of higher order resonances like the 7:3 and 5:2 in the main asteroid belt that give a ratio of periods of 1.0699 or the 4:7 and 3:5 in the trans-Neptunian belt that give a ratio of 1.0501. Interior and exterior resonances are possible if there are multiple trans-Plutonian planets. Regarding the impact of this resonant scenario on the Planet Nine hypothesis, it is clearly compatible with it.

If the mechanisms responsible for inducing dynamical structure in the main and trans-Neptunian belts are also at work in the region occupied by the ETNOs then we should expect that for two given ETNOs in near commensurability:

$$\left(\frac{a_p}{a_j}\right)^{3/2} \left(\frac{a_i}{a_p}\right)^{3/2} = \left(\frac{m}{n}\right) \left(\frac{k}{l}\right), \quad (1)$$

where  $a_p$  is the semimajor axis of the orbit of the perturber and  $k, l, m, n$  are all small integers. This expression can be applied to two of the most clear clusterings in semimajor axis in Table 2. The average value of the barycentric semimajor axis of (90377) Sedna and 2007 TG<sub>422</sub> is 504 au. On the other hand, the equivalent mean value for 2004 VN<sub>112</sub>, 2010 GB<sub>174</sub> and 2013 RF<sub>98</sub> is 332 au. These five objects are part of the set of six singled out by Batygin & Brown (2016). The associated period ratio for these two sets of ETNOs is 1.87. In the main asteroid belt, this ratio is obtained for objects trapped in the 5:3 mean motion resonance with Jupiter and those in the 3:1, that is one of the main resonances in the outer belt (Holman & Murray 1996). Making a dynamical analogy between the two situations and decomposing equation (1) in two we have:  $(a_p/504)^{3/2} = 5/3$  and  $(332/a_p)^{3/2} = 1/3$ . The average of the two values of  $a_p$  is  $\sim 700$  au which is the favoured value for the semimajor axis of Planet Nine in Batygin & Brown (2016). The 1.8 commensurability has a statistical significance of  $239\sigma$  for heliocentric orbits, the 1.89 commensurability has  $51\sigma$  for barycentric orbits (see Fig. 3). This is unlikely to be mere coincidence.

Another example of the potential implications of our findings arises when we focus on 2003 HB<sub>57</sub>, 2015 SO<sub>20</sub>, 2005 RH<sub>52</sub>,

(445473) 2010 VZ<sub>98</sub> and 2013 GP<sub>136</sub>, the first two could be in a 3:2 resonance with a hypothetical planet at  $a = 213$  au, with the other three in a 5:3 resonance with the same planet. In this framework, the pair 2003 HB<sub>57</sub> and 2015 SO<sub>20</sub> would be in a 10:9 accidental resonance with the other three ETNOs. A ratio of periods  $\sim 1.1$  is present in both the main asteroid belt and the trans-Neptunian belt. The 1.1 commensurability has a statistical significance of  $76\sigma$  for heliocentric orbits and  $61\sigma$  for barycentric orbits (see Fig. 3). On the other hand, the 1.65 commensurability is present for both heliocentric ( $192\sigma$ ) and barycentric ( $131\sigma$ ) orbits; a similar analysis focusing on 2003 HB<sub>57</sub>, 2013 GP<sub>136</sub>, (82158) 2001 FP<sub>185</sub> and 2002 GB<sub>32</sub> is compatible with a hypothetical planet at  $a = 329$  au considering resonances  $(3/1)(5/9)=5/3\sim 1.66$ . With the currently available data, degenerate solutions are possible, but they still hint at a multi-planet scenario. Trans-Plutonian planets may have been scattered out of the region of the giant planets early in the history of the Solar system (see e.g. Bromley & Kenyon 2014, 2016), but planets similar to Uranus or Neptune may also form at 125–750 au from the Sun (Kenyon & Bromley 2015, 2016).

## 5 CONCLUSIONS

In this Letter, we have explored the existence of commensurabilities between the known ETNOs. This analysis has been inspired by the hypothesis and discussion presented in Malhotra et al. (2016). Given the fact that these objects are not massive enough to induce orbit–orbit coupling by themselves, the existence of statistically significant commensurabilities could only signal the presence of massive unseen perturbers. Summarizing, our conclusions are as following.

(i) The clustering in orbital parameter space observed for the heliocentric orbits of the known ETNOs is also present when considering their barycentric orbits. In particular,  $e$  clumps about  $0.81 \pm 0.06$ ,  $i$  about  $20^\circ \pm 8^\circ$ ,  $\Omega$  about  $134^\circ \pm 72^\circ$ , and  $\omega$  about  $-26^\circ \pm 49^\circ$ . Statistical outliers do exist.

(ii) The frequency distribution of the ratio of orbital periods of known ETNOs is statistically incompatible with that of an unperturbed asteroid population following heliocentric/barycentric orbits. In contrast, it resembles the ones present in the main and trans-Neptunian belts. The existence of statistically significant accidental commensurabilities between ETNOs strongly suggests that external perturbers induce the observed dynamical structure.

(iii) The fraction of known ETNOs in 1:1 commensurabilities is dozens of times the expected value for a random population.

(iv) A number of known ETNOs may be trapped in the 5:3 and 3:1 mean motion resonances with a putative Planet Nine with semimajor axis  $\sim 700$  au.

We must stress that our results are based on small number statistics, but it is also true that the deviations from what is expected for an unperturbed asteroid population following heliocentric/barycentric

orbits are so strong that what is observed is unlikely to be the result of chance alone. Gravitational perturbations with sources in the region travelled by the ETNOs are the most probable explanation for the observed patterns.

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